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Multimodality in multi-representational environments

Ferdinando Arzarello, Ornella Robutti

Dipartimento di Matematica, Università di Torino

ferdinando.arzarello@unito.it, ornella.robutti@unito.it

Summary

The main issue of the paper concerns the way new technologies (TI-Nspire and TI-Navigator) influence the different students' multimodal production, in terms of words, gestures, inscriptions, actions on the artefacts. Specifically, it investigates how they can modify students' processes of mathematics learning, what descriptors are the most suitable for grasping such changes, and what are the new opportunities for the teacher in designing and managing mathematical activities within such environments. Three different lenses (instrumentation, humans-with-media, multimodality) are used to analyse some classroom activities, where students employ such new technologies. It is so shown how the multi-representations present in the two technological environments can support the students' multimodal production, interaction and communication, when they are engaged in constructing mathematical meanings. In particular the article underlines new features of the "new technologies", such as hand-held environments, compared with the older ones.

Multimodality, multi-representation, representational infrastructure, communication infrastructure, handheld technology, semiotic approach

Introduction

When working with calculators or computers, teachers and students are faced with new types of environments, where the usual constraints of paper and pencil are radically changed. In fact the technological environments today offer more than one world where students can move and act, with the possibility of integrating and simultaneously using either different registers (such as numerical, symbolic, graphical) with the same tool (the screen of the calculator), or different tools (the screens of the various calculators present in the classroom). Moreover the technology can support the communicational and interpersonal interactions between students and between students and the teacher according to affordances, which are consonant with the common technological environments that people meet in everyday life.

These possibilities open new frontiers and enlarge the usual boundaries of technology used in mathematics education; insofar as it overlaps the technologies of everyday life. Any research about learning processes in technological environments today should also be sensitive to the integration of technologies of "old" and "new" generation (Robutti 2010). In fact, if it is a matter of *re-mediation* (Bolter and Grusin 2000) from one side (because new media achieve their cultural significance by paying homage to earlier media: e.g., a DGE re-mediate static geometry with compass and ruler); from the other side the kind of use of the technology is important. Using a technology does not mean a repetition of traditional activities made in paper and pencil, rather introducing new methodology, which can give different approaches to mathematics.

For this reason, we need a theoretical framework that allows us to frame these new features of technologies used in didactics of mathematics. Therefore, today it is necessary to update the classical *institutional* and *cultural* dimensions, according to which mathematics learning within technological environments must be looked at. For example it is necessary to enlarge the *instrumental* approach (Verillion and Rabardel 1995), to focus the new instances that come from the technology that is entering the classrooms at the beginning of the new century.

Therefore our main reference in this paper is the model of *Representation and Communication Infrastructures*, as described in Hegedus and Moreno-Armella (2009), and a consequent reformulation and integration of the following frames:

- the *instrumental approach,* coming from cognitive ergonomy of Verillion and Rabardel (1995);
- the *humans-with-media approach*, introduced by Borba and Villarreal (2005);
- the *multimodal approach,* coming from cognitive science (Kress et al. 2001; Williamson 2005);

We support our analysis by considering two kinds of handheld technology: TI-Nspire, a symbolic-graphic software implemented on calculators (but also on computers), and TI-Navigator, a software that support wireless communication between students' graphic calculators (TI-84) and teacher's computer (we call them TI-technologies in short). Both of them offer multi-representational environments, in the sense that students have more representations (numerical, graphical, and symbolic) of the same mathematical objects at their disposal.

Moreover in TI-Nspire such representations are dynamically connected. On its side the TI-Navigator offers another possibility, namely the possibility of sharing these representations on the teacher's screen, which is visible to the whole class if it is connected to a projector. This has consequences for the different modality of behaviours simultaneously activated by students during their learning processes: for example on their verbal, written, gesture productions.

Such features represent a novelty in the panorama of technological environments and it is natural to wonder if and how such novelties modify processes and products of people who learn mathematics and the way they might change teachers' practices.

Hence we start posing some "naïve" questions that spring out in a natural way when using a new technological environment in the class of mathematics:

a) How do the TI-environments modify students' processes of mathematics learning?
b) What descriptors are the most suitable for grasping such changes? What are the most useful for entering into the specific *mediations* allowed by the TI-technologies?
c) What are the new challenges and opportunities for the teacher in designing and managing mathematical activities within such environments?

In this paper we first sketch out a description of how we have transformed them into a more scientific form; we then describe the research we have developed in order to answer them with the support of some data.

To achieve these aims we use the following methodology:

- First, moving from the theoretical analysis of Hegedus and Moreno-Armella (2009), we point out two main dimensions that feature the most recent technological environments:

i. the *dynamical representations* of mathematical concepts (typically supported by TI-Nspire multi-representational environment);
ii. the *specific interactions* between the students and between the teacher and the students (typically supported by the TI-Navigator device).

We describe these two dimensions, pointing out their main specific features and comparing each other.

- Second, within such a theoretical frame, we use the three different approaches listed above (instrumentation, humans-with-media, multimodality) as theoretical lens suitable for transforming the naïve questions above into a more scientific

form. We get some variables as descriptors of teaching and learning processes within technological environments.

- Third, we investigate them through empirical research in some classrooms, where the new technologies are used to learn mathematics. We are so able to describe some didactical situations that are particular to these environments; analysing classroom activities where students interact and work together in a social way, with the modality of a mathematics laboratory. Our examples will be used emblematically to illustrate the features of the "new technologies", such as hand-held environments, compared with the previous ones.

This paper is divided in three main parts: a short description of the theoretical frames with consequent reformulation of the research questions (# 1); the description of two classrooms episodes, where, using our theoretical lenses, we illustrate two different technological environments (# 2); the discussion about similarities and differences between the two environments and a first answer to our questions (# 3).

1. Theoretical framework

Representational and Communication Infrastructures

Hegedus and Moreno-Armella (2009) describe the recent evolution of educational technology developing further the analysis made by Kaput and others through the idea of *representational infrastructures* (Kaput, Noss & Hoyles 2002). They observe that "Software has become more visual, interactive and … more dynamic. Hardware has evolved to allow more complex programs to be executed for work to be done at a distance (both proximally and longitudinally) through the advances of networks (in particular wireless), and to be more portable in terms of its hand-heldability." (ibid., p.399).

They point out that such new affordances impact two types of infrastructure:

1. Representational Infrastructures
2. Communication Infrastructures.

Their analysis shows how "technology has offered and afforded representations and interactions between representations ... in terms of symbolic manipulators where computational duties are offloaded to the microprocessor and new actions are linked to traditional notations systems." The novelty of last generation software means that now "there is ... support for new interactive notation systems

such as programming languages underlying mathematics packages (e.g. Maple) and spreadsheets, enhanced interactivity and expressibility of new phenomenon by linking traditional notations systems and representations to new ones (e.g. simulations), and finally digital communication." (ibid., p.400). These last affordances have made new capabilities accessible to students, namely the ability to see through abstract constructs or symbolic figures (ibid.). Hence "two unique technological ingredients are integrated—dynamic, interactive software that works with multiple representations simultaneously (RI) and wireless networks to create networked classrooms (CI)." (ibid., p. 409).

The Representational and Communication Infrastructures trigger specific practices in the classroom and support new processes of instrumentation (Verillion and Rabardel 1995), which consist of ways of working that can be added to more traditional ones:

1) A change in the *multimodal behaviours* of students, supported not only by a traditional environment such as paper and pencil, but also by Representational and Communication Infrastructures, which can offer the possibility to enlarge the multimodal production, through the multi-representations they offer.
2) A wider notion of the teacher's *instrumental orchestration* (Trouche 2004); the role of the teacher changes and must be analysed according to a wider approach to take into account not only his relation with students, but also the use of various technologies in the classroom.

In the paragraph 2 we will sketch two scenarios that show the differences between a good Representational-only environment and one where there exists integration between the Representational- and Communication- aspects.

Instrumental approach

Rabardel introduces the concept of *instrumental genesis* to describe the process by which an artefact becomes an instrument. It indicates the two directions in which this process takes place: towards the subject and towards outside reality. The first meaning of appropriation requires the artefact to be integrated within one's own cognitive structure (e.g., one's existing representations, available action schemes, etc.). That, in general, requires adaptation (instrumentation). The second meaning indicates that the artefact has to be appropriated to an outside context. Specific ends and functional properties –some not necessarily intended by design– are

attributed to it by the user (instrumentalisation).

For example, students learn to properly use data-capture in TI-Nspire through an instrumentation process. But they generate an instrumentalisation process, when they refine their actions for producing a more uniform set of data with data-capture through an animation or may decide to use data-capture to explore the relationships among the sides and an angle in a variable triangle, subject to some constraints.

Rephrasing our naïve questions within the instrumental frame we get the following issues:

a') What is the specificity of instrumented actions in these TI-technological environments?

b') What instrumented actions in these technologies help (or do not help) students learning processes?

More specifically: what are its instrumentation/instrumentalisation aspects? How the activated instrumented actions can (or cannot) support student learning?

Humans-with-media

Humans-with-media is a theoretical approach that takes both the subject and the tool involved in a mathematical activity into account (Borba and Villarreal 2005) and it is grounded on two ideas. First, the construction of knowledge is made in a social way by subjects working together; second, the media involved are part of this construction because they collaborate to re-organise thinking with a different role than the one assumed by written or oral language. Borba and Villarreal introduce a point of view that contains and enlarges previous instrumental approaches because they particularly focus on the community of learners (small groups, as well as the whole class or bigger groups), along with the tools. This point of view overcomes the deep-rooted dichotomy between humans and technology, in that 'it' considers learning as a process of interaction amongst humans as a community including tools. Media interact with humans in the double sense that technologies transform and modify humans' reasoning, as well as humans are continuously transforming technologies according to their purposes. This theoretical issue is not far from the instrumental approach, on one hand because it considers the relationships and interaction between human and technologies with mutual influences. Nor it is not far from Representational and Communication Infrastructures frame. On the other hand, because it analyses the

features of communication at distance (with the double possibility to be synchronous or a-synchronous), thanks to modern platforms on the web.

Rephrasing the research questions within the humans-with-media frame we get the following issues:

a'') What are the features introduced by the TI-technologies in the teaching-learning processes, which offer new opportunities of interaction among students, and with the teacher, not previously observed with other technologies?

c'') What is the teacher's role in the classroom within these environments?

Multimodal approach

The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in these last years (Wilson 2002). Embodiment is a current movement in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested in how the body is involved in thinking and learning. It emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. Concepts are thus analysed not on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese and Lakoff 2005, p.455), but considering the *multimodality* of our cognitive performances as well. Multimodality is the use of two or more forms of communication from the two main modalities, namely auditory and visual (Loncke et al. 2006) and is deeply intertwined with perceptuo-motor activities. Interests in multimodality in education have been generated by the increasing use of multimedia in the classroom, from image manipulation software to electronic music-making packages, to science simulations, and to virtual theatres that exist on computers. Studies in neuroscience tell us that the sensory-motor system of the brain is multimodal rather than modular (Gallese and Lakoff 2005): "an action like grasping ... (1) is neurally enacted using neural substrates used for both action and perception, and (2) the modalities of action and perception are integrated at the level of the sensory-motor system itself and not via higher association areas." (ibid., p. 459). "Accordingly, language is inherently multimodal in this sense, that is, it uses many modalities linked together—sight, hearing, touch, motor actions, and so on. Language exploits the pre-existing multimodal character of the

sensory-motor system." (ibid., p. 456). If the sensory-motor system of the human brain is multimodal, then human activity is also multimodal, and therefore we can analyse all the modalities in order to understand cognitive processes. (Arzarello and Edwards 2005). During the mathematical activities with media, students produce a variety of signs as words, gestures, and actions on the tools, interactions, written or oral signs of whatever nature.

Instrumented activity in technological settings is multimodal, because action is not only directed towards objects, but also towards people. In such situations teacher and students simultaneously use a wide array of verbal, gestural, and graphic registers to communicate their thought. This multimodal production is particularly rich when subjects use technology such as symbolic-graphic calculators with TI-Nspire or Navigator, which amplify multimodality with their multiple environments.

The experience of learning together (learning to be with others in mathematics, as written by Radford, 2006) with the use of a technological tool, can be described by a frame that takes the multimodal production of the students, the teacher and the technology itself into account. In this approach learning mathematics is a matter of being-in-mathematics (Radford 2006), living in a classroom as a community, working together and sharing activities and results.

The previous research questions can be then rephrased:

a''') To what extent do technologies modify the usual multimodal production of students engaged in a task in paper and pencil?

c''') To what extent do technologies modify the usual multimodal production of teachers engaged in a task in paper and pencil?

Therefore we have different formulations of our naïve questions: a'), a''), a'''), b'), c''), c'''). The research questions written at the beginning are general and, taking into account the theoretical elements, we have specified them, first according to the instrumental approach, then to humans-with-media frame, and finally to the multimodal paradigm. These are three lenses that focus our questions from three different points of view and allow us to elaborate an integrated frame. All these questions give us a set of lines of investigation to refer

to, in order to carry out our research. They are not to be intended as mutually separate, but integrated together and sinergically related.

In the next paragraph we will describe some emblematic episodes from different teaching experiments that allow us to answer such questions through empirical data.

2. The teaching experiments

Methodology

The two concrete and emblematic teaching episodes described below allow us to describe the main features of processes that are triggered and supported within two technological environments: TI-Nspire, TI-Navigator. They represent the two main dimensions that feature the most recent technological environments, sketched out above: the dynamical representations of mathematical concepts (TI-Nspire); the different specific interactions in the classroom (TI-Navigator).

The episodes are taken from two different teaching experiments developed in two different classes. The first is a 9th grade scientific class using TI-Nspire, with a mature male experienced teacher, and the second is a 10th grade commercial class using TI-Navigator, with a young female teacher with some years of experience. The classes have about 20 students each.

The didactical methodology follows the approach of the *mathematics laboratory*, developed in the Italian mathematics education community and presented at ICME10 in various events: discussion group (Chapman and Robutti 2008), CD-ROM, presentations, and a booklet of the recent Italian mathematics education research. A mathematics laboratory is a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. A mathematics laboratory activity involves people, structures, ideas, just as in a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, and communicating with each other; namely practicing. In the activities the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together. This way of working is typical of perceptuo-motor activities described in literature (e.g.,

Nemirovsky 2003), where students are involved in solving mathematical problems individually or in groups.

In the teaching experiments we present below, students work together in small groups (two, or maximum three members) with technology (respectively TI-Nspire and TI-Navigator), and have to fill in a worksheet. Each group uses one TI-technology and one worksheet. We made this choice in the research group (made of the authors and the teachers of the classes), in order to favour social interaction among friends in each group as much as possible. Groups are located in the class on separate tables (made of 2-3 desks), or in computer laboratory. They have to read the task, think of a solution, discuss together and operate for the solution using technology, then to fill the worksheet in, as final report of the group. During the group activity, teacher and observer (one of the authors) observe students working together and possibly interact with them, never giving the solution, but stimulating them in finding it.

The group's activities are followed by collective discussions orchestrated by the teacher (Bartolini Bussi 1996), who leave the students free to discuss, compare, conjecture, imagine, and connect various ideas and concepts. His/her role is to ask questions as "why", "explain", "what … if …" and so on, to stimulate argumentation and explanation.

The topic developed in the teaching experiments presented here is related to the so-called *mathematics of change* (Kaput and Roschelle 1998), integrated by the frames of *representational and communication infrastructures* and *humans-with-media*, in order to consider students and media as a whole community with mutual influences. The didactical aims are directed to developing competences related to the *number sense* (Sowder 1992), the *graph sense* (Robutti 2006) and the *symbol sense* (Arcavi 1994) in an integrated approach, as in other studies we carried out at various school levels (e.g. Robutti 2010).

At the 9th and 10th grade (first and second year of secondary school) some contents of the curricula are: functions of first and second degree, with their representations, polynomials, algebraic fractions and algebraic manipulation. We developed the topic of functions at 9th grade with numerical explorations, using tables with abscissas and ordinates, and their finite differences (first and second),

in the TI-Nspire environment. At the 10th grade, we used numerical sequences, where students have to find patterns, formulas and functions and to represent them on graphs in the TI-84 environment, with a connection of calculators through TI-Navigator.

As cognitive roots (Tall 1989) for the description of functions we choose the qualitative concept of invariance (of shape on a graph) and the quantitative concept of slope on an interval (as ratio of increments) and its variation (first and second). Related to these roots we also use other concepts, such as: domain, sign, intersection, zero, parallelism, and so on. The activities are centred on families of functions, principally linear, quadratic, cubic, and exponential, and the construction of meaning starts from modelling problems.

Teachers of the two classes participated with researchers in planning activities, use and orchestration of TI-technologies and didactical methodologies. In both classes a master degree student in mathematics education used a video-camera to record the activities, in order to analyse videos and write protocols. Students in both classes knew that they were selected for teaching experiments with research purposes and participated totally to the activities. Parents agreed to let us film students for research aims and gave their consent in doing it.

We chose to film with one video-camera, moving in the class, in order to follow activities of students in groups, interventions of the teacher or of students during the collective discussions. Data are made of films, worksheets, pictures extracted from the films and files from the technology used.

Students-with-Representational Infrastructures

Our first example illustrates a very interesting aspect of Representational Infrastructures (the first dimension mentioned at the beginning of # 2), namely how TI-Nspire can trigger and support what we call students' *symbol sense.* It is "a tool for understanding, expressing, and communicating generalization, for revealing structure, and for establishing connections and formulating mathematical arguments" (see Arcavi 1994, p.24). Students have symbol sense if they are able, for example, to call on symbols in the process of solving a problem, and conversely, to abandon a symbolic treatment for better tools; to recognise the

meaning of a symbolic expression, and to sense the different roles symbols can play in different contexts.

One specific feature, which distinguishes TI-Nspire from other software, is the instrumented actions that students develop using the symbolic spreadsheet of TI-Nspire. It is considered a strong didactical innovation and we show it through an example from the 9th grade students, who were studying functions using the tables of finite differences.

These students are approaching the so-called *mathematics of change*, in a scientific high school in Italy. They already know that for first degree functions the first differences are constant. They also know how to write a first degree function and how to represent it in a Cartesian plane. But they do not know algebraic manipulation, second degree functions ($y=ax^2+bx+c$), their representations and the role of their parameters.

They are asked to make conjectures on what families of functions have the first differences that change linearly. Their conjecture is that quadratic functions (as generalisation of first degree functions) have this property, and they arrange a spreadsheet like in Figure 1a, where they utilise:

- Columns A, B, C, D to indicate respectively the values of the variable x, of the function $f(x)$ (in B_i there is the value of $f(A_i)$) and of its related first and second differences (namely in C_i there is the value $f(A_{i+1})$-$f(A_i)$ and D_j there is the value C_{j+1}- C_j);
- Variable numbers in cells E2, F2, …, I2 to indicate respectively: the values x_0 (the first value for the variable x to put in A2); a, b, c for the coefficients of the second degree function $y=ax^2+bx+c$; the step h of which the variable in column A is incremented each time for passing to A_i to A_{i+1}.

Students work in pairs at this task and the teacher goes around the classroom, possibly helping those having difficulties with the task. At the end, a balance discussion orchestrated by the teacher is the occasion to discuss the results and share them in the classroom.

During the activity, the students do their explorations modifying the values of E2, F2, …, I2. This is a practice students have learnt, which gradually becomes a habit in the classroom, because of the interventions of the teacher, who stresses it's value as an instrumented action (Verillion and Rabardel 1995), which supports explorations in the numerical environment.

It is interesting to observe that such a practice reveals it's didactical power if analysed through a semiotic lens. Using the terminology in Duval (2006), this

instrumented action supports a systematic *treatment* of numbers, scaffolded according to the formula of the second-degree function. These types of treatments, in Duval language, are one of the roots for developing algebraic thinking in students. Hence the instrumented actions of this type seem to be promising for learning algebra. In this treatment, TI-Nspire is not different from an artefact as usual spreadsheets, and shares with them the same potentialities for learning algebra.

	A	B	C	D	E	F	G	H	I
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	3	-1	-4	0	-2	1	3	1
3	1	2	-5	-4					
4	2	-3	-9	-4					
5	3	-12	-13	-4					
6	4	-25	-17	-4					
7	5	-42	-21	-4					
8	6	-63	-25	-4					
9	7	-88	-29	-4					
10	8	-117	-33	-4					
11	9	-150	-37	-4					
12	10	-187	-41	-4					
13	11	-228	-45	-4					
14	12	-273	-49	-4					
15	13	-322	-53	-4					
16	14	-375	-57						
17	15	-432							

	A	B	C	D	E	F	G	H
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	x0	a*x0^2+b*x0...	a*(h^2+2*h*x0)+...	2*a*h^2	x0	a	b	c	h
3	h+x0	a*(h+x0)^2+...	a*(3*h^2+2*h*x0...	2*a*h^2					
4	2*h+x0	a*(2*h+x0)^2...	a*(5*h^2+2*h*x0...	2*a*h^2					
5	3*h+x0	a*(3*h+x0)^2...	a*(7*h^2+2*h*x0...	2*a*h^2					
6	4*h+x0	a*(4*h+x0)^2...	a*(9*h^2+2*h*x0...	2*a*h^2					
7	5*h+x0	a*(5*h+x0)^2...	a*(11*h^2+2*h*x...	2*a*h^2					
8	6*h+x0	a*(6*h+x0)^2...	a*(13*h^2+2*h*x...	2*a*h^2					
9	7*h+x0	a*(7*h+x0)^2...	a*(15*h^2+2*h*x...	2*a*h^2					
10	8*h+x0	a*(8*h+x0)^2...	a*(17*h^2+2*h*x...	2*a*h^2					
11	9*h+x0	a*(9*h+x0)^2...	a*(19*h^2+2*h*x...	2*a*h^2					
12	10*h...	a*(10*h+x0)^...	a*(21*h^2+2*h*x...	2*a*h^2					
13	11*h...	a*(11*h+x0)^...	a*(23*h^2+2*h*x...	2*a*h^2					
14	12*h...	a*(12*h+x0)^...	a*(25*h^2+2*h*x...	2*a*h^2					
15	13*h...	a*(13*h+x0)^...	a*(27*h^2+2*h*x...	2*a*h^2					
16	14*h...	a*(14*h+x0)^...	a*(29*h^2+2*h*x...						
17	15*h...	a*(15*h+x0)^...							

Figure 1a – 1b Numerical and symbolic spreadsheet in TI-Nspire

There is a more interesting second paragraph, which happens because of the symbolic calculations that TI-Nspire can support with its spreadsheet.

In fact, students realise that:

- If they change only the coefficient *c*, only column B changes, while the columns C and D of the first and second differences do not change; hence they argue that a function increases/decreases *not* in dependence of the coefficient *c* (in this way, they find an *invariant* in a changing situation);
- If they change the coefficient *b*, then columns B and C change, but column D does not (another invariant); many students conjecture that the coefficient *b* determines *if* a function increases or decreases but *not* its concavity;
- If they change the coefficient *a*, then columns B, C and D change; hence the coefficient *a* determines a change of the concavity of the function.

A difficult point for the students here is to understand *why* such relationships hold. The tables of numbers do not suggest anything by themselves. It is the symbolic power of the spreadsheet which is useful in this case. The epistemic instrumented action is very interesting and consists of substituting letters to the numbers: see Figure 1b. In most cases the teacher has suggested this practice, but a two students did it autonomously. The spreadsheet shows clearly that the value of the second difference is $2ah^2$. Here the letters in the formula condense the symbolic meaning of the explorations developed before in the numerical environment.

The teacher has stressed this power of the symbolic spreadsheet in the final discussion and again a new practice has entered the classroom.

In this sense, TI-Nspire allows an early exposure of students to the symbolic aspects of the mathematical language supporting suitable instrumented actions, which are particularly apt to trigger the symbolic function of the algebraic language. This is possible since TI-Nspire is an excellent Representational Infrastructure[1], perfectly integrated with the mathematical culture to be taught and consonant with the cognitive processes of students: symbol sense is approached here in a very natural way, without the cognitive load which seems unavoidable in paper and pencil environments.

It is apparent from this example that the symbolic spreadsheet has a strong value in pushing students towards the understanding why the second differences are constant and depend only on the first coefficient *a* and the step *h*. This would have been very difficult without the two steps done with the spreadsheet: first the numerical representation, second the transition to the letters. It has been the multi-representational feature of the software which has triggered and allowed this in a process, which is a "jeu des cadres" (game of frames, according to the terminology of R. Douady 1986), namely a dynamic interaction and evolution

between the numerical and the symbolical frames. In this process the role of the teacher is almost outside the humans-with-media environment determined by the relationships between the students and the TI-technology. We shall come back to this point later, in the final discussion.

Students-with-Representational and Communication Infrastructures

Our second example illustrates the added value of Communication Infrastructure (the second dimension recalled at the beginning of # 2), namely an episode from a classroom where TI-Navigator is used. In TI-Navigator, the public display consists of a common Cartesian plane (called Activity Center), to which each student and the teacher can give their personal contribution, inserting mathematical objects as points, lines, and so on (Figure 2). Another environment is the Screen Capture (Figure 3), through which all the screens of the students' calculators are captured simultaneously on the teacher's screen.

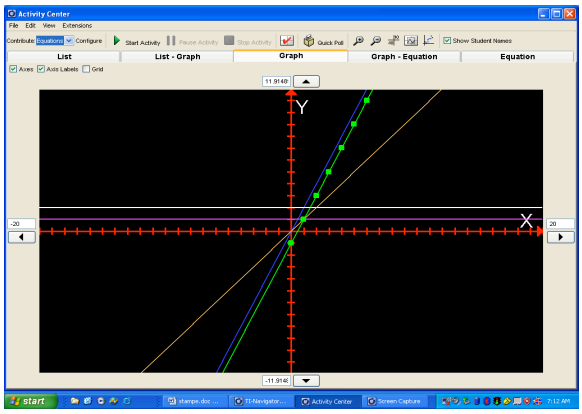
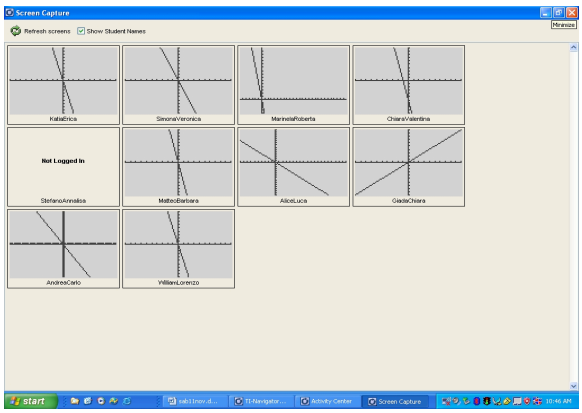


Figure 2: Activity Center

Both the environments can be projected on a big screen, if the teacher's computer is connected to a video-projector.



[1] TI-Nspire is a good RI in this sense, for many functions it offers: the most interesting case is that

Figure 3: Screen Capture

The tool TI-Navigator is substantially different from usual equipment, made of computers or calculators used by groups (or individuals), who can follow the screen of their calculator and do not have any information of what is happening in the other groups. With usual equipment, if the teacher wants to have information on the processing made by the students, has to pass from one group to the other. With TI-Navigator, each group may follow his work, and simultaneously also other groups' work, looking at a public screen that can be Activity Center or Screen Capture. So the teacher herself may remain in a central position, following all the works on the big screen, discussing with a single group, or guiding a class discussion where everyone can take part, because information is shared.

The activity presented here is one of the first of the teaching experiment with TI-Navigator, after an introduction on the software and some exercises with the environments of the calculator (Robutti 2010). Here the students have to find the various terms of a numerical sequence as coordinates of points, and to send them to the public screen, where they are represented altogether. Pairs of students carry out the activity with one calculator (TI-84) connected to the public screen and one worksheet, to be filled in.

Consider the point P_0 (0, -1). Find the coordinates of P_1 , by adding 1 to the abscissa of P_0, 2 to its ordinate. Represent the point on the Cartesian plane. Find P_2, adding 1 to the abscissa of P_1, 2 to its ordinate and represent P_2. Now find P_3, P_4 and so on. Write the sequence of the points P_0, …, P_6. How do you pass from one point to the subsequent? What are the coordinates of P_{10}? Explain how to determine P_{100} and what the rule is.

The aim of the activity is the model (linear) of the situation, expressed not only in a recursive form ($x_n=x_{n-1}+1$, $y_n=y_{n-1}+2$; with $x_0=0$, $y_0=-1$), where each element of the sequence is written in function of the previous element, but also with a formula ($x_n=n$, $y_n=2n-1$), where each element of the sequence is determined in function of its position in the sequence. The important point is not only the symbolic expression, but the meaning of the relation among the abscissas and ordinates of the points in the sequence.

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of *data-capture*, which gives this software an almost-empirical feature (see Arzarello, 2009).

We can observe the progression in the construction of this meaning, both in the working groups and in the plenary discussion. In the following section we have the excerpts reported from one group, made by the students Ca and An (pseudonymous). They are medium achievers, well integrated together and in the classroom.

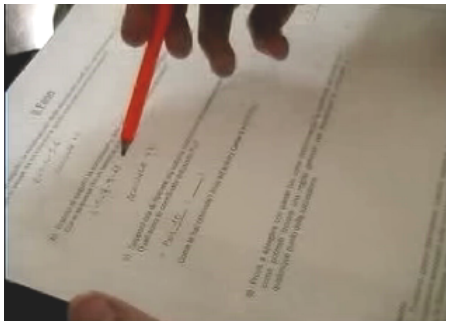

Figure 4: Gesture with the pencil

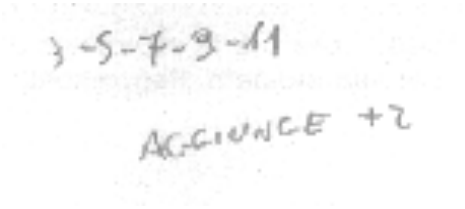

Figure 5: “It adds +2”

At the beginning of the working group students try to construct a meaning for the sequence, in order to understand that with the abscissas they have to add 1 and with the ordinates they have to add 2. The signs they introduce in this activity are: actions with the pencil (Figure 4), inscriptions on the paper (Figure 5), gestures with the hands, words, and show the multimodality of their production.

To understand the evolution of this multimodal production over time, we can observe the role played by the elements of this production in the various steps:

- The first step in the construction of meaning is telling the sequence of ordinates, understanding that the rule for passing from one to the other is adding 2. This step is marked by different signs: the beating of pencil on paper (Figure 4); the sequence of numbers pronounced by the two students, the same sequence written on the worksheet (Figure 5); the rule, first made explicit orally and then written on the paper (Figure 5). At this

step, the meaning they have of the sequence of ordinates is the rule of "adding 2", in order to pass from one ordinate to the subsequent one. But they have no idea of how to find any elements of the sequence, knowing the position it has in the sequence.

- The second step is marked by a recursive interpretation of the rule, applied to four points instead of one, but it is not the formula they can apply to find whatever point, only knowing its place in the sequence. The task is constructed in such a way in order to gradually mediate this construction, passing first through the recursive law ($x_n=x_{n-1}+1,\ y_n=y_{n-1}+2;$ with $x_0=0,\ y_0=-1$), and then through the formula ($x_n=n,\ y_n=2n-1$). This passage is not simple for the students, because they have never met it before and they never had to make this explicit it in such a way. These students are not able to find the coordinates of P_{100}, so they randomly conjecture that the ordinate of P_{100} is 190, for a sort of similarity to $P_{10}(10,19)$. An asks (himself and the mate) if the rule could be "times 10", but he is not convinced, nor is Ca.

In this working session, some groups of students were able to construct a meaning for the sequence, other groups found it difficult to do that, and made various kinds of mistakes.

At the end of the group work, a discussion takes place. In order to share the results and discuss them from a critical point of view, the role of the teacher here is not to give answers, but to stimulate the participation of the students with their observations, justifications and conjectures, with the methodology of the mathematical discussion. First the discussion has the aim of writing the formula (this is the last question on the worksheet), then of describing the model from a graphical point of view.

Until now the activity has been similar to any other activity made by students working in groups with a handheld technology, like graphic calculators of the same kind they used, namely there is no difference between what happens here and with a good Representational Infrastructure. If in the previous protocol nothing "new" with respect to the use of technology was introduced, during the collective discussion a "new" technological Communication Infrastructure

environment enters the scene: the Activity Center (Figure 6), to which every group has previously sent the results of their work, namely the coordinates of the points of the sequence, from P_1 to P_6 and moreover P_{10}. In the following protocol, we describe the construction of meaning of the whole class, during the collective discussion. This is an emblematic example of the value that a Communication Infrastructure can add to a Representational Infrastructure.

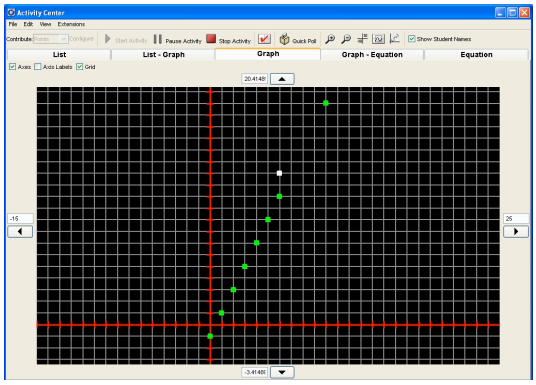

Figure 6: the Activity Center

Figure 7: Gesture in the discussion

The Activity Center makes the difference in the discussion, because it is the catalyst of gazes, gestures and words of students, and supports the teacher in the mediation of meaning construction. The teacher starts the discussion with an attention to the objects on the Cartesian plane (Figure 6).

In the following, ***Th*** means teacher, ***St*** a group of students answering together, and other names denote particular students.

1. ***Th:*** *What do you observe in the points you found?*

2. ***Ca:*** *They are a straight line* (*Figure 6*).

3. ***Th:*** *Yes, they are a straight line. Except that one, which seems to be out of place. Why is it out of its place?*

4. ***Ma:*** *We calculated incorrectly.*

5. ***Th:*** *You calculate wrongly. What are the coordinates of the point, which seems out of place?*

6. ***Ma:*** *(6,13).*

7. ***Th:*** *Why doesn't it work?*

8. ***Ma:*** *Because I added ... I had to put (6,14), then it resulted to be more in this direction.* [with a gesture he shows the direction, which is wrong (Figure 8)].

9. ***Th:*** *(6,14) do you agree? You also put (6,14)?*

10. ***St:*** *(6,11).*

...

17. ***Th:*** *Why 11 and not 14?*

18. ***Ma:*** *The last point before was (5,12).*

19. ***Th:*** *(5,12) Do you also have (5,12)?*

20. ***St:*** *No, (5,9).*

21. ***Th:*** *And then, the one before how much was that?*

22. ***St:*** *(4,7).*

23. ***Ma:*** *(4,10).*

24. ***Th:*** *We are coming back. Let's start from the beginning. What is the first point?*

25. ***St:*** *(0,-1).*

26. ***Th:*** *(0,-1). This was the point we called* P_0.

The teacher goes back to the first point of the sequence, in order to understand the process followed by Ma and his classmate, to obtain such a wrong value (6,13). Along with the class, the teacher comes to the point: Ma and Ba always added 2 to the abscissa and 1 to the ordinate, then exchanged x and y, obtaining a wrong table of numbers, with a pattern in itself that makes sense even if not correct. The discussion let students understand the reasoning of this group.

Here there are two more variables with respect to the previous protocol: the teacher involved in the discussion and the public screen (Activity Center). For this reason, the multimodality also involves the teacher's production, and the humans-

with-media is intended as: all the students, the teacher and the technologies (calculators and software) used. The public screen offers the environment to share the previous experiences made in the small group activity, where every pair of students worked with their calculator, without knowing what the other groups were doing. So the discussion is a process of sharing results and justifying them through a rule written by the students. The teacher has the role of mediator in this discussion and supports the process of construction of meaning, sharing results and justifying them. We can trace an evolution in this process, marked by various cognitive steps, revealed by corresponding signs.

At the beginning of the discussion, the teacher's question (#1) calls attention to the public graph of the Activity Center, constructed with the points sent (as coordinates) by the groups to the teacher computer, and visible on the screen in different colours. The students answer looking at the signs on the public screen (the points) and introduce what they see among these points: a straight line.

The steps in the evolution of meaning are:

- The straight line introduced by the students; this represents the graphical pattern of the points sent to the Activity Center. Another sign is commented on by the teacher, who says that one of the points seems to be "out of place" (#3). The group responsible of this point says that it is not correct (#4), because they wrongly calculated it. The graphical representation on the public screen actually gives feedback that can be used by the teacher not to comment on a mistake, but to ask why it is out of the pattern (#3, 5 and 7).
- A new sign gives the second step: Ma's gesture that correctly shows the direction in which he wants to move his point in order to put it in the right place. This gesture is possible thanks to the public screen that represents all the points given by the other groups. Through the comparison between the other points and the one put by his group, he can be aware of the mistake and try to correct it. So, the direction of his gesture is correct (the correct point should be aligned with the others), but he gives wrong coordinates, influenced by the process followed during the group activity.

Here the signs introduced by the teacher are strategic in order to understand why the point is wrong.

- The third step begins with a question directed to the audience: "do you agree?" This question, along with the repetition of the wrong coordinates (6,14), and the word "why" again (#17), are the signs introduced by the teacher, which mark the beginning of a new reasoning. In a process of going back, form the end, until arriving to the first point of the sequence, the teacher supports the students in comparing the correct with the wrong coordinates, discovering the pattern followed by Ma and Ba. This comparison is useful not only to this pair of students, but also to the others, who can be aware of the processes.

All these signs are related each other, because one is substituted by another or is transformed into another. The role of the public screen here is to make what the students did in the groups visible and represent a novelty with respect to work done with the calculators alone. The public screen is also a vehicle for signs, because it contains the points sent by all the groups of students. Therefore, the public screen can be considered part of the community of humans, in the sense that Borba and Villarreal (2005). So its presence is not neutral, and it gives the students the possibility of sharing results, having an immediate feedback, positive or negative, or of introducing a new sign (think of the straight line). The differences between the group results offer the teacher the occasion to discuss why there are such differences and to analyse the students' processes in obtaining them. With the calculators alone, without TI-Navigator, this sharing would have been more difficult to obtain, for both time and space reasons.

Then the discussion goes on, with two aims: first, having a symbolic way to write the pattern of the points; second, finding a general way to represent a straight line, linking its graphical and symbolic features.

The teacher's aim is to follow the students and their processes in the group work, to express first in word and then in symbols, the recursive rule that links the x and y. The students are able to: say this expression: "you always add 1 to x and 2 to y"; to calculate some specific points; to write the rules in symbols. After those

results, the teacher guides the discussion towards a formula that gives the coordinates of whatever point, knowing its place in the sequence.

60. **Th:** *15. So, if I call this point P_{15}, it has 15 as x. Can we express this fact in words? That the abscissa of the point P_{tot}, is …?*

…

63. **St:** *tot*

…

70. **Th:** *and now let's speak about the ordinate. The ordinate of the point P_{15} will be?*

71. **Lu:** *15+14*

72. **Th:** *15+14. Why?*

73. **Ma:** *We have to take off 1, because going on of 2, say, …*

74. **Lu:** *It increases always of the previous number…*

75. **Ma:** *Yes of course, going on from 2 it does one more jump, so…*

76. **Th:** *So it is as if we double it? Is it right to say that it is as if we do the double? But the double would be 30.*

77. **St:** *The double minus 1!*

78. **Se:** *Minus 1 because we started from -1.*

The teacher uses the sign *tot*, in order to give generality to the reasoning. This sign mediates the construction of the law for a point P_{tot} (x_{tot}= *tot*), particularly simple in the case of *x*. Then the teacher starts from a number, P_{15}, to ask for the corresponding *y*, asking "Why", to stimulate the reasoning. The strategies followed by the students are two: to add to the abscissa of the point, the abscissa minus one (Lu, who justifies it saying: "it always increases from the previous number"), or to double the abscissa and then subtract 1 (Ma). The final strategy reported by more than one student (St) is: "The double minus 1", justified with: "because we started from -1".

With the mediation of a sign (*tot*) and the use of the words, the teacher guides the construction of the formula in the discussion, asking also for its justification: here the usual semiotic mediation (Bartolini and Mariotti 2008) of the teacher is integrated within the Communication *Infrastructure* environment. Then she wants to check if the class has understood the formula, expressed in natural language, so she asks to apply it to other points, far from 15.

83. ***Th:*** *We have expressed a general rule in words. Now, we want to apply these words, through which we have stated the rule, to one point far from these ones. We write a rule likely to always be applied. So we write, for example,* P_{90}. *It has as abscissa ...*

84. ***St:*** *90.*

85. ***Se:*** *Minus 1 because we started from -1.*

86. ***Th:*** *And it has as ordinate ...*

87. ***St:*** *179.*

88. ***Th:*** *Why?*

89. ***St:*** *Because it is the double minus 1.*

The passage from the natural language to the symbolic expression of the formula (Figure 8) is quite simple, because the construction of the meaning has been made before. So the students are able to apply the formula to a various set of points, knowing their position in the sequence.

$$\boxed{P_m(m, 2m-1)}$$

Figure 8: the formula written by a group

After that, the discussion goes on, focusing the students' attention on the public screen, where the representation of the points is projected. The aim now is to transfer the construction of meaning to the graph itself, in relation to the formula discovered before (Figure 9). The discussion begins with some considerations about the number of conditions to give, in order to identify a unique straight line. The students themselves say they need two points or one point and a rule, as we can see below.

The straight line (not present on the public screen) is what the students imagine passing through the points represented on the Activity Center (Figure 9), and the representation of the points is the material to start a discussion. The teacher introduces this new piece of discussion and the students are consonant with the requests. The words: "How *x* and *y* increase" are the first sign that starts a chain of other signs. They are immediately re-used by the teacher, with a question: "How do they increase?", in order to construct other signs for the meaning of a straight line. Ma introduces the second sign (a gesture in the air), also repeated later, to show "how" (Figure 10). The gesture is substituted with a word: "positive", which should explain "how". But it needs more explanation, so the teacher asks for a

meaning of “positive”. Some students introduce other words about “positive”: “Because I always add positive numbers”.

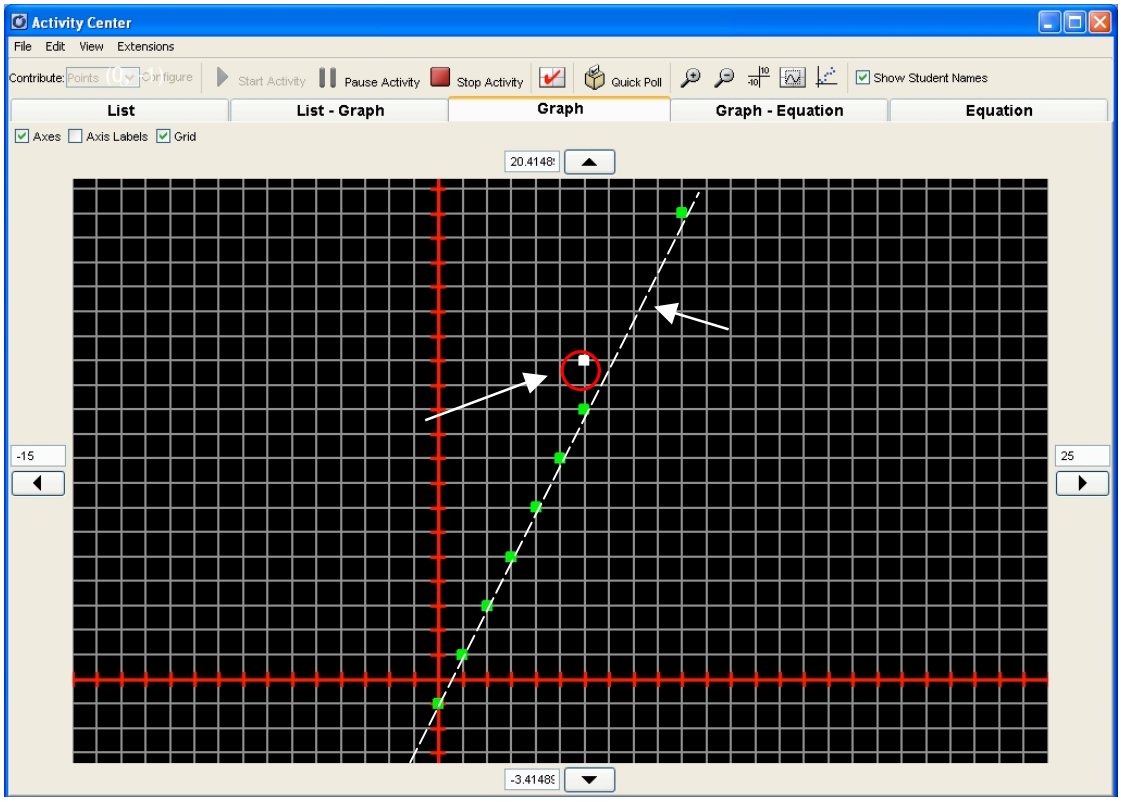

Figure 9: The Activity Center with the sequence of points

Figure 10: Ma’s gestures for the straight line

The teacher, who outlines those signs, in order to distinguish between an increasing and a decreasing straight line, captures these words. Again Ma introduces a gesture corresponding to a decreasing straight line (Figure 10). This evolution of signs starts from the public screen of the Activity Center, where all the students see the points aligned (except one), and it remains strictly linked to the representation on the public screen, where all the students and the teacher are directing their gazes. As a general comment we observe that during the process with the Activity Center the teacher is completely integrated within the humans-with-media environment developed in the classroom.

3. Discussion

In this paragraph we discuss the emblematic episodes described in # 3 in order to answer the questions we have reformulated at the end of # 1:

a') What is the specificity of instrumented actions in these TI-technological environments?

a'') What are the features introduced by the TI-technologies in the teaching-learning processes, which offer new opportunities of interaction among students, and with the teacher, not previously observed with other technologies?

a''') To what extent do technologies modify the usual multimodal production of students engaged in a task in paper and pencil?

b') What instrumented actions in these technologies help (or not) students learning processes?

c'') What is the teacher's role in the classroom within these environments?

c''') To what extent do technologies modify the usual multimodal production of teachers engaged in a task in paper and pencil?

The answers are interdependent for simplicity's sake, we will indicate the paragraphs separately where they are more involved, even if one can get the complete answer to each question only from the whole discussion.

Questions *a'', c''*. Let us reconsider the two categories, Representational and Communication *Infrastructure* which we have used to frame the two episodes. They correspond to the two dimensions sketched out at the beginning of # 1: respectively the dynamical representations of mathematical concepts; the specific

interactions in the classroom. The first dimension is emblematic of processes that happen with Representational *Infrastructure*: the *multi-representations* in the TI-Nspire environments triggered and supported students' processes of mathematics learning (through a suitable didactical design), focusing their strong (and weak) features. The specific *mediation* of the TI-Nspire instrument has suitably supported students' learning processes. The role of the teacher in this process has been "disconnected" from the strong interactions between the students and the different representations within the TI-Nspire software. He was outside the human-with-media structure, in this case represented by the TI-Nspire environment with the pairs of students working at the task. His main role consisted in designing a "working" didactical situation, in pushing students towards the use of suitable tools and frames (the spreadsheet, first using numbers then using letters).

A very different scenario appears when we look at the second example situation: the role of the teacher and the consequent interactions of students are very different. While the scenario in the first phase is quite similar to the first example, it changes in the second phase, when Activity Center is used. The human-with-media structure has widened; the teacher enters it in a very decisive way and the interactions between the students and with the teacher herself change substantially. Also the analysis of dialogues in the two cases changes radically. In the first example, the pair typically goes on with short periods of exploration on the screen, followed by peer reflections/discussions, sometimes with the production of abductions (Arzarello 2009). Students' productions are generally verbal and written (mostly using the Note environment of the software). Processes are different in the second case (Activity Center phase). The ways students interact with each other, strongly mediated by the tool and by the teacher, become richer with respect to their multimodal features: their non verbal productions increase (e.g. there is a high production of gestures, which were rare in the other example), and communication features deeply mark the dynamics of these episodes. All the richness of the previously used Representational Infrastructures is still present, but the Communication Infrastructure now amplifies its power.

Questions *a'*, *b'*. The orchestration coached by the teacher in the second example intertwines with the Activity Center and it is different from the one in the first example: it becomes an *instrumented orchestration* (Trouche 2004).

It may be interesting to picture the different structures of the two examples, the first is typical of pure Representational Infrastructures, the second is emblematic of Representational *Infrastructure* + Communication *Infrastructure*. The key point is that the different functions which representations assume in the two cases (Figures 11 and 12).

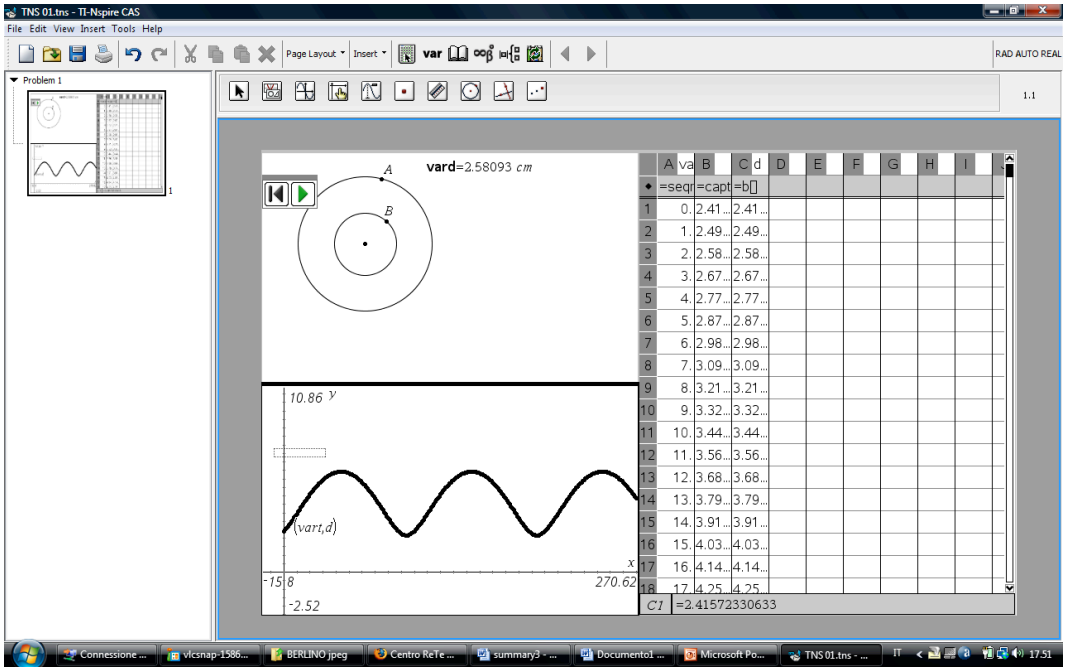

Figure 11 TI-Nspire multirepresentations

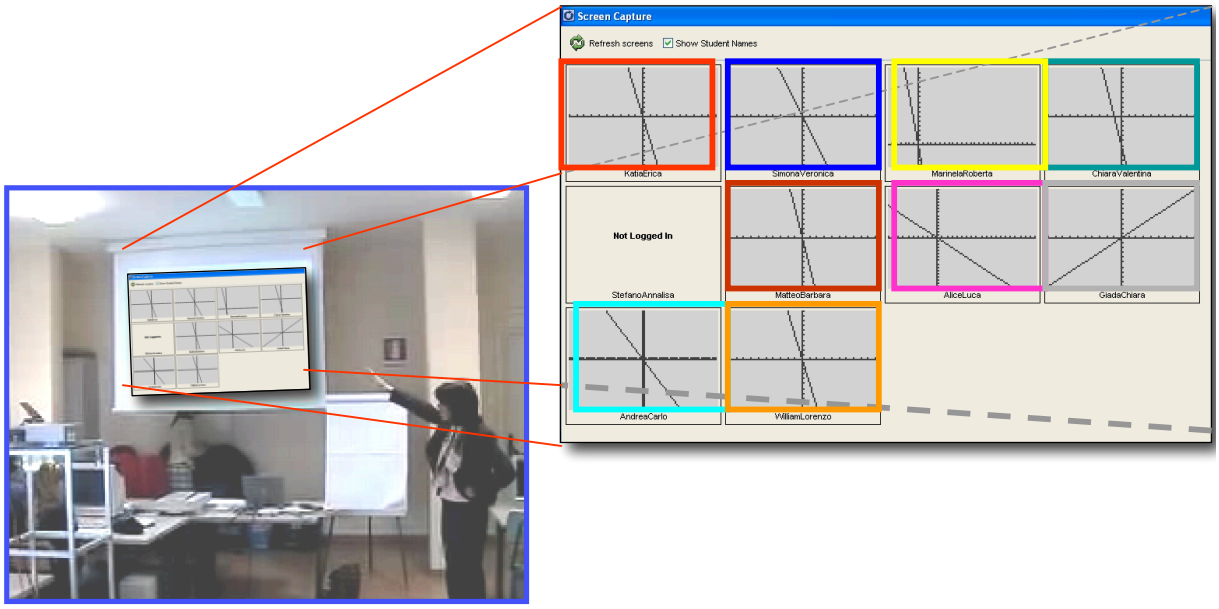

Figure 12 TI-Navigator multirepresentations

TI-*ns*

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In Figures 11 and 12 two main features of TI-Nspire and of TI-Navigator are shown namely two dual ways of implementing multi-representations. In fact we may have:

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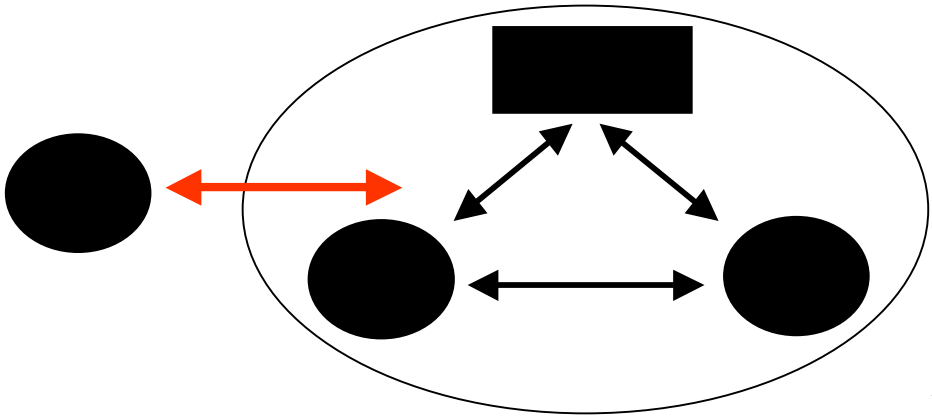
- different representations in different registers produced by the same subject (TI-Nspire, Figure 11)
- different shared representations produced by different subjects within the same register (TI-Navigator, Figure 12).



b

Figure 13 TI-Nspire environment:within-multi-representations

Figures 13 and 14 illustrate the different interactive situations that we have in the two environments. Correspondently we have two different functions of representations in the classroom interactions:

Within-multi-representations: these focus on the interactions of students with the multi-representations supported by the software itself (e.g. TI-Nspire): typically, the geometrical, the algebraic and the spreadsheet treatment of the same problem produced by a single student or by a small group of students using the same device

Between-multi-representations: these principally focus on the interactions that the instrument (e.g. TI-Navigator) triggers and supports among students in the classroom, because of the simultaneous access on the shared screen to the solutions produced by different students for the same task.

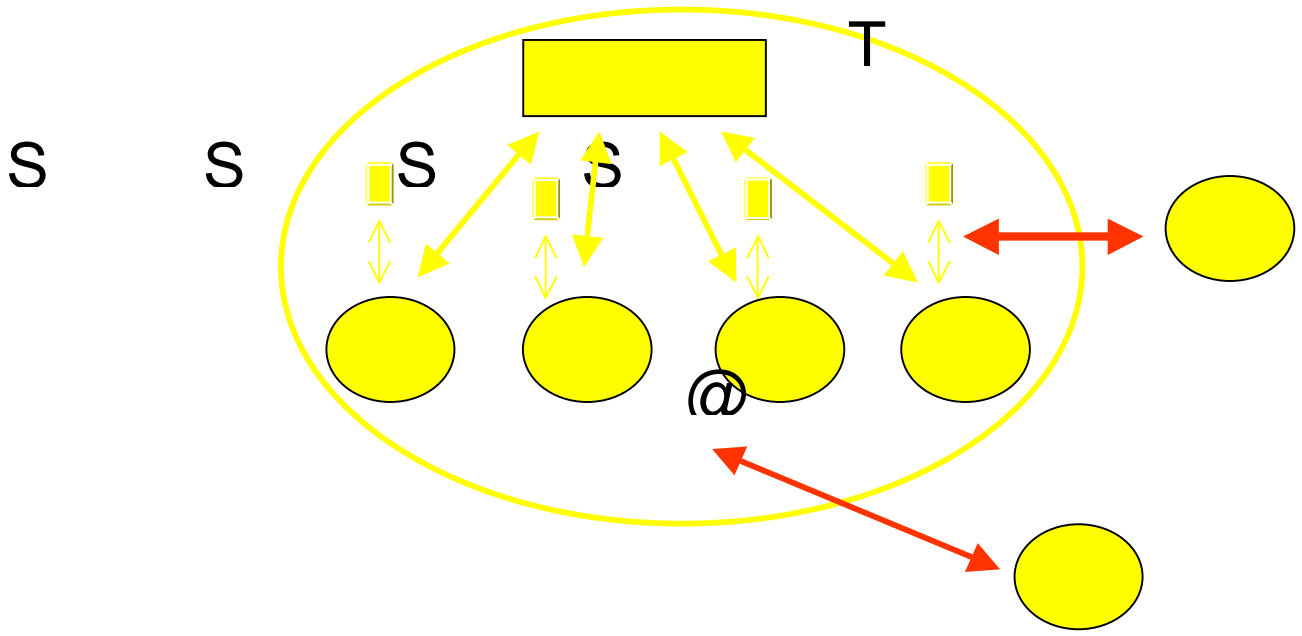


Figure 14 TI-Navigator environment: between-multi-representations

While the *Within-multi-representations* are very well triggered and supported by the TI-Nsire environment, it is not so for the *Between-multi-representations*. The handheld integrated with the Connect to Class software offers both the functions (Figure 15).

Figure 15 The TI-Nspire + TI-Navigator device

Questions *a'''*, *c'''*. Whereas the *Within-multi-representations* can support the multi-representations and the consequent multimodal behaviours of subjects while interacting with a software like TI-Nspire (giving different representations of the same problem and supporting their relationships), the *Between-multi-representations* can trigger and support the interactive and social aspects of knowledge construction. Intertwined together, the *Within-multi-representations* and *Between-multi-representations* constitute a Representation and Communication Infrastructure like those described by Hegedus and Moreno-Armella (2009). Hence, they are an environment that fosters what is called

"representational expressivity" (Hegedus and Moreno-Armella 2009, p.400). As such, *Within-multi-representations* + *Between-multi-representations* while promoting and supporting this specific form of social construction of knowledge, can contribute to a reorganisation of thinking, with a different role from the semiotic resources usually used in the classroom, such as paper and pencil. These offer a community of humans-with-media in the sense of Borba.

We tried to overcome this missing aspect of the TI-Nspire technology we had during the experiment, using the opportunities that the Internet connections and the use of a shared platform in our laboratory could offer. Table 1 illustrates the different features of the two interactive modalities.

	Registers/ Subjects	Interactions
Within m-repres (TI-Nspire)	Many/One	One-Few subjects Vs One Instrument
Between m-repres. (TI-Navigator)	One/Many	Between all subjects each with one Instrument

Table 1 Within- Vs Between- main features

In general the *Within-multi-representations* support more the so called "jeu des cadres" (Douady 1986), e.g. from the geometrical to the algebraic, as illustrated above, and consequently the conversions between registers, e.g. from the numerical environment of the spreadsheet to the geometrical on of Graphs and Geometry. On the other hand, the *Between-multi-representations* better support the so-called "mathematical discussion", which develop in the classroom when different solutions are compared or contrasted. Moreover, the different multi-representations support different cognitive processes. *Within-multi-representations* fosters more "*surfing*" modalities among the different environments and students are looking more for connections between them. *Between-multi-representations* foster more "*chatting*" modalities among the students and they are looking for more sharing, contrasting, comparing their different points of view. Both modalities are important and very useful in learning processes and illustrate two different "dual" modalities, according to which humans and media co-exist and intertwine in the Borba and Villarreal frame. It is

worth highlighting that both modalities are consonant with those supported by the new technologies (such as i-phone, i-pad etc.) in everyday life. Such practices are becoming more common in Western societies: hence their instrumentation is at zero cost in the school. Therefore, when a teacher has both modalities at his/her disposal, through a suitable instrumented orchestration developed in the classroom, it is easier for him/her: "to promote a positive attitude towards mathematics", "to foster the acquisition of new knowledge", "to make the point" ,and "to institutionalise" the achieved new knowledge in the classroom.

Based on such findings, we are now carrying out further research in a technological environment, where the two dimensions (dynamical representations of mathematical concepts; different interactions in the classroom) are integrated in a deeper manner (the new TI-Nspire + TI-Navigator). We will investigate what the added value of such integration is, and consequently if and how the answers above change the analysis of the teaching-learning processes of the new technological environment.

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