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Numerical Comparison of Different Weights in Shepard's Interpolants on the Sphere

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Abstract

In this paper we present a computational analysis of different weights in the global and local Shepard's method on the sphere. More precisely, we used a modified Shepard's interpolant with nodal functions given by zonal basis functions [8], and successively we presented an efficient algorithm associated [6]. Here, we propose the use of exponential weights in spherical Shepard's formulas and we perform a comparison of them with the classical one. In particular, numerical results highlight that an appropriate choice of weights permits to improve accuracy and it is also of considerable interest in applications.

Mathematics Subject Classification: 65D05, 65D15, 65D17

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1 Introduction

Given a set of N distinct points $\mathcal{X}_N = \{P_i\}_{i=1}^N$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ centered at the origin and a set of N real values $\mathcal{F}_N = \{f_i\}_{i=1}^N$, the problem that we address is the construction of a function F defined on $\mathbb{S}^2$ satisfying the interpolation conditions $F(P_i) = f_i$, for $i = 1, \ldots, N$. Applications include modeling closed surfaces in CAGD and representing scalar functions which estimate geophysical or meteorological quantities at all points on the surface of the Earth based on a discrete sample of values taken at arbitrary locations.

Several methods for spherical interpolation have been proposed (see [9] for an overview). In [8] a local interpolation method was presented. It is a mixed technique combining the Shepard's method and the zonal basis function method [4, 11, 14], in order to achieve stability and accuracy. A parallel version

based on the very same approach was presented in [7]. In [6] we presented an efficient and accurate algorithm which implements the method proposed in [8].

In this paper we present an extension of the method proposed in [8, 6]. In fact, we substitute to the classical Shepard's weight some exponential weights in the global and local versions of Shepard's interpolants defined on the sphere. The idea was already formulated in the bivariate setting, see [1, 2, 3]. The aim is here to find if there are weights in the spherical Shepard's interpolants which furnish better approximants than the classical Shepard's one. Indeed, Shepard's method has recently gained a new interest, since it can be considered as a partition of unity method (see [10]). A recent application which is based on the use of exponential weights in a partition of unity can be found in [13]. Thus, the possibility of using different weights constituites an important novelty compared to traditional weights. The computational analysis suggests to users improvements of the accuracy both in global and local formulas, due to the appropriate choice of the parameters in classical and exponential Shepard's type weights. Moreover, the numerical comparison is in our opinion very meaningful for applications, where exponential weights can be successfully used.

The paper is organized as follows. In Section 2 we present the classical Shepard's method formulated in the spherical setting with different weights. Section 3 is devoted to the local scheme using zonal basis functions as nodal functions in the Shepard's type interpolants. In Section 4 numerical results are presented.

2 Shepard's formula on the sphere

Let us consider a linear space $\phi(D)$, $D \in \mathbb{S}^2$, spanned by the functions $g_k : D \to \mathbb{R}$, $k = 1, \ldots, N$, such that

$$g_k(P) \geq 0, \quad \sum_{k=1}^{N} g_k(P) = 1, \quad g_k(P_i) = \delta_{ki}. \tag{1}$$

We define *Shepard's interpolant* the function

$$F(P) = \sum_{k=1}^{N} f_k g_k(P). \tag{2}$$

F satisfies the interpolation conditions and achieves the characteristic properties of a weighted arithmetic means. Let us consider now functions w_k, $k = 1, \ldots, N$, satisfying $w_k(P) = 0$, for $P = P_j$, $j \neq k$, and $w_k(P) > 0$, for $P = P_k$. We can define

$$g_k(P) = \frac{w_k(P)}{\sum_{j=1}^{N} w_j(P)}, \tag{3}$$

and these g_k can be interpreted as the basis functions of the space $\phi(D)$. Let be α a real and continuous function such that

$$\alpha(P,Q) > 0,\ P \neq Q; \quad \alpha(P,P) = 0,\ \forall P,Q \in D. \tag{4}$$

Then, set $\alpha(P,Q) = \alpha(d_g(P,Q))$, where $d_g(P,Q) = \arccos(P^TQ)$ denotes the geodesic distance between P and Q, we consider

$$w_k(P) = \prod_{i=1, i\neq k} \alpha(d_g(P,P_i)) \tag{5}$$

and these weights satisfy the properties above. Possible choices of α can be

$$\begin{aligned}
\alpha_p(P,Q) &= (d_g(P,Q))^p, && p > 0,\\
\alpha_\gamma(P,Q) &= \frac{\exp\left(\gamma \cdot (d_g(P,Q))^\mu\right)}{(d_g(P,Q))^\mu}, && \gamma \geq 0,\ \mu > 0,\\
\alpha_\delta(P,Q) &= \exp\left(\delta^2 \cdot d_g(P,Q)\right), && \delta > 0.
\end{aligned}$$

The function α_p, for $p = 2$, gives the classical *spherical Shepard's formula*, considered for instance in [8] and [6]. The use of a power weight α_p, with $p > 2$, and of exponential weights α_γ and α_δ, is justified by the rapidly decreasing behaviour of these functions, which makes unnecessary the use of compactly supported functions, and it is also of remarkable interest in applications. Moreover, we remark that exponential weights constitute a partition of unity (see, e.g., [10]). Some results concerning exponential-type weights in Shepard's method formulated in the bivariate setting can be found in [1] and [2].

3 Local Shepard's formula using ZBFs

In this section we briefly recall the local Shepard's method using zonal basis functions as nodal functions, referring to [8, 5, 6] for the details. At first we introduce the following definition.

Definition 3.1 *Given a set of distinct nodes $\mathcal{X}_N = \{P_i\}_{i=1}^N$ lying on $\mathbb{S}^2$, and the corresponding set of function values $\mathcal{F}_N = \{f_i\}_{i=1}^N$, a modified spherical Shepard's interpolant $\tilde{F} : \mathbb{S}^2 \to \mathbb{R}$ takes the form*

$$\tilde{F}(P) = \sum_{j=1}^N w_j(P) Z_j(P) \Big/ \sum_{k=1}^N w_k(P), \tag{6}$$

where the nodal functions $Z_j(P)$, $j = 1, \ldots, N$, are local approximations to f in P_j, and the relative weight functions are defined by

$$w_j(P) = \frac{\tau(P,P_j)}{\alpha(d_g(P,P_j))}, \tag{7}$$

for

$$\tau(P,P_j)=\begin{cases}1, & \text{if} \quad P_j\in\mathcal{K}(P,r),\\ 0, & \text{if} \quad P_j\notin\mathcal{K}(P,r),\end{cases} \tag{8}$$

with $\mathcal{K}(P,r)$ *denoting a spherical cap of centre at* P *and spherical radius* r*, and* $d_g(P,P_j)=\arccos(P^TP_j)$*.*

To define the local interpolation method we need an appropriate space in which to choose the local approximants. To this end we recall something about the zonal basis functions (ZBFs) method.

Definition 3.2 *Given a set of distinct nodes* $\mathcal{X}_N=\{P_i\}_{i=1}^N$ *lying on* $\mathbb{S}^2$*, and the corresponding set of function values* $\mathcal{F}=\{f_i\}_{i=1}^N$*, a zonal basis function interpolant* $s:\mathbb{S}^2\to\mathbb{R}$ *has the form*

$$s(P)=\sum_{j=1}^{N}a_j\psi(d_g(P,P_j)), \tag{9}$$

where $\psi:[0,\pi]\to\mathbb{R}$ *is called a zonal basis function, and* $s(P_i)=f_i$*,* $1\le i\le N$*, i.e.,* s *satisfies the interpolation conditions.*

Let us now consider the following interpolation space given by $T_\psi=\text{span}\{\psi(d_g(\cdot,P_1)),\dots,\psi(d_g(\cdot,P_N))\}$. The interpolation is unique in T_ψ if and only if the associated interpolation matrix $A\in\mathbb{R}^{N\times N}$, defined by $A_{i,j}=\psi(d_g(P_i,P_j))$, $1\le i,j\le N$, is nonsingular. A sufficient condition for nonsingularity is that A be positive definite, see [4, 5, 10] and references therein. One can use this condition to generate the zonal basis functions to be used.

In the following we list the analytic expression of some zonal basis functions:

Logarithmic Spline (LS): $\psi_2(t)=\frac{1}{\beta}\log\left(1+\frac{2\beta}{\sqrt{1+\beta^2-2\beta c}+1-\beta}\right)$

spherical Gaussian (G): $\psi_3(t)=\mathrm{e}^{-\alpha(2-2c)}$

where $\alpha>0$, $\beta\in(0,1)$, $c=\cos(t)$ and t measures geodesic distance on the sphere.

Finally, to define the local interpolation method, we simply consider Z_j in (6) given by (9), namely

$$\tilde{s}(P)=\sum_{j=1}^{N}w_j(x)Z_j(x)\Big/\sum_{k=1}^{N}w_k(P), \tag{10}$$

where Z_j is the zonal basis function interpolant relative to the subset $\mathcal{X}_j = \{P_i \in \mathcal{X}_N, i \in I_j\}$, where I_j is the set of indexes of n_Z neighbours of P_j. The influence of the ZBF $Z_j(P)$ is then limited by means of a weight function which decreases with the inverse of the geodesic distance from P_j. To control the localization of the ZBF, a certain number n_w of nodes close to P_j is considered.

The above interpolation method achieves a very good accuracy (see [8, 5]). Moreover, in [6] a fast algorithm implementing the method was presented, which is comparable, and sometimes better, with the standard routine of Renka [12].

4 Numerical experiments

In order to test if accuracy improves using different weights in global and local Shepard's methods, we used some sets of scattered Halton nodes [6] on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, namely $N = 5000, 10000, 20000$, evaluating Shepard's interpolants on a set of 600 spiral points. The latter are generated by the spiral method, which gives a fairly good point distribution over the sphere. These points uniformly fill up the sphere by tracing out an imaginary spiral from the south pole P_1 to the north pole P_N [11].

Data values were taken from the restriction of the following two trivariate test functions $f(x, y, z)$ onto $\mathbb{S}^2$:

$$f_1(x, y, z) = \frac{\mathrm{e}^x + 2\mathrm{e}^{y+z}}{10}, \qquad f_2(x, y, z) = \sin x \sin y \sin z.$$

Then, we report in the following the root mean square errors (RMSEs) achieved in global and local interpolation processes. In both cases, the choice of the optimal values of Shepard's weight parameters p, γ and δ, for the weights α_p, α_γ and α_δ, respectively, is carried out by analyzing the behaviour of the RMSEs.

4.1 Accuracy: global case

In Tables 1 and 2 we list RMSEs for f_1 and f_2, respectively, obtained considering the different weights. In line 1 of the two Tables we report errors computed using the classical Shepard's weight α_2, while in the following lines we list RMSEs obtained using the other weights. Note that together with errors the best choice of the parameter in the weight is indicated; it represents the parameter value used to compute the error. In Figure 1 the behaviour of RMSEs is represented for all values of the parameters in the intervals $[0, 5]$ for p and γ, and $[0, 10]$ for δ, respectively.

We remark that in the global case the choice of optimal parameters permits to achieve a significant improvement of accuracy. In fact, in comparison with

the standard Shepard's weight, errors decrease by about 2 orders of magnitude for all sets and test functions used.

N	5000	10000	20000
α_2	3.6498E − 2	3.2773E − 2	3.0840E − 2
α_p	3.1407E − 3	2.0741E − 3	1.4530E − 3
p	3.5	3.5	3.5
α_γ	2.8308E − 3	1.9069E − 3	1.3453E − 3
γ	3.1	3.1	3.1
α_δ	1.7977E − 3	1.1846E − 3	7.6240E − 4
δ	5.5	6.2	6.7

Table 1: Global case: RMSEs for f_1.

N	5000	10000	20000
α_2	2.6292E − 2	2.3961E − 2	2.2466E − 2
α_p	2.8666E − 3	1.9132E − 3	1.3399E − 3
p	3.7	3.7	3.7
α_γ	2.7554E − 3	1.8521E − 3	1.2998E − 3
γ	3.5	3.5	3.5
α_δ	1.9297E − 3	1.2414E − 3	8.4883E − 4
δ	6.2	7.0	7.7

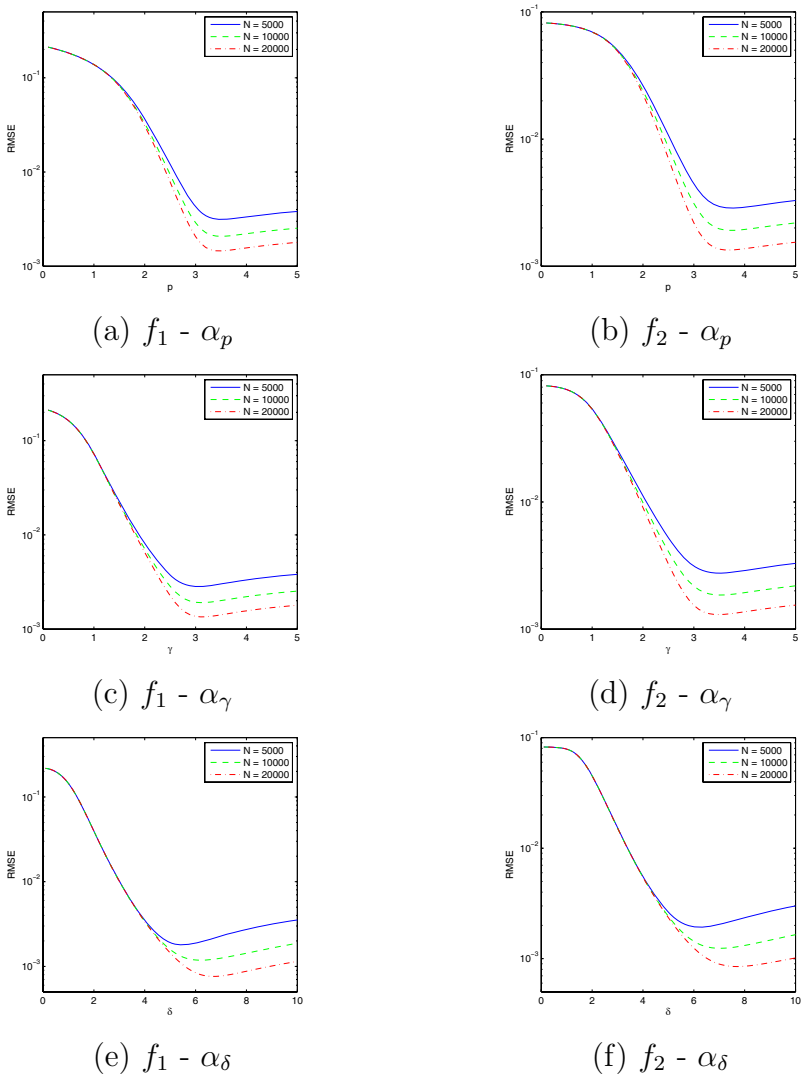
Table 2: Global case: RMSEs for f_2.

4.2 Accuracy: local case

Before analyzing the results obtained using different weights in the local Shepard's method some remarks concerning the choice of the other parameters in the local scheme are to be pointed out.

◇ **Selection of localization parameters.** The choice of the appropriate numbers n_Z and n_w is obviously a non-trivial problem, since it determines the efficiency of the local scheme. Numerical experiments have shown that in some cases good values for the parameters can be $n_Z = 16$ and $n_w = 9$. These values are not obviously the only allowable; in fact, there are many elements which influence the final results, such as the data point distribution (in particular the separation distance), the kind of basis function, the value of ZBF's shape parameter and the behaviour of test functions [5].

◇ **Choice of the ZBF's parameters.** We made the choice of the parameters in zonal basis functions taking into consideration the computational results



(a) f_1 - α_p (b) f_2 - α_p

(c) f_1 - α_γ (d) f_2 - α_γ

(e) f_1 - α_δ (f) f_2 - α_δ

Figure 1: Global case: RMSEs behaviour varying weight parameters.

obtained by Hubbert in [11]. The results suggest that the selection of appropriate values for the shape parameters should also depend on the function to be interpolated. Hovewer, taking into account that a good compromise between accuracy and stability is needed, and analyzing the behaviour of errors, we choose to use the values $\beta = 0.5$ for the Poisson spline and the logarithmic spline, and $\alpha = 4$ for the spherical Gaussian. Here, for brevity we reported only numerical results for the logarithmic spline ψ_2 (as it gave slightly better results), although both the Poisson spline ψ_1 and the spherical Gaussian ψ_3 were successfully tested as well.

In Tables 3 - 4 we listed the RMSEs achieved by using ψ_2 in the local

N	5000	10000	20000
α_2	5.5510E − 7	9.8136E − 8	6.0174E − 8
α_p	5.4634E − 7	9.7998E − 8	3.3721E − 8
p	2.6	1.9	0.1
α_γ	5.4620E − 7	9.7960E − 8	3.3729E − 8
γ	2.6	1.8	0.1
α_δ	5.0356E − 7	9.4130E − 8	3.3260E − 8
δ	7.6	7.6	3.3

Table 3: Local case: RMSEs for f_1.

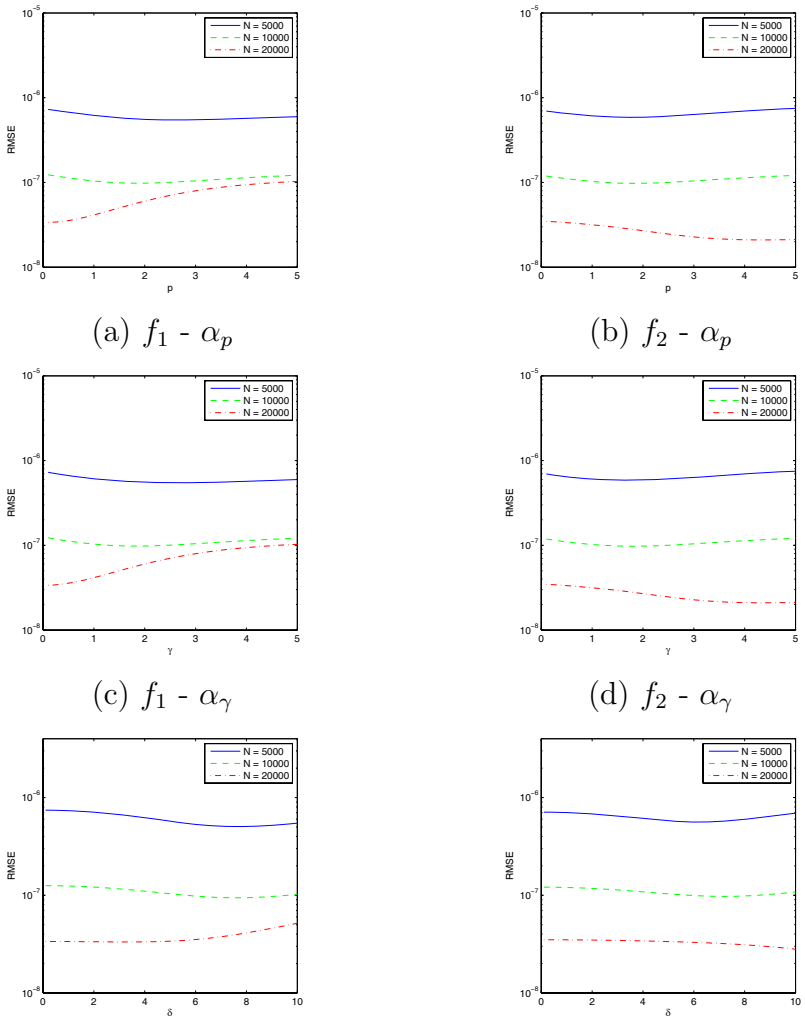
N	5000	10000	20000
α_2	5.8923E − 7	9.7743E − 8	2.6926E − 8
α_p	5.8695E − 7	9.7554E − 8	2.0977E − 8
p	1.8	1.8	4.3
α_γ	5.8641E − 7	9.7542E − 8	2.0977E − 8
γ	1.7	1.8	4.3
α_δ	5.6142E − 7	9.7349E − 8	2.8028E − 8
δ	6.3	7.2	10.0

Table 4: Local case: RMSEs for f_2.

scheme for each test function. Also for the local case we compare results obtained with the different weights. We observe that when we take optimal values of the parameters in the weights the errors are approximatively equal. However, we point out that these optimal values are often very different, and they depend on the number of nodes and the test function, in opposition with the results given by the global case. For example, the optimal values for γ are 2.6, 1.8, 0.1 for $N = 5000, 10000, 20000$, respectively, for the test function f_1, and are 1.7, 1.8, 4.3 for f_2. Figure 2 highlights the smooth behaviour of RMSEs when the weight parameters vary. We remark that this behaviour is quite obvious, since the importance of the weight is less in the local case than in the global one.

4.3 Computational convergence

Finally, since the aim of experiments is also to examine how errors change as the interpolation nodes double, we experimentally estimate convergence orders. For this reason, we recall the definitions of the separation distance, which measures the closest pair of points in the data set, and of the fill-distance, which gives measure of the density of the nodes with respect to the sphere,



(a) f_1 - α_p (b) f_2 - α_p

(c) f_1 - α_γ (d) f_2 - α_γ

(e) f_1 - α_δ (f) f_2 - α_δ

Figure 2: Local case: RMSEs behaviour varying weight parameters.

namely

$$q = \frac{1}{2} \min_{i \neq j} d_g(P_i, P_j), \qquad h = \sup_{P \in \mathbb{S}^2} \inf_{1 \leq i \leq N} d_g(P, P_i). \tag{11}$$

Being able to control q and h, i.e. the distribution of the interpolation nodes on $\mathbb{S}^2$, is important in order to investigate the accuracy of the method. Thus, denoting by e_N the RMSE obtained using N interpolation nodes and by h_N the related fill distance (see Table 5), we know that $e_N/e_{2N} \sim (h_N/h_{2N})^r$, where $r = r_{N/2N}$ is the convergence order. The results, obtained considering test functions f_1 and f_2 and evaluating the convergence order with respect to

RMSEs, are shown in Tables 6 - 7. Here for each weight the optimal value for parameters, previously determined, was used.

As we expected, the convergence order for the global Shepard's method is very low, say near 1, and the convergence is therefore slow, while it is substantially higher when we use the local method.

N	5000	10000	20000
h_N	$5.5661\text{E}-2$	$3.9331\text{E}-2$	$2.5306\text{E}-2$

Table 5: Fill distance h_N for Halton nodes on the sphere.

	α_p		α_γ		α_δ	
N	$r_{5000/10000}$	$r_{10000/20000}$	$r_{5000/10000}$	$r_{10000/20000}$	$r_{5000/10000}$	$r_{10000/20000}$
global	1.1948	0.8071	1.1377	0.7911	1.2011	0.9994
local	4.9481	2.4193	4.9484	2.4178	4.8292	2.3592

Table 6: Convergence orders for f_1.

	α_p		α_γ		α_δ	
N	$r_{5000/10000}$	$r_{10000/20000}$	$r_{5000/10000}$	$r_{10000/20000}$	$r_{5000/10000}$	$r_{10000/20000}$
global	1.1644	0.8077	1.1439	0.8030	1.2703	0.8620
local	5.1676	3.4854	5.1653	3.4852	5.0456	2.8235

Table 7: Convergence orders for f_2.

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