Cauchy problems for hyperbolic systems in $\$ \mathbf{R}^{\wedge} \mathbf{n} \mathbf{\$}$ with irregular principal symbol in time and for \$|x| \rightarrow \infty\$.

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## UNIVERSITÀ DEGLI STUDI DI TORINO

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# Cauchy problems for hyperbolic systems in $\mathbb{R}^{n}$ with irregular principal symbol in time and for $|x| \rightarrow \infty$ 

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#### Abstract

The aim of this paper is to present an approach for the study of well posedness for diagonalizable hyperbolic systems of (pseudo)differential equations with characteristics which are not Lipschitz continuous with respect to both the time variable $t$ (locally) and the space variables $x \in \mathbb{R}^{n}$ for $|x| \rightarrow \infty$. We introduce optimal conditions guaranteeing the well-posedness in the scale of the weighted Sobolev spaces $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)$, cf. Introduction, with finite or arbitrarily small loss of regularity. We give explicit examples for ill-possedness of the Cauchy problem in the Schwartz spaces when the hypotheses on the growth for $|x| \rightarrow \infty$ fail.


Key words: Cauchy problem, hyperbolic systems, non-Lipschitz coefficients, superlinear growth, global solutions.

## 1 Introduction and main results

We study the Cauchy problem for the first order hyperbolic systems

$$
\left\{\begin{array}{l}
\partial_{t} u=i A\left(t, x, D_{x}\right) u+B\left(t, x, D_{x}\right) u+f(t, x)  \tag{1.1}\\
u(0, x)=u_{0}(x)
\end{array} \quad, \quad t \in \mathbb{R}, x \in \mathbb{R}^{n}\right.
$$

in the scale of the two-indexed weighted Sobolev spaces $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)$ introduced by Cordes [9], Parenti [21] and defined for $s_{1}, s_{2} \in \mathbb{R}$ as follows

$$
\begin{equation*}
H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)=\left\{u \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right):\|u\|_{s_{1}, s_{2}}=\left\|\langle x\rangle^{s_{2}}\langle D\rangle^{s_{1}} u\right\|_{L^{2}\left(\mathbb{R}^{n}\right)}<+\infty\right\}, \tag{1.2}
\end{equation*}
$$

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(here $\langle x\rangle=\left(1+|x|^{2}\right)^{1 / 2}$ and $\langle D\rangle^{s_{1}}$ denotes the Fourier multiplier with symbol $\langle\xi\rangle^{s_{1}}$ ). We assume that $A\left(t, x, D_{x}\right)=\left\{A_{j k}\left(t, x, D_{x}\right)\right\}_{j, k=1}^{m}$ is a $m \times m$ matrix of first order (pseudo)differential operators and $B\left(t, x, D_{x}\right)=\left\{B_{j k}\left(t, x, D_{x}\right)\right\}_{j, k=1}^{m}$ is a $m \times m$ matrix of lower order terms and we allow irregular behaviour of the principal part $A$, namely, nonLipschitz continuous symbols simultaneously with respect to the time variable $t$ (locally) and to the space variables $x \in \mathbb{R}^{n}$ for $|x| \rightarrow \infty$ (i.e. superlinear growth for $|x| \rightarrow \infty$ ).
Strictly hyperbolic systems with unbounded coefficients in the space variables were studied by Cordes [9] assuming the coefficients smooth in $t$ and admitting linear growth with respect to $x$ in the principal part $A$. The author derived energy estimates in the weighted Sobolev spaces (1.2). As a consequence of the obvious identities

$$
\begin{equation*}
\bigcap_{s_{1}, s_{2} \in \mathbb{R}} H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)=\mathcal{S}\left(\mathbb{R}^{n}\right), \quad \bigcup_{s_{1}, s_{2} \in \mathbb{R}} H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)=\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \tag{1.3}
\end{equation*}
$$

he obtained well posedness for (1.1) in the Schwartz spaces $\mathcal{S}\left(\mathbb{R}^{n}\right), \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$. Other authors extended the results of [9] to more general systems using Fourier integral operators and global $S\left(\mathbb{R}^{n}\right)$-type wavefront sets, cf. Cappiello [4], Coriasco [10], Coriasco and Rodino [11], Coriasco and Maniccia [12], Ichinose [20], Ruzhansky and Sugimoto [24], [25]. In a recent paper [16], Gramchev and Gourdin relaxed the assumptions on the growth of the coefficients with respect to $x$ admitting characteristics with superlinear growth, showing global well-posedness in $\mathcal{S}\left(\mathbb{R}^{n}\right)$ and in more general weighted time depending spaces in $\mathbb{R}^{n}$. Examples exhibiting a loss with respect to the second index $s_{2}$ in $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)$ for $t \neq 0$ have been given. However, the issue of a critical "threshold" in order to have well posedness with respect to the second index $s_{2}$ has not been addressed. On the other hand, Ascanelli and Cappiello [2] have investigated hyperbolic systems of p.d.o.s with Log-Lipschitz symbols with respect to $t$, smooth in $x \in \mathbb{R}^{n}$ with characteristics admitting linear growth for $|x| \rightarrow \infty$, deriving a loss with respect to both indexes $s_{1}$ and $s_{2}$ in $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)$ for $t \neq 0$. We recall that a function $a:[0, T] \rightarrow \mathbb{R}$ is Log-Lipschitz continuous if

$$
\sup _{\substack{t, s \in[0, T] \\ 0<|t-s|<1 / 2}} \frac{|a(t)-a(s)|}{|t-s||\log | t-s| |}<+\infty .
$$

Log-Lipschitz regularity appeared for the first time in the celebrated paper by Colombini, De Giorgi and Spagnolo [7] concerning second order strictly hyperbolic equations with time depending coefficients and was identified as the minimal regularity in $t$ to be required in order to obtain $C^{\infty}$-well-posedness, or more precisely, well-posedness in Sobolev spaces with a finite loss of derivatives. This result has been extended to more general classes of hyperbolic equations and systems with coefficients depending also on $x$ but uniformly bounded on $\mathbb{R}^{n}$ by Colombini and Lerner [8], Cicognani [5], Ascanelli [1], Reissig [22,23], Colombini and Cicognani [6]. Ascanelli and Cappiello in [2] first analyzed the effect of Log-Lipschitz regularity on the growth/decay of the solution, cf. also [3].

The first issue of the present paper is the following: can we relax the usual assumptions of Cordes on the linear growth for $x \rightarrow \infty$ and on the Lipschitz regularity with respect to the time variable in order to obtain well-posedness in the scale $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right), s_{1}, s_{2} \in \mathbb{R}$ with an arbitrarily small loss in the two Sobolev indexes, namely if the initial data

$$
\begin{equation*}
u_{0} \in H^{s_{1}^{-}, s_{2}^{-}}\left(\mathbb{R}^{n}\right)=\bigcap_{\varepsilon_{1}>0, \varepsilon_{2}>0} H^{s_{1}-\varepsilon_{1}, s_{2}-\varepsilon_{2}}\left(\mathbb{R}^{n}\right) \tag{1.4}
\end{equation*}
$$

then the unique solution

$$
\begin{equation*}
u(t, \cdot) \in H^{s_{1}^{-}, s_{2}^{-}}\left(\mathbb{R}^{n}\right), \quad t \in \mathbb{R} ? \tag{1.5}
\end{equation*}
$$

A second issue consists in extending the results obtained in [2] for systems with LogLipschitz coefficients in $t$ to the case in which the principal part admits superlinear growth with respect to $x$ and the results of [16] to systems with non-Lipschitz coefficients in $t$.
A last minor issue concerns the lower order symbol $B\left(t, x, D_{x}\right)=\left\{B_{j k}\left(t, x, D_{x}\right)\right\}_{j, k=1}^{m}$ in (1.1). In the previous papers it is assumed to be a continuous in $t$ differential or pseudodifferential operator in $x$ of order 0 (cf. [2]) or smooth in $t$ admitting at most logarithmic growth as in [16]. We want to allow stronger singularities in $t$ with respect to the previous papers and even we will allow some nonclassical unbounded symbols in $\xi$ as well.
The present paper solves completely the issues stated above.
Let us fix our assumptions on the principal part $A$ of the system (1.1). Here we use the traditional notation $D_{x}=i^{-1} \partial_{x}$. We assume that $A_{j k}(t, x, \xi), j, k=1, \ldots, m$, are smooth in $x, \xi$ and that for every $T>0, \alpha, \beta \in \mathbb{Z}_{+}^{n}$, there exists a positive constant $C_{\alpha \beta T}$ such that

$$
\begin{equation*}
\left|D_{\xi}^{\alpha} D_{x}^{\beta}\left(A_{j k}(t, x, \xi)-A_{j k}\left(t^{\prime}, x, \xi\right)\right)\right| \leq C_{\alpha \beta T}\left|t-t^{\prime}\right| \cdot \sigma\left(\left|t-t^{\prime}\right|^{-1}\right) \rho(\langle x\rangle)\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{1-|\beta|} \tag{1.6}
\end{equation*}
$$

for every $\alpha, \beta \in \mathbb{Z}_{+}^{n}, x, \xi \in \mathbb{R}^{n}, 0<\left|t-t^{\prime}\right|<1 / 2$, where $\rho(z)$ and $\sigma(z)$ are positive smooth nondecreasing functions defined for $z>0$ and satisfying the following conditions:

$$
\begin{array}{ll}
\sup _{z>0} \frac{\rho(z)}{\ln (2+z)}<+\infty \quad \text { and } & \sup _{z>0}\left(\langle z\rangle^{k}\left|D_{z}^{k} \rho(z)\right|\right)<+\infty, \\
\sup _{z>0} \frac{\sigma(z)}{\ln (2+z)}<+\infty \quad \text { and } \quad \sup _{z>0}\left(\langle z\rangle^{k}\left|D_{z}^{k} \sigma(z)\right|\right)<+\infty, \tag{1.8}
\end{array}
$$

for $k \geq 1$. We also assume that there exists a positive constant $C$ such that

$$
\begin{equation*}
\sigma\left(z_{1} \cdot z_{2}\right) \leq C\left(\sigma\left(z_{1}\right)+\sigma\left(z_{2}\right)\right), \quad z_{1}, z_{2}>0 . \tag{1.9}
\end{equation*}
$$

We note that the condition (1.7) implies the Wintner type condition in [16] (cf. also [27]):

$$
\int_{1}^{+\infty} \frac{1}{y \rho(y)} d y=+\infty
$$

Example 1.1 Let $r>0$. The function $\phi(z)=\ln ^{r}(2+z)$ satisfies the conditions (1.7), (1.9) for every $r \in[0,1]$.

We shall also assume that the system (1.1) is diagonalizable, i.e. there exists a $m \times m$ matrix $S(t, x, \xi)$ invertible with inverse $S^{-1}(t, x, \xi)$ such that $S^{-1}(t, x, \xi) A(t, x, \xi) S(t, x, \xi)$ $=\Lambda(t, x, \xi)=\operatorname{diag}\left\{\lambda_{1}, \ldots, \lambda_{m}\right\}$, with $\lambda_{j}(t, x, \xi)$ real-valued functions continuous in $t$, smooth in $x, \xi$ and satisfying the estimate

$$
\begin{equation*}
\sup _{t \in[0, T]}\left|D_{\xi}^{\alpha} D_{x}^{\beta} \lambda_{j}(t, x, \xi)\right| \leq C_{\alpha \beta T} \rho(\langle x\rangle)\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{1-|\beta|} \tag{1.10}
\end{equation*}
$$

for every $x, \xi \in \mathbb{R}^{n}, \alpha, \beta \in \mathbb{Z}_{+}^{n}, j=1, \ldots, m$. It is natural to require that $S$ and $S^{-1}$ have the same regularity as $A$ with respect to $t, x, \xi$. Namely we shall assume that

$$
\left|D_{\xi}^{\alpha} D_{x}^{\beta}\left(S_{j k}(t, x, \xi)-S_{j k}\left(t^{\prime}, x, \xi\right)\right)\right| \leq C_{\alpha \beta T}\left|t-t^{\prime}\right| \cdot \sigma\left(\left|t-t^{\prime}\right|^{-1}\right)\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}
$$

for $j, k=1, \ldots, m$, and the same estimate holds for $S^{-1}$.
Concerning the lower order term $B$ we propose various generalizations (or weaker restrictions) in comparison to the preceding literature. More precisely, we allow sum of $L^{1}$ functions in $t$ or unboundedness for $x \rightarrow+\infty$ or even lower order symbols which are not bounded with respect to $\xi$. Namely we assume that $B$ is smooth in $x, \xi$ and satisfies the following estimates: for every $\alpha, \beta \in \mathbb{Z}_{+}^{n}, T>0$, one can find $C=C_{T \alpha \beta}>0$ and $\psi=\psi_{\alpha \beta} \in L^{1}([0, T])$ such that

$$
\begin{equation*}
\left|D_{\xi}^{\alpha} D_{x}^{\beta} B_{j k}(t, x, \xi)\right| \leq C(\rho(\langle x\rangle)+\psi(t)+\sigma(\langle\xi\rangle)+\sigma(\langle x\rangle))\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \tag{1.11}
\end{equation*}
$$

for all $t \in[0, T], x \in \mathbb{R}^{n}, \xi \in \mathbb{R}^{n}, j, k=1, \ldots, m$. The novelty is twofolded. First, we treat larger classes of lower order terms in the presence of the $L^{1}$ term $\psi(t)$ (see Gourdin and Mechab [17], Gourdin and Gramchev [15], where such type of $L^{1}$ singularities with respect to $t$ appear in the coefficients of evolution PDEs). Secondly, we allow perturbations with "nonclassical" p.d.o. whose order is less than every $\varepsilon>0$, e.g.,

$$
\left.\left.\sigma(\langle\xi\rangle)=\ln ^{r}(1+\langle\xi\rangle), \quad r \in\right] 0,1\right] .
$$

We stress that the use of pseudodifferential operators yields the latter possibility, in the case of differential equations we have to stick to zero order terms.

Let us now state our main result.
Theorem 1.2 Suppose that $A$ (respectively, B) satisfies (1.6) (respectively (1.11)) for some functions $\rho, \sigma$ satisfying the conditions (1.7), (1.8), (1.9). Then, for every $T>$ $0, s_{1}, s_{2} \in \mathbb{R}$ there exist $\delta>0, \mu>0$ such that, setting

$$
\begin{equation*}
\mathcal{W}(t,\langle x\rangle,\langle\xi\rangle)=\mathcal{W}_{1}(t,\langle\xi\rangle)+\mathcal{W}_{2}(t,\langle x\rangle), \tag{1.12}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{W}_{1}(t, \eta)=\delta t \sigma(\eta) \tag{1.13}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{W}_{2}(t, z)=\left(e^{\mu t}-1\right)(\rho(z)+\sigma(z)), \tag{1.14}
\end{equation*}
$$

the Cauchy problem (1.1) is globally well-posed in

$$
C H_{\mathcal{W}, T}^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)=\left\{u:\|u\|_{C H_{W, T}^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)}:=\sup _{0 \leq t \leq T}\left\|e^{-\mathcal{W}(t,(\cdot),\langle D\rangle)} u(t, \cdot)\right\|_{s_{1}, s_{2}}<+\infty\right\} .
$$

Moreover, for every $T>0$ we can find $C=C_{T}>0$ such that the solution $u$ of (1.1) satisfies the following energy inequality:

$$
\begin{equation*}
\|u\|_{C H_{W, T}^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)} \leq C_{T}\left(\left\|u_{0}\right\|_{s_{1}, s_{2}}+\int_{0}^{T}\|f\|_{C H_{W, \tau}^{s_{1}, s_{2}\left(\mathbb{R}^{n}\right)}} d \tau\right) \tag{1.15}
\end{equation*}
$$

In particular, the Cauchy problem (1.1) is well posed in $\mathcal{S}\left(\mathbb{R}^{n}\right), \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$.

Remark 1.3 We note that the result in the previous assertion is not symmetric with respect to the two indexes $s_{1}$ and $s_{2}$. In fact, the loss with respect to the second index $s_{2}$ depends on $\sigma$ and $\rho$ while the loss with respect to $s_{1}$ depends only on $\sigma$.

The second result is in part inspired by the results obtained by Cicognani and Colombini in [6] (see also Reissig [22]), where it was proved that a sub-logarithmic growth of the modulus of continuity $\sigma$ determines an arbitrarily small loss of regularity in the solution. Here we prove that the same effect appears also in the behaviour at infinity of the solution as a consequence of Theorem 1.2 under further restrictions on the growth of $\rho$ and $\sigma$.

Theorem 1.4 Assume the same hypotheses of Theorem 1.2. Suppose moreover that

$$
\begin{equation*}
\lim _{z \rightarrow+\infty} \frac{\rho(z)}{\ln z}=0 \quad \text { and } \quad \lim _{z \rightarrow+\infty} \frac{\sigma(z)}{\ln z}=0 \tag{1.16}
\end{equation*}
$$

Then the Cauchy problem (1.1) is well posed in the scale $H^{s_{1}^{-}, s_{2}^{-}}\left(\mathbb{R}^{n}\right)$. Namely, for every $T>0$ we can find $C=C_{T}>0$ such that for any positive $\varepsilon_{1}, \varepsilon_{2}$ :

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\|u(t, \cdot)\|_{s_{1}-\varepsilon_{1}, s_{2}-\varepsilon_{2}} \leq C\left(\left\|u_{0}\right\|_{s_{1}, s_{2}}+\int_{0}^{T} \sup _{|\tau| \leq t}\|f(\tau, \cdot)\|_{s_{1}-\varepsilon_{1}, s_{2}-\varepsilon_{2}} d t\right) \tag{1.17}
\end{equation*}
$$

Remark 1.5 We observe that under the assumptions (1.7), (1.8), (1.16), we have

$$
\begin{equation*}
C\left([0, T]: H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)\right) \subset C H_{s_{1}, s_{2}}^{\mathcal{W}, T}\left(\mathbb{R}^{n}\right) \subset C\left([0, T]: H^{s_{1}-\varepsilon_{1}, s_{2}-\varepsilon_{2}}\left(\mathbb{R}^{n}\right)\right), \tag{1.18}
\end{equation*}
$$

for any $\varepsilon_{1}, \varepsilon_{2}>0$. Hence, it is sufficient to prove Theorem 1.2 to deduce the proof of Theorem 1.4.

Remark 1.6 Note that if $\sigma(z)=\ln z$, then Theorem 1.2 provides an extension of the results of [2] to systems with principal part admitting superlinear growth with respect to $x$ and of the results of [16] to the case of systems with Log-Lipschitz coefficients. In this case, we can take $\mathcal{W}_{1}(t,\langle\xi\rangle)=\delta t \ln (\langle\xi\rangle)$ and $\mathcal{W}_{2}(t,\langle x\rangle)=\left(e^{\mu t}-1\right) \ln (\langle x\rangle)$ in the proof. In terms of weighted Sobolev spaces the loss turns out to be finite with respect to both indexes $s_{1}$ and $s_{2}$. On the other hand, when $\rho(z)=\ln z$ and $\sigma$ satisfies (1.8), (1.9), (1.16), then we have an arbitrarily small loss of derivatives but a finite loss with respect to the second index $s_{2}$. This gives an extension of the results in [6] to systems with principal part admitting superlinear growth for $|x| \rightarrow \infty$.

The paper is organized as follows. In Section 2 we recall briefly some basic notions on SG-pseudodifferential operators which will be instrumental in the proofs of our results. Section 3 contains the proof of Theorem 1.2. Finally, in Section 4, we provide several examples showing the sharpness of our assumptions on the regularity in $t$ and growth in $x$ of the coefficients of (1.1) and we outline new phenomena related to the presence of a superlinear growth in $x$ in the principal part.

## 2 Preliminaries

In the proofs of our results we shall use arguments from the theory of pseudodifferential operators of SG-type. We recall here briefly their definition and basic properties. For a detailed exposition on this subject we refer the reader to [9], [13], [21], [26].

Definition 2.1 For any $m_{1}, m_{2} \in \mathbb{R}$, we shall denote by $S G^{m_{1}, m_{2}}$ the space of all functions $p(x, \xi) \in C^{\infty}\left(\mathbb{R}^{2 n}\right)$ such that

$$
\sup _{(x, \xi) \in \mathbb{R}^{2 n}}\langle\xi\rangle^{-m_{1}+|\alpha|}\langle x\rangle^{-m_{2}+|\beta|}\left|D_{\xi}^{\alpha} D_{x}^{\beta} p(x, \xi)\right|<+\infty
$$

for all $\alpha, \beta \in \mathbb{Z}_{+}^{n}$. We shall write $S G^{0}$ for $S G^{0,0}$.
Given any $p \in S G^{m_{1}, m_{2}}$, we can consider the pseudo-differential operator $P=O p(p)$ with symbol $p$, defined as standard by

$$
\begin{equation*}
P u(x)=p(x, D) u=(2 \pi)^{-n} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} p(x, \xi) \hat{u}(\xi) d \xi, \quad u \in \mathcal{S}\left(\mathbb{R}^{n}\right) \tag{2.19}
\end{equation*}
$$

where $\hat{u}$ denotes the Fourier transform of $u$. We denote by $L G^{m_{1}, m_{2}}$ the space of all operators of the form (2.19) with symbol in $S G^{m_{1}, m_{2}}$ and by $\mathcal{K}$ the space of all operators (2.19) with symbol in $\mathcal{S}\left(\mathbb{R}^{2 n}\right)=\bigcap_{m_{1}, m_{2} \in \mathbb{R}} S G^{m_{1}, m_{2}}$.

Proposition 2.2 Given $p \in S G^{m_{1}, m_{2}}$, the operator $P$ is linear and continuous from $\mathcal{S}\left(\mathbb{R}^{n}\right)$ to $\mathcal{S}\left(\mathbb{R}^{n}\right)$ and it extends to a linear continuous map from $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ to itself. Precisely, $P$ is linear and continuous from $H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)$ to $H^{s_{1}-m_{1}, s_{2}-m_{2}}\left(\mathbb{R}^{n}\right)$ for every $s_{1}, s_{2} \in \mathbb{R}$.

Proposition 2.3 Every $P \in \mathcal{K}$ can be extended to a linear and continuous map from $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ to $\mathcal{S}\left(\mathbb{R}^{n}\right)$. An operator $P \in \mathcal{K}$ will be called regularizing.

Proposition 2.4 Let $p \in S G^{m_{1}, m_{2}}, q \in S G^{m_{1}^{\prime}, m_{2}^{\prime}}$. Then, the following statements hold:
(1) There exists $s \in S G^{m_{1}+m_{1}^{\prime}, m_{2}+m_{2}^{\prime}}$ such that $P Q=s(x, D)+K$ for some $K \in \mathcal{K}$. Moreover, for every $N \in \mathbb{Z}_{+}$we have:

$$
s(x, \xi)-\sum_{|\alpha|<N} \alpha!^{-1} \partial_{\xi}^{\alpha} p(x, \xi) D_{x}^{\alpha} q(x, \xi) \in S G^{m_{1}+m_{1}^{\prime}-N, m_{2}+m_{2}^{\prime}-N} .
$$

(2) Denoting by $R$ the commutator $[P, Q]$, we have $R=r(x, D)+K^{\prime}$ for some $r \in$ $S G^{m_{1}+m_{1}^{\prime}-1, m_{2}+m_{2}^{\prime}-1}, K^{\prime} \in \mathcal{K}$. Moreover, for every $N \in \mathbb{Z}_{+}$we have:

$$
r(x, \xi)-\sum_{0 \neq|\alpha|<N} \alpha!^{-1}\left(\partial_{\xi}^{\alpha} p(x, \xi) D_{x}^{\alpha} q(x, \xi)-\partial_{\xi}^{\alpha} q(x, \xi) D_{x}^{\alpha} p(x, \xi)\right) \in S G^{m_{1}+m_{1}^{\prime}-1-N, m_{2}+m_{2}^{\prime}-1-N} ;
$$

(3) Denoting by $P^{*}$ the $L^{2}$-adjoint of $P$, we have $P^{*}=p^{*}(x, D)+K^{\prime \prime}$ for some $p^{*} \in$ $S G^{m_{1}, m_{2}}, K^{\prime \prime} \in \mathcal{K}$. Moreover, for every $N \in \mathbb{Z}_{+}$we have:

$$
p^{*}(x, \xi)-\sum_{|\alpha|<N} \alpha!^{-1} \partial_{\xi}^{\alpha} D_{x}^{\alpha} \overline{p(x, \xi)} \in S G^{m_{1}-N, m_{2}-N} .
$$

Definition 2.5 $A$ symbol $p \in S G^{m_{1}, m_{2}}$ is said to be $S G$-elliptic (or md-elliptic) if there
exists a positive constant $R$ such that

$$
\inf _{|x|+|\xi| \geq R}\langle\xi\rangle^{-m_{1}}\langle x\rangle^{-m_{2}}|p(x, \xi)|=: C_{1}>0 .
$$

Proposition 2.6 A symbol $p \in S G^{m_{1}, m_{2}}$ is $S G$-elliptic if and only if there exists an operator $E \in L G^{-m_{1},-m_{2}}$ such that

$$
E P=I+R_{1}, \quad P E=I+R_{2},
$$

where $I$ is the identity operator and $R_{1}, R_{2} \in \mathcal{K}$. The operator $E$ is said to be a parametrix of $P$.

In the following we shall also consider symbols satisfying estimates expressed in terms of the functions $\rho$ and $\sigma$ defined in the Introduction, cf. (1.6), (1.11). We stress the fact that the conditions (1.7), (1.8) imply that such symbols have at most polynomial growth and can then be included in a class of SG-type. Composition of the related operators would require the formulation of a specific calculus for each type of symbols or else it can be performed in the setting of the SG calculus described above taking into account the weight functions involved case by case. For the sake of simplicity we shall adopt the latter approach.

## 3 Energy estimates in weighted spaces

In this section we prove Theorem 1.2.
Proof of Theorem 1.2. We first prove the result for a system with diagonal principal part of the form

$$
\left\{\begin{array}{l}
\partial_{t} v=i \Lambda v+\widetilde{B} v+\tilde{f}  \tag{3.20}\\
v(0, x)=g(x)
\end{array}\right.
$$

where $\Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ with $\lambda_{j}(t, x, \xi), j=1, \ldots, m$, satisfying the condition (1.10) and $\widetilde{B}$ satisfies the condition (1.11). After that we shall prove that the system (1.1) can be reduced to one of the form (3.20).
Let us set

$$
v^{\mathcal{W}}(t, x)=e^{-\mathcal{W}(t,\langle x\rangle,\langle D\rangle)} v(t, x)
$$

where $\mathcal{W}$ is defined by $(1.12),(1.13),(1.14)$ for some positive $\delta, \mu$ to be chosen later on. We first consider the case $s_{1}=s_{2}=0$ corresponding to $L^{2}$-estimates. We have

$$
\begin{aligned}
\frac{d}{d t}\left(\frac{1}{2}\left\|v^{\mathcal{W}}(t, \cdot)\right\|_{L^{2}}^{2}\right)= & \operatorname{Re}\left(\left(-\partial_{t} \mathcal{W}_{2}(t,\langle\cdot\rangle) v^{\mathcal{W}}-\delta \sigma(\langle D\rangle) v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}}\right. \\
& +\operatorname{Re}\left(e^{-\mathcal{W}_{2}(t, \cdot)} e^{-\delta t \sigma(\langle D\rangle)}(i \Lambda) e^{\delta t \sigma(\langle D\rangle)} e^{\mathcal{W}_{2}(t,)} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} \\
& +\operatorname{Re}\left(e^{\left.-\mathcal{W}_{2}(t,)\right)} e^{-\delta t \sigma(\langle D\rangle)} \widetilde{B} e^{\delta t \sigma(\langle D\rangle)} e^{\mathcal{W}_{2}(t, \cdot)} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} \\
& +\operatorname{Re}\left(f^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}}
\end{aligned}
$$

We observe at this point that by (1.7), (1.8), the symbol $e^{ \pm \mathcal{W}(t,\langle x\rangle,\langle\xi\rangle)} \in C\left([0, T], S G^{\varepsilon, \varepsilon}\right)$ for any $\varepsilon>0$. Hence, by (1.10), (1.11) we have

$$
e^{-\delta t \sigma(\langle D\rangle)}(i \Lambda+\widetilde{B}) e^{\delta t \sigma(\langle D\rangle)}=i \Lambda+\widetilde{B}+R,
$$

for some remainder $R$ such that

$$
\begin{equation*}
\left|D_{\xi}^{\alpha} D_{x}^{\beta} r(t, x, \xi)\right| \leq C_{\alpha \beta T}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \rho(\langle x\rangle) \tag{3.21}
\end{equation*}
$$

for all $t \in[0, T], x, \xi \in \mathbb{R}^{n}$. Similarly we have

$$
e^{-\mathcal{W}_{2}(t, \cdot)}(\widetilde{B}+R) e^{\mathcal{W}_{2}(t, \cdot)}=\widetilde{B}+R+R_{0}
$$

for some remainder $R_{0} \in C\left([0, T], L G^{0}\right)$. Moreover, we have

$$
\begin{aligned}
\operatorname{Re}\left(i e^{-\mathcal{\mathcal { W } _ { 2 } ( t , )}} \Lambda e^{\mathcal{W}_{2}(t,)} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} & =\frac{1}{2}\left(i\left(e^{-\mathcal{W}_{2}(t,)} \Lambda e^{\mathcal{\mathcal { W } _ { 2 }}(t, \cdot)}-e^{\mathcal{W}_{2}(t, \cdot)} \Lambda^{*} e^{-\mathcal{W}_{2}(t, \cdot)}\right) v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} \\
& =\left(A_{1} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}}+\left(R_{1} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}},
\end{aligned}
$$

where $A_{1}=a_{1}(t, x, D)$, with

$$
a_{1}(t, x, \xi)=\sum_{j=1}^{n} \partial_{\xi_{j}} \Lambda(t, x, \xi) \partial_{x_{j}} \mathcal{W}_{2}(t,\langle x\rangle)-\frac{1}{2} \sum_{\ell=1}^{n} \partial_{x_{\ell} \xi_{\ell}}^{2} \Lambda(t, x, \xi)
$$

and $R_{1} \in C\left([0, T], L G^{0}\right)$. Then we get

$$
\begin{aligned}
& \frac{d}{d t}\left(\frac{1}{2}\left\|v^{\delta, \mathcal{W}}(t, \cdot)\right\|_{L^{2}}^{2}\right)= \\
& \quad=\left(\left(-\partial_{t} \mathcal{W}_{2}(t,\langle\cdot\rangle)-\delta \sigma(\langle D\rangle)+A_{1}+\frac{1}{2}\left(\widetilde{B}+\widetilde{B}^{*}\right)+\frac{1}{2}\left(R+R^{*}\right)\right) v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} \\
& \quad+\left(R_{2} v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}}+\operatorname{Re}\left(f^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}}
\end{aligned}
$$

with $R_{2} \in C\left([0, T], L G^{0}\right)$. To obtain the desired energy estimates, it is sufficient to prove that for some positive $\delta, \mu$ we have

$$
\left(\left(-\partial_{t} \mathcal{W}_{2}(t,\langle\cdot\rangle)-\delta \sigma(\langle D\rangle)+A_{1}+\frac{1}{2}\left(\widetilde{B}+\widetilde{B}^{*}\right)+\frac{1}{2}\left(R+R^{*}\right) v^{\mathcal{W}}, v^{\mathcal{W}}\right)_{L^{2}} \leq 0\right.
$$

applying the sharp Gårding inequality for SG symbols (see [19], Thm. 18.6.14 for the $\left.\underset{\widetilde{B}^{*}}{\text { metric }} g=|d x|^{2} /\langle x\rangle^{2}+|d \xi|^{2} /\langle\xi\rangle^{2}\right)$ to the matrix $-\partial_{t} \mathcal{W}_{2}(t,\langle\cdot\rangle)-\delta \sigma(\langle D\rangle)+A_{1}+\frac{1}{2}(\widetilde{B}+$ $\left.\widetilde{B}^{*}\right)+\frac{1}{2}\left(R+R^{*}\right)$. To do this, by (1.9), (1.10), (1.11), (3.21), it is sufficient to choose $\delta, \mu$ so large that the following differential inequality holds

$$
\begin{equation*}
\partial_{t} \mathcal{W}_{2}(t, r)+\delta \sigma(\eta)-C_{1} r \rho(r) \partial_{r} \mathcal{W}_{2}(t, r)-C_{2} \sigma(\eta)-C_{3} \rho(r)-C_{4} \sigma(r) \geq 0 \tag{3.22}
\end{equation*}
$$

for some positive constants $C_{j}, j=1,2,3,4$, depending on $\Lambda, \widetilde{B}$ and independent of $\delta$ and $\mu$. Denote now $\phi(r)=\rho(r)+\sigma(r)$. Choosing $\delta>C_{2}$ and taking into account the fact that $0 \leq C_{1} r \rho(r) \phi^{\prime}(r) \leq C_{5} \phi(r)$ for some positive constant $C_{5}$, we have:

$$
\begin{aligned}
& \partial_{t} \mathcal{W}_{2}(t, r)+ \delta \sigma(\eta)-C_{1} r \rho(r) \partial_{r} \mathcal{W}_{2}(t, r)-C_{2} \sigma(\eta)-C_{3} \rho(r)-C_{4} \sigma(r) \\
&=\mu e^{\mu t} \phi(r)+\delta \sigma(\eta)-C_{1}\left(e^{\mu t}-1\right) r \rho(r) \phi^{\prime}(r) \\
& \quad-C_{2} \sigma(\eta)-C_{3} \rho(r)-C_{4} \sigma(r) \\
& \geq \mu e^{\mu t} \phi(r)-C_{5} e^{\mu t} \phi(r)-\max \left\{C_{3}, C_{4}\right\} \phi(r) \\
& \geq\left(\frac{\mu}{2}-C_{5}\right) e^{\mu t} \phi(r)+\left(\frac{\mu}{2}-\max \left\{C_{3}, C_{4}\right\}\right) \phi(r) \geq 0
\end{aligned}
$$

taking $\mu \geq 2 \max \left\{C_{3}, C_{4}, C_{5}\right\}$. We finally obtain the following estimate

$$
\frac{d}{d t}\left\|v^{\delta, \mathcal{W}}(t, \cdot)\right\|_{L^{2}}^{2} \leq C\left(\left\|f^{\mathcal{W}}\right\|_{L^{2}}^{2}+\left\|v^{\mathcal{W}}\right\|_{L^{2}}^{2}\right)
$$

and we can conclude applying Gronwall lemma. For generic $s_{1}, s_{2} \in \mathbb{R}$, let us denote

$$
\tilde{v}=\langle x\rangle^{s_{2}}\langle D\rangle^{s_{1}} v^{\mathcal{W}} .
$$

We observe that

$$
\begin{gathered}
\langle x\rangle^{s_{2}}\langle D\rangle^{s_{1}}\left(-\partial_{t} \mathcal{W}_{2}-\delta \sigma(\langle D\rangle)+i \Lambda+A_{1}+\widetilde{B}+R+R_{2}\right)\langle D\rangle^{-s_{1}}\langle x\rangle^{-s_{2}} \\
=\left[\langle x\rangle^{s_{2}}\langle D\rangle^{s_{1}}, i \Lambda\right]\langle D\rangle^{-s_{1}}\langle x\rangle^{-s_{2}}-\partial_{t} \mathcal{W}_{2}-\delta \sigma(\langle D\rangle)+i \Lambda+A_{1}+\widetilde{B}+R+R_{3},
\end{gathered}
$$

where $R_{3} \in C\left([0, T], L G^{0}\right)$. Now,

$$
\left[\langle x\rangle^{s_{2}}\langle D\rangle^{s_{1}}, i \Lambda\right]\langle D\rangle^{-s_{1}}\langle x\rangle^{-s_{2}}=i \Lambda_{s}(t, x, D)+R_{4},
$$

with

$$
\begin{gathered}
\Lambda_{s}(t, x, \xi)=\sum_{j=1}^{n}\left(\langle x\rangle^{s_{2}} D_{x_{j}} \Lambda(t, x, \xi) \partial_{\xi_{j}}\langle\xi\rangle^{s_{1}}-\langle\xi\rangle^{s_{1}} \partial_{\xi_{j}} \Lambda(t, x, \xi) D_{x_{j}}\langle x\rangle^{s_{2}}\right)\langle\xi\rangle^{-s_{1}}\langle x\rangle^{-s_{2}} \\
=\sum_{j=1}^{n}\left(D_{x_{j}} \Lambda(t, x, \xi) s_{1} \xi_{j}\langle\xi\rangle^{-2}-D_{\xi_{j}} \Lambda(t, x, \xi) s_{2} x_{j}\langle x\rangle^{-2}\right)
\end{gathered}
$$

and $R_{4} \in C\left([0, T], L G^{0}\right)$. Hence by (1.10) we get

$$
\left|D_{\xi}^{\alpha} D_{x}^{\beta} \Lambda_{s}(t, x, \xi)\right| \leq C_{\alpha \beta}\left(\left|s_{1}\right|+\left|s_{2}\right|\right)\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \rho(\langle x\rangle) .
$$

Then, arguing as for the $L^{2}$-estimates, we are reduced to prove that for $\delta$ and $\mu$ sufficiently large, the following inequality is satisfied:

$$
\partial_{t} \mathcal{W}_{2}(t, r)+\delta \sigma(\eta)-C_{1} r \rho(r) \partial_{r} \mathcal{W}_{2}(t, r)-C_{2} \sigma(\eta)-C_{3}\left(\left|s_{1}\right|+\left|s_{2}\right|+1\right) \rho(r)-C_{4} \sigma(r) \geq 0
$$

which is of the same form as (3.22). Hence, we can choose $\delta>C_{2}$ and $\mu$ sufficiently large depending on both $s_{1}$ and $s_{2}$ and we conclude as for the $L^{2}$-estimates. The first part of the theorem is then proved. To conclude the proof we will now show that a system of the form (1.1), with $A, B$ satisfying (1.6), (1.11) can be reduced to a system of the form (3.20). To do this we first need to regularize the diagonalizer $S$ with respect to the time variable. First of all we extend $s(t, x, \xi)$ on $\mathbb{R}_{t}$ defining

$$
s(t, x, \xi)=s(0, x, \xi) \text { for } t<0, \quad s(t, x, \xi)=s(T, x, \xi) \text { for } t>T
$$

Then we consider a function $\varphi \in C_{0}^{\infty}(\mathbb{R})$ such that $0 \leq \varphi \leq 1, \int \varphi(\tau) d \tau=1$, and we define the symbol $\tilde{s}(t, x, \xi)$ by

$$
\tilde{s}(t, x, \xi)=\int s\left(t-\frac{t^{\prime}}{\langle x\rangle \rho(x)\langle\xi\rangle}, x, \xi\right) \varphi\left(t^{\prime}\right) d t^{\prime} .
$$

We claim now that the following estimates hold:

$$
\begin{equation*}
\left|D_{\xi}^{\alpha} D_{x}^{\beta}(s-\tilde{s})(t, x, \xi)\right| \leq C_{\alpha \beta} \frac{\langle\xi\rangle^{-1-|\alpha|}\langle x\rangle^{-1-|\beta|}}{\rho(\langle x\rangle)} \sigma(\langle x\rangle \rho(\langle x\rangle)\langle\xi\rangle), \tag{3.23}
\end{equation*}
$$

$$
\begin{equation*}
\left|D_{\xi}^{\alpha} D_{x}^{\beta} \partial_{t} \tilde{s}(t, x, \xi)\right| \leq C_{\alpha \beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \sigma(\langle x\rangle \rho(\langle x\rangle)\langle\xi\rangle) . \tag{3.24}
\end{equation*}
$$

In fact, using the property $\int \varphi(\tau) d \tau=1$ we can write

$$
\begin{aligned}
&(s-\tilde{s})(t, x, \xi)=\int\left(s(t, x, \xi)-s\left(t-\frac{t^{\prime}}{\langle x\rangle \rho(x)\langle\xi\rangle}, x, \xi\right)\right) \varphi\left(t^{\prime}\right) d t^{\prime} \\
&=\langle x\rangle \rho(x)\langle\xi\rangle \int(s(t, x, \xi)-s(\tau, x, \xi)) \varphi((t-\tau)\langle x\rangle \rho(x)\langle\xi\rangle) d \tau
\end{aligned}
$$

Moreover, since $\int \varphi^{\prime}(\tau) d \tau=0$ we have

$$
\begin{aligned}
\partial_{t} \tilde{s}(t, x, \xi) & =\int s(\tau, x, \xi) \varphi^{\prime}((t-\tau)\langle x\rangle \rho(x)\langle\xi\rangle) d \tau \\
& =\langle x\rangle^{2} \rho^{2}(x)\langle\xi\rangle^{2} \int(s(\tau, x, \xi)-s(t, x, \xi)) \varphi^{\prime}((t-\tau)\langle x\rangle \rho(x)\langle\xi\rangle)\langle x\rangle \rho(x)\langle\xi\rangle d \tau .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \left|\partial_{\xi}^{\alpha} \partial_{x}^{\beta}(s-\tilde{s})(t, x, \xi)\right| \leq \sum_{\substack{\alpha_{1}+\alpha_{2}+\alpha_{3}=\alpha \\
\beta_{1}+\beta_{2}+\beta_{3}=\beta}} c_{\alpha_{j}, \beta_{j}} \\
& \cdot \int\left|\partial_{\xi}^{\alpha_{1}} \partial_{x}^{\beta_{1}}(s(t, x, \xi)-s(\tau, x, \xi))\right| \cdot\left|\partial_{\xi}^{\alpha_{2}} \partial_{x}^{\beta_{2}} \varphi((t-\tau)\langle x\rangle \rho(x)\langle\xi\rangle)\right| \cdot\left|\partial_{x}^{\beta_{3}}(\langle x\rangle \rho(x))\right| \cdot\left|\partial_{\xi}^{\alpha_{3}}\langle\xi\rangle\right| d \tau \\
& \leq c_{\alpha \beta}^{\prime}\langle x\rangle^{1-|\beta|} \rho(x)\langle\xi\rangle^{1-|\alpha|} \int|t-\tau| \cdot \sigma\left(|t-\tau|^{-1}\right) d \tau \\
& =c_{\alpha \beta}^{\prime} \frac{\langle x\rangle^{-1-|\beta|}}{\rho(x)}\langle\xi\rangle^{-1-|\alpha|} \int\left|t^{\prime}\right| \sigma\left(\frac{\langle x\rangle \rho(x)\langle\xi\rangle}{t^{\prime}}\right) d t^{\prime} \\
& \\
& \quad \leq c_{\alpha \beta} \frac{\langle x\rangle^{-1-|\beta|}}{\rho(x)}\langle\xi\rangle^{-1-|\alpha|} \sigma(\langle x\rangle \rho(x)\langle\xi\rangle) .
\end{aligned}
$$

Then (3.23) is proved. The proof of (3.24) is similar. Furthermore, by (3.23), we have

$$
\begin{align*}
|\tilde{s}(t, x, \xi)| \geq|s(t, x, \xi)|-\mid(s-\tilde{s}) & (t, x, \xi) \mid \\
& \geq C_{1}-C_{2}\langle\xi\rangle^{-1}(\langle x\rangle \rho(x))^{-1} \sigma(\langle x\rangle \rho(x)\langle\xi\rangle) \geq \frac{C_{1}}{2} \tag{3.25}
\end{align*}
$$

if $|x|+|\xi|$ is sufficiently large. Thus $\tilde{S}$ is invertible modulo a regularizing operator. Now we introduce the new variable $v=\tilde{S} u$. We have

$$
\begin{equation*}
u=\tilde{S}^{-1} v+J u \quad \text { for some } \quad J \in C([0, T], \mathcal{K}) \tag{3.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{S}^{-1} \in C\left([0, T], L G^{0}\right) \tag{3.27}
\end{equation*}
$$

denotes the left parametrix of $\tilde{S}$. From (1.1) we obtain

$$
\begin{align*}
& \partial_{t} v=\tilde{S} \partial_{t} u+\left[\partial_{t}, \tilde{S}\right] u=i \tilde{S} A u+\left(\partial_{t} \tilde{S}\right) u+\tilde{S} B u+\tilde{S} f \\
& \quad=i S A u+i(\tilde{S}-S) A u+\left(\partial_{t} \tilde{S}\right) u+\tilde{S} B u+\tilde{S} f \\
& \quad=i \Lambda S u+i(\tilde{S}-S) A u+\left(\partial_{t} \tilde{S}\right) u+\tilde{S} B u+\tilde{S} f \\
& =i \Lambda \tilde{S} u+i \Lambda(S-\tilde{S}) u+i(\tilde{S}-S) A u+\left(\partial_{t} \tilde{S}\right) u+\tilde{S} B u+\tilde{S} f \\
& =i \Lambda v+\left[i \Lambda(S-\tilde{S}) \tilde{S}^{-1}+i(\tilde{S}-S) A \tilde{S}^{-1}+\left(\partial_{t} \tilde{S}\right) \tilde{S}^{-1}+\tilde{S} B \tilde{S}^{-1}\right] v+J^{\prime} u+\tilde{S} f \tag{3.28}
\end{align*}
$$

for some $J^{\prime} \in C([0, T], \mathcal{K})$. Thus, we come to a system of the form

$$
\partial_{t} v=i \Lambda v+\widetilde{B} v+\tilde{f},
$$

where $\tilde{f}=J^{\prime} u+\tilde{S} f$ and $\tilde{B}=i \Lambda(S-\tilde{S}) \tilde{S}^{-1}+i(\tilde{S}-S) A \tilde{S}^{-1}+\left(\partial_{t} \tilde{S}\right) \tilde{S}^{-1}+\tilde{S} B \tilde{S}^{-1}$. The conditions (1.6), (1.10), (1.11), (3.23), (3.24), (3.27) together give that the symbol of $\widetilde{B}$ satisfies (1.11). We are then reduced to a system of type (3.20).

Remark 3.1 By the proof of Theorem 1.2, we notice that in fact the assumptions on the lower order term $B\left(t, x, D_{x}\right)$ can be further relaxed assuming $B=B_{1}+i B_{2}, B_{j}=B_{j}^{*}$, with $B_{1}$ satisfying the condition (1.11) and $B_{2}$ admitting a growth of type $\langle x\rangle \rho(\langle x\rangle)$ for $|x| \rightarrow \infty$, cf. [20].

## 4 Examples and concluding remarks

In this section we provide explicit examples showing the sharpness of the assumptions (1.7), (1.8), (1.16) in Theorems 1.2 and 1.4. The first example shows that the conditions for the (lack of) loss in the scales $H^{s_{1}^{-}, s_{2}^{-}}\left(\mathbb{R}^{n}\right)$ are sharp. This is achieved by a change of variables which allows to reduce to a problem with bounded in $x$ and singular in $t$ coefficients for which the sharpness of the assumptions is already known from [7] and [8].

We consider the Cauchy problem for the one dimensional second order hyperbolic equation

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u=\lambda(t)\left(a(x) \partial_{x}\right)^{2} u,  \tag{4.29}\\
\left.u\right|_{t=0}=u^{0},\left.\quad u_{t}\right|_{t=0}=u_{1}
\end{array} \quad, \quad t \in \mathbb{R}, x \in \mathbb{R}\right.
$$

where $\lambda$ is a positive continuous function and $a$ is a positive smooth function satisfying for some $q \in] 0,1]$

$$
\begin{equation*}
a(x)= \pm x \ln ^{q}|x|, \quad \pm x \gg 1 . \tag{4.30}
\end{equation*}
$$

We note that if $q=0$ we recapture the Cordes type linear growth for $x \rightarrow \infty$ in [2], while $q>0$ in (4.30) is a novelty even for the autonomous wave equation $(\lambda \equiv 1)$ with the wave speed or the metric depending quadratically on $x$ for $x \rightarrow \infty$ (see the books of Georgiev [14] and Hebey [18] and the references therein).

As a particular case of the Liouville type theorem in [16], we obtain from (4.30) that the map

$$
y=\varphi(x)=\int_{0}^{x} \frac{1}{a(\eta)} d \eta
$$

defines a global diffeomorphism of $\mathbb{R}$, and, for some $C_{q}^{ \pm}>0$, the following identities hold

$$
\varphi(x)=\left\{\begin{array}{clc} 
\pm C_{0}^{ \pm} \pm \ln |x| & \text { if } \quad q=0  \tag{4.31}\\
\pm C_{q}^{ \pm} \pm \frac{\ln ^{1-q}|x|}{1-q} & \text { if } q \in] 0,1[, \quad \pm x \gg 1 \\
\pm C_{1}^{ \pm} \pm \ln (\ln |x|) & \text { if } & q=1
\end{array}\right.
$$

The inverse function $\psi:=\varphi^{-1}$ satisfies, for some $K_{0}^{ \pm}>0$,

$$
\psi(y)=\left\{\begin{array}{cl} 
\pm e^{|y|-C_{0}^{ \pm}} & \text {if } q=0  \tag{4.32}\\
\pm e^{\left((1-q)\left(|y|-C_{q}^{ \pm}\right)\right)^{1 /(1-q)}} & \text { if } q \in] 0,1\left[, \quad \pm y \geq K_{0}^{ \pm} .\right. \\
e^{|y|-C_{1}^{ \pm}} & \text {if } q=1
\end{array}\right.
$$

Set

$$
\begin{aligned}
v_{0}(y) & =\psi^{*} u_{0}(y)=u_{0}(\psi(y)), \quad v_{1}(y)=\psi^{*} u_{1}(y)=u_{1}(\psi(y)), \\
v(t, y) & =\psi^{*} u(t, y)=u(t, \psi(y)) .
\end{aligned}
$$

Clearly

$$
\begin{aligned}
u_{0}(x) & =\varphi^{*} v_{0}(x)=v_{0}(\varphi(x)), \quad u_{1}(x)=\varphi^{*} v_{1}(x)=v_{1}(\varphi(x)), \\
u(t, x) & =\varphi^{*} v(t, x)=v(t, \varphi(x)) .
\end{aligned}
$$

We define

$$
\begin{aligned}
E_{q}[u](t) & :=\sup _{|\tau| \leq|t|}\left\|\langle x\rangle^{1 / 2} \ln ^{q / 2}\langle x\rangle \partial_{x} u(\tau, \cdot)\right\|_{L^{2}} \\
& +\sup _{|\tau| \leq|t|}\left\|\langle x\rangle^{-1 / 2} \ln ^{-q / 2}\langle x\rangle u_{t}(\tau, \cdot)\right\|_{L^{2}}, \quad|t| \leq T,
\end{aligned}
$$

and

$$
\begin{aligned}
E_{q, s}[u](t) & :=\sup _{|\tau| \leq|t|}\left\|\langle x\rangle^{1 / 2} \ln ^{q / 2}\langle x\rangle\left\langle\langle x\rangle \ln ^{q}\langle x\rangle D\right\rangle^{s} \partial_{x} u(\tau, \cdot)\right\|_{L^{2}} \\
& +\sup _{|\tau| \leq|t|}\left\|\langle x\rangle^{-1 / 2} \ln ^{-q / 2}\langle x\rangle\left\langle\langle x\rangle \ln ^{q}\langle x\rangle D\right\rangle^{s} u_{t}(\tau, \cdot)\right\|_{L^{2}}, \quad|t| \leq T,
\end{aligned}
$$

Theorem 4.1 We have that $u(t, x)$ solves the Cauchy problem (4.29) iff $v(t, y)$ satisfies

$$
\left\{\begin{array}{l}
\partial_{t}^{2} v=\lambda(t) \partial_{y}^{2} v,  \tag{4.33}\\
\left.v\right|_{t=0}=v_{0},\left.\quad v_{t}\right|_{t=0}=v_{1}
\end{array}\right.
$$

Moreover, for every $T>0$ one can find $C_{T}>0$ such that

$$
\begin{equation*}
E_{q}[u](t) \leq C_{T} E_{q}[u](0), \quad|t| \leq T \tag{4.34}
\end{equation*}
$$

iff there exists $\tilde{C}_{T}>0$ such that $v$ satisfies

$$
\begin{equation*}
\sup _{|\tau| \leq t}\left\|\partial_{y} v(\tau, \cdot)\right\|_{L^{2}}+\sup _{|\tau| \leq t}\left\|v_{\tau}(\tau, \cdot)\right\|_{L^{2}} \leq \tilde{C}\left(\left\|\partial_{y} v_{0}\right\|_{L^{2}}+\left\|v_{1}\right\|_{L^{2}}\right), \tag{4.35}
\end{equation*}
$$

for all $|t| \leq T$. Finally, for every $T$ and every (respectively, some) $\varepsilon>0$ one can find $C_{\varepsilon, T}>0$ such that

$$
\begin{equation*}
\left.E_{q}[u](t) \leq C_{\varepsilon, T} E_{q, \varepsilon|t|}[u](0), \quad|t| \leq T\right) \tag{4.36}
\end{equation*}
$$

iff $v$ satisfies for some $\tilde{C}_{\varepsilon, T}>0$

$$
\begin{equation*}
\sup _{|\tau| \leq t}\left\|\partial_{y} v(\tau, \cdot)\right\|_{L^{2}}+\sup _{|\tau| \leq t}\left\|v_{\tau}(\tau, \cdot)\right\|_{L^{2}} \leq \tilde{C}_{\varepsilon, T}\left(\left\|\partial_{y} v_{0}\right\|_{H^{\varepsilon|t|}}+\left\|v_{1}\right\|_{H^{\varepsilon|t|}}\right) \tag{4.37}
\end{equation*}
$$

for all $|t| \leq T$.
Proof. The equivalence of the systems (4.29) and (4.33) directly follows by the transformation rule for linear PDEs under coordinate changes. As it concerns the equivalence of the estimates (4.34) and (4.35), we observe that

$$
\begin{gathered}
\left\|\psi^{*} f\right\|_{L^{2}}^{2}=\int_{\mathbb{R}}|f(\psi(y))|^{2} d y=\int_{\mathbb{R}}|f(x)|^{2} a^{-1}(x) d x \\
\left\|\partial_{y} \psi^{*} f\right\|_{L^{2}}^{2}=\int_{\mathbb{R}}\left|\partial_{y}(f(\psi(y)))\right|^{2} d y=\int_{\mathbb{R}}\left|\partial_{x} f(x)\right|^{2} a(x) d x
\end{gathered}
$$

(we have used the identity $\partial_{y} \psi^{*} f(y)=a(\psi(y))\left(\partial_{x} f\right)(\psi(y))$, which yield

$$
\| v_{t}\left(t, \cdot\left\|_{L^{2}}^{2}=\right\| a^{-1 / 2} u_{t}(t, \cdot) \|_{L^{2}}^{2} \quad \text { and } \quad\left\|\partial_{y} v(t, \cdot)\right\|_{L^{2}}^{2}=\left\|a^{1 / 2} \partial_{x} u(t, \cdot)\right\|_{L^{2}}^{2}\right.
$$

Hence the inequalities

$$
C^{-1}\langle x\rangle \ln ^{q}\langle x\rangle \leq a(x) \leq C\langle x\rangle \ln ^{q}\langle x\rangle, \quad x \in \mathbb{R}
$$

imply the equivalence of (4.34) and (4.35).
The proof of the equivalence of the estimates (4.36) and (4.37) is somewhat more involved. Let $s \in \mathbb{N}$. Then we can use a $H^{s}$ Sobolev norm for $g(y)=\psi^{*} f(y)$ defined by

$$
\|g\|_{H^{s}}^{2}=\|g\|_{L^{2}}^{2}+\left\|\partial_{x}^{s} g\right\|_{L^{2}}^{2}=\left\|a^{-1 / 2} \cdot f\right\|_{L^{2}}^{2}+\left\|a^{-1 / 2}\left(a(\cdot) \partial_{x}\right)^{s} f\right\|_{L^{2}}^{2}
$$

which yields the equivalence if $s=\varepsilon|t|$ is positive integer. The case $s \in] k, k+1\left[, k \in \mathbb{Z}_{+}\right.$, is obtained by interpolation.

If $q=0$ (the linear growth case), another approach is to consider the coordinate change as a FIO, namely

$$
f(x)=\varphi^{*} g=\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i \varphi(x) \xi} \hat{g}(\xi) \mathrm{d} \xi
$$

and derive global $L^{2}$ estimates for FIOs following the results of Ruzhansky and Sugimoto [24], [25].

In the proof of Theorem 1.2 it is possible, as standard, to derive energy estimates in the interval $[-T, T]$ instead of $[0, T]$, obtaining more generally a loss of regularity and decay for $t \neq 0$. Next, we outline new interesting phenomena of "gaining" a decay provided we
investigate the Cauchy problem either for $t \geq 0$ or $t \leq 0$ and we make suitable assumptions on the sign of the characteristics. We stress the fact that this phenomenon is a novelty since it appears only when the principal part presents a superlinear growth in $x$. Here we illustrate this situation in a simple example. Our purpose will be to prove more general statements in a forthcoming paper.

Consider the Cauchy problem for the first order scalar equation

$$
\left\{\begin{array}{l}
\partial_{t} u+\mu(x) \partial_{x} u=0  \tag{4.38}\\
\left.u\right|_{t=0}=u_{0}(x)
\end{array}\right.
$$

where $\mu$ is a smooth function satisfying for some $q \in[0,1], c_{ \pm} \neq 0$

$$
\begin{equation*}
\mu(x)=c_{ \pm} x \ln ^{q}|x|, \quad \pm x \gg 1 \tag{4.39}
\end{equation*}
$$

Proposition 4.2 There exists a positive continuous function $\Theta(t, x)$ such that the unique solution $u(t, x)$ of (4.39) satisfies

$$
\begin{equation*}
\left\|\Theta^{1 / 2} u(t, \cdot)\right\|_{L^{2}}=\left\|u_{0}\right\|_{L^{2}}, \quad t \in \mathbb{R} . \tag{4.40}
\end{equation*}
$$

In particular, if $q \in] 0,1]$ and $c_{+}$and $c_{-}$are both positive (respectively, negative), then the solution $u(t, x)$ gains decay for $t>0$ (respectively, $t<0$ ) and loses decay for $t<0$ (respectively, $t>0$ ) while if $c_{+} c_{-}<0$ then the solution loses decay for every $t \neq 0$. Finally, if $q=1$, the estimate (4.40) implies

$$
\left\|\langle x\rangle^{-1 / 2+e^{-c|t|} / 2} u(t, \cdot)\right\|_{L^{2}} \leq C_{T}\left\|u_{0}\right\|_{L^{2}}, \quad|t| \leq T
$$

if $c_{+} c_{-}<0$ with $c=\max \left\{\left|c_{+}\right|,\left|c_{-}\right|\right\}$,

$$
\begin{array}{ll}
\left\|\langle x\rangle^{-1 / 2+e^{\min \left\{c_{+}, c-\right\} t} / 2} u(t, \cdot)\right\|_{L^{2}} \leq C_{T}\left\|u_{0}\right\|_{L^{2}}, \quad 0 \leq t \leq T \\
\left\|\langle x\rangle^{-1 / 2+e^{\max \left\{c_{+}, c-\right\} t} / 2} u(t, \cdot)\right\|_{L^{2}} \leq C_{T}\left\|u_{0}\right\|_{L^{2}}, \quad-T \leq t \leq 0
\end{array}
$$

if $c_{+}>0, c_{-}>0$, and

$$
\begin{aligned}
\left\|\langle x\rangle^{-1 / 2+e^{\max \left\{c_{+}, c_{-}\right\} t} / 2} u(t, \cdot)\right\|_{L^{2}} \leq C_{T}\left\|u_{0}\right\|_{L^{2}}, & 0 \leq t \leq T \\
\left\|\langle x\rangle^{-1 / 2+e^{\min \left\{c_{+}, c_{-}\right\} t} / 2} u(t, \cdot)\right\|_{L^{2}} \leq C_{T}\left\|u_{0}\right\|_{L^{2}}, & -T \leq t \leq 0,
\end{aligned}
$$

if $c_{+}<0, c_{-}<0$.
Proof. Using the method of the characteristics and the Fourier transformation, we can represent the solution $u$ as follows

$$
\begin{equation*}
u(t, x)=u_{0}\left(\gamma^{-1}(t, x)\right)=(2 \pi)^{-1} \int_{\mathbb{R}} e^{i \gamma^{-1}(t, x) \xi} \hat{u}_{0}(\xi) d \xi \tag{4.41}
\end{equation*}
$$

where $x=\gamma(t, y)$ is the forward characteristic and $y=\gamma^{-1}(t, x)$ is the backward characteristic. Therefore,

$$
\begin{align*}
\left\|\Theta^{1 / 2}(t, \cdot) u(t, \cdot)\right\|_{L^{2}}^{2} & =\int_{\mathbb{R}} \Theta(t, x)\left|u_{0}\left(\gamma^{-1}(t, x)\right)\right|^{2} d x \\
& =\int_{\mathbb{R}} \Theta(t, \gamma(t, y)) \partial_{y} \gamma(t, y)\left|u_{0}(y)\right|^{2} d y=\left\|u_{0}\right\|_{L^{2}}^{2} \tag{4.42}
\end{align*}
$$

provided

$$
\begin{equation*}
\Theta(t, x)=\frac{1}{\partial_{y} \gamma\left(t, \gamma^{-1}(t, x)\right)} . \tag{4.43}
\end{equation*}
$$

The conclusion follows easily by the explicit calculation of $\gamma$ and $\gamma^{-1}$ for $\pm x \gg 1$.

Remark 4.3 One notes that if $q \in] 0,1]$, the phase function $\gamma^{-1}(t, x) \xi$ in (4.41) does not satisfy the hypotheses in the work of Ruzhansky and Sugimoto [24]. Our estimates imply, first, the sharpness of the global $L^{2}$ estimates for FIO in [24] and secondly, refinements of such estimates.

The next example concerns the sharpness of the $|x| \log |x|$ growth condition for the principal part of (1.1). Namely, we show that when this condition is not satisfied, then the well posedness in $\mathcal{S}\left(\mathbb{R}^{n}\right)$ fails in general.

Consider the Cauchy problem for the scalar equation

$$
\left\{\begin{array}{l}
\partial_{t} u+a(x) \partial_{x} u=0,  \tag{4.44}\\
u(0, x)=u_{0}(x)
\end{array}, \quad t \in \mathbb{R}, x \in \mathbb{R}\right.
$$

where $a \in C^{\infty}(\mathbb{R}, \mathbb{R})$ and

$$
\begin{equation*}
a(x)=c_{ \pm} x \ln |x|(\ln \ln |x|)^{\sigma}, \quad \pm x \gg 1, c_{ \pm} \neq 0,0<\sigma \leq 1 \tag{4.45}
\end{equation*}
$$

We have
Proposition 4.4 Let $\varkappa \in C^{\infty}(\mathbb{R},[0,+\infty[)$ satisfy

$$
\begin{gather*}
\varkappa(x)=0, \quad|x| \leq e^{2},  \tag{4.46}\\
\varkappa(x)=\left\{\begin{array}{ccc}
(\ln \ln |x|)^{1-\sigma} & \text { if } 0<\sigma<1 \\
\ln \ln |x| & \text { if } & \sigma=1
\end{array} \quad|x| \geq e^{2} .\right. \tag{4.47}
\end{gather*}
$$

Then

$$
\begin{equation*}
u_{0}(x):=e^{-\varkappa(x) \ln |x|} \in S(\mathbb{R}) \tag{4.48}
\end{equation*}
$$

and the Cauchy problem (4.44) admits a unique solution $u \in C^{\infty}\left(\mathbb{R}^{2}\right)$ having the following properties.

- Suppose that $c_{+} c_{-}<0$. Then for every $t \neq 0,0<\varepsilon \ll 1$ we have

$$
\begin{equation*}
\limsup _{x \rightarrow \infty}\left(|x|^{\varepsilon}|u(t, x)|\right) \geq 1 \tag{4.49}
\end{equation*}
$$

i.e., the decay to zero of $u(t, x), t \neq 0$, for $x \gg 1$ or $-x \gg 1$ is slower than any polynomial decay. In particular,

$$
\begin{equation*}
u(t, \cdot) \notin S(\mathbb{R}), \quad t \neq 0 \tag{4.50}
\end{equation*}
$$

- Let $c_{+} c_{-}>0$. Then

$$
\begin{equation*}
u(t, \cdot) \in S(\mathbb{R}) \tag{4.51}
\end{equation*}
$$

for $t>0$ (respectively, $t<0$ ) and

$$
\begin{equation*}
\limsup _{x \rightarrow \infty}\left(|x|^{\varepsilon} \mid u(t, x \mid) \geq 1, \quad \forall 0<\varepsilon \ll 1\right. \tag{4.52}
\end{equation*}
$$

for $t<0$ (respectively, $t>0$ ), provided $c_{+}, c_{-}<0$ (respectively, $\left.c_{+}, c_{-}>0\right)$.
Proof. The conditions (4.46), (4.47) imply

$$
\begin{equation*}
\left|D^{k} \varkappa(x)\right|=o\left(|x|^{-(k-\varepsilon)}\right), \quad x \rightarrow \infty, \quad \text { for } k \in \mathbb{N}, 0<\varepsilon \ll 1 \tag{4.53}
\end{equation*}
$$

which leads to (4.48).
By the method of the characteristics,

$$
\begin{equation*}
u(t, x)=u_{0}(\psi(t, x))=e^{-\varkappa(\psi(t, x)) \ln |\psi(t, x)|}, \quad|t| \leq T,|x| \gg 1 \tag{4.54}
\end{equation*}
$$

where $\psi(t, \cdot)=\varphi^{-1}(t, \cdot)$, with $\varphi(t, y)$ being the characteristic defined by

$$
\varphi^{\prime}=a(\varphi),\left.\quad \varphi\right|_{t=0}=y
$$

In view of (4.44), one obtains that for all $T>0$ there exists $M=M_{T}>0$ such that

$$
\psi(t, x)=\operatorname{sign}(x) \exp \left(\exp \left(\left((\ln \ln |x|)^{1-\sigma}-c_{ \pm}(1-\sigma) t\right)^{1 /(1-\sigma)}\right)\right), \quad \pm x \geq M,|t| \leq T
$$

if $0<\sigma<1$ and

$$
\begin{align*}
\psi(t, x) & =\operatorname{sign}(x) \exp \left(\exp \left(\exp \left(\ln \ln \ln |x|-c_{ \pm} t\right)\right)\right) \\
& =\operatorname{sign}(x) \exp \left(\exp \left(\exp \left(-c_{ \pm} t\right) \ln \ln |x|\right)\right) \\
& =\operatorname{sign}(x) \exp \left((\ln |x|)^{\exp \left(-c_{ \pm} t\right)}\right) \quad \pm x \geq M,|t| \leq T \tag{4.55}
\end{align*}
$$

if $\sigma=1$. Clearly, if $t \neq 0$ is fixed, we have

$$
\begin{align*}
\ln |\psi(t, x)| & =\exp \left(\left((\ln \ln |x|)^{1-\sigma}-c_{ \pm}(1-\sigma) t\right)^{1 /(1-\sigma)}\right) \\
& =\exp \left(\ln \ln |x|\left(1-c_{ \pm}(1-\sigma) t /(\ln \ln |x|)^{1-\sigma}\right)^{1 /(1-\sigma)}\right) \\
& =\exp \left(\ln \ln |x|-c_{ \pm} t(\ln \ln |x|)^{\sigma}+O\left((\ln \ln |x|)^{2 \sigma-1}\right)\right) \\
& =\exp \left(-c_{ \pm} t(\ln \ln |x|)^{\sigma}(1+o(1))\right) \ln |x| \\
& =\ln \left(|x|^{e^{-c_{ \pm} t(\ln \ln |x|)^{\sigma}(1+o(1))}}\right), \quad|x| \gg 1,|t| \leq T \tag{4.56}
\end{align*}
$$

if $0<\sigma<1$ and

$$
\begin{equation*}
\ln |\psi(t, x)|=(\ln |x|)^{\exp \left(-c_{ \pm} t\right)}, \quad \pm x \geq M,|t| \leq T \tag{4.57}
\end{equation*}
$$

if $\sigma=1$. Clearly, the choice of $\varkappa$ yields

$$
\varkappa(\psi(t, x))=\left\{\begin{array}{cc}
(\ln \ln |x|)^{1-\sigma}-c_{ \pm}(1-\sigma) t & \text { if } 0<\sigma<1  \tag{4.58}\\
e^{-c_{ \pm} t} \ln \ln |x| & \text { if } \quad \sigma=1
\end{array}, \quad|x| \gg 1,|t| \leq T,\right.
$$

which implies that

$$
|u(t, x)|=\left\{\begin{array}{ccc}
\exp \left(-\tilde{\varkappa}_{\sigma}(t, x) \ln |x|\right) & \text { if } 0<\sigma<1  \tag{4.59}\\
\exp \left(-\tilde{\varkappa}_{1}(t, x)(\ln |x|)^{\exp \left(-c_{ \pm} t\right)}\right) & \text { if } \quad \sigma=1
\end{array},\right.
$$

for $|x| \gg 1,|t| \leq T$,

$$
\tilde{\varkappa}_{\sigma}(t, x)=\left\{\begin{array}{cc}
\left((\ln \ln |x|)^{1-\sigma}-c_{ \pm}(1-\sigma) t\right) e^{-c_{ \pm} t(\ln \ln |x|)^{\sigma}(1+o(1))} & \text { if } 0<\sigma<1  \tag{4.60}\\
e^{-c_{ \pm t}} \ln \ln |x| & \text { if } \quad \sigma=1
\end{array}\right.
$$

for $|x| \gg 1,|t| \leq T$.
Now we readily obtain that for fixed $t \neq 0$ we have

$$
\lim _{x \rightarrow \pm \infty} \tilde{\varkappa}_{\sigma}=\left\{\begin{array}{ccc}
+\infty & \text { if } & c_{ \pm} t<0  \tag{4.61}\\
0 & \text { if } & c_{ \pm} t>0
\end{array},\right.
$$

provided $\sigma \in] 0,1[$. Hence, the proposition is proved for $0<\sigma<1$.
Finally, if $\sigma=1$, we observe that if $q \in] 0,1[$

$$
\begin{equation*}
\lim _{z \rightarrow+\infty} z^{N} e^{-(\ln \ln z)(\ln z)^{q}}=+\infty, \quad \forall N>0 \tag{4.62}
\end{equation*}
$$

while if $q>1$ we have

$$
\begin{equation*}
\lim _{z \rightarrow+\infty} z^{N} e^{-(\ln \ln z)(\ln z)^{q}}=0, \quad \forall N>0 . \tag{4.63}
\end{equation*}
$$

Therefore, for $q=\exp \left(-c_{ \pm} t\right)$, one obtains that for $\sigma=1$ (4.62) implies (4.51) and (4.63) yields (4.50). The proof is complete.

We conclude the section with an example on the role of regularity assumptions on the lower order terms. The result above emphasizes the fact that even a strong singularity in $t$ in the lower order term does not yield a loss of regularity or decay in the solution if the principal part is regular.

Consider the Cauchy problem for the second order wave equation

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u=\mu(t) u  \tag{4.64}\\
\left.u\right|_{t=0}=u^{0},\left.\quad u_{t}\right|_{t=0}=u_{1}
\end{array}\right.
$$

where $\mu(t)$ is an atomic measure defined as follows: given an increasing sequence of positive numbers $0<t_{1}<t_{2}<\ldots<t_{k}<\ldots$ we have

$$
\mu(t)=\sum_{j=1}^{\infty} \mu_{j} \delta\left(t-t_{j}\right), \quad \text { with } \quad \sum_{j=1}^{\infty}\left|\mu_{j}\right|<+\infty .
$$

We have
Proposition 4.5 The Cauchy problem (4.64) admits a unique solution u satisfying

$$
u \in\left(\bigcap _ { j = 1 } ^ { \infty } C ^ { j } \left([ 0 , t _ { 1 } [ : H ^ { s _ { 1 } - j , s _ { 2 } } ( \mathbb { R } ^ { n } ) ) ) \bigcap C \left(\left[0, \infty\left[: H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right)\right)\right.\right.\right.\right.
$$

and

$$
u_{t} \in L^{\infty}\left(\left[0,+\infty\left[: H^{s_{1}-1, s_{2}}\left(\mathbb{R}^{n}\right)\right)\right.\right.
$$

for all $u_{0} \in H^{s_{1}, s_{2}}\left(\mathbb{R}^{n}\right), u_{1} \in H^{s_{1}-1, s_{2}}\left(\mathbb{R}^{n}\right)$.
Proof. Using the Fourier transform, we are reduced to

$$
\left\{\begin{array}{l}
\partial_{t}^{2} \hat{u}(t, \xi)+|\xi|^{2} \hat{u}(t, \xi)=\mu(t) \hat{u}(t, \xi) \\
\left.\hat{u}\right|_{t=0}=\hat{u_{0}},\left.\quad \hat{u}_{t}\right|_{t=0}=\hat{u_{1}}
\end{array}\right.
$$

In view of the action of the Dirac delta function

$$
\delta\left(t-t_{0}\right) a(t)=a\left(t_{0}\right) \delta\left(t-t_{0}\right), \quad a \in C^{\infty}
$$

we have to solve the nonlocal equation

$$
\partial_{t}^{2} \hat{u}(t, \xi)+|\xi|^{2} \hat{u}(t, \xi)=\sum_{j=1}^{\infty} \mu_{j} \delta\left(t-t_{j}\right) \hat{u}\left(t_{j}, \xi\right)
$$

Clearly, for $t \in\left[0, t_{1}[\right.$ we have

$$
\hat{u}(t, \xi)=\cos (t|\xi|) \hat{u}_{0}(\xi)+\frac{\sin (t|\xi|)}{|\xi|} \hat{u}_{1}(\xi)
$$

while for $t \in\left[0, t_{2}[\right.$ (in view of the continuity with respect to $t$ ) we are reduced to the inhomogeneous problem

$$
\left\{\begin{array}{l}
\partial_{t}^{2} \hat{u}(t, \xi)+|\xi|^{2} \hat{u}(t, \xi)=\mu_{1} \delta\left(t-t_{1}\right)\left[\cos \left(t_{1}|\xi|\right) \hat{u_{0}}(\xi)+\frac{\sin \left(t_{1}|\xi|\right)}{|\xi|} \hat{u_{1}}(\xi)\right] \\
\hat{u}(0, \xi)=\hat{u}_{0}(\xi), \quad \hat{u}_{t}(0, \xi)=\hat{u_{1}}(\xi)
\end{array}\right.
$$

Hence, using the Green function identity

$$
\int_{0}^{t} \frac{\sin ((t-\tau)|\xi|)}{|\xi|} \delta\left(\tau-t_{1}\right) d \tau=H\left(t-t_{1}\right) \frac{\sin \left(\left(t-t_{1}\right)|\xi|\right)}{|\xi|}
$$

where $H$ denotes the Heaviside function, we obtain, for $t \in\left[0, t_{2}[\right.$ :

$$
\begin{aligned}
\hat{u}(t, \xi)= & \cos (t|\xi|) \hat{u_{0}}(\xi)+\frac{\sin (t|\xi|)}{|\xi|} \hat{u}_{1}(\xi) \\
& +\mu_{1} H\left(t-t_{1}\right) \frac{\sin \left(\left(t-t_{1}\right)|\xi|\right)}{|\xi|}\left[\cos \left(t_{1}|\xi|\right) \hat{u}_{0}(\xi)+\frac{\sin \left(t_{1}|\xi|\right)}{|\xi|} \hat{u}_{1}(\xi)\right] .
\end{aligned}
$$

Then we can conclude by iteration.
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## References

[1] A. Ascanelli, Quasilinear hyperbolic operators with Log-Lipschitz coefficients, J. Math. Anal. Appl. 295 (2004) 70-79.
[2] A. Ascanelli and M. Cappiello, Log-Lipschitz regularity for SG hyperbolic systems, J. Differential Equations 230 (2006), 556-578.
[3] A. Ascanelli and M. Cappiello, Hölder continuity in time for $S G$ hyperbolic systems. J. Differential Equations 244 (2008), 2091-2121.
[4] M. Cappiello, Fourier integral operators of infinite order and applications to SG-hyperbolic equations, Tsukuba J. Math. 28 (2004) n. 2, 311-361.
[5] M. Cicognani, A remark about hyperbolic equations with Log-Lipschitz coefficients, J. Partial Diff. Eqs. 10 n. 3 (1997), 284-288.
[6] M. Cicognani and F. Colombini, Modulus of continuity of the coefficients and loss of derivatives in the strictly hyperbolic Cauchy problem. J. Differential Equations 221 (2006), no. 1, 143-157.
[7] F. Colombini, E. De Giorgi and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temp, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 n. 6 (1979), 511-559.
[8] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, Duke Math. J. 77 n. 3 (1995), 657-698.
[9] H.O. Cordes, The technique of pseudodifferential operators, LMS Lecture Note Series, 202, Cambridge Univ. Press, Cambridge, 1995.
[10] S. Coriasco, Fourier integral operators in SG classes.II. Application to SG hyperbolic Cauchy problems, Ann. Univ. Ferrara Sez VII, 44 (1998), 81-122.
[11] S. Coriasco and L. Rodino, Cauchy problem for SG-hyperbolic equations with constant multiplicities, Ricerche Mat. 48 (1999), suppl., 25-43.
[12] S. Coriasco and L. Maniccia, Wave front set at infinity and hyperbolic linear operators with multiple characteristics, Ann. Global Anal. Geom. 24 (2003), no. 4, 375-400.
[13] Y.V. Egorov and B.-W. Schulze, Pseudo-differential operators, singularities, applications, Operator Theory: Advances and Applications, 93 Birkhäuser Verlag, Basel, 1997.
[14] V. Georgiev, Semilinear hyperbolic equations. With a preface by Y. Shibata, MSJ Memoirs, 7 (2nd ed.). Mathematical Society of Japan, Tokyo, 2005.
[15] D. Gourdin and T. Gramchev, Global in time solutions of evolution equations in scales of Banach function spaces in $\mathbb{R}^{n}$, Bull. Sci. Math. 131 (2007) n. 8, 761-786.
[16] D. Gourdin and T. Gramchev, Global Cauchy problems for hyperbolic systems with characteristics admitting superlinear growth for $|x| \rightarrow \infty$, C.R. Acad. Sci. Paris, Ser. I 347 (2009), 49-54.
[17] D. Gourdin and M. Mechab, Solutions globales d'un problème de Cauchy linèaire, J. Funct. Anal. 202:1 (2003), 123-146.
[18] E. Hebey, Sobolev spaces on Riemannian manifolds, Lecture Notes in Mathematics, 1635, Springer-Verlag, Berlin, 1996.
[19] L. Hörmander, The analysis of linear partial differential operators III. Pseudodifferential operators, Springer-Verlag, Berlin, (1985).
[20] W. Ichinose, The continuity of solutions with respect to a parameter to symmetric hyperbolic systems, to appear in Proc. ISAAC 2009.
[21] C. Parenti, Operatori pseudodifferenziali in $\mathbb{R}^{n}$ e applicazioni, Ann. Mat. Pura Appl. 93, 359-389 (1972).
[22] M. Reissig, Hyperbolic equations with non-Lipschitz coefficients, Rend. Sem. Mat. Univ. Politec. Torino 61 n. 2 (2003) 135-181.
[23] M. Reissig, About strictly hyperbolic operators with non-regular coefficients, Pliska Stud. Math. Bulgar. 15 (2003) 105-130.
[24] M. Ruzhansky and M. Sugimoto, Global L²-boundedness theorems for a class of Fourier integral operators, Comm. Partial Differential Equations 31 (2006), 547-569.
[25] M. Ruzhansky and M. Sugimoto, Global calculus of Fourier integral operators, weighted estimates, and applications to global analysis of hyperbolic equations. In "Operator Theory: Advances and Applications", 164, 65-78, Birkhäuser, Basel, 2006.
[26] E. Schrohe, Spaces of weighted symbols and weighted Sobolev spaces on manifolds. In "Pseudodifferential Operators", Proceedings Oberwolfach 1986. H. O. Cordes, B. Gramsch and H. Widom editors, Springer LNM, 1256 New York, 360-377 (1987).
[27] A. Wintner, The nonlocal existence problem of ordinary differential equations, Amer. J. Math. 67 (1945), 277-284.

