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The Euler Characteristic as a Polynomial in the Chern Classes

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Abstract

In this paper we obtain some explicit expressions for the Euler characteristic of a rank n coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ and of its twists $\mathcal{F}(t)$ as polynomials in the Chern classes $c_i(\mathcal{F})$, also giving algorithms for the computation. The employed methods use techniques of umbral calculus involving symmetric functions and Stirling numbers.

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Introduction

The aim of this paper is to find a general polynomial expression for the Euler characteristic $\chi(\mathcal{F})$ of a rank n coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^N$ on the field $\mathbb{K}$, in terms of the Chern classes $c_i(\mathcal{F})$ of $\mathcal{F}$.

For fixed N (the dimension of the projective space) and n (the rank of the sheaf), we explicitly obtain the polynomial $P(C_1, \ldots, C_N) \in \mathbb{Q}[C_1, \ldots, C_N]$ such that

$$P(c_1(\mathcal{F}), \ldots, c_N(\mathcal{F})) = \chi(\mathcal{F}) = \sum_{j=0}^{N} (-1)^j h^j \mathcal{F}$$

where we consider the Chern classes $c_i(\mathcal{F})$ on $\mathbb{P}^N$ as integers and $h^j\mathcal{F}$ is the dimension of the i-th cohomology module $H^i(\mathcal{F})$ as a $\mathbb{K}$-vector space.

More generally we obtain a general polynomial expression for the Euler characteristic of every twist of $\mathcal{F}$ in terms of the Chern class $c_i(\mathcal{F})$ of $\mathcal{F}$ and on t, namely:

$$G(c_1(\mathcal{F}), \ldots, c_N(\mathcal{F}), t) = \chi(\mathcal{F}(t)) = \sum_{j=0}^{N} (-1)^j h^j \mathcal{F}(t)$$

It is well known that such polynomials exist and that they do not depend on $\mathcal{F}$ (see for instance [1], Theorem 2.3), so that they can be computed by means of the special cases given by the totally split sheaves $\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)$ with $a_i \geq 0$: this argument is usually called the "splitting principle".

For our purposes those free sheaves are very easy to manage because their Euler characteristic is only given by the 0-cohomology, $\chi(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = h^0(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i))$, so that it can be easily expressed as a sum of n binomials involving N and the a_i's; moreover also their Chern classes can be easily written in terms of the a_i's, as symmetric functions $c_j(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = \sum a_{i_1}\dots a_{i_j}$, where the sum is over every sequence of j indexes $i_1 < \cdots < i_j$.

However in practice a general computation necessary involves "changes of basis" for polynomials, mainly for those which are invariant with respect to the action of the permutation group on the variables, that are not so easy to express in a suitable way. In fact we have to expand polynomials expressed through binomials into their expansion as a sum of monomials and then as a sum of elementary symmetric functions.

We will divide the solution of the problem in three steps.

1. We first write the polynomial $Q_N(x_1,\dots,x_n) \in \mathbb{Q}[x_1,\dots,x_n]$ such that $\chi(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = h^0(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = \frac{1}{N!}Q_N(a_1,\dots,a_n)$ for every $a_i \geq 0$. As $a_1,\dots,a_n$ is not an ordered sequence, the polynomial Q_N must be invariant under the action of the permutation group on the variable x_i.

2. Then we "change the basis" that is we substitute the variables x_i (corresponding to the a_i's) by the variables C_j given by their symmetric functions (corresponding to the Chern classes c_i) so obtaining the polynomial $P(C_1,\dots,C_N)$.

The first step involves the Stirling numbers of first kind (that we introduce in §1); here we show that they give the coefficients of the expansion of a the polynomial $R_N(x)$ such that $h^0\mathcal{O}_{\mathbb{P}^N}(a) = 1/(N!)R_N(a)$ (see Theorem 1.1 and Corollary 1.3).

The second step is closely related to the *umbral calculus* (see [5] for an overview of the subject). We also use the well-known *Newton-Girard formulas* (see [6]) in order to obtain a faster algorithm to compute the polynomial P.

Finally using the relations between the Chern classes of a sheaf $\mathcal{F}$ and those of the twists $\mathcal{F}(t)$ (see (6)), we compute the polynomial $G(C_1,\dots,C_N,T) \in \mathbb{Q}[C_1,\dots,C_N,T]$ such that $\chi(\mathcal{F}(t)) = G(c_1(\mathcal{F}),\dots,c_N(\mathcal{F}),t)$.

Beyond the theoretical results, we also present some procedures for the explicit computations of the polynomials $P(C_1,\dots,C_N)$ and $G(C_1,\dots,C_N,t)$, for a fixed dimension of the projective space N and a fixed rank n for the sheaf (see §3).

1 Stirling Numbers of first kind

The *Stirling number of first kind* $\begin{bmatrix} N \\ m \end{bmatrix}$ is the number of permutations of N elements which contain exactly m distinct cycles.
As a direct consequence of the definition, one can immediately see that:

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ but $\begin{bmatrix} 0 \\ m \end{bmatrix} = 0$ if $m > 0$ and $\begin{bmatrix} N \\ 0 \end{bmatrix} = 0$ if $N > 0$.

- $\begin{bmatrix} N \\ m \end{bmatrix} = 0$ if $m > n$ and $\begin{bmatrix} N \\ N \end{bmatrix} = 1$

- $\begin{bmatrix} N \\ N-1 \end{bmatrix} = \binom{N}{2}$ because a permutation of N elements which contain $N-1$ cycles is determined by its only 2-cycle.

Consider a "square"table whose entries are the integers $\begin{bmatrix} N \\ m \end{bmatrix}$, where each row is associated to a value for N and each column is associated to a value for m. The above properties of Stirling numbers say that in such a table:

- the triangle above the main diagonal is completely 0;

- on the main diagonal the entries are all 1's;

- on the "second"diagonal, the entries are the binomials $\binom{N}{2}$.

The following *recurrence relation* for the Stirling numbers of the first kind allows to complete the table:

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} + (N-1)\begin{bmatrix} N-1 \\ m \end{bmatrix} \tag{1}$$

This relation easily follow from the definition. In fact if we fix an element α among the N, there are two kinds of permutations of N elements with m cycles: the first addendum in the right side of (1) is the number of permutations containing (α) as a 1-cycle; the second one corresponds to permutations not containing the 1-cycle (α): for every permutation of the other $N-1$ elements with m cycles, the element α can be introduced in $(N-1)$ different ways (in fact there are j different ways to put a new element in a cycle of j elements).

Now we can complete the table of the Stirling numbers of first kind

	0	1	2	3	4	5	6	7	8	...
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	11	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The above considered Stirling numbers of the first kind are often called *unsigned*, in opposition to the *signed* Stirling numbers of first kind, which we denote by $s(N,m)$ and that are simply recovered from the unsigned ones by the rule

$$s(N,m) = (-1)^{N-m}\begin{bmatrix} N \\ m \end{bmatrix}.$$

The original definition of signed Stirling numbers of first kind comes from a particular polynomial, the *falling factorial* $(x)_N$, which is "similar" to the one we are interested in:

$$(x)_N = x(x-1)\cdots(x-N+1)$$

The signed Stirling numbers of first kind are defined as the coefficients of the expansion

$$(x)_N := \sum_{k=0}^{N} s(N,k)x^k.$$

We are mainly interested in the *rising factorial polynomial*

$$(x)^{(N)} := x(x+1)\cdots(x+N-1).$$

or, more precisely to the polynomial

$$R_N(x) := (x+1)(x+2)\cdots(x+N) = (x+1)^{(N)}.$$

Theorem 1.1.

$$R_N(x) = (x+1)^{(N)} = \sum_{k=0}^{N}\begin{bmatrix} N+1 \\ k+1 \end{bmatrix}x^k. \tag{2}$$

Proof: We proceed by induction on N.

If $N = 1$, the thesis is immediately verified.

Then assume that the formula holds for $R_{N-1}(x)$. Applying the inductive hypothesis to

$$R_N(x) = (x+N)R_{N-1}(x) = (x+N)\left((x+1)^{(N-1)}\right)$$

we obtain:

$$R_N(x) = (x+N)\left(\sum_{k=0}^{N-1}\begin{bmatrix}N\\k+1\end{bmatrix}x^k\right) = \sum_{k=0}^{N-1}\begin{bmatrix}N\\k+1\end{bmatrix}x^{k+1} + \sum_{k=0}^{N-1}N\begin{bmatrix}N\\k+1\end{bmatrix}x^k.$$

The change $k+1 \to k$ in the first sum of the right side and the recurrence relation (1) give:

$$R_N(x) = \begin{bmatrix}N\\N\end{bmatrix}x^N + \sum_{k=1}^{N-1}\left(\begin{bmatrix}N\\k\end{bmatrix} + N\begin{bmatrix}N\\k+1\end{bmatrix}\right)x^k + N\begin{bmatrix}N\\1\end{bmatrix} =$$

$$= \begin{bmatrix}N+1\\N+1\end{bmatrix}x^N + \sum_{k=1}^{N-1}\begin{bmatrix}N+1\\k+1\end{bmatrix}x^k + \begin{bmatrix}N+1\\1\end{bmatrix}$$

which is (2) (note that for every $r > 0$, $\left[\begin{smallmatrix}r\\r\end{smallmatrix}\right] = 1$). $\diamond$

Remark 1.2. *For a different proof of Theorem 1.1, we could refer to [2], formula (6.11), and use the equality* $x^{(N+1)} = xR_N(x)$.

Corollary 1.3. *For every* $a \geq 0$ *the dimension of the vector space of the degree* a *hypersurfaces in* $\mathbb{P}^N$ *is given by:*

$$h^0\mathcal{O}_{\mathbb{P}^N}(a) = \frac{1}{N!}R_N(a) = \frac{1}{N!}\sum_{k=0}^{N}\begin{bmatrix}N+1\\k+1\end{bmatrix}a^k.$$

2 Invariant Polynomials and the Main Theorem

Now let $x_1, \ldots, x_n$ be n variables and consider the polynomial $Q_N(x_1, \ldots, x_n) := \sum_j R_N(x_j) \in \mathbb{Q}[x_1, \ldots, x_n]$. This polynomial is closely related to our problem, because for every choice of n positive integers $a_1, \ldots, a_n$, we have $\chi(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = h^0(\oplus\mathcal{O}_{\mathbb{P}^N}(a_i)) = \frac{1}{N!}Q_N(a_1, \ldots, a_n)$.

It is quite evident that Q_N does not change under permutation of the variables, that is it is *invariant* for symmetric group $\mathcal{S}_n$. We just recall some basic definition and properties of the invariant polynomials; for more details one can see, for instance, [3].

The action of the symmetric group $\mathcal{S}_n$ on $\mathbb{Q}[x_1, \ldots, x_n]$ is given in the following way.
If $p(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ and $\sigma \in \mathcal{S}_n$, then:

$$(\sigma \cdot p)(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We say that $p \in \mathbb{Q}[x_1, \ldots, x_n]$ is *invariant* for the action of $\mathcal{S}_n$ if

$$\sigma \cdot p = p \quad \forall \sigma \in \mathcal{S}_n.$$

It is easy to prove that the set of invariant polynomials that we denote by $\mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ is an algebra, called the *algebra of symmetric polynomials*.

Since $\mathcal{S}_n$ is a reductive linear algebraic group (see [3]), there is a set of algebraically independent polynomials $\{f_1, \ldots, f_n\}$, $f_i \in \mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n}$, such that the polynomial ring they generate on $\mathbb{Q}$ is exactly $\mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n}$, that is

$$\mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n} = \mathbb{Q}[f_1, \ldots, f_n].$$

We call $\{f_1, \ldots, f_n\}$ a set of *basic invariants*.

There are of course many sets of basic invariants for $\mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n}$, but we will be interested only in two of these:

- the elementary symmetric polynomials:

$$C_0 = 1; \quad C_j := \sum_{\lambda_1 < \cdots < \lambda_j} x_{\lambda_1} \cdots x_{\lambda_j} \quad j = 1, \ldots, n.$$

- the power sum symmetric polynomials:

$$B_k := \sum_{i=1}^{n} a_i^k \quad k = 0, \ldots, n.$$

Since both $\{C_1, \ldots, C_n\}$ and $\{B_1, \ldots, B_n\}$ are sets of basic invariants and so their elements are algebraically independent, we can consider them as indeterminates.

Every invariant polynomial, included $Q_N(x_1, \ldots, x_n) = \sum_i R_N(x_i)$, can be written using either of the two sets of basic invariants: as a polynomial in the indeterminates C_j's and as a polynomial in the indeterminates B_k's.

Lemma 2.1. *In the above notation:*

$$Q_N(x_1, \ldots, x_n) = \sum_{k=0}^{N} \begin{bmatrix} N+1 \\ k+1 \end{bmatrix} B_k.$$

Proof: Applying Theorem 1.1, we immediately obtain

$$Q_N(x_1,\dots,x_n)=\sum_{i=1}^{n}\left(\sum_{k=0}^{N}\begin{bmatrix}N+1\\k+1\end{bmatrix}x_i^k\right)=\sum_{k=0}^{N}\begin{bmatrix}N+1\\k+1\end{bmatrix}\left(\sum_{i=1}^{n}x_i^k\right).$$

where we can commute the two summations because they are independent. ⋄

If we know the expression of an invariant polynomial in terms of a set of basic invariants and want to obtain its expression in terms of the other one, we have to manage the not so easy problem of the "change of basis". For instance:

$$B_0=nC_0 \quad , \quad B_1=C_1 \quad , \quad B_2=C_1^2-2C_2.$$

In order to find a general expression of B_k as a function of the C_j's, we recall the Newton-Girard formula (see [6]):

$$(-1)^r B_r+\sum_{l=1}^{r}(-1)^{r+l}B_lC_{r-l}=0$$

Note that in fact this formula holds for every $r\in\mathbb{N}$, with the convention that B_k is the sum of powers x_i^k and $C_k=0$ if $k\geq n+1$.

With these notations, we can then prove

Lemma 2.2. *For every* $1\leq r\leq n$ $B_r=\det(M_r)$ *where*

$$M_r=\begin{pmatrix} C_1 & 1 & 0 & 0 & \dots & 0 & 0\\ 2C_2 & C_1 & 1 & 0 & \dots & 0 & 0\\ 3C_3 & C_2 & C_1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ (r-1)C_{r-1} & C_{r-2} & C_{r-3} & C_{r-4} & \dots & C_1 & 1\\ rC_r & C_{r-1} & C_{r-2} & C_{r-3} & \dots & C_2 & C_1 \end{pmatrix} \tag{3}$$

Proof: We proceed by induction on r. If $r=1$ there is nothing to prove.

Assume $r\geq 2$ and the thesis true for B_l, $l\leq r-1$. From Newton-Girard formula we have

$$B_r=B_{r-1}C_1-B_{r-2}C_2+\dots+(-1)^{r-2}B_1C_{r-1}+(-1)^{r-1}rC_r. \tag{4}$$

Observe that since the thesis is true for B_l, $l\leq r-1$, we can write the second term of (4) as

$$B_r=\sum_{l=1}^{r-1}(-1)^{l-1}C_l\det(M_{r-l})+(-1)^{r-1}rC_r. \tag{5}$$

Thanks to the presence of the 1's above the main diagonal of the matrix (3), this is exactly the determinant of M_r. ⋄

Remark 2.3. *For a different proof of the previous result one can see [4], Chapter V. However the proof we present is more constructive and gives rise to a faster algorithm, that we will present in §3.*

Finally, we obtain the main Theorem as an application of Lemmas 2.1 and 2.2

Theorem 2.4. *Let $\mathcal{F}$ be rank n reflexive sheaf on $\mathbb{P}^N$. Consider*

$$P = \frac{1}{N!} \sum_{k=1}^{N} \begin{bmatrix} N+1 \\ k+1 \end{bmatrix} \det(M_k) + n$$

where M_k is the $k \times k$ matrix previously defined and its determinant is a polynomial in the variables $C_1, \dots, C_k$.
Then

$$P(c_1(\mathcal{F}), \dots, c_N(\mathcal{F})) = \chi(\mathcal{F})$$

.

Proof: First, using the "splitting principle", we know there is a polynomial $P \in \mathbb{Q}[c_1, \dots, c_N]$, depending only on N and n, such that $\chi(\mathcal{F}) = P(c_1(\mathcal{F}), \dots, c_N(\mathcal{F}))$ for any rank n coherent sheaf $\mathcal{F}$.

It is then sufficient to find such a polynomial for the sheaf $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i)$, $a_i \geq 0$, with $n \geq N$.

Thanks to Lemma 2.2, we can pass from the B_k's to the C_j's in the expression of Lemma 2.1 for $Q_N(x_1, \dots, x_n)$:

$$Q_N(x_1, \dots, x_n) = \sum_{k=0}^{N} \begin{bmatrix} N+1 \\ k+1 \end{bmatrix} B_k = \sum_{k=1}^{N} \begin{bmatrix} N+1 \\ k+1 \end{bmatrix} \det(M_k) + n(N!).$$

Using Corollary 1.3, we obtain

$$\chi(\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i)) = h^0(\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i)) = \frac{1}{N!} Q_N(a_1, \dots, a_n) =$$
$$= \frac{1}{N!} \sum_{k=1}^{N} \begin{bmatrix} N+1 \\ k+1 \end{bmatrix} \det(M_k)(c_1, \dots, c_k) + n$$

where $\det(M_k)(c_1, \dots, c_k)$ means evaluating the polynomial in the Chern classes of $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i)$.

$\diamond$

Remark 2.5. *In the proof of Theorem 2.4, the assumption $n \geq N$ for the completely split bundle is not a lost in generality; in fact, a coherent sheaf $\mathcal{F}$ of rank $n \leq N$ may have $c_i(\mathcal{F}) \neq 0$ for $i \geq n+1$.*

3 Implementation

In the previous paragraphs we obtained the following result:

> Let $\mathcal{F}$ be a rank n coherent sheaf on $\mathbb{P}^N$. Then
>
> $$\chi(\mathcal{F}) = \frac{1}{N!}\sum_{k=1}^{N}\begin{bmatrix} N+1 \\ k+1 \end{bmatrix}\det M_k(\mathcal{F}) + n$$
>
> where M_k is a $k \times k$ matrix whose definition is (3), $\det(M_k) \in \mathbb{Z}[c_1, \ldots, c_k]$ and with $\det M_k(\mathcal{F})$ we mean $\det M_k(c_1(\mathcal{F}), \ldots, c_N(\mathcal{F}))$.

The polynomial for $\chi(\mathcal{F})$ is not too easy to handle, since it contains some determinants.

Anyway, if we fix the dimension N, it is quite easy to write a procedure to compute the polynomial $\chi(\mathcal{F})$.

Here we write some procedures for Maple. Probably they are not the best implementations for the algorithms we wish to expose, they are just intended to be examples.

3.1 A first algorithm for $\chi(\mathcal{F})$

First we write a procedure to write the r-th row of the matrix M_k

```
Row:=proc(r,k)
v:=[c[1]];
if (r=1)
  then v:=[op(v),1];
      for j from 2 to n-r do v:=[op(v),0] od;
      return v;
   else
    if (r=2) then
      if (k=2)
          then return  [2*c[2], c[1]];
          else v:=[2*c[2],c[1],1]  fi;
    for j from 2 to k-r do v:=[op(v),0] od;
    return v ;
    else
      for i from 2 to r-1 do v:=[c[i],op(v)] od
    fi
fi ;
v:=[r*c[r],op(v)];
```

```
if (r=k)
   then return v;
   else v:=[op(v),1]
fi;
for j from 2 to k-r do v:=[op(v),0] od;
return v;
end proc;
```

Then we write the procedure that outputs the matrix M_k

```
MatrixM:=proc(k)
if (k=1)
   then return matrix(1,1,[c[1]])
fi;
V:=[];
for i from 1 to k do V:=[op(V),op(Row(i,k))]
od;
return matrix(k,k,[op(V)]);
end proc;
```

Finally, we write the procedure that returns the polynomial for $\chi(\mathcal{F})$ once that we have fixed N

```
chi:=proc(n,N)
for i from 1 to N do S[i]:=linalg[det](MatrixM(i))
od;
return 1/N!*sum(abs(stirling1(N+1,k+1))*S[k],k=1..N)+n;
end proc;
```

With the procedure `chi(n,N)` one can then easily obtain the Euler characteristic for a rank n coherent sheaf on $\mathbb{P}^N$, just fixing N.

The polynomial expression for the Euler Characteristic for a rank n sheaf on $\mathbb{P}^3$ is known: one can see [1], Theorem 2.3. We give as examples the polynomial expressions for $\chi(\mathcal{F})$ for a rank n sheaf on $\mathbb{P}^4$ and $\mathbb{P}^5$.

$chi(n,4)$;

$$\frac{1}{24}\left[{c_1}^4 + 10\,{c_1}^3 - 4\,{c_1}^2 c_2 + 35\,{c_1}^2 - 30\,c_1 c_2 + 4\,c_1 c_3 + \right.$$

$$\left. 2\,{c_2}^2 + 50\,c_1 - 70\,c_2 + 30\,c_3 - 4\,c_4\right] + n$$

$chi(n,5)$;

$$\frac{1}{120}\left[{c_1}^5 + 15\,{c_1}^4 - 5\,{c_1}^3 c_2 + 85\,{c_1}^3 - 60\,{c_1}^2 c_2 + 5\,{c_1}^2 c_3 + 5\,c_1 {c_2}^2 + 225\,{c_1}^2 + \right.$$

$$-255\, c_1 c_2 + 60\, c_1 c_3 - 5\, c_1 c_4 + 30\, {c_2}^2 - 5\, c_2 c_3 + 274\, c_1 +$$

$$-450\, c_2 + 255\, c_3 - 60\, c_4 + 5\, c_5] + n$$

3.2 A faster algorithm for $\chi(\mathcal{F})$

The procedure `chi(n,N)` is quite expensive from the computational viewpoint: Maple 11 on a personal computer (Intel Pentium CPU 3.00 Ghz, 992 mb RAM) took more than 20 seconds for the case $N = 18$.

We can improve the procedure because actually we do not need to construct the matrices M_k to compute their determinant. We can just construct a recursive procedure using formula (5).

First the procedure to compute $\det M_k$:

```
detM:=proc(k)
if k=1 then return c[1]
fi;
if k=2 then return c[1]^2-2*c[2]
fi;
M:=(-1)^(k-1)*k*c[k];
for i from 1 to k-1 do M:=M+(-1)^(i-1)*c[i]*detM(k-i)
od;
return expand(M) end proc;
```

Then we rewrite the procedure `chi(n,N)`, but using `detM(k)`:

```
chifast:=proc(n,N)
for i from 1 to N do M[i]:=detM(i)
od;
return 1/N!*sum(abs(stirling1(N+1,k+1))*M[k],k=1..N)+n;
end proc;
```

This last procedure is much faster than `chi(n,N)`: for instance, we computated the polynomial for $\chi(\mathcal{F})$ for a rank n coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{20}$, on a personal computer (Intel Pentium CPU 3.00 Ghz, 992 mb RAM) using Maple 11:

- `chi(n,20)` took 172.14 sec to output the polynomial;

- `chifast(n,20)` took only 7.72 sec to output the polynomial.

3.3 An algorithm for $\chi(\mathcal{F}(t))$

Since we have already a polynomial form for $\chi(\mathcal{F})$, we can easily obtain the polynomial associated to $\chi(\mathcal{F}(t))$ for every $t \in \mathbb{Z}$. It is sufficient to remember that, if $\mathcal{F}$ is a rank n coherent sheaf on $\mathbb{P}^N$ and Chern classes c_i, then

$$c_i(\mathcal{F}(t)) = c_i + (n-i+1)tc_{i-1} + \binom{n-i+2}{2}t^2c_{i-2} + \cdots + \binom{n-1}{i-1}t^{i-1}c_1 + \binom{n}{i}t^i. \tag{6}$$

So we substitue $C_i(T) = C_i + (n-i+1)TC_{i-1} + \binom{n-i+2}{2}T^2C_{i-2} + \cdots + \binom{n-1}{i-1}T^{i-1}C_1 + \binom{n}{i}T^i$ to C_i in $P(C_1,\dots,C_N)$ obtaining

$$P(C_1(T),\dots,C_N(T)) = G(C_1,\dots,C_N,T) \in \mathbb{Q}[C_1,\dots,C_N,T].$$

With some little changes, we can rewrite the procedure `chifast(n,N)` for any twist $\mathcal{F}(t)$, $t \in \mathbb{Z}$: the procedure outputs a polynomial in the variable T. First, we write a procedure to obtain a "twisted"Chern class

```
ct:=proc(j,N)
cT:=c[j];
for i from 1 to j-1 do
cT:=cT+binomial(N-j+i,i)*T^i*c[j-i]
od;
cT:=cT+binomial(N,j)*T^j;
return cT;
end proc;
```

Then we simply rewrite the procedure `detM(k)`

```
detMT:=proc(k,N)
if (k=1) then return ct(1,N)
fi;
if k=2 then return ct(1,N)^2-2*ct(2,N)
fi;
Mt:=(-1)^(k-1)*k*ct(k,N);
for i from 1 to k-1 do Mt:=Mt+(-1)^(i-1)*ct(i,N)*detMt(k-i,N)
od;
return expand(Mt)
end proc;
```

Finally, we rewrite `chifast(r,N)` using the "twisted"determinants

```
chit:=proc(n,N)
for i from 1 to N do M[i]:=detMT(i,N)
```

```
od;
return
 sort(collect(1/N!*sum(abs(stirling1(N+1,k+1))*M[k],k=1..N)+n,T),T);
end proc;
```

For instance, we obtain the polynomial associated to $\chi(\mathcal{F}(t))$ for a coherent sheaf on $\mathbb{P}^6$

$chit(n,6);$

$$\frac{1}{240}\left[6\,T^6+(6\,c_1+126)\,T^5+\left(15\,{c_1}^2+105\,c_1+1050-30\,c_2\right)T^4+\right.$$
$$+\left(60\,c_3+210\,{c_1}^2+20\,{c_1}^3+700\,c_1+4410-60\,c_1c_2-420\,c_2\right)T^3+$$
$$+\left(-60\,c_4+1050\,{c_1}^2-60\,{c_1}^2c_2-2100\,c_2-630\,c_1c_2+2205\,c_1+\right.$$
$$\left.+630\,c_3+9744+210\,{c_1}^3+60\,c_1c_3+15\,{c_1}^4+30\,{c_2}^2\right)T^2+$$
$$+\left(420\,c_1c_3-4410\,c_2-30\,{c_1}^3c_2+30\,c_1{c_2}^2+2205\,{c_1}^2+2100\,c_3+30\,c_5+\right.$$
$$+700\,{c_1}^3+105\,{c_1}^4+6\,{c_1}^5+3248\,c_1+210\,{c_2}^2-420\,c_4+10584-30\,c_2c_3+30\,{c_1}^2c_3+$$
$$\left.-30\,c_1c_4-2100\,c_1c_2-420\,{c_1}^2c_2\right)T+2205\,c_3-3248\,c_2-700\,c_4+r+350\,{c_2}^2+$$
$$+175\,{c_1}^4+1764\,c_1-105\,c_1c_4+105\,{c_1}^2c_3-105\,{c_1}^3c_2+105\,c_5-105\,c_2c_3+1624\,{c_1}^2+$$
$$-2205\,c_1c_2+105\,c_1{c_2}^2+700\,c_1c_3-700\,{c_1}^2c_2-6\,{c_1}^2c_4+6\,{c_1}^3c_3-6\,{c_1}^4c_2+9\,{c_1}^2{c_2}^2+$$
$$\left.+{c_1}^6-2\,{c_2}^3+6\,c_2c_4+3\,{c_3}^2+21\,{c_1}^5-6\,c_6+735\,{c_1}^3-12\,c_1c_2c_3+6\,c_1c_5\right]+n.$$

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