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# A NOTE ON THE LINEARIZED FINITE THEORY OF ELASTICITY

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**Abstract:** In this paper we solve a simple dead load problem for an isotropic constrained material according to the linearized finite theory of elasticity. We show that the solution of such a problem can be obtained by linearizing with respect to the displacement gradient the solution of the corresponding problem in finite elasticity for an isotropic material subject to the same constraint, exactly as occurs for the constitutive equations of the two theories. On the contrary, the solution of the same dead load problem provided by the classical linear elasticity for constrained materials can be obtained by the solution of the corresponding problem for the unconstrained linear elastic material for limiting behaviour of suitable elastic moduli. The same applies for the constitutive equations of the classical linear elasticity for constrained materials: they are derived by those of the linear elasticity for unconstrained materials for limiting values of some elastic modulus. Finally we compare the solutions in finite elasticity, linearized finite theory of elasticity, classical linear elasticity for constrained materials and we show that they are in agreement with different hypotheses on the prescribed loads.

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**Key Words:** constrained elastic materials, linearized finite elasticity

## 1. Introduction

In this paper we study a dead load problem for an incompressible isotropic elastic material according to the linearized finite theory of elasticity proposed

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by Hoger and Johnson in [2], [3] and Marlow in [4]. For brevity, in the following such a theory is denoted by LFTE. The LFTE is the best choice in order to describe the behaviour of constrained linear hyperelastic materials, even if usually for such materials the classical linear theory of elasticity is adopted.

In fact, in LFTE the linear constitutive equations differ from those of the classical linear elasticity by some terms that are first order in the strain, so that only LFTE provides the accuracy required by a linear theory. In [2], [3], [4] the LFTE is applied to static problems, while in [5], [6] is used to solve dynamical problems, as wave propagation; in both cases unexpected results are obtained.

The first aim of this paper is to obtain and compare the solution of a dead load problem for an incompressible isotropic elastic material in LFTE with the solution provided by the finite theory of elasticity for incompressible isotropic materials. The second aim is to obtain and compare the solution provided by the classical linear elasticity for incompressible isotropic materials with the solution obtained in classical linear elasticity for compressible isotropic materials.

First we show that the solution provided by LFTE can be obtained by linearization of the solution of the corresponding problem for incompressible materials in finite elasticity. Moreover, we show that the solution provided by the classical linear elasticity for incompressible materials can be obtained by the solution appropriate for compressible linear elastic materials for limiting behaviour of the bulk modulus.

The same occurs for the constitutive equations: the stress-strain relations for an incompressible isotropic material used in LFTE are obtained by linearizing with respect to the displacement gradient the stress-strain relations for the corresponding incompressible material in finite elasticity, while the constitutive relations of the classical linear elasticity for incompressible isotropic materials are obtained from those for compressible linear elastic materials when the bulk modulus goes to infinity.

Finally we show that under different hypotheses on the prescribed loads both the solution of the dead load problem in LFTE and the solution in classical linear elasticity for constrained materials can be obtained by linearization of the solution in finite elasticity: the former holds when the loading creates a large pressure, while the latter holds when the loading is small.

## 2. The Linearized Finite Theory of Elasticity

In this section we summarize the field equations of the so-called linearized finite theory of elasticity for constrained materials, derived by Hoger and Johnson in [2], [3].

In LFTE the linear constitutive equations are obtained by linearization with respect to the displacement gradient of the corresponding finite constitutive equations for the material subject to the same constraint and the same material symmetry. As extensively shown in [2], [3], this procedure of linearization provides constitutive equations which differ from those of the classical linear elasticity for constrained materials by terms that are first order in the strain; moreover, such terms play an important role in boundary value problems for incompressible materials, as shown in [2].

Denote by $\mathcal{B}_0$ a fixed reference configuration and by $\mathcal{B} = \mathbf{f}(\mathcal{B}_0)$ the deformed configuration, where $\mathbf{f}$ is the deformation function which carries point $\mathbf{X} \in \mathcal{B}_0$ into point $\mathbf{x} = \mathbf{f}(\mathbf{X}) \in \mathcal{B}$.

Then,

$$\mathbf{u}(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \mathbf{X}, \tag{1}$$

$$\mathbf{F} = \mathrm{Grad}\mathbf{f}, \tag{2}$$

$$\mathbf{H} = \mathrm{Grad}\mathbf{u} = \mathbf{F} - \mathbf{I} \tag{3}$$

denote the displacement, the deformation gradient and the displacement gradient, respectively; in the previous formulas Grad is the gradient operation taken with respect to $\mathbf{X}$ and $\mathbf{I}$ is the identity tensor.

If we linearize the finite Green strain tensor

$$\mathbf{E}_G = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \tag{4}$$

about the zero strain state, we obtain the infinitesimal strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right). \tag{5}$$

For an elastic material subject to a single constraint, the finite constraint equation is

$$\hat{c}\left(\mathbf{E}_G\right) = 0. \tag{6}$$

Now we use the expansion $\hat{c}\left(\mathbf{E}_G\right) \cong \hat{c}(\mathbf{O}) + \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \mathbf{E}$, where $\mathbf{O}$ is the zero tensor, moreover we take into account that $\hat{c}(\mathbf{O}) = 0$ and we denote by $\tilde{c}(\mathbf{E}) = \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \mathbf{E}$ the linear constraint function; then the linearized constraint equation is

$$\tilde{c}(\mathbf{E}) = 0. \tag{7}$$

For hyperelastic constrained materials with strain energy function $W = \hat{W}(\mathbf{E}_G)$, the finite constitutive equation for the Cauchy stress $\mathbf{T}$ is

$$\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{F} \frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T + q\mathbf{F}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E_G})\mathbf{F}^T, \tag{8}$$

where $q$ is a Lagrange multiplier, and the Piola-Kirchhoff stress is

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-T}. \tag{9}$$

By means of an accurate procedure of linearization of $\mathbf{T}$ in which the linearization of the derivative of $\hat{W}$ parallels that of the derivative of $\hat{c}$ and using the conditions

$$\det \mathbf{F} \cong 1 + \mathrm{tr}\mathbf{E}, \tag{10}$$

$$\mathbf{F}^{-T} \cong \mathbf{I} - \mathbf{H}^T, \tag{11}$$

we obtain from (9) the following expression for $\mathbf{S}$ according to the LFTE

$$\begin{aligned}\mathbf{S} = &\left.\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_c \mathbf{E} + q(1 + \mathrm{tr}\mathbf{E})\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \\ &+ q\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + q\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\mathbf{E};\end{aligned} \tag{12}$$

in (12) the symbol $|_c$ indicates evaluation on the linearized constraint equation (7).

As detailed shown in [3], Section 3, in LFTE the Cauchy stress and the Piola-Kirchhoff stress for constrained materials differ by terms that are first order in the strain, while in classical linear elasticity it is *assumed* that they coincide, as occurs for unconstrained materials.

For an incompressible material the finite and the linearized constraint equations (6) and (7) become (see [3], Section 4)

$$\det(2\mathbf{E}_G + 1) = 1, \tag{13}$$

$$\mathrm{tr}\mathbf{E} = 0, \tag{14}$$

respectively; moreover, for isotropy, $W$ is a function of polynomial invariants of $\mathbf{E}_G$

$$W = \hat{W}(\mathbf{I} \cdot \mathbf{E}_G, \mathbf{I} \cdot \mathbf{E}_G^2, \mathbf{I} \cdot \mathbf{E}_G^3). \tag{15}$$

Therefore, for an incompressible isotropic material in LFTE the Piola-

Kirchhoff stress (12) reduces to

$$\mathbf{S} = 2\mu\mathbf{E} + 2q\left(\mathbf{I} - \mathbf{H}^T\right), \tag{16}$$

while the Cauchy stress takes the following form

$$\mathbf{T} = 2\mu\mathbf{E} + 2q\mathbf{I}, \tag{17}$$

where $\mu$ is the shear modulus.

Coincidently, the expression for $\mathbf{T}$ in classical linear elasticity is just (17), although for other constraints the classical constitutive equations for $\mathbf{T}$ are not accurate.

Finally, we list the basic field equations for incompressible isotropic materials according to LFTE

$$\begin{aligned} &\mathbf{H} = \mathrm{Grad}\mathbf{u}, \\ &\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right), \\ &\mathrm{tr}\mathbf{E} = 0, \\ &\mathbf{S} = 2\mu\mathbf{E} + 2q\left(\mathbf{I} - \mathbf{H}^T\right), \\ &\mathrm{Div}\mathbf{S} + \mathbf{b} = \mathbf{0}; \end{aligned} \tag{18}$$

in $(18)_5$, Div is the divergence operator taken with respect to $\mathbf{X}$ and $\mathbf{b}$ is the body force density measured per unit volume of $\mathcal{B}_0$.

## 3. A Piola-Kirchhoff Traction Problem in LFTE

In this section we briefly expose the dead load boundary value problem formulated and solved in [2], Section 5.

Let the undeformed configuration $\mathcal{B}_0$ be a unit cube centered on a rectangular coordinate system, with unit normals $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$, $\pm\mathbf{e}_3$ to the faces. The material is taken to be incompressible and isotropic, so that equations (18) hold. The prescribed Piola-Kirchhoff tractions are:

$\pm\alpha\mathbf{e}_1$ on the face with unit normal $\pm\mathbf{e}_1$,
$\pm\alpha\mathbf{e}_2$ on the face with unit normal $\pm\mathbf{e}_2$,
$\pm\beta\mathbf{e}_3$ on the face with unit normal $\pm\mathbf{e}_3$;

moreover, in $(18)_5$ the body force density $\mathbf{b}$ is taken to be zero.

By substituting the linearized constraint condition $(18)_3$ and the Piola-Kirchhoff tractions into $(18)_4$, we obtain $q = \frac{1}{6}(2\alpha + \beta)$; then, if $6\mu \neq 2\alpha + \beta$,

$(18)_4$ provides the following homogeneous strain

$$\mathbf{E} = \frac{\alpha - \beta}{6\mu - (2\alpha + \beta)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{19}$$

Apart from a translation, the displacement $\mathbf{u}$ from the origin is $\mathbf{u} = \mathbf{EX}$; then by (19) we have

$$\mathbf{u} = \frac{\alpha - \beta}{6\mu - (2\alpha + \beta)} \begin{bmatrix} X_1 \\ X_2 \\ -2X_3 \end{bmatrix}. \tag{20}$$

## 4. The Dead Load Problem in Finite Elasticity

In this section the dead load boundary value problem in Section 3 is formulated within the framework of the finite theory of elasticity for constrained materials.

In finite elasticity for an isotropic material subject to the constraint of incompressibility $\det \mathbf{F} = 1$, the Cauchy stress (8) takes the form (see [7], formula $(49.5)_1$)

$$\mathbf{T} = \varphi_1 \mathbf{F}\mathbf{F}^T + \varphi_{-1} \mathbf{F}^{-T}\mathbf{F}^{-1} + 2q\mathbf{I}, \tag{21}$$

where the coefficients $\varphi_1$ and $\varphi_{-1}$ are functions of the two invariants $I_\mathbf{B}$ and $II_\mathbf{B}$ of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

It follows by (9) and (21) that the Piola-Kirchhoff stress is

$$\mathbf{S} = \varphi_1 \mathbf{F} + \varphi_{-1} \mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{F}^{-T} + 2q\mathbf{F}^{-T}. \tag{22}$$

As in Section 3, the prescribed Piola-Kirchhoff tractions on the faces of the cube are

$$\mathbf{S} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \tag{23}$$

the body force density $\mathbf{b}$ is taken to be zero, so that the Piola-Kirchhoff stress (22) must satisfy the equilibrium equation $\mathrm{Div}\mathbf{S} = \mathbf{0}$.

It is well known that for non linear elastic materials it is rarely possible to obtain analytical solutions of boundary value problems, even if simple geometries of the reference configuration and simple forms of strain energy function are used. This situation just occurs in the present problem. For this reason, we adopt an inverse method, that is we *assume* an explicit form for the solution, suggested by the geometry of the problem, in this case a simple extension with stretch $\Lambda$ along the axis $\mathbf{e}_3$ (see [8], Section 44). For this homogeneous strain the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 & 0 \\ 0 & \Lambda^{-\frac{1}{2}} & 0 \\ 0 & 0 & \Lambda \end{bmatrix}. \tag{24}$$

By combining (22) with (23) and using (24), we obtain

$$\begin{aligned} \alpha &= \varphi_1 \Lambda^{-\frac{1}{2}} + \varphi_{-1} \Lambda^{\frac{3}{2}} + 2q\Lambda^{\frac{1}{2}}, \\ \alpha &= \varphi_1 \Lambda^{-\frac{1}{2}} + \varphi_{-1} \Lambda^{\frac{3}{2}} + 2q\Lambda^{\frac{1}{2}}, \\ \beta &= \varphi_1 \Lambda + \varphi_{-1} \Lambda^{-3} + 2q\Lambda^{-1}, \end{aligned} \tag{25}$$

where:

$$\varphi_1 = \varphi_1(2\Lambda^{-1} + \Lambda^2, 2\Lambda + \Lambda^{-2}), \;\; \varphi_{-1} = \varphi_{-1}(2\Lambda^{-1} + \Lambda^2, 2\Lambda + \Lambda^{-2}).$$

Equations (25) provide

$$\begin{aligned} 2q = \Lambda \left(1 + 2\Lambda^{\frac{3}{2}}\right)^{-1} \Big[ & 2\alpha + \beta - \varphi_1 \Lambda \left(1 + 2\Lambda^{-\frac{3}{2}}\right) \\ & -\varphi_{-1} \Lambda^{\frac{2}{3}} \left(2 + \Lambda^{-\frac{9}{2}}\right) \Big]. \end{aligned} \tag{26}$$

If we substitute (26) into (25) we can eliminate the Lagrange multiplier in such equations. From $(25)_{1,2}$ we have

$$\begin{aligned} \alpha = \varphi_1 \Lambda^{-\frac{1}{2}} + \varphi_{-1} \Lambda^{\frac{3}{2}} + \Lambda^{\frac{3}{2}} \left(1 + 2\Lambda^{\frac{3}{2}}\right)^{-1} & [2\alpha + \beta \\ -\varphi_1 \Lambda \left(1 + 2\Lambda^{-\frac{3}{2}}\right) - \varphi_{-1} \Lambda^{\frac{2}{3}} \left(2 + \Lambda^{-\frac{9}{2}}\right) & \Big], \end{aligned} \tag{27}$$

while $(25)_3$ provides

$$\begin{aligned} \beta = \varphi_1 \Lambda + \varphi_{-1} \Lambda^{-3} + \left(1 + 2\Lambda^{\frac{3}{2}}\right)^{-1} & [2\alpha + \beta \\ -\varphi_1 \Lambda \left(1 + 2\Lambda^{-\frac{3}{2}}\right) - \varphi_{-1} \Lambda^{\frac{2}{3}} \left(2 + \Lambda^{-\frac{9}{2}}\right) & \Big]. \end{aligned} \tag{28}$$

Equations (27), (28) show that in finite elasticity the components of the deformation gradient are defined in implicit form in terms of the prescribed tractions and the constitutive coefficients $\varphi_1$ and $\varphi_{-1}$, which again depend on the components of the deformation gradient.

## 5. Comparison of the Solutions in LFTE and Finite Elasticity

In this section we compare the solution of the dead load problem obtained by means of LFTE in Section 3, formula (20), with the solution of the same problem obtained in implicit form by the finite theory of elasticity in Section 4, formulas (27), (28).

To this aim, first we observe that by (3) and (24) we have

$$\mathbf{H} = \begin{bmatrix} \Lambda^{-\frac{1}{2}} - 1 & 0 & 0 \\ 0 & \Lambda^{-\frac{1}{2}} - 1 & 0 \\ 0 & 0 & \Lambda - 1 \end{bmatrix}; \tag{29}$$

then we substitute (29) into (27), (28) and we linearize such equations with respect to the displacement gradient. Finally we take into account that the shear modulus $\mu$ for isotropic incompressible materials in linear elasticity is connected to the coefficients $\varphi_1$ and $\varphi_{-1}$ by the relation

$$\mu = \varphi_1(3,3) - \varphi_{-1}(3,3) \tag{30}$$

(see [7], formula (50.14)).

As final result we obtain (19) for the strain tensor $\mathbf{E}$ and then (20) for the displacement $\mathbf{u}$.

Then the solution of the dead load problem in LFTE can be obtained by linearizing with respect to the displacement gradient the solution of the corresponding problem in finite elasticity; the same occurs for the constitutive equations of the two theories.

## 6. The Dead Load Problem in Classical Linear Elasticity for Constrained Materials

In this section we formulate the same dead load problem according to the classical linear elasticity for constrained materials. We only recall the basic equations of such a theory and the corresponding solution of the dead load problem.

We refer to [2], Sections 7, 8, for more details. In classical linear elasticity for incompressible isotropic materials the following basic equations for the undeformed configuration hold

$$\begin{aligned} &\mathbf{H} = \mathrm{Grad}\mathbf{u}, \\ &\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right), \\ &\mathrm{tr}\mathbf{E} = 0, \\ &\mathbf{S} = 2\mu\mathbf{E} + 2q\mathbf{I}, \\ &\mathrm{Div}\mathbf{S} + \mathbf{b} = \mathbf{0}. \end{aligned} \tag{31}$$

Note that in classical linear elasticity for constrained materials it is stated that the Piola-Kirchhoff stress $\mathbf{S}$ and the Cauchy stress $\mathbf{T}$ coincide.

For our dead load problem the Piola-Kirchhoff tractions are given by (23), the basic equations are listed in (31) and the body force $\mathbf{b}$ is taken to be zero. We obtain the strain

$$\mathbf{E} = \frac{\alpha - \beta}{6\mu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{32}$$

and then the displacement

$$\mathbf{u} = \frac{\alpha - \beta}{6\mu} \begin{bmatrix} X_1 \\ X_2 \\ -2X_3 \end{bmatrix}, \tag{33}$$

apart from a translation.

## 7. The Dead Load Problem in Classical Linear Elasticity for Unconstrained Materials

In this section we solve the dead load problem according to the classical linear elasticity for unconstrained materials. In classical linear elasticity for isotropic materials the following basic equations for the undeformed configuration hold

$$\begin{aligned} &\mathbf{H} = \mathrm{Grad}\mathbf{u}, \\ &\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right), \\ &\mathbf{S} = 2\mu\mathbf{E} + \lambda\left(\mathrm{tr}\mathbf{E}\right)\mathbf{I}, \\ &\mathrm{Div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \end{aligned} \tag{34}$$

where $\mu$ and $\lambda$ are the Lamé moduli of the material. For our dead load problem the Piola-Kirchhoff tractions are given by (23), the basic equations are listed in (34) and moreover in $(34)_4$ we set $\mathbf{b} = \mathbf{0}$. We obtain the strain

$$\mathbf{E} = \begin{bmatrix} \frac{\alpha\mu/k}{3\mu} + \frac{\alpha-\beta}{6\mu}\left(1 - \frac{2}{3}\frac{\mu}{k}\right) & 0 & 0 \\ 0 & \frac{\alpha\mu/k}{3\mu} + \frac{\alpha-\beta}{6\mu}\left(1 - \frac{2}{3}\frac{\mu}{k}\right) & 0 \\ 0 & 0 & \frac{\beta\mu/k}{3\mu} - \frac{\alpha-\beta}{3\mu}\left(1 - \frac{2}{3}\frac{\mu}{k}\right) \end{bmatrix} \tag{35}$$

and the displacement

$$\mathbf{u} = \begin{bmatrix} \left(\dfrac{\alpha\mu/k}{3\mu} + \dfrac{\alpha-\beta}{6\mu}\left(1 - \dfrac{2}{3}\dfrac{\mu}{k}\right)\right) X_1 \\ \left(\dfrac{\alpha\mu/k}{3\mu} + \dfrac{\alpha-\beta}{6\mu}\left(1 - \dfrac{2}{3}\dfrac{\mu}{k}\right)\right) X_2 \\ \left(\dfrac{\beta\mu/k}{3\mu} - \dfrac{\alpha-\beta}{3\mu}\left(1 - \dfrac{2}{3}\dfrac{\mu}{k}\right)\right) X_3 \end{bmatrix}, \tag{36}$$

apart from a translation.

In the previous formulas, the components of $\mathbf{E}$ and $\mathbf{u}$ are written in terms of the shear modulus $\mu$ and the bulk modulus $k = \frac{2}{3}\mu + \lambda$, in order to facilitate the comparison of the displacement (36) for the unconstrained linear elastic material with the displacement (33) for the constrained linear elastic material.

## 8. Comparison of the Solutions in Classical Linear Elasticity for Constrained and Unconstrained Materials

In this section the solution of the dead load problem in classical linear elasticity for the unconstrained material (36) is compared with the solution in classical linear elasticity for the constrained material (33).

When the bulk modulus $k$ is large with respect to the shear modulus $\mu$, so that $\mu/k \to 0$, we see from (36) that the displacement field (36) for the unconstrained material reduces to the displacement field (33) for the incompressible material.

In one sense we can claim that the behaviour of the two solutions (36) and (33) parallels that of the two constitutive equations $(34)_3$ and $(31)_4$. In fact, the constitutive equation $(34)_3$ reduces to the form $(31)_4$ in the limiting case as $\mu/k \to 0$ and then $\lambda \to \infty$, while correspondently $\mathrm{tr}\mathbf{E} \to 0$, in such a way that their product is finite, so that for incompressible isotropic materials the stress $\mathbf{S}$ is determined only to a within a hydrostatic pressure.

In classical linear elasticity this limiting behaviour can be observed in other problems for constrained materials: for instance, England in [1], Section 2, compares the solution of a plane problem for an orthotropic material reinforced by two orthogonal families of inestensible fibres with the solution of the corresponding problem for an orthotropic unconstrained material.

## 9. Comparison of the Solutions in Finite Elasticity, LFTE and Classical Linear Elasticity for Constrained Materials

In Section 5 we have shown how the solution (27), (28) appropriate for the constrained material in finite elasticity can provide the solution (20) for LFTE.

We briefly summarize such a procedure. From (24) we have $\Lambda^{-\frac{1}{2}} = \mathrm{F}_{11}$, so that (27) takes the form $G(\mathrm{F}_{11}) = 0$; then at the first order of approximation we have

$$G(\mathrm{F}_{11}) \cong G(1) + G'(1)(\mathrm{F}_{11} - 1). \tag{37}$$

From (27) it follows that $G(1) = \frac{1}{3}(\beta - \alpha)$ and moreover, by using formula (30), $G'(1) = 2\mu - \frac{1}{3}(2\alpha + \beta)$; then (37) provides for the component $\mathrm{E}_{11}$ of the infinitesimal strain tensor $\mathbf{E}$ defined in (5) the expression

$$\mathrm{E}_{11} = \frac{\alpha - \beta}{6\mu - (2\alpha + \beta)}; \tag{38}$$

for $\mathrm{E}_{22}$ the same expression holds.

The previous procedure applied to (28) yields

$$\mathrm{E}_{33} = \frac{2(\beta - \alpha)}{6\mu - (2\alpha + \beta)}. \tag{39}$$

Formulae (38), (39) exactly give the strain tensor $\mathbf{E}$ represented in (19).

In this section our aim is to obtain from the solution (27), (28) appropriate for the finite elasticity the solution (32) provided by the classical linear elasticity for constrained materials.

To this aim, we write (27) in the form $L(\alpha, \beta, \mathrm{F}_{11}) = 0$ and then we apply to such an equation the implicit function theorem in order to obtain $\mathrm{F}_{11}$ as a function of $\alpha$ and $\beta$. If suitable analytical conditions are satisfied in a neighborhood of $(0, 0, 1)$, at the first order of approximation we have

$$\mathrm{F}_{11} \cong 1 - \frac{\left.\frac{\partial L}{\partial \alpha}\right|_{(0,0,1)}}{\left.\frac{\partial L}{\partial \mathrm{F}_{11}}\right|_{(0,0,1)}}\alpha - \frac{\left.\frac{\partial L}{\partial \beta}\right|_{(0,0,1)}}{\left.\frac{\partial L}{\partial \mathrm{F}_{11}}\right|_{(0,0,1)}}\beta; \tag{40}$$

from (27) it follows that

$$\left.\frac{\partial L}{\partial \alpha}\right|_{(0,0,1)} = -\frac{1}{3}, \qquad \left.\frac{\partial L}{\partial \beta}\right|_{(0,0,1)} = \frac{1}{3}$$

and, by using formula (30), $\left.\frac{\partial L}{\partial \mathrm{F}_{11}}\right|_{(0,0,1)} = 2\mu$, so that (40) becomes

$$\mathrm{F}_{11} \cong 1 + \frac{\alpha}{6\mu} - \frac{\beta}{6\mu}. \tag{41}$$

Moreover, by applying a similar procedure to (28) we have

$$\mathrm{F}_{33} \cong 1 - \frac{\alpha}{3\mu} + \frac{\beta}{3\mu}. \tag{42}$$

In terms of the components of the infinitesimal strain tensor $\mathbf{E}$, (41) and (42) take the form

$$\mathrm{E}_{11} = \frac{\alpha - \beta}{6\mu}, \tag{43}$$

$$\mathrm{E}_{33} = \frac{\beta - \alpha}{3\mu}, \tag{44}$$

respectively, that is the solution (32) provided by the classical linear elasticity for constrained materials.

It is worth noting that both (38), (39) and (43), (44) are in agreement with a linear theory, because $\mathbf{H}$ is assumed to be small and only terms that are at most linear in $\mathbf{H}$ are retained.

However a considerable difference must be remarked: (37) and then (38), (39) have been attained without hypotheses concerning the smallness of $\alpha$ and $\beta$, that is (19) holds when the loading creates a large pressure, while (41), (42) and then (43), (44) have been obtained under the hypothesis that $\alpha$ and $\beta$ are small, and then also the pressure $-\frac{1}{3}(2\alpha + \beta)$ is small.

Moreover we note that in LFTE, as extensively shown in [2], Section 5, $\alpha$ and $\beta$ may be arbitrarily large, but the difference $|\alpha - \beta|$ must satisfy a suitable restriction defining the range over which $|\alpha - \beta|$ can vary for LFTE to be applicable; a different restriction on $|\alpha - \beta|$ is placed in classical linear elasticity (see [2], Section 8).

Of course (38), (39) reduce to (43), (44) when the pressure is small. This behaviour of the solutions for LFTE and classical linear elasticity parallels the behaviour of the corresponding constitutive equations $(18)_4$, and $(31)_4$: if the pressure is small, $(18)_4$ reduces to $(31)_4$ because the term $2q\mathbf{H}^T$ is neglected. This is the usual hypothesis adopted in classical linear elasticity for constrained materials, since in the current literature concerning such materials almost all authors assume that small strains correspond to small prescribed loads.

On the contrary, LFTE is based on a more reasonable hypothesis: for constrained materials the strain can be small even under large loads, and this is just a characteristic of the presence of the constraint.

## 10. Conclusions

In this paper a dead load problem for a constrained isotropic material is solved both in linearized finite theory of elasticity and classical linear elasticity, with the aim to enlighten an interesting property of the two corresponding solutions. The main feature of the two theories is the method followed in order to obtain the constitutive equations for a material subject to a constraint: in LFTE the constitutive equations are obtained by linearizing with respect to the displacement gradient the constitutive equations for the corresponding constrained material in finite elasticity, while in classical linear elasticity the constitutive equations are obtained by those appropriate for the corresponding unconstrained linear elastic material for limiting value of some elastic modulus.

In this paper we show that for a dead load problem the same behaviour characterizes the solutions provided by the two theories.

Finally, we show that both the solution appropriate for LFTE and the solution provided by the classical linear elasticity for constrained materials can be obtained by linearizing the solution of the corresponding problem in finite elasticity, under different hypotheses on the prescribed load.

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## References

[1] A.H. England, Plane problems for fibre-reinforced linearly elastic solids, In: *Continuum Theory of the Mechanics of Fibre-Reinforced Composites* (Ed. A.J.M. Spencer), Springer, Wien (1984).

[2] A. Hoger, B.E. Johnson, Linear elasticity for constrained materials: Incompressibility, *J. Elasticity*, **38** (1995), 69-93.

[3] A. Hoger, B.E. Johnson, Linear elasticity for constrained materials: General theory for hyperelasticity, *J. Elasticity*, **38** (1995), 95-120.

[4] R.S. Marlow, On the stress in an internally constrained elastic material, *J. Elasticity*, **27** (1992), 97-131.

[5] M.L. Tonon, Waves in constrained linear elastic materials, *J. Elasticity*, **69** (2002), 15-39.

[6] M.L. Tonon, Wave propagation according to the linearized finite theory of elasticity, *International J. of Pure and Applied Mathematics*, **40**, No. 1 (2007), 135-152.

[7] C. Truesdell, W. Noll, The nonlinear field theories of mechanics, In: *Hand. der Physik*, Volume III/3 (Ed. S. Flügge), Springer, Berlin (1965).

[8] C. Truesdell, R.A. Toupin, The classical field theories, In: *Hand. der Physik*, Volume III/1 (Ed. S. Flügge), Springer, Berlin (1960).