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# UNIVERSITÀ DEGLI STUDI DI TORINO 

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# Finite-part integrals over polygons by an 8-node quadrilateral spline finite element 

Chong-Jun Li • Vittoria Demichelis • Catterina Dagnino


#### Abstract

In this paper we consider the numerical integration on a polygonal domain $\Omega$ in $\mathbb{R}^{2}$ of a function $F(x, y)$ which is integrable except at a point $P_{0}=\left(x_{0}, y_{0}\right) \in$ $\stackrel{\circ}{\Omega}$, where $F$ becomes infinite of order two. We approximate either the finite-part or the two-dimensional Cauchy principal value of the integral by using a spline finite element method combined with a subdivision technique also of adaptive type. We prove the convergence of the obtained sequence of cubatures. Finally, to illustrate the behaviour of the proposed method, we present some numerical examples.


Keywords Finite part integral • Cauchy principal value • Spline finite element method • Bivariate splines

Mathematics Subject Classification (2000) 65D05 65D07 • 65D30 • 65D32

## 1 Introduction

This paper deals with the numerical evaluation of certain hypersingular integrals on a polygonal domain $\Omega$ in $\mathbb{R}^{2}$, i.e. a domain with the boundary composed of piecewise straight lines.

[^0][^1]The above integrals arise in several engineering problems and in particular in applied mechanics ([8] and references therein).

Firstly, we consider the integration of a function $F(x, y)$ on a convex domain $D$, with $F$ integrable except at a point $P_{0}=\left(x_{0}, y_{0}\right)$ where it becomes infinite of order two. We set $P_{0}$ as the origin of polar coordinates $(r, \theta)$, with $r=\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}$ and denote by $r=R(\theta)$ the polar equation of the contour of $D$. Following Tricomi's presentation [10], we suppose

$$
\begin{equation*}
F(x, y)=\frac{\phi(\theta)}{r^{2}}+F_{1}(x, y) \tag{1.1}
\end{equation*}
$$

where $F_{1}$ denotes a function at most infinite of (algebraic) order less than two at isolated points.

We set

$$
\begin{equation*}
f_{D} F(x, y) d x d y:=\int_{D} F_{1}(x, y) d x d y+\int_{0}^{2 \pi} \phi(\theta) \log R(\theta) d \theta \tag{1.2}
\end{equation*}
$$

If $\int_{0}^{2 \pi} \phi(\theta) d \theta=0$ then (1.2) defines the two-dimensional Cauchy principal value integral [10]; otherwise, it defines the Hadamard finite-part [7].

The numerical evaluation of (1.2), with

$$
\begin{equation*}
F(x, y)=\frac{f(x, y)}{r^{2}} \tag{1.3}
\end{equation*}
$$

and $D$ a triangle, a rectangle or, more in general, a convex polygon, has been studied in literature.

A classical approach consists in the decomposition of $D$ in triangles, each one with the singular point $P_{0}$ at one vertex. Then, the integral on each triangle is evaluated by a cubature rule of sufficiently high degree of exactness.

Since $P_{0}$ is the origin of polar coordinates, $f(x, y)$ can be expressed by

$$
\begin{equation*}
f(x, y)=f\left(x_{0}+r \cos \theta, y_{0}+r \sin \theta\right)=\bar{f}(r, \theta) \tag{1.4}
\end{equation*}
$$

Cubatures based on product of univariate Gaussian rules on each triangle of the decomposition of $D$, are proposed in [6]. Convergence results are proved for $\bar{f}(r, \theta) \in$ $H_{\mu, \mu}^{s}, s \geq 1,0<\mu \leq 1$, where $H_{\mu, \mu}^{s}$ denotes the space of functions with continuous partial derivatives up to the order $s$ and such that each partial derivative of order $s$ satisfies a Hölder condition of order $\mu$, i.e.

$$
\begin{equation*}
\left|\bar{f}^{(s)}(\bar{r}, \bar{\theta})-\bar{f}^{(s)}(r, \theta)\right| \leq C\left(|\bar{r}-r|^{\mu}+|\bar{\theta}-\theta|^{\mu}\right), \quad C>0 . \tag{1.5}
\end{equation*}
$$

Approximations for (1.2), based on local bivariate quadratic quasi-interpolating splines on each triangle of the decomposition of $D$, are studied in [1], where convergence results are proved for $\bar{f}(r, \theta)$ satisfying the condition (1.5), with $s=0$.

Gaussian rules and bivariate splines are combined in [11] for evaluating

$$
f_{D} \frac{g\left(P_{0}, \theta\right)}{r^{p}} \Phi(P) d P, \quad p=2,3
$$

where $D$ is a rectangle. Convergence results are proved assuming that $g\left(P_{0}, \theta\right)$ is continuous and $\Phi(P) \in \mathbb{C}^{p+2}(D)$.

In this paper, we consider the problem of the numerical evaluation of

$$
\begin{equation*}
I=f_{\Omega} F(x, y) d x d y, \quad P_{0} \in \stackrel{\circ}{\Omega}, \tag{1.6}
\end{equation*}
$$

where $\Omega$ is a not necessarily convex polygonal domain and $F$ is defined by (1.3) with $f(x, y) \in \mathbb{C}\left(\Omega \backslash P_{0}\right)$.

We propose a method based on a special spline quadrilateral finite element ([3, 12]), reproducing all bivariate polynomials of total degree at most two, and applied by a subdivision technique also of adaptive kind.

We denote by $D$ a convex subset of $\Omega$, such that the singular point $P_{0} \in \stackrel{\circ}{D}$. If $\bar{f}$, given by (1.4), satisfies the following condition on $D$

$$
\begin{equation*}
|\bar{f}(r, \theta)-\bar{f}(0, \theta)| \leq A(\theta) r^{v}, \quad 0<v \leq 1, \quad 0 \leq A(\theta)<\infty \tag{1.7}
\end{equation*}
$$

then we can express $F$ on $D$ in the form (1.1), with

$$
\begin{equation*}
\phi(\theta)=\bar{f}(0, \theta) \text { and } F_{1}(x, y)=\frac{\bar{f}(r, \theta)-\bar{f}(0, \theta)}{r^{2}} \tag{1.8}
\end{equation*}
$$

We set $I=I_{1}+I_{2}$, where $I_{1}=f_{D} F(x, y) d x d y$ is defined by (1.2), with $F_{1}$ and $\phi$ given in (1.8), and $I_{2}=\int_{\Omega \backslash D} F(x, y) d x d y$, with $F$ defined by (1.3), is a regular integral. We evaluate $I_{2}$ by the cubature formula over polygons, having degree of exactness 2 [5] and based on the spline finite element method presented in [3], combined with the subdivision technique also of adaptive kind ([4]). Then, we apply the same finite element method to evaluate $I_{1}$.

For $\bar{f}$ satisfying the condition (1.7), we prove the convergence to $I$ of the obtained cubature sequence. Such condition is weaker than conditions on $\bar{f}$ required by previous methods in [1,6,11].

The above performances make our composite strategy suitable when $\bar{f}(r, \theta)$ and $f(x, y)$ are not smooth in $D$ and $\Omega \backslash D$, respectively, for example with singularities of the gradient. Moreover, the adaptive subdivision technique concentrates nodes in regions of difficulty, reducing the computational cost with respect to a non adaptive one. However, when $\Omega$ is a convex polygon and $\bar{f}(r, \theta)$ is a smooth function, our composite strategy cannot be competitive with more classical ones, based on high degree polynomial approximation at Gaussian.

The integration nodes of our method lie on the boundary of each quadrilateral element and, in the subdivision procedure, they are shared by several elements, i.e. the subdivision procedure generates an embedded sequence of integration nodes. This feature makes the computational cost of our method lower than that one of another similar composite strategy, using a basic (polynomial) interpolatory type cubature, whose interior nodes are not kept in the procedure of subdivision, as remarked in [5].

The paper is organized as follows. In Section 2 we review some results related to the spline finite element method and to the cubature for regular integrals based
on it. In Section 3 we present the cubature for (1.6), we analyse the convergence and we discuss its computational cost. Finally, Section 4 proposes some numerical examples illustrating the behaviour of our cubatures. Some structural characteristics of our algorithm and a possible extension to other cases of singular integrals are also discussed.

## 2 L8 spline operator and cubature

In this section, we review some results on the interpolation operator and cubature by L8 element basis presented in [3] and [5].


Fig. 2.1 The 8 nodes and 13 domain points on a quadrilateral element.

For a convex quadrilateral element $Q$, divided by four triangles $\Delta_{1}, \ldots, \Delta_{4}$ as shown in Fig. 2.1(a), we consider the eight nodes $V_{1}, \ldots, V_{8}$, where

$$
V_{5}=\left(V_{1}+V_{2}\right) / 2, V_{6}=\left(V_{2}+V_{3}\right) / 2, V_{7}=\left(V_{3}+V_{4}\right) / 2, V_{8}=\left(V_{4}+V_{1}\right) / 2,
$$

and the quadratic spline space defined on the quadrangle $Q$, with $C^{1}$ smoothness on both diagonals $\overline{V_{1} V_{3}}$ and $\overline{V_{2} V_{4}}$. The dimension of the spline space is eight, as given in [3].

It is well known ([2]) that a polynomial $p$ of total degree two on a triangle $\Delta$ can be represented in the local Bernstein basis as

$$
p(\lambda)=\sum_{|\alpha|=2} \gamma_{\alpha} b_{\alpha}(\lambda)
$$

where $b_{\alpha}(\lambda)=\frac{2}{\alpha!} \lambda^{\alpha}, \lambda=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ are the barycentric coordinates of $\Delta, \alpha=$ $\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right),|\alpha|=\alpha_{1}+\alpha_{2}+\alpha_{3}=2, \alpha!=\alpha_{1}!\alpha_{2}!\alpha_{3}!$ and $\lambda^{\alpha}=\lambda_{1}^{\alpha_{1}} \lambda_{2}^{\alpha_{2}} \lambda_{3}^{\alpha_{3}}$. The $\gamma_{\alpha}$ are called Bézier ordinates or Bézier coefficients of $p$. The points with barycentric coordinates $\alpha / 2$ are called domain points in $\Delta$. The piecewise linear interpolant to the points $\left(\alpha / 2, \gamma_{\alpha}\right)$ is called Bézier net or B-net or control net of $p$. Such a B-net uniquely defines the patch, a fact which is made use of in the so called BernsteinBézier technique, where all information about the patch is extracted from this net.

Then, by the B-net method, there are thirteen domain points in the quadrilateral element, as shown in Fig. 2.1(b). We denote by L8 basis the eight quadratic nodal
basis splines $L_{i}(i=1, \ldots, 8)$ interpolating the eight nodes $V_{1}, \ldots, V_{8}$, ([3]). The Bézier coefficients of each base $L_{i}(i=1, \ldots, 8)$, according to the thirteen domain points, are given in the following matrix

$$
\begin{aligned}
& \left(L_{1} L_{2} L_{3} L_{4} L_{5} L_{6} L_{7} L_{8}\right)^{T} \\
& =\left(\begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{b}{2} & 0 & -\frac{b}{2} & -\frac{b}{2} \\
0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{a}{2} & -\frac{1}{2} & -\frac{a}{2} & 0 & -\frac{a}{2} \\
0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{d}{2} & -\frac{1}{2} & -\frac{d}{2} & -\frac{d}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{c}{2} & 0 & -\frac{c}{2} & -\frac{1}{2} & -\frac{c}{2} \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 a & 2 b & 0 & 0 & 2 a b \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 d & 2 a & 0 & 2 a d \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 c & 2 d & 2 c d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 c & 0 & 0 & 2 b & 2 b c
\end{array}\right)
\end{aligned}
$$

where $a, b, c, d$ are defined by the following ratios:

$$
\begin{equation*}
a=\frac{\left|\overline{V_{4} V_{0}}\right|}{\left|\overline{V_{4} V_{2}}\right|}, b=\frac{\left|\overline{V_{3} V_{0}}\right|}{\left|\overline{V_{3} V_{1}}\right|}, c=1-a, d=1-b . \tag{2.1}
\end{equation*}
$$

The spline basis consists of piecewise quadratic polynomials $\mathbb{C}^{1}$ continuous on the two diagonals and $\mathbb{C}^{0}$ on the boundary of $Q$. By the barycentric coordinates in each triangle, it is easy to obtain the piecewise polynomial form $L_{i}(x, y), i=1, \ldots, 8$ in Cartesian coordinates by the above B-net form.

The Fig. 2.2 shows the representations in Cartesian coordinates of the L8 basis on $[-1,1]^{2}$.

The interpolation operator by the L8 basis on $Q$ is defined by ([3])

$$
\begin{equation*}
L_{Q} f(x, y)=\sum_{i=1}^{8} f\left(V_{i}^{Q}\right) L_{i}^{Q}(x, y) \tag{2.2}
\end{equation*}
$$

where $V_{i}^{Q}$ and $L_{i}^{Q}(x, y)(i=1,2, \ldots, 8)$ are the eight nodes $V_{i}$ and basis splines $L_{i}(x, y)$ defined on $Q$ specifically. $L_{Q} f(x, y)$ interpolates $f$ at the eight nodes and reproduces polynomials of degree two ([3]), i.e.,

$$
L_{Q} f\left(V_{i}^{Q}\right)=f\left(V_{i}^{Q}\right), i=1, \ldots, 8
$$

and

$$
L_{Q} f=f, \forall f \in \mathbb{P}_{2}
$$

Since $L_{Q} f \in \mathbb{C}^{1}(Q)$, it is easy to obtain its partial derivatives. In particular, when $Q=[-1,1]^{2}$, then $V_{0}=(0,0), V_{1}=(-1,-1), V_{2}=(1,-1), V_{3}=(1,1), V_{4}=$ $(-1,1)$. Let $f_{i}=f\left(V_{i}\right), i=1, \ldots, 8$, then $L_{[-1,1]^{2}} f(x, y)=\sum_{i=1}^{8} f_{i} L_{i}(x, y)$, where $L_{i}(x, y)$ are shown in Fig. 2.2 in piecewise polynomial form. Differentiating the polynomials in Fig. 2.2 at $V_{i}(i=0,1, \ldots, 4)$, we obtain the values of the first partial derivatives of $L_{[-1,1]^{2}} f$ at $V_{0}, V_{1}, \ldots, V_{4}$, as shown in Table 2.1.

Then, for a polygonal domain $\Omega$, divided by $N$ convex quadrilateral elements $Q_{k}(k=1, \ldots, N)$, the L 8 cubature on $\Omega$ is defined by ([5])

$$
\tilde{I}_{\Omega} f=\sum_{k=1}^{N} \tilde{I}_{Q_{k}} f
$$

$\Delta_{1}$
$\frac{1}{4}\left(-1+x+2 x^{2}+y+3 x y+2 y^{2}\right)$
$\frac{1}{4}\left(-1-x+2 x^{2}+y-3 x y+2 y^{2}\right)$
$-\frac{1}{4}(1+x)(1+y)$
$\frac{1}{4}(-1+x)(1+y)$
$\frac{1}{2}\left(1-2 x^{2}-2 y-y^{2}\right)$
$\frac{1}{2}(1+2 x-y)(1+y)$
$\frac{1}{2}(1+y)^{2}$
$-\frac{1}{2}(1+y)(-1+2 x+y)$
$\Delta_{3}$
$-\frac{1}{4}(-1+x)(-1+y)$
$\frac{1}{4}(1+x)(-1+y)$
$\frac{1}{4}\left(-1-x+2 x^{2}-y+3 x y+2 y^{2}\right)$
$\frac{1}{4}\left(-1+x+2 x^{2}-y-3 x y+2 y^{2}\right)$
$\frac{1}{2}(-1+y)^{2}$
$-\frac{1}{2}(-1+y)(1+2 x+y)$
$\frac{1}{2}\left(1-2 x^{2}+2 y-y^{2}\right)$
$-\frac{1}{2}(-1+y)(1-2 x+y)$

$$
\begin{aligned}
& \Delta_{z} \\
& -\frac{1}{4}(-1+x)(-1+y) \\
& \frac{1}{4}\left(-1-x+2 x^{2}+y-3 x y+2 y^{2}\right) \\
& \frac{1}{4}\left(-1-x+2 x^{2}-y+3 x y+2 y^{2}\right) \\
& \frac{1}{4}(-1+x)(1+y) \\
& -\frac{1}{2}(-1+x)(1+x-2 y) \\
& \frac{1}{2}\left(1+2 x-x^{2}-2 y^{2}\right) \\
& -\frac{1}{2}(-1+x)(1+x+2 y) \\
& \frac{1}{2}(-1+x)^{2} \\
& \frac{\Delta 4}{4} \\
& \frac{1}{4}\left(-1+x+2 x^{2}+y+3 x y+2 y^{2}\right) \\
& \frac{1}{4}(1+x)(-1+y) \\
& -\frac{1}{4}(1+x)(1+y) \\
& \frac{1}{4}\left(-1+x+2 x^{2}-y-3 x y+2 y^{2}\right) \\
& -\frac{1}{2}(1+x)(-1+x+2 y) \\
& \frac{1}{2}(1+x)^{2} \\
& -\frac{1}{2}(1+x)(-1+x-2 y) \\
& \frac{1}{2}\left(1-2 x-x^{2}-2 y^{2}\right)
\end{aligned}
$$

Fig. 2.2 The piecewise polynomials of L 8 basis on each triangles $\Delta_{1}, \ldots, \Delta_{4}$ in $[-1,1]^{2}$.

Table 2.1 The partial derivatives of $L_{[-1,1]^{2}} f$ at $V_{0}, V_{1}, \ldots, V_{4}$.

|  | $V_{0}$ | $V_{1}$ | $V_{2}$ | $V_{3}$ | $V_{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{\partial}{\partial x}$ | $\frac{f_{1}-f_{2}-f_{3}+f_{4}+4 f_{6}-4 f_{8}}{4}$ | $\frac{4 f_{5}-3 f_{1}-f_{2}}{2}$ | $\frac{f_{1}+3 f_{2}-4 f_{5}}{2}$ | $\frac{3 f_{3}+f_{4}-4 f_{7}}{2}$ | $\frac{4 f_{7}-f_{3}-3 f_{4}}{2}$ |
| $\frac{\partial}{\partial y}$ | $\frac{f_{1}+f_{2}-f_{3}-f_{4}-4 f_{5}+4 f_{7}}{4}$ | $\frac{4 f_{8}-3 f_{1}-f_{4}}{2}$ | $\frac{4 f_{6}-3 f_{2}-f_{3}}{2}$ | $\frac{f_{2}+3 f_{3}-4 f_{6}}{2}$ | $\frac{f_{1}+3 f_{4}-4 f_{8}}{2}$ |

where

$$
\begin{equation*}
\tilde{I}_{Q_{k}} f=\int_{Q_{k}} L_{Q_{k}} f(x, y) d x d y=\sum_{i=1}^{8} C_{i}^{Q_{k}} f\left(V_{i}^{Q_{k}}\right) \tag{2.3}
\end{equation*}
$$

The coefficients $C_{i}^{Q}=\int_{Q} L_{i}^{Q}(x, y) d x d y$, with $Q=Q_{k}(k=1, \ldots, N)$, can be computed as follows:

$$
\begin{align*}
& C_{1}^{Q}=-\frac{1}{6} b\left(S_{1}+S_{2}+S_{3}+S_{4}\right), \\
& C_{2}^{Q}=-\frac{1}{6} a\left(S_{1}+S_{2}+S_{3}+S_{4}\right), \\
& C_{3}^{Q}=-\frac{1}{6} d\left(S_{1}+S_{2}+S_{3}+S_{4}\right), \\
& C_{4}^{Q}=-\frac{1}{6} c\left(S_{1}+S_{2}+S_{3}+S_{4}\right),  \tag{2.4}\\
& C_{5}^{Q}=\frac{1}{3}\left((1+a+b+a b) S_{1}+(b+a b) S_{2}+a b S_{3}+(a+a b) S_{4}\right), \\
& C_{6}^{Q}=\frac{1}{3}\left((d+a d) S_{1}+(1+a+d+a d) S_{2}+(a+a d) S_{3}+a d S_{4}\right), \\
& C_{7}^{Q}=\frac{1}{3}\left(c d S_{1}+(c+c d) S_{2}+(1+c+d+c d) S_{3}+(d+c d) S_{4}\right), \\
& C_{8}^{Q}=\frac{1}{3}\left((c+b c) S_{1}+b c S_{2}+(b+b c) S_{3}+(1+b+c+b c) S_{4}\right),
\end{align*}
$$

where $S_{1}, \ldots, S_{4}$ are the areas of the four triangles $\Delta_{1}, \ldots, \Delta_{4}$ in $Q=Q_{k}$ and $a, b, c, d$ are defined in (2.1).

It is clear that the formula (2.3) and its coefficients (2.4) only depend on the four vertices $V_{i}^{Q}, i=1, \ldots, 4$. In particular, if $Q$ is a rectangle or a parallelogram with area $S_{Q}$ then ([5])

$$
\begin{align*}
& a=b=c=d=\frac{1}{2}, \quad S_{1}=S_{2}=S_{3}=S_{4}=\frac{S_{Q}}{4} \quad \text { and } \\
& C_{1}^{Q}=C_{2}^{Q}=C_{3}^{Q}=C_{4}^{Q}=-\frac{1}{12} S_{Q}, \quad C_{5}^{Q}=C_{6}^{Q}=C_{7}^{Q}=C_{8}^{Q}=\frac{1}{3} S_{Q} \tag{2.5}
\end{align*}
$$

The cubature is exact for quadratic polynomials on arbitrary convex quadrangulations, and for cubic polynomials on rectangulations ([5]).

## 3 The algorithm for finite-part integral evaluation on a polygonal domain

We consider integrals of the form (1.6) for $F(x, y)$ defined by (1.3), with $f(x, y) \in$ $\mathbb{C}\left(\Omega \backslash P_{0}\right)$. In order to evaluate (1.6), we use the following subdivision strategy.

1) The whole polygonal domain $\Omega$ is divided into several initial quadrilateral elements, where the singular point $P_{0}$ is at the center of a square element $D_{0}$, as shown in Fig. 3.1(a).
2) In subdivision, the square $D_{0}$ is divided into one small square $D_{1}$ and four symmetric trapezoidal elements, as shown in Fig. 3.1(b) and so on, step by step, obtaining the squares $D_{k} \subset D_{k-1} \subset \cdots \subset D_{0}$.
3) Each quadrilateral element, except the square $D_{k}$, is divided into two or four quadrilateral elements by equal subdivision or adaptive subdivision [4].


Denote by $D_{0}=P_{0}+\left[-h_{0}, h_{0}\right]^{2}$ the square with edge $2 h_{0}$ and $P_{0}=\left(x_{0}, y_{0}\right)$ as center, where $h_{0}$ is a positive constant such that $D_{0} \subset \Omega$. Then the contour of $D_{0}$ is

$$
R_{0}(\theta)=\left\{\begin{array}{l}
\frac{-h_{0}}{\sin \theta}, \theta \in\left[-\frac{3 \pi}{4},-\frac{\pi}{4}\right]  \tag{3.1}\\
\frac{h_{0}}{\cos \theta}, \theta \in\left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\
\frac{h_{0}}{\sin \theta}, \theta \in\left[\frac{\pi}{4}, \frac{3 \pi}{4}\right] \\
\frac{h_{0}}{\cos \theta}, \theta \in\left[\frac{3 \pi}{4}, \frac{5 \pi}{4}\right]
\end{array}\right.
$$

Set $D_{k}=P_{0}+\left[-h_{k}, h_{k}\right]^{2}$, where $h_{k}=h_{0} / 2^{k}, k=1,2, \ldots$. Denote by $R_{k}(\theta)$ the contour of $D_{k}$, obtained by replacing $h_{0}$ by $h_{k}$ in (3.1).

The above subdivision includes one square $D_{k}$ and other quadrilateral elements in the quadrangulation of $\Omega \backslash D_{k}=\left\{\Omega \backslash D_{0}\right\} \cup\left\{D_{0} \backslash D_{k}\right\}$. Denote by $\delta$ the length of the longest diagonal or edge in all elements $Q \subset \Omega \backslash D_{k}$. Then the subdivision procedure can be described by considering $k \rightarrow \infty$ and $\delta \rightarrow 0$.

The integral (1.6) can be expressed as follows

$$
I=f_{\Omega} \frac{f(x, y)}{r^{2}} d x d y=I_{1}+I_{2}
$$

where

$$
I_{1}=f_{D_{0}} \frac{f(x, y)}{r^{2}} d x d y
$$

and

$$
I_{2}=\int_{\Omega \backslash D_{0}} \frac{f(x, y)}{r^{2}} d x d y
$$

At the $k$ th step, for $k=1,2, \ldots$, with reference to $I_{1}$, using (1.2) and (1.8), we write

$$
\begin{aligned}
I_{1} & =\int_{D_{0}} F_{1}(x, y) d x d y+\int_{0}^{2 \pi} \bar{f}(0, \theta) \log R_{0}(\theta) d \theta \\
& =\int_{0}^{2 \pi} \int_{0}^{R_{k}(\theta)} \frac{\bar{f}(r, \theta)-\bar{f}(0, \theta)}{r} d r d \theta+\int_{D_{0} \backslash D_{k}} F_{1}(x, y) d x d y+\int_{0}^{2 \pi} \bar{f}(0, \theta) \log R_{0}(\theta) d \theta \\
& =I_{1,1}^{(k)}+I_{1,2}^{(k)}+I_{1,3},
\end{aligned}
$$

where $I_{1,3}$ can be computed exactly by

$$
\begin{align*}
I_{1,3} & =\int_{0}^{2 \pi} \bar{f}(0, \theta) \log R_{0}(\theta) d \theta \\
& =\log h_{0} \int_{0}^{2 \pi} \bar{f}(0, \theta) d \theta-\int_{-\pi / 4}^{\pi / 4} \sum_{i=-1}^{2} \bar{f}\left(0, \theta+\frac{i \pi}{2}\right) \log \cos \theta d \theta \tag{3.2}
\end{align*}
$$

since $R_{0}(\theta)$ is fixed and given in Eq. (3.1).
Integrals $I_{1,1}^{(k)}, I_{1,2}^{(k)}$ and $I_{2}$ can be computed numerically by the spline interpolation operator $L_{Q}$ defined in (2.2) on each quadrilateral element $Q$ of the quadrangulation on $\Omega \backslash D_{k}$, as follows.

In order to evaluate $I_{1,1}^{(k)}$, we approximate $\bar{f}(r, \theta)$ on the square $D_{k}$ by

$$
\begin{equation*}
L_{D_{k}} \bar{f}(r, \theta)=\sum_{i=1}^{8} f\left(V_{i}^{D_{k}}\right) \bar{L}_{i}^{D_{k}}(r, \theta) \tag{3.3}
\end{equation*}
$$

where $V_{i}^{D_{k}}(i=1, \ldots, 8)$ are the eight nodes on $D_{k}$ and $\bar{L}_{i}^{D_{k}}(r, \theta)(i=1, \ldots, 8)$ are the eight basis splines defined on $D_{k}$, by replacing $x$ and $y$ by $\left(x-x_{0}\right) / h_{k}=r \cos \theta / h_{k}$ and $\left(y-y_{0}\right) / h_{k}=r \sin \theta / h_{k}$ in the representations in Fig. 2.2, respectively.

Therefore

$$
\begin{equation*}
I_{1,1}^{(k)} \simeq \tilde{I}_{1,1}^{(k)}=\int_{0}^{2 \pi} \int_{0}^{R_{k}(\theta)} \frac{L_{D_{k}} \bar{f}(r, \theta)-L_{D_{k}} \bar{f}(0, \theta)}{r} d r d \theta=\sum_{i=1}^{8} \hat{C}_{i}^{(k)} f\left(V_{i}^{D_{k}}\right) \tag{3.4}
\end{equation*}
$$

where the cubature coefficients are

$$
\begin{equation*}
\hat{C}_{i}^{(k)}=\int_{0}^{2 \pi} \int_{0}^{R_{k}(\theta)} \frac{\bar{L}_{i}^{D_{k}}(r, \theta)-\bar{L}_{i}^{D_{k}}(0, \theta)}{r} d r d \theta \tag{3.5}
\end{equation*}
$$

By some algebra, it is easy to obtain

$$
\begin{equation*}
\left\{\hat{C}_{i}^{(k)}\right\}_{i=1, \ldots, 8}=\{1,1,1,1,-1,-1,-1,-1\} \tag{3.6}
\end{equation*}
$$

Hence

$$
I_{1,1}^{(k)} \simeq \tilde{I}_{1,1}^{(k)}=\sum_{i=1}^{4} f\left(V_{i}^{D_{k}}\right)-\sum_{i=5}^{8} f\left(V_{i}^{D_{k}}\right)
$$

In order to evaluate $I_{1,2}^{(k)}$, denoting by $N_{k}$ the number of quadrilateral elements in the subdivision of $D_{0} \backslash D_{k}$, we write

$$
\begin{align*}
I_{1,2}^{(k)} \simeq \tilde{I}_{1,2}^{(k, \delta)} & =\sum_{j=1}^{N_{k}} \int_{Q_{j}} L_{Q_{j}} F_{1}(x, y) d x d y \\
& =\sum_{j=1}^{N_{k}} \sum_{i=1}^{8} C_{i}^{Q_{j}} F_{1}\left(V_{i}^{Q_{j}}\right) \tag{3.7}
\end{align*}
$$

where $C_{i}^{Q_{j}}$ are the cubature coefficients (2.4) on each element $Q=Q_{j} \subset D_{0} \backslash D_{k}$.
Finally, denoting by $M$ the number of quadrilateral elements in $\Omega \backslash D_{0}$,

$$
\begin{align*}
I_{2} \simeq \tilde{I}_{2}^{(\delta)} & =\sum_{j=1}^{M} \int_{Q_{j}} L_{Q_{j}} F(x, y) d x d y \\
& =\sum_{j=1}^{M} \sum_{i=1}^{8} C_{i}^{Q_{j}} F\left(V_{i}^{Q_{j}}\right) \tag{3.8}
\end{align*}
$$

where $C_{i}^{Q_{j}}$ are the cubature coefficients (2.4) on each element $Q=Q_{j} \subset \Omega \backslash D_{0}$.
In conclusion,

$$
\begin{equation*}
I \simeq \tilde{I}^{(k, \delta)}=\tilde{I}_{1,1}^{(k)}+\tilde{I}_{1,2}^{(k, \delta)}+\tilde{I}_{2}^{(\delta)}+I_{1,3} . \tag{3.9}
\end{equation*}
$$

In order to study the convergence of proposed cubatures, we need the following lemma.

Lemma 3.1 For any $P_{0}$ as the origin of polar coordinates, if (1.7) holds on $D_{0}$, then for $k=1,2, \ldots$

$$
\left|L_{D_{k}} \bar{f}(r, \theta)-L_{D_{k}} \bar{f}(0, \theta)\right| \leq 8 A(\theta) r^{v}
$$

Proof By the definition of $D_{k}$ and the operator $L_{D_{k}}$ in (3.3), $L_{D_{k}} \bar{f}(r, \theta)=L_{D_{k}} f(P) \in$ $\mathbb{C}^{1}\left(D_{k}\right)$ and $L_{D_{k}} f(P)=L_{[-1,1]^{2}} f\left(\frac{P-P_{0}}{h_{k}}\right)$ is a piecewise quadratic polynomial on $D_{k}$. Note that only the values of $f$ at the nodes $V_{i}^{D_{k}}(i=1, \ldots, 8)$ on the boundary of $D_{K}$ are used in $L_{D_{k}} f(P)$. Then, by Table 2.1 and assumption conditions we have

$$
\begin{align*}
\left\{\left|\frac{\partial L_{D_{k}} f(P)}{\partial x}\right|,\left|\frac{\partial L_{D_{k}} f(P)}{\partial y}\right|\right\} & \leq \frac{4}{h_{k}} \max _{i=1, \ldots, 8}\left|f\left(V_{i}^{D_{k}}\right)-\bar{f}(0, \theta)\right| \\
& \leq \frac{4}{h_{k}} \max _{(r, \theta) \in D_{k}}|\bar{f}(r, \theta)-\bar{f}(0, \theta)| \\
& \leq \frac{4}{h_{k}} A(\theta)\left(\sqrt{2} h_{k}\right)^{v} . \tag{3.10}
\end{align*}
$$

Note that $r=\left|P-P_{0}\right| \leq \sqrt{2} h_{k}$, for $P \in D_{k}$. Therefore, from (3.10)

$$
\begin{aligned}
& \left|L_{D_{k}} \bar{f}(r, \theta)-L_{D_{k}} \bar{f}(0, \theta)\right| \\
\leq & r \max _{(r, \theta) \in D_{k}}\left|\frac{\partial L_{D_{k}} \bar{f}(r, \theta)}{\partial r}\right| \\
\leq & r^{v}\left(\sqrt{2} h_{k}\right)^{1-v} \max _{P \in D_{k}}\left|\frac{\partial L_{D_{k}} f(P)}{\partial x} \cos \theta+\frac{\partial L_{D_{k}} f(P)}{\partial y} \sin \theta\right| \\
\leq & 8 A(\theta) r^{v} .
\end{aligned}
$$

We state and prove the following convergence result.
Theorem 3.1 For any $P_{0} \in \stackrel{\circ}{\Omega}$ as the origin of polar coordinates, if $f \in \mathbb{C}\left(\Omega \backslash P_{0}\right)$ and (1.7) holds on $D_{0}$, then

$$
\begin{equation*}
\tilde{I}^{(k, \delta)} \rightarrow I \text { as } k \rightarrow \infty \text { and } \delta \rightarrow 0 \tag{3.11}
\end{equation*}
$$

Proof Let $E^{(k, \delta)}=\left|I-\tilde{I}^{(k, \delta)}\right|$. From (3.9) we can write

$$
E^{(k, \delta)} \leq E_{1,1}^{(k)}+E_{1,2}^{(k, \delta)}+E_{2}^{(\delta)}
$$

where

$$
E_{1,1}^{(k)}=\left|I_{1,1}^{(k)}-\tilde{I}_{1,1}^{(k)}\right|, \quad E_{1,2}^{(k, \delta)}=\left|I_{1,2}^{(k)}-\tilde{I}_{1,2}^{(k, \delta)}\right|, \quad E_{2}^{(\delta)}=\left|I_{2}-\tilde{I}_{2}^{(\delta)}\right| .
$$

From (1.7) and Lemma 3.1,

$$
\begin{aligned}
E_{1,1}^{(k)} & \leq \int_{0}^{2 \pi} \int_{0}^{R_{k}(\theta)} \frac{\left|(\bar{f}(r, \theta)-\bar{f}(0, \theta))-\left(L_{D_{k}} \bar{f}(r, \theta)-L_{D_{k}} \bar{f}(0, \theta)\right)\right|}{r} d r d \theta \\
& \leq \int_{0}^{2 \pi} \int_{0}^{R_{k}(\theta)} \frac{A(\theta) r^{v}+8 A(\theta) r^{v}}{r} d r d \theta \\
& \leq \int_{0}^{2 \pi} \hat{A}(\theta)\left(\int_{0}^{R_{k}(\theta)} r^{v-1} d r\right) d \theta \\
& =\int_{0}^{2 \pi} \hat{A}(\theta)\left(R_{k}(\theta)\right)^{v} / v d \theta .
\end{aligned}
$$

Since $\hat{A}(\theta)=9 A(\theta)$ is bounded and $R_{k}(\theta) \rightarrow 0$ as $k \rightarrow \infty$, for any $\varepsilon>0, \exists k_{0} \in \mathbb{N}$, such that $E_{1,1}^{\left(k_{0}\right)}<\varepsilon / 3$.

Then, fix $k_{0}$ and $D_{k_{0}}$. By the error estimate on the operator $L_{Q}$ in [5], from (3.7) we have

$$
\begin{align*}
E_{1,2}^{\left(k_{0}, \delta\right)} & \leq \sum_{j=1}^{N_{k_{0}}} \int_{Q_{j}}\left|F_{1}(x, y)-L_{Q_{j}} F_{1}(x, y)\right| d x d y \\
& =\int_{D_{0} \backslash D_{k_{0}}}\left|F_{1}(x, y)-L_{D_{0} \backslash D_{k_{0}}} F_{1}(x, y)\right| d x d y \\
& \leq 2 \omega_{D_{0} \backslash D_{k_{0}}}\left(F_{1}, \delta\right) \cdot S_{D_{0}}=8 h_{0}^{2} \omega_{D_{0} \backslash D_{k_{0}}}\left(F_{1}, \delta\right) \tag{3.12}
\end{align*}
$$

where

$$
L_{D_{0} \backslash D_{k_{0}}} F_{1}(x, y)=\sum_{j=1}^{N_{k_{0}}} L_{Q_{j}} F_{1}(x, y),
$$

$\omega_{D_{0} \backslash D_{k_{0}}}\left(F_{1}, \delta\right)$ is the modulus of continuity of $F_{1} \in \mathbb{C}\left(D_{0} \backslash D_{k_{0}}\right), S_{D_{0}}=4 h_{0}^{2}$ is the area of $D_{0}$ and, especially, $\delta$ is the length of the longest diagonal or edge in all elements $Q \subset D_{0} \backslash D_{k_{0}}$.

Hence, for the same above $\varepsilon, \exists \delta_{0}>0$, such that $\omega_{D_{0} \backslash D_{k_{0}}}\left(F_{1}, \delta_{0}\right)<\varepsilon /\left(24 h_{0}^{2}\right)$. If the subdivision of $D_{0} \backslash D_{k_{0}}$ is such that $\delta<\delta_{0}$, then from (3.12) we can deduce $E_{1,2}^{\left(k_{0}, \delta\right)}<\varepsilon / 3$.

Similarly, from (3.8)

$$
\begin{align*}
E_{2}^{(\delta)} & \leq \sum_{j=1}^{M} \int_{Q_{j}}\left|F(x, y)-L_{Q_{j}} F(x, y)\right| d x d y \\
& =\int_{\Omega \backslash D_{0}}\left|F(x, y)-L_{\Omega \backslash D_{0}} F(x, y)\right| d x d y \\
& \leq 2 \omega_{\Omega \backslash D_{0}}(F, \delta) \cdot S_{\Omega \backslash D_{0}}, \tag{3.13}
\end{align*}
$$

where

$$
L_{\Omega \backslash D_{0}} F(x, y)=\sum_{j=1}^{M} L_{Q_{j}} F(x, y),
$$

$\omega_{\Omega \backslash D_{0}}(F, \delta)$ is the modulus of continuity of $F \in \mathbb{C}\left(\Omega \backslash D_{0}\right), S_{\Omega \backslash D_{0}}$ is the area of $\Omega \backslash D_{0}$ and, especially, $\delta$ is the length of the longest diagonal or edge in all elements $Q \subset \Omega \backslash D_{0}$.

For the same above $\varepsilon, \exists \delta_{1}>0$, such that $\omega_{\Omega \backslash D_{0}}\left(F, \delta_{1}\right)<\varepsilon /\left(6 S_{\Omega \backslash D_{0}}\right)$. If the subdivision of $\Omega \backslash D_{0}$ is such that $\delta<\delta_{1}$, then from (3.13) we get $E_{2}^{(\delta)}<\varepsilon / 3$.

In conclusion, for $k>k_{0}$ and $\delta<\min \left\{\delta_{0}, \delta_{1}\right\}$, it holds

$$
\begin{equation*}
E^{(k, \delta)}<\varepsilon \tag{3.14}
\end{equation*}
$$

Therefore, (3.11) follows from (3.14).
Now, we consider the computational cost of our cubature due both to the number of integration nodes in (3.9) and to the construction of its coefficients.

From Fig. 2.1(a) and from the description of our algorithm, it is clear that the subdivision strategy generates an embedded sequence of integration nodes. Consequently, the total number of cubature nodes is $V+E$, where $V$ is the number of the vertices of the quadrangulation, $E$ is the number of the edges, respectively. We outline that the number of function evaluations one has to perform is lower than that one of another similar composite strategy using a basic (polynomial) interpolatory type cubature with nodes interior to each quadrilateral element. Comparisons with the tensor product $2 \times 2$ Gauss-Legendre cubature and the tensor product Simpson rule are presented in [5].

With reference to the computational cost due to the construction of the basic rule coefficients, defined either by (2.4), in case of a general quadrilateral element, or by (2.5), in case of rectangles and parallelograms, we can remark that it is comparable with the computational cost of another similar composite strategy applied on quadrilateral elements and based on classical rules, like those ones above mentioned. Indeed, classical basic rules require bilinear transformations from general quadrilateral elements to rectangular ones.

## 4 Numerical examples and conclusions

In this section, we propose some numerical examples to test our method for integrals of the form (1.6), with $F$ defined by (1.3). We consider several functions $f$, given in Table 4.1, and two kinds of integration domains $\Omega$ : the square $[-1,1]^{2}$ and the nonconvex polygon in Fig. 4.1.

For $\Omega=[-1,1]^{2}$ and $f=f_{1}$, we can compare numerical results of our cubature with those ones presented in $[6,9]$ and obtained by classical methods based on the decomposition of $\Omega$ in triangles having a common vertex at $P_{0}$ and on tensor product of univariate integration rules.

Table 4.1 The test functions $f$.

| $i$ | $f_{i}(x, y)=\bar{f}_{i}(r, \theta)$ | $F_{1}$ |
| :---: | :---: | :---: |
| 1 | $\frac{x-x_{0}}{r}=\cos \theta$ | 0 |
| 2 | $\frac{\left(x-x_{0}\right) e^{x}}{r}=e^{x} \cos \theta$ | $\frac{\left(e^{x}-e^{x_{0}}\right) \cos \theta}{r^{2}}$ |
| 3 | $\frac{r^{3 / 2}+\left\|x-x_{0}\right\|}{r}=\sqrt{r}+\|\cos \theta\|$ | $r^{-3 / 2}$ |
| 4 | $\sqrt{\|x-y\|}$ | $\frac{\sqrt{\|x-y\|}-\sqrt{\left\|x_{0}-y_{0}\right\|}}{r^{2}}$ |

By (3.2), for each function $\bar{f}=\bar{f}_{i}(i=1, \ldots, 4)$, the integrals $I_{1,3}$ are

$$
\begin{aligned}
& \int_{0}^{2 \pi} \bar{f}_{1}(0, \theta) \log R_{0}(\theta) d \theta=\int_{0}^{2 \pi} \cos \theta \log R_{0}(\theta) d \theta=0 \\
& \int_{0}^{2 \pi} \bar{f}_{2}(0, \theta) \log R_{0}(\theta) d \theta=e^{x_{0}} \int_{0}^{2 \pi} \cos \theta \log R_{0}(\theta) d \theta=0, \\
& \int_{0}^{2 \pi} \bar{f}_{3}(0, \theta) \log R_{0}(\theta) d \theta=\int_{0}^{2 \pi}|\cos \theta| \log R_{0}(\theta) d \theta
\end{aligned}
$$

$$
\begin{aligned}
& =4 \log h_{0}-\int_{0}^{\pi / 4} 4(\sin \theta+\cos \theta) \log \cos \theta d \theta \\
& =4 \log h_{0}-8 \operatorname{arctanh}\left(\tan \frac{\pi}{8}\right)+4 \\
\int_{0}^{2 \pi} \bar{f}_{4}(0, \theta) \log R_{0}(\theta) d \theta & =\sqrt{\left|x_{0}-y_{0}\right|} \int_{0}^{2 \pi} \log R_{0}(\theta) d \theta \\
& =\sqrt{\left|x_{0}-y_{0}\right|}\left(2 \pi \log h_{0}-4 \int_{-\pi / 4}^{\pi / 4} \log \cos \theta \mathrm{~d} \theta\right) \\
& =\sqrt{\left|x_{0}-y_{0}\right|}\left(2 \pi \log \left(2 h_{0}\right)-4 \text { Catalan }\right)
\end{aligned}
$$

where Catalan $=\sum_{k=0}^{\infty}(-1)^{k}(2 k+1)^{-2} \simeq 0.915966$ ([14]).
For the function $f=f_{1}$, we know the following explicit formula of the integral $I$ on $\Omega=[-1,1]^{2}$ ([6])

$$
\begin{aligned}
I & =f_{\Omega} \frac{f_{1}(x, y)}{r^{2}} d x d y \\
& =\log \frac{\left[1-y_{0}+\sqrt{\left(1+x_{0}\right)^{2}+\left(1-y_{0}\right)^{2}}\right]\left[-1-y_{0}+\sqrt{\left(1+x_{0}\right)^{2}+\left(1+y_{0}\right)^{2}}\right]}{\left[-1-y_{0}+\sqrt{\left(1+x_{0}\right)^{2}+\left(1+y_{0}\right)^{2}}\right]\left[1-y_{0}+\sqrt{\left(1-x_{0}\right)^{2}+\left(1-y_{0}\right)^{2}}\right]}
\end{aligned}
$$

and in Table 4.2 we present the relative errors of our cubature (3.9), combined with an equal subdivision, for several singular points $P_{0}$. In the Table, 'EleN', 'NodN' and 'Rel-Err' denote the number of elements, the number of nodes and the relative error, respectively. Such results show the convergence of our method.

Then, in order to improve its performance, we combine (3.9) with the adaptive subdivision scheme proposed in [4]. Each quadrilateral element is checked, and the elements with the largest estimated error are selected automatically to be subdivided into two or four sub-elements according to the differences of integrand function values in next step. The termination condition is the successive step error less than the given tolerance. The Table 4.3 shows the results, in terms of relative errors/total number of nodes, when the adaptive algorithm stops with tolerance $10^{-4}$ and $10^{-5}$, where 'Step-Err' and 'Approximate Value' denote the successive step error and the approximate value of the integral, respectively.

We compare the above results with those presented in $[6,9]$. With reference to numerical results of rules based on tensor product of Gaussian quadratures ([6], Table 2), we note they show a better convergence rate, as we espected, because $\bar{f}_{1}(r, \theta)=$ $\cos (\theta)$ is a smooth function. With reference to results of rules based on tensor product of composite trapezoidal and Gaussian quadratures ([9], Table 1), the relative errors, with respect to the total number of nodes, seem to be comparable with ours.

However, we can remark that a significant difference of all such cubatures is in the node location. In the classical approach $([6,9])$, the node location is fixed and almost all nodes change when the accuracy degree increases. The advantage of our rule with respect to the other considered ones, is that at any step the previous nodes are kept in the subdivision procedure, since they are vertices of the finer quadrilateral subdivision, which the new nodes belong to. Moreover, our method can be efficiently combined with other numerical algorithms based on quadrilateral element boundary nodes. Finally, it is suitable in case of a nonconvex polygonal integration domain
$\Omega$ and the adaptive subdivision allows to dynamically concentrate the computational work in the subregions of $\Omega$ where the integrand is more irregular.

Table 4.2 $\Omega=[-1,1]^{2}, f=f_{1},\left(x_{0}, y_{0}\right)=$ (a) $(0.4,0.1)$, (b) $(0.6,0.2)$, (c) $(0.8,0.4)$, (d) $(0.9,0.9)$.

| Equal subdivision |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: |
| EleN | NodN | Rel-Err (a) | Rel-Err (b) | Rel-Err (c) | Rel-Err (d) |
| 5 | 20 | $1.57 \mathrm{E}+00$ | $3.30 \mathrm{E}+00$ | $1.09 \mathrm{E}+01$ | $4.54 \mathrm{E}+01$ |
| 13 | 44 | $4.62 \mathrm{E}-01$ | $9.05 \mathrm{E}-01$ | $3.11 \mathrm{E}+00$ | $1.22 \mathrm{E}+01$ |
| 29 | 100 | $3.78 \mathrm{E}-02$ | $9.94 \mathrm{E}-02$ | $2.87 \mathrm{E}-01$ | $1.36 \mathrm{E}+00$ |
| 61 | 196 | $1.15 \mathrm{E}-02$ | $3.82 \mathrm{E}-02$ | $5.50 \mathrm{E}-02$ | $4.90 \mathrm{E}-01$ |
| 125 | 404 | $4.13 \mathrm{E}-03$ | $1.05 \mathrm{E}-02$ | $6.06 \mathrm{E}-02$ | $3.46 \mathrm{E}-01$ |
| 253 | 788 | $9.74 \mathrm{E}-04$ | $4.49 \mathrm{E}-04$ | $1.57 \mathrm{E}-02$ | $1.43 \mathrm{E}-01$ |
| 509 | 1588 | $2.75 \mathrm{E}-04$ | $1.07 \mathrm{E}-03$ | $6.89 \mathrm{E}-03$ | $3.74 \mathrm{E}-02$ |
| 1021 | 3124 | $4.85 \mathrm{E}-05$ | $1.54 \mathrm{E}-04$ | $3.26 \mathrm{E}-05$ | $4.47 \mathrm{E}-03$ |

Table $4.3 \Omega=[-1,1]^{2}, f=f_{1},\left(x_{0}, y_{0}\right)=($ a) $(0.4,0.1)$, (b) $(0.6,0.2)$, (c) $(0.8,0.4)$, (d) $(0.9,0.9)$.

| Adaptive subdivision with tolerance $10^{-4}$ |  |  |  |  |  |
| :---: | ---: | ---: | ---: | ---: | ---: |
| $\left(x_{0}, y_{0}\right)$ | EleN | NodN | Rel-Err | Step-Err | Approximate Value |
| (a) | 265 | 952 | $1.20 \mathrm{E}-04$ | $1.54 \mathrm{E}-05$ | $-1.234431115739747 \mathrm{E}+00$ |
| (b) | 281 | 1020 | $1.61 \mathrm{E}-04$ | $1.57 \mathrm{E}-05$ | $-2.087387475120013 \mathrm{E}+00$ |
| (c) | 345 | 1244 | $1.39 \mathrm{E}-04$ | $1.01 \mathrm{E}-05$ | $-3.419420090286125 \mathrm{E}+00$ |
| (d) | 445 | 1574 | $2.79 \mathrm{E}-04$ | $1.17 \mathrm{E}-05$ | $-3.586671224186048 \mathrm{E}+00$ |
| Adaptive subdivision with tolerance $10^{-5}$ |  |  |  |  |  |
| (a) | 559 | 1960 | $9.44 \mathrm{E}-05$ | $3.59 \mathrm{E}-07$ | $-1.234462106679414 \mathrm{E}+00$ |
| (b) | 589 | 2072 | $7.73 \mathrm{E}-05$ | $9.14 \mathrm{E}-07$ | $-2.087561495861009 \mathrm{E}+00$ |
| (c) | 727 | 2574 | $9.79 \mathrm{E}-05$ | $7.81 \mathrm{E}-07$ | $-3.419560736761150 \mathrm{E}+00$ |
| (d) | 923 | 3222 | $3.02 \mathrm{E}-05$ | $6.09 \mathrm{E}-07$ | $-3.585779065300728 \mathrm{E}+00$ |

Another integration domain $\Omega$ is shown in Fig. 4.1 with six initial quadrilateral elements. The coordinates of the ten vertices are $(0,0.5),(0.1,0),(0.8,0.2),(1,0.85)$, $(0.6,0.8),(0.4,1),(0.3,0.3),(0.7,0.3),(0.7,0.6),(0.3,0.6)$. The singular point is $P_{0}=(0.5,0.45)$ and the test functions $f_{i}, i=1, \ldots, 4$ are given in Table 4.1.

The Table 4.4 shows the results when the adaptive algorithm stops with tolerance $10^{-4}$, where 'Step-Err' and 'Approximate Value' are defined as in Table 4.3. The Fig. 4.2 presents meshes and nodes when the adaptive procedure stops for the considered test functions.

All computations were carried out by Matlab ([13]).

In conclusion, we can remark the following further advantage of the proposed approach. By (3.9), the finite part integral $I$ is approximated by the sum of four regular integrals $\tilde{I}_{1,1}^{(k)}, \tilde{I}_{1,2}^{(k, \delta)}, \tilde{I}_{2}^{(\delta)}, I_{1,3}$, where the first three are $2 D$ integrals. By Eqs. (3.4), (3.7) and (3.8), $\tilde{I}_{1,1}^{(k)}, \tilde{I}_{1,2}^{(k, \delta)}, \tilde{I}_{2}^{(\delta)}$ are the same kind of cubature on quadrangle with the same kind of nodes, i.e., the eight boundary nodes on the quadrangle, as shown in Fig. 2.1(a). It means that we can use only one procedure to evaluate the three integrals


Fig. 4.1 A non-convex domain $\Omega$ with initial quadrilateral elements.

Table 4.4 $\Omega$ is the polygon given in Fig. 4.1, $P_{0}=(0.5,0.45)$.

|  | Adaptive subdivision with tolerance $10^{-4}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $f$ | EleN | NodN | Step-Err | Approximate Value |
| $f_{1}$ | 563 | 1988 | $2.09 \mathrm{E}-05$ | $-3.015424731852383 \mathrm{E}-01$ |
| $f_{2}$ | 601 | 2054 | $1.03 \mathrm{E}-05$ | $1.804458159572488 \mathrm{E}+00$ |
| $f_{3}$ | 661 | 2172 | $6.90 \mathrm{E}-05$ | $5.786696849043063 \mathrm{E}+00$ |
| $f_{4}$ | 967 | 3474 | $3.03 \mathrm{E}-05$ | $5.666572548642983 \mathrm{E}-01$ |

just with different cubature weights (3.6) and (2.4) according to the integrands in (3.4), (3.7) and (3.8). Besides, the computational cost, equivalent to the number of nodes, can be easily obtained as that one of regular integration. In fact, the proposed algorithm for finite part integrals is compatible and consistent with the algorithms for regular cases presented in $[4,5]$. Therefore, it is easy to extend the approach to other cases of singular integrals and by cubatures of high accuracy defined on various kinds of elements. This topic will be discussed in our future work.

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Fig. 4.2 Meshes and nodes when the adaptive procedure stops.
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