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ON THE DIMENSION OF THE MINIMAL VERTEX COVER SEMIGROUP RING OF AN UNMIXED BIPARTITE GRAPH

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In a paper in 2008, Herzog, Hibi and Ohsugi introduced and studied the semigroup ring associated to the set of minimal vertex covers of an unmixed bipartite graph. In this paper we relate the dimension of this semigroup ring to the rank of the Boolean lattice associated to the graph.

1. Introduction

Let G be a finite graph without loops, multiple edges and isolated vertices and let $\mathcal{M}(G)$ be the set of minimal vertex covers of G. In [2, Section 3] the authors introduce and study the semigroup ring associated to the minimal vertex covers of an unmixed and bipartite graph G.

In this paper we relate the dimension of this semigroup ring to the rank of the Boolean lattice associated to G.

In Section 2, we recall the concept of an unmixed bipartite graph G and we give some preliminaries about the Boolean lattice associated to G. In particular, we characterize those sublattices of the Boolean lattice which are associated to G (cf. Theorem 2.3). Then, in the particular case of bipartite graphs, we concentrate on the concept of vertex cover algebra.

In Section 3 we define the semigroup ring associated to the minimal vertex covers of an unmixed and bipartite graph G and we prove that its dimension

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equals the rank of $\mathscr{L}_G$ plus one (cf. Theorem 3.3). As a particular case of this result we get that the dimension of the semigroup ring associated to the minimal vertex covers of bipartite and Cohen-Macaulay graphs on $2n$ vertices is equal to $n+1$ (cf. Corollary 3.4).

2. Preliminaries

Throughout this paper, graphs are assumed to be finite, loopless, without multiple edges and isolated vertices. We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G.

Definition 1. For a graph G, a subset C of the set of vertices $V(G)$ is called a **vertex cover** for G if every edge of $E(G)$ is incident to at least one vertex from C. C is a **minimal** vertex cover if for any $C' \subsetneq C$, C' is not a vertex cover for G.

Let $\mathscr{M}(G)$ denote the set of minimal vertex covers of G. In general, the minimal vertex covers of a graph do not have the same cardinality.

Example 2.1. Let G be the graph with $V(G) = \{1,2,3,4,5\}$ and

$$E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{4,5\}\}.$$

Then $\mathscr{M}(G) = \{\{2,4\},\{1,3,5\}\}$.

Definition 2. A graph G is **unmixed** if all the elements of $\mathscr{M}(G)$ have the same cardinality.

Example 2.2. The graph G with $V(G) = \{1,2,3,4\}$ and $E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ is unmixed as $\mathscr{M}(G) = \{\{2,4\},\{1,3\}\}$.

Definition 3. A graph G is **bipartite** if its set of vertices $V(G)$ can be divided in two disjoint subsets U and V such that, for all $l \in E(G)$, we have $|l \cap U| = 1 = |l \cap V|$.

In what follows G will be assumed to be bipartite and unmixed with respect to the partition $V(G) = U \cup V$ of its vertices, where $U = \{x_1,\dots,x_m\}$ and $V = \{y_1,\dots,y_n\}$.

Since G is unmixed and U and V are both minimal vertex cover for G, then $n = m$.

Furthermore, let $U' \subseteq U$ and $N(U')$ be the set of those vertices $y_j \in V$ for which there exist a vertex $x_i \in U'$ such that $\{x_i, y_j\} \in E(G)$; then (cf.[1, p. 300]), since $(U \setminus U') \cup N(U')$ is a vertex cover of G for all subset U' of U and since G is unmixed, it follows that $|U'| \leq |N(U')|$ for all subset U' of U. Thus, the marriage theorem enable us to assume that $\{x_i, y_i\} \in E(G)$ for $i = 1, \dots, n$. We can also assume that each minimal vertex cover of G is of the form

$$\{x_{i_1}, \dots, x_{i_s}, y_{i_{s+1}}, \dots, y_{i_n}\}$$

where $\{i_1, \dots, i_n\} = [n] = \{1, \dots, n\}$.

For a minimal vertex cover $C = \{x_{i_1}, \dots, x_{i_s}, y_{i_{s+1}}, \dots, y_{i_n}\}$ of G, we set $C' = \{x_{i_1}, \dots, x_{i_s}\}$. Let $\mathscr{L}_n$ denote the Boolean lattice of all the subset of $\{x_1, \dots, x_n\}$ and let $\mathscr{L}_G = \{C' \mid C$ is a minimal vertex cover of G $\}$. One easily checks this is a sublattice of $\mathscr{L}_n$. Since $\mathscr{L}_n$ is a distributive lattice, any sublattice is distributive as well.

Actually, there is a one to one correspondence between the graphs we are studying and the sublattices of $\mathscr{L}_n$ containing $\emptyset$ and $\{x_1, \dots, x_n\}$:

Theorem 2.3. *[2, Theorem 1.2] Let $\mathscr{L}$ be a subset of $\mathscr{L}_n$. Then there exists a (unique) unmixed bipartite graph G on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ such that $\mathscr{L} = \mathscr{L}_G$ if and only if $\emptyset$ and $\{x_1, \dots, x_n\}$ belong to $\mathscr{L}$ and $\mathscr{L}$ is a sublattice of $\mathscr{L}_n$.*

2.1. Cohen-Macaulay bipartite graphs

Let A be the polynomial ring $K[z_1, \dots, z_N]$ over a field K. To any graph G on vertex set $[N]$, let $I(G)$ be the ideal of A, called the *edge ideal* of G, generated by the quadratic monomials $z_i z_j$ such that $\{i, j\} \in E(G)$.

Definition 4. A graph G is **Cohen-Macaulay** if the quotient ring $A/I(G)$ is Cohen-Macaulay.

Let, as before, $\mathscr{L}_n$ denote the Boolean sublattice on $\{x_1, \dots, x_n\}$.

Definition 5. The **rank** of a sublattice $\mathscr{L}$ of $\mathscr{L}_n$, rank$\mathscr{L}$, is the non-negative integer l where $l+1$ is the maximal cardinality of a chain of $\mathscr{L}$. A sublattice $\mathscr{L}$ of $\mathscr{L}_n$ is called **full** if rank$\mathscr{L} = n$.

Theorem 2.4. *[2, Theorem 2.2] A subset $\mathscr{L}$ of $\mathscr{L}_n$ is a full sublattice of $\mathscr{L}_n$ if and only if there exists a Cohen-Macaulay bipartite graph G on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ with $\mathscr{L} = \mathscr{L}_G$.*

2.2. Vertex cover algebra

Let G be a bipartite and unmixed graphs on the set of vertices $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and with minimal vertex cover $C = \{x_{i_1}, \dots, x_{i_s}, y_{i_{s+1}}, \dots, y_{i_n}\}$. It is useful to notice that $x_i \in C$ if and only if $y_i \notin C$.

We can identify C with the $(0,1)$-vector, $b_C \in \mathbb{N}^{2n}$ such that

$$b_C(j) = \begin{cases} 1 \text{ if } 1 \le j \le n \text{ and } x_j \in C \\ 1 \text{ if } n+1 \le j \le 2n \text{ and } y_{j-n} \in C \\ 0 \text{ otherwise} \end{cases}$$

where $b_C(j)$ denotes the j-th coordinate of the vector b_C.

In this way, we can associate to each minimal vertex cover C of G a square-free monomial in the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ with $\deg x_i = \deg y_i = 1$; in fact, we first associate to C its vector b_C and then we consider the monomial $u_C = x_1^{b_C(1)} \cdots x_n^{b_C(n)} y_1^{b_C(n+1)} \cdots y_n^{b_C(2n)}$.

Definition 6. The **vertex cover algebra** of the bipartite graph G is the subalgebra $A(G)$ of $S[t]$ generated, over S, by the monomials $u_C t$ for every minimal vertex cover C of G, that is $A(G) = S[u_C t,\ C \in \mathscr{M}(G)]$.

By [3, Theorem 4.2 and Corollary 4.4], we have, in particular, that $A(G)$ is a finitely generated, graded, normal, Gorenstein S-algebra.

Moreover, in [3, Theorem 5.1], the authors show that $A(G)$ is generated in degree ≤ 2 and that it is standard graded.

3. The dimension of $\overline{A(G)}$

We now introduce the object of our study in this paper.

Let $\mathfrak{m}$ be the maximal graded ideal of S. For a bipartite unmixed graph G, we consider the standard graded K-algebra

$$\overline{A(G)} := A(G)/\mathfrak{m}A(G) \cong K[u_C t,\ C \in \mathscr{M}(G)] \cong K[u_C,\ C \in \mathscr{M}(G)].$$

Hence $\overline{A(G)}$ is the semigroup ring generated by all monomials u_C such that $C \in \mathscr{M}(G)$. This object has been introduced and studied in [2, Section 3], where the authors proved, in particular, that $\overline{A(G)}$ is a normal and Koszul semigroup ring (cf. [2, Corollary 3.2]).

The aim of the paper is to relate the dimension of $\overline{A(G)}$ to $\operatorname{rank} \mathscr{L}_G$ (cf. Theorem 3.3).

Let $d = |\mathscr{M}(G)|$ and B_G be the $d \times 2n$ matrix whose rows are exactly the vectors b_C. Since $C_1 = \{x_1, \dots, x_n\}$ and $C_2 = \{y_1, \dots, y_n\}$ are always in $\mathscr{M}(G)$, we can assume that the first and the last rows of B_G are b_{C_1} and b_{C_2} respectively. Finally, let $\widetilde{b_C}$ be the n-vector containing only the first n entries of b_C and let $\widetilde{B_G}$ be the $d \times n$ matrix whose rows are the vectors $\widetilde{b_C}$.

Lemma 3.1. $\operatorname{rank} B_G = \operatorname{rank} \widetilde{B_G} + 1$.

Proof. Let $C_1,\dots,C_{2n}$ be the column vectors of B_G (note that the columns of $\widetilde{B_G}$ are exactly $C_1,\dots,C_n$) and let $\widetilde{C}$ be the column vector with d entries each equals to 1. Since $C_{n+j} = \widetilde{C} - C_j$ for every $j = 1,\dots,n$, then we have that

$$\langle C_1,\dots,C_n,\widetilde{C}\rangle_K = \langle C_1,\dots,C_n,C_{n+1},\dots,C_n\rangle_K.$$

as K-vector spaces.

Finally, since the last entry in each column $C_1,\dots,C_n$ is 0, it follows that $\widetilde{C} \notin \langle C_1,\dots,C_n\rangle_K$, that is

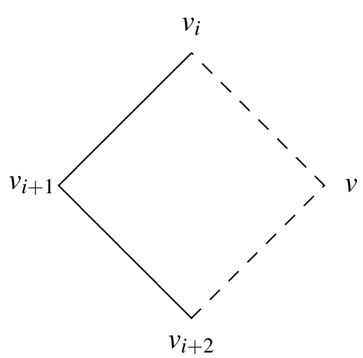
$$\dim_K\langle C_1,\dots,C_n,\widetilde{C}\rangle_K = \dim\langle C_1,\dots,C_n\rangle_K + 1.$$

□

Lemma 3.2. $\operatorname{rank}\widetilde{B_G} = \operatorname{rank}\mathscr{L}_G$

Proof. Let $\operatorname{rank}\mathscr{L}_G = m$ and consider a chain of maximal length $m+1$ in the sublattice $\mathscr{L}_G$. We note that, by Theorem 2.3, $\emptyset$ and $[n]$ are in this chain. Each element of this maximal chain corresponds to a row of the matrix $\widetilde{B_G}$. Let denote with $v_1,\dots,v_{m+1}$ the row vectors associated to this maximal chain, where v_1 is the vector associated to the element at the top of the chain, v_2 is the vector associated to the element of the chain just below the top, and so on for the remaining vectors $v_3,\dots,v_{m+1}$. With this notation we have that v_1 is the vector with all 1's and v_{m+1} is the vector with all 0's. We note that if $i > j$, then the numbers of 1's in v_i is strictly less than the number of 1's in v_j and that if 0 is the l-th coordinate of v_i, then 0 is the l-th coordinate of v_j. This two facts imply that $v_1,\dots,v_m$ are linearly indipendent. So we have $\operatorname{rank}\widetilde{B_G} \geq m$.

In order to prove that equality holds, we show that all the other rows of $\widetilde{B_G}$ are linear combination of the m rows associated to $v_1,\dots,v_m$. With an abuse of notation, we now identify the elements of the lattice $\mathscr{L}_G$ with their associated vectors. Since $\mathscr{L}_G$ is a lattice containing $[n]$ and $\emptyset$, following the maximal chain in the lattice containing the vectors $v_1,\dots,v_m$, we have, at a certain height, the situation depicted in the picture



where v_i, v_{i+1}, v_{i+2} are in the maximal chain.

But $\mathscr{L}_G$ is a distributive lattice and, in terms of the vectors, this means that we can obtain v from the other vectors in the picture: in fact (vectorially)

$$v = v_i - v_{i+1} + v_{i+2}.$$

Repeating this in each analogous situation, we have that all the possible vectors representing elements of the lattice which are not in the chosen maximal chain, can be obtained by a linear combination of the vectors $v_1, \dots, v_m$. In terms of the matrix $\widetilde{B_G}$, this means that $\operatorname{rank} \widetilde{B_G} \leq m$.

□

Theorem 3.3. *Let G be an unmixed, bipartite graph on $2n$ vertices with no isolated vertices and let $\mathscr{L}_G$ be the associated sublattice of $\mathscr{L}_n$. Then*

$$\dim \overline{A(G)} = \operatorname{rank} \mathscr{L}_G + 1.$$

Proof. By [4, Proposition 7.1.17], we have that $\dim \overline{A(G)} = \operatorname{rank} B_G$. By Lemmas 3.1 and 3.2, we get the proof. □

Corollary 3.4. *Let G be a Cohen-Macaulay bipartite graph on $2n$ vertices. Then*

$$\dim \overline{A(G)} = n + 1$$

Proof. By [4, Proposition 6.1.21], G is unmixed. Furthermore, by Theorem 2.4, Cohen-Macaulay graphs correspond to full sublattices. Hence, by Theorem 3.3, we get the thesis. □

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