Sum of weighted Lebesgue spaces and nonlinear elliptic equations

(Article begins on next page)

03 January 2025

UNIVERSITÀ DEGLI STUDI DI TORINO

Sum of wheighted Lebesgue spaces and nonlinear elliptic equations

Marino Badiale – Lorenzo Pisani – Sergio Rolando

Dipartimento di Matematica
Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: marino.badiale@unito.it, sergio.rolando@unito.it

Dipartimento di Matematica
Università degli Studi di Bari, Via Orabona 4, 70125 Bari, Italy
e-mail: pisani@dm.uniba.it

Abstract

We study the sum of wheighted Lebesgue spaces, by considering an abstract measure space $(\Omega, \mathcal{A}, \mu)$ and investigating the main properties of both the Banach space

$$L(\Omega) = \{u_1 + u_2 : u_1 \in L^{q_1}(\Omega), u_2 \in L^{q_2}(\Omega)\}, \quad L^{q_i}(\Omega) := L^{q_i}(\Omega, d\mu),$$

and the Nemytskiĭ operator defined on it. Then we apply our general results to prove existence and multiplicity of solutions to a class of nonlinear p-laplacian equations of the form

$$-\triangle_p u + V(|x|)|u|^{p-2}u = f(|x|, u) \quad \text{in } \mathbb{R}^N$$

where V is a nonnegative measurable potential, possibly singular and vanishing at infinity, and f is a Carathéodory function satisfying a double-power growth condition in u.

1. Introduction

The recent mathematical literature has seen a growing interest in what we may call, borrowing a terminology from [19], the *zero mass* case of noncritical elliptic problems of the form

$$-\triangle u + V(x) u = g(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \tag{1.1}$$

namely, the case of nonlinearities different from the critical power and such that $g'(0) = 0$, and potentials satifying $\liminf_{|x| \to \infty} \mathcal{V}(x) = 0$ (which is also a particular *critical frequency* case, as termed in [22, 23]).

These problems are variational in nature and the main difficulty in studying existence is that the related energy space

$$H^1\left(\mathbb{R}^N, V\right) := \left\{ u \in D^{1,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V\left(x\right) u^2 dx < +\infty \right\}$$

is not necessarily contained in any Lebesgue space $L^q(\mathbb{R}^N)$ with $q \neq 2^* := 2N/\left(N-2\right)$ and thus, not only the standard compact embeddings of $H^1(\mathbb{R}^N)$ are not avaliable, but also a single-power growth condition of the form $|g\left(u\right)| \leq \left(\text{const.}\right)|u|^q$ (with $q \neq 2^*$) does not ensure the finiteness of the energy functional of the equation on $H^1(\mathbb{R}^N, V)$.

The existence of solutions for a null potential $V = 0$ was obtained by Berestycki and Lions in [19], where the authors probably first used the so-called *double power* growth condition on g, namley, $g\left(u\right)$ behaves as a subcritical power u^{q_1-1} at infinity and a supercritical power u^{q_2-1} near the origin, where $q_1 < 2^* < q_2$.

More recently, the zero mass case of equations (1.1) with noncritical nonlinearities behaving as a single power has been widely studied in both the autonomous and nonautonomous cases (see e.g. [10, 12, 21, 24, 25, 36] and [1, 3, 20, 30, 36] respectively, and the references therein), showing essentially that the existence of solutions relies on suitable compatibility conditions between the power of u and the growth and decaying rates of $V\left(x\right)$ (and possibly of the nonlinearity) at zero and infinity.

Besides, many authors resumed the study of equation (1.1) under the double power growth condition, after it was successfully exploited in [14, 15] in dealing with the semilinear Maxwell equations (see also [4, 5] for other recent works using the double power assumption). The autonomous zero mass case of (1.1) has been considered, e.g., in [7, 8, 9, 11, 13, 16, 17, 18], where it is seen that the double power assumption allows the potential V to be very general and no compatibility condition is needed in order to get existence. For instance, the radial existence and multiplicity results of [11] only require the mild integrability assumption (**V**) below, in such a way that no behaviour is prescribed either at infinity or at the origin to the potential, which may also have a nonempty, even continuous, set of singularities. As far as we know, the nonautonomous zero mass case of equation (1.1) is only studied in [6], where (1.1) is considered with $V = 0$ and $g\left(x, u\right) = K\left(x\right) f\left(u\right)$, and in [27], where the authors assume $V < 0$ and deal with nonlinearities of the form $g\left(x, u\right) = f\left(u\right) + K\left(x\right)$.

Here we study the nonautonomous radial case of equation (1.1) with nonnegative potentials and double power nonlinearities, by actually considering the more general p-laplacian problem

$$-\triangle_p u + V\left(|x|\right) |u|^{p-2} u = f\left(|x|, u\right) \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where $1 < p < N$ and $\triangle_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$. More precisely, we assume that $V : \left(0, +\infty\right) \to \left[0, +\infty\right]$ and $f : \left(0, +\infty\right) \times \mathbb{R} \to \left[0, +\infty\right)$ are, respectively, a measurable and a Carathéodory function, both nonnegative; then we define the space

$$W^{1,p}\left(\mathbb{R}^N, V\right) := \left\{ u \in D^{1,p}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V\left(|x|\right) |u|^p dx < +\infty \right\}$$

2

and prove existence and multiplicity of solutions to (1.2) in the following weak sense: we say that $u \in W^{1,p}(\mathbb{R}^N, V)$ is a *weak solution* to equation (1.2) if and only if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla h + V(|x|) |u|^{p-2} uh \right) dx = \int_{\mathbb{R}^N} f(|x|, u) h \, dx \qquad (1.3)$$

for all $h \in W^{1,p}(\mathbb{R}^N, V)$.

Denoting $F(x,t) := \int_0^t f(x,s) \, ds$ and $p^* := pN/(N-p)$, we will exploit the following hypotheses, where $(\mathbf{f})$ is a double power growth condition:

$(\mathbf{V})$ $V \in L^1((a,b))$ *for some open bounded interval* (a,b) *with* $b > a > 0$;

$(\mathbf{f})$ *there exist* $\gamma > p$ *such that for almost every* $r > 0$ *and all* $t \geq 0$ *one has*

$$0 \leq \gamma F(r,t) \leq f(r,t) t \leq M \max\{r^{\theta_1}, r^{\theta_2}\} \min\{t^{q_1}, t^{q_2}\} \qquad (1.4)$$

for some constant $M > 0$ *and* $\theta_1, \theta_2, q_1, q_2 \in \mathbb{R}$ *such that*

$$p < q_1 < p^* + \frac{\theta_1 p}{N-p} \leq p^* + \frac{\theta_2 p}{N-p} < q_2. \qquad (1.5)$$

Note that (1.4) implies $f(\cdot, 0) = 0$ almost everywhere and observe that the inequality $p < p^* + \frac{\theta_1 p}{N-p} \leq p^* + \frac{\theta_2 p}{N-p}$ of (1.5) is equivalent to

$$\theta_2 \geq \theta_1 > -p. \qquad (1.6)$$

The requirement $q_1 > p$ is not restrictive and may also be avoided in (1.5).

Our existence and multiplicity results are the following Theorems 1.1 and 1.2, which, as far as we know, are also new for the nonautonomous semilinear case $p = 2$. Note that no assumptions neither on the regularity of V nor on its behaviour at zero or infinity are made, and that $V = 0$ is allowed in both Theorems 1.1 and 1.2.

Theorem 1.1. *Assume* $(\mathbf{V})$ *and* $(\mathbf{f})$. *If there exists* $t_* > 0$ *such that*

$$F(r,t) > 0 \quad \text{for almost every } r > 0 \text{ and all } t \geq t_*, \qquad (1.7)$$

then equation (1.2) has a nontrivial nonnegative radial weak solution.

Theorem 1.2. *Assume* $(\mathbf{V})$ *and* $(\mathbf{f})$. *If for almost every* $r > 0$ *and all* $t \geq 0$ *one has*

$$f(r,t) = -f(r,-t), \qquad (1.8)$$

$$F(r,t) \geq m \max\{r^{\theta_1}, r^{\theta_2}\} \min\{t^{q_1}, t^{q_2}\} \qquad (1.9)$$

for some constant $m > 0$ *(with the same exponents* $\theta_1, \theta_2, q_1, q_2$ *of assumption* $(\mathbf{f})$), *then equation (1.2) has infinitely many radial weak solutions.*

For instance, the assumptions of both Theorems 1.1 and 1.2 hold true for

$$f\left(\left|x\right|,u\right) = \left|x\right|^{\theta} \frac{\left|u\right|^{q_2-2} u}{1 + \left|u\right|^{q_2-q_1}} \quad \text{with } \theta > -p \text{ and } q_1 < p^* + \frac{\theta p}{N-p} < q_2.$$

More generally, Theorem 1.1 applies for example to nonlinearities of the form

$$f\left(\left|x\right|,u\right) = K\left(\left|x\right|\right) g\left(u\right)$$

provided that $K : (0, +\infty) \to (0, +\infty)$ is measurable and such that

$$\limsup_{r\to 0^+} \frac{K\left(r\right)}{r^{\theta_1}} < +\infty, \quad \limsup_{r\to+\infty} \frac{K\left(r\right)}{r^{\theta_2}} < +\infty \quad \text{for some } \theta_1, \theta_2 \text{ satisfying (1.6),}$$

and $g : \mathbb{R} \to [0, +\infty)$ is continuous and satisfying

$$\limsup_{t\to 0^+} \frac{g\left(t\right)}{t^{q_2-1}} < +\infty, \quad \limsup_{t\to+\infty} \frac{g\left(t\right)}{t^{q_1-1}} < +\infty \quad \text{for some } q_1, q_2 \text{ as in (1.5)}$$

together with

$$\int_0^{t_*} g\left(s\right) ds > 0 \quad \text{for some } t_* > 0$$

and the so-called Ambrosetti-Rabinowitz condition:

$$\exists \gamma > p, \quad \forall t \geq 0, \quad \gamma \int_0^t g\left(s\right) ds \leq g\left(t\right) t.$$

Theorems 1.1 and 1.2 will be proved in Section 5 by applying some abstract results from the previous sections, where we study of the sum $L^{q_1} + L^{q_2}$ of Lebesgue spaces L^{q_i}. In fact, for a complete treatment, variational problems involving double power nonlinearities need the use of such a particular type of functional framework, to such an extent that its main properties have been investigated by different authors in different works, such as [7, 11, 13, 15, 18, 34] and the unpublished note [32], which is a preliminary version of our Sections 2 and 3. These studies, though occasional, have clarified the main features of the $L^{q_1} + L^{q_2}$ spaces, showing essentially that they share several important properties with the usual Lebesgue spaces and that, in many respects, they play for the Sobolev space $D^{1,2}(\mathbb{R}^N)$ the same role that the usual Lebesgue spaces play for $H^1(\mathbb{R}^N)$ (see for instance [15, Lemma 3] and the compactness results of [7]). Here, owing to nonautonomous nonlinearities, we need to consider the sum of wheighted Lebesgue spaces instead of the sum of the usual ones, so that the interest of our Sections 2, 3 and 4 is actually twofold: on the one hand, we collect some general results which have already been used in many of the above-mentioned papers, but which are presently expounded in the unpublished works [32, 34] only; on the other, we extend the investigation to the sum of wheighted Lebesgue spaces, by considering an abstract measure space $(\Omega, \mathcal{A}, \mu)$ and studying the space

$$L(\Omega) := \{u_1 + u_2 : u_1 \in L^{q_1}(\Omega), u_2 \in L^{q_2}(\Omega)\}, \quad L^{q_i}(\Omega) := L^{q_i}(\Omega, d\mu). \quad (1.10)$$

Going into more detail, the paper is organized as follows.

Section 2 is devoted to the study of the basic properties of $L(\Omega)$. Some characterizations of the set $L(\Omega)$ are given in Proposition 2.3, showing in particular that it is made up by the measurable functions which are q_1-integrable on some measurable subset and q_2-integrable on its complement. Then we structure $L(\Omega)$ as a Banach space by introducing a natural family of equivalent norms and proving an isometrical identification between $L(\Omega)$ and the dual space of $L^{q'_1}(\Omega) \cap L^{q'_2}(\Omega)$, $q'_i = q_i/(q_i - 1)$, endowed with the norm $\|\cdot\|_{L^{q'_1}} + \|\cdot\|_{L^{q'_2}}$. The related topology is briefly studied and some useful inequalities are given in Subsection 2.3. In the last part of Section 2, we show that $L(\Omega)$ can be characterized as an Orlicz space (cf. [29, 28, 33]), whose Orlicz norm induces the same topology. We observe that some of the results of Section 2 could be deduced from the theory presented in [29, Chapter 12] for the sum of Orlicz spaces; nevertheless, the proofs are often simple and direct, so we give them anyway here, for sake of completeness.

The aim of Section 3 is the investigation of the Nemytskiĭ operator $\mathcal{N}$ defined on $L(\Omega)$, which is a central topic in nonlinear analysis. Unfortunately, the characterization of $L(\Omega)$ as an Orlicz space is not very helpful in this direction, since, at least to our knowledge, not many results on the subject are avaliable in the literature; for instance, the classical monograph [28] only gives some results for Orlicz spaces on a base set of finite measure, while the Nemytskiĭ operator is considered in [26] just from the point of view of its good definition (see Remark 3 below). On the other hand, the case

$$\mathcal{N} : L(\Omega) \to L^q(\Omega)$$

can be studied through direct arguments, and we will show that, in this respect, $L(\Omega)$ behaves exactly as the usual Lebesgue spaces (cf. [37, Theorem 19.1]): whenever the Nemytskiĭ operator is well defined, it is continuous and (under a suitable continuity assumption on the measure μ) it is also bounded (see Theorems 3.1 and 3.4 for precise statements). A differentiability result will also be given (Proposition 3.7).

In Section 4 we prove a new compactness result (Theorem 4.1) involving $L(\Omega)$ with $\Omega = \mathbb{R}^N$ and $d\mu = \omega(x)\,dx$. Some consequences are pointed out at the end of the section (Corollaries 4.5 and 4.6), recovering a compactness lemma of [15] as a particular case.

Section 5 is finally devoted to the proofs of Theorems 1.1 and 1.2.

Notations. We conclude this introductory section by defining some notations of frequent use throughout the paper.

- $\mathbb{N}$ is the set of natural numbers, including 0.
- For every $r > 0$, we set $B_r := \left\{ x \in \mathbb{R}^N : |x| < r \right\}$.
- For any subset E of an ambient set Ω (which will be understood from the context), we set $E^c := \Omega \setminus E$ and denote the characteristic function of E by χ_E.
- $\|\cdot\|_X$ and X' denote a norm and the dual space of a Banach space X, in which $\to$ and $\rightharpoonup$ mean *strong* and *weak* convergence respectively.
- $\hookrightarrow$ denotes *continuous* embeddings.
- $p' := p/(p-1)$ is the Hölder-conjugate exponent of p.
- $p^* := pN/(N-p)$ is the Sobolev exponent related to p.

• For any mapping $u : \Omega \to \mathbb{R}$, we denote $\Lambda_u := \Lambda_u(\Omega) := \{x \in \Omega : |u(x)| > 1\}$.

2. The space $L(\Omega)$

Let $(\Omega, \mathcal{A}, \mu)$ be a nonempty σ-finite measure space and fix $1 < q_1 \leq q_2 < \infty$.

For any measurable set $E \subseteq \Omega$, we will omit the indication of the measure μ in the Lebesgue space notation $L^p(E, d\mu)$, simply writing $L^p(E) = L^p(E, d\mu)$.

2.1. Definitions and basic properties of $L(\Omega)$

We denote by $\mathcal{M}(\Omega)$ the linear space of the real valued measurable functions defined on Ω, in which equality is meant in the μ-a.e. sense, and, as in (1.10), we define

$$L(\Omega) := L^{q_1}(\Omega) + L^{q_2}(\Omega) := \{u \in \mathcal{M}(\Omega) : u = u_1 + u_2, \, u_1 \in L^{q_1}(\Omega), \, u_2 \in L^{q_2}(\Omega)\},$$

which clearly contains both $L^{q_1}(\Omega)$ and $L^{q_2}(\Omega)$. Observe that the set $L(\Omega)$ is actually of interest only if $q_1 < q_2$ and the base set Ω has infinite measure, since $\mu(\Omega) < +\infty$ or $q_1 = q_2$ implies $L^{q_2}(\Omega) \subseteq L^{q_1}(\Omega)$ and thus $L(\Omega) = L^{q_1}(\Omega)$. Nevertheless, we do not require such restrictions, for future convenience in encompassing particular cases.

Proposition 2.1. *Let $u \in L(\Omega)$ and let $E \subseteq \Omega$ be a measurable set. Then*

$$\mu(E) < +\infty \Rightarrow u \in L^{q_1}(E) \qquad and \qquad u \in L^\infty(E) \Rightarrow u \in L^{q_2}(E).$$

Proof. Let $u_1 \in L^{q_1}(\Omega)$ and $u_2 \in L^{q_2}(\Omega)$ be such that $u = u_1 + u_2$. Then $\mu(E) < +\infty$ implies $L^{q_2}(E) \subseteq L^{q_1}(E)$ and thus $u = u_1 + u_2 \in L^{q_1}(E)$. Now assume $u \in L^\infty(E)$. Since $u_1 \in L^{q_1}(E)$ and $u_2 \in L^{q_2}(E)$, in order to get $u \in L^{q_2}(E)$ we need only to show that $u_1 \in L^{q_2}(E)$. Setting $\Lambda_{u_1} = \{x \in \Omega : |u_1(x)| > 1\}$, we write

$$E = (E \cap \Lambda_{u_1}) \cup \left(E \cap \Lambda_{u_1}^c\right).$$

Since $q_1 < q_2$ and $|u_1| \leq 1$ μ-a.e. in $\Lambda_{u_1}^c$, we get

$$\int_{E \cap \Lambda_{u_1}^c} |u_1|^{q_2} \, d\mu = \int_{E \cap \Lambda_{u_1}^c} |u_1|^{q_2 - q_1} |u_1|^{q_1} \, d\mu \leq \int_{E \cap \Lambda_{u_1}^c} |u_1|^{q_1} \, d\mu < +\infty.$$

On the other hand, one has

$$|u_1| = |u - u_2| \leq \|u\|_{L^\infty(E)} + |u_2| \quad \mu\text{-a.e. in } E,$$

so that

$$\begin{aligned}
\int_{E \cap \Lambda_{u_1}} |u_1|^{q_2} \, d\mu &\leq 2^{q_2 - 1} \int_{E \cap \Lambda_{u_1}} \left(\|u\|_{L^\infty(E)}^{q_2} + |u_2|^{q_2}\right) d\mu \\
&\leq 2^{q_2 - 1} \left(\|u\|_{L^\infty(E)}^{q_2} \mu(\Lambda_{u_1}) + \int_{E \cap \Lambda_{u_1}} |u_2|^{q_2} \, d\mu\right) < +\infty
\end{aligned}$$

since Λ_{u_1} has finite measure. ∎

6

Corollary 2.2. *One has $L(\Omega) \cap L^\infty(\Omega) = L^{q_2}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Since the inclusion $L^{q_2}(\Omega) \cap L^\infty(\Omega) \subseteq L(\Omega) \cap L^\infty(\Omega)$ is obvious, the claim readily follows from Proposition 2.1. ∎

We now give some first characterizations of the functions in $L(\Omega)$. In particular, we show that $L(\Omega)$ is the set of the functions $u \in \mathcal{M}(\Omega)$ which are q_1-integrable on some measurable subset $E \subseteq \Omega$ (possibly depending on u) and q_2-integrable on its complement E^c. For any $u \in L(\Omega)$, one of such subsets is

$$\Lambda_u := \Lambda_u(\Omega) := \{x \in \Omega : |u(x)| > 1\}, \tag{2.1}$$

which is defined (up to null measure sets) for every $u \in \mathcal{M}(\Omega)$ and will play an important role hereafter.

Proposition 2.3. *For any $u \in \mathcal{M}(\Omega)$, the following propositions are equivalent:*

(i) $u \in L(\Omega)$

(ii) $u \in L^{q_1}(E) \cap L^{q_2}(E^c)$ *for some measurable subset $E \subseteq \Omega$*

(iii) $u \in L^{q_1}(\Lambda_u) \cap L^{q_2}(\Lambda_u^c)$ *and $\mu(\Lambda_u) < +\infty$*

(iv) $|u| \in L(\Omega)$

(v) $|u| \leq v$ *for some $v \in L(\Omega)$.*

Proof. Since the implications **iii $\Rightarrow$ ii** and **iv $\Rightarrow$ v** are obvious, we need only to show that **i $\Rightarrow$ iii**, **ii $\Rightarrow$ i**, **i $\Rightarrow$ iv** and **v $\Rightarrow$ i**.

(i $\Rightarrow$ iii) If $u \in L(\Omega)$ then $u \in L^{q_2}(\Lambda_u^c)$ by Proposition 2.1 and $u = u_1 + u_2$ for some $u_1 \in L^{q_1}(\Omega)$ and $u_2 \in L^{q_2}(\Omega)$ by definition. Since $1 < |u| \leq |u_1| + |u_2|$ implies $|u_j| \geq 1/2$ for some $j \in \{1, 2\}$, we get

$$+\infty > \int_\Omega |u_j|^{q_j}\, d\mu \geq \int_{\Lambda_u} |u_j|^{q_j}\, d\mu \geq \frac{1}{2^{q_j}} \int_{\Lambda_u} d\mu = \frac{1}{2^{q_j}} \mu(\Lambda_u),$$

which also yields that $u \in L^{q_1}(\Lambda_u)$ by Proposition 2.1 again.

(ii $\Rightarrow$ i) If $u \in L^{q_1}(E) \cap L^{q_2}(E^c)$ for some measurable set $E \subseteq \Omega$ then $u = u\chi_E + u\chi_{E^c}$ with $u\chi_E \in L^{q_1}(\Omega)$ and $u\chi_{E^c} \in L^{q_2}(E^c)$, which means $u \in L(\Omega)$.

(i $\Rightarrow$ iv) It follows from from the already proved equivalence **i $\Leftrightarrow$ iii**, since $u \in L^{q_1}(\Lambda_u) \cap L^{q_2}(\Lambda_u^c) \Leftrightarrow |u| \in L^{q_1}(\Lambda_u) \cap L^{q_2}(\Lambda_u^c)$ and $\Lambda_u = \Lambda_{|u|}$.

(v $\Rightarrow$ i) If $v \in L(\Omega)$ then $v \in L^{q_1}(\Lambda_v) \cap L^{q_2}(\Lambda_v^c)$ by implication **i $\Rightarrow$ iii**, so that $|u| \leq v$ implies $u \in L^{q_1}(\Lambda_v) \cap L^{q_2}(\Lambda_v^c)$. This gives that $u \in L(\Omega)$ since **ii $\Rightarrow$ i**. ∎

2.2. The Banach structure of $L(\Omega)$

The set $L(\Omega)$ has a natural linear structure as a subspace of $\mathcal{M}(\Omega)$ and can be equipped with a family of equivalent norms by setting

$$\|u\| := \|u\|_1 := \inf_{u_1+u_2=u} \left(\|u_1\|_{L^{q_1}(\Omega)} + \|u_2\|_{L^{q_2}(\Omega)} \right) \tag{2.2}$$

$$\|u\|_t := \inf_{u_1+u_2=u} \left(\|u_1\|_{L^{q_1}(\Omega)}^t + \|u_2\|_{L^{q_2}(\Omega)}^t \right)^{1/t} \quad \text{for } 1 < t < \infty \tag{2.3}$$

$$\|u\|^* := \|u\|_\infty := \inf_{u_1+u_2=u} \max \left\{ \|u_1\|_{L^{q_1}(\Omega)}, \|u_2\|_{L^{q_2}(\Omega)} \right\}. \tag{2.4}$$

Notice that

$$\|u\|_t = \inf_{u_1+u_2=u} \|(u_1, u_2)\|_t \quad \text{for every } 1 \le t \le \infty,$$

where

$$\|(u_1, u_2)\|_t := \begin{cases} \left(\|u_1\|_{L^{q_1}(\Omega)}^t + \|u_2\|_{L^{q_2}(\Omega)}^t \right)^{1/t} & \text{if } 1 \le t < \infty \\ \max \left\{ \|u_1\|_{L^{q_1}(\Omega)}, \|u_2\|_{L^{q_2}(\Omega)} \right\} & \text{if } t = \infty \end{cases} \tag{2.5}$$

defines the usual family of equivalent norms of $L^{q_1}(\Omega) \times L^{q_2}(\Omega)$.

Proposition 2.4. $\{\|\cdot\|_t\}_{1 \le t \le \infty}$ *is a family of equivalent norms on $L(\Omega)$. Moreover, $\|u\|_t = \| |u| \|_t$ for every $u \in \bar{L}(\Omega)$ and $1 \le t \le \infty$.*

Proof. The fact that the functionals (2.2)-(2.4) define a family of norms of $L(\Omega)$ is trivial. We just observe that such functionals are positive definite since $\|u\|_t = 0$ implies the existence of $(u_{1,n}, u_{2,n}) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ such that $u = u_{1,n} + u_{2,n} \to 0$ μ-a.e. in Ω as $n \to \infty$. The equivalence readily follows from the fact that the right hand sides of (2.2)-(2.4) are the infima of equivalent norms of $L^{q_1}(\Omega) \times L^{q_2}(\Omega)$. Finally, if $u \in L(\Omega)$, $u_1 \in L^{q_1}(\Omega)$, $u_2 \in L^{q_2}(\Omega)$ are such that $u = u_1 + u_2$, letting $\text{sign}(u) := \chi_{\{u>0\}} - \chi_{\{u<0\}}$ we have $|u| = \text{sign}(u) u = \text{sign}(u) u_1 + \text{sign}(u) u_2$ with $\|\text{sign}(u) u_1\|_{L^{q_1}(\Omega)} = \|u_1\|_{L^{q_1}(\Omega)}$ and $\|\text{sign}(u) u_2\|_{L^{q_2}(\Omega)} = \|u_2\|_{L^{q_2}(\Omega)}$, which implies $\| |u| \|_t \le \|u\|_t$. Similarly, if $\bar{u}_1 \in L^{q_1}(\Omega)$ and $\bar{u}_2 \in L^{q_2}(\Omega)$ are such that $|u| = \bar{u}_1 + \bar{u}_2$, then $u = \text{sign}(u) |u| = \text{sign}(u) \bar{u}_1 + \text{sign}(u) \bar{u}_2$ yields $\|u\|_t \le \| |u| \|_t$. ∎

Proposition 2.5. *The infimum in $\|\cdot\|_t$ is attained for every $1 \le t \le \infty$.*

Proof. Let $1 \le t \le \infty$, fix $u \in L(\Omega)$ and consider a minimizing sequence for $\|u\|_t$, namely $\|(u_{1,n}, u_{2,n})\|_t \to \|u\|_t$ and $u = u_{1,n} + u_{2,n}$ μ-a.e. in Ω. Since $L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ is a reflexive Banach space with respect to the norm (2.5), up to a subsequence there exists $(u_1, u_2) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ such that $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ and

$$\|(u_1, u_2)\|_t \le \liminf_{n \to \infty} \|(u_{1,n}, u_{2,n})\|_t = \|u\|_t. \tag{2.6}$$

Now we observe that the linear mapping defined by $(v, w) \mapsto v + w$ is continuous from $L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ into $L(\Omega)$, as one has

$$\|v + w\|_t \le \|v\|_t + \|w\|_t \le \|v\|_{L^{q_1}(\Omega)} + \|w\|_{L^{q_2}(\Omega)} = \|(v, w)\|_1.$$

Hence $u_{1,n} + u_{2,n} \rightharpoonup u_1 + u_2$ in $L(\Omega)$. Therefore, $u_{1,n} + u_{2,n} = u$ implies $u = u_1 + u_2$ by uniqueness of the weak limit, so that $\|u\|_t \leq \|(u_1, u_2)\|_t$ by definition of $\|u\|_t$. Together with (2.6), this means $\|u\|_t = \|(u_1, u_2)\|_t$. $\blacksquare$

Proposition 2.6. *The norm $\|\cdot\|_t$ is uniformly convex for $1 < t < \infty$.*

Proof. Let $1 < t < \infty$. From the abstract theory of product spaces, the norm (2.5) is uniformly convex, that is, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$
\left.
\begin{array}{l}
\|(u_1, u_2)\|_t, \|(v_1, v_2)\|_t \leq 1 \\
\|(u_1 - v_1, u_2 - v_2)\|_t > \varepsilon
\end{array}
\right\}
\Rightarrow \frac{1}{2} \|(u_1 + v_1, u_2 + v_2)\|_t < 1 - \delta.
$$

Now, if $u, v \in L(\Omega)$ satisfy $\|u\|_t \leq 1$, $\|v\|_t \leq 1$ and $\|u - v\|_t > \varepsilon$, then by Proposition 2.5 there exist $(u_1, u_2), (v_1, v_2) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ such that $\|(u_1, u_2)\|_t = \|u\|_t \leq 1$, $\|(v_1, v_2)\|_t = \|v\|_t \leq 1$ and $\varepsilon < \|u - v\|_t \leq \|(u_1 - v_1, u_2 - v_2)\|_t$. Hence $\|u + v\|_t \leq \|(u_1 + v_1, u_2 + v_2)\|_t < 2(1 - \delta)$. $\blacksquare$

According to Proposition 2.4, we will henceforth consider $L(\Omega)$ as a normed topological vector space, equipped with the equivalent norms (2.2)-(2.4).

Proposition 2.7. *Let $\{u_n\} \subseteq L(\Omega)$ be such that for every $\varepsilon > 0$ there exist $n_\varepsilon > 0$ and a sequence of measurable sets $E_{\varepsilon, n} \subseteq \Omega$ satisfying*

$$
\forall n > n_\varepsilon, \quad \int_{E_{\varepsilon, n}} |u_n|^{q_1} \, d\mu + \int_{E_{\varepsilon, n}^c} |u_n|^{q_2} \, d\mu < \varepsilon. \tag{2.7}
$$

Then $u_n \to 0$ in $L(\Omega)$.

Proof. Let $E = E_{\varepsilon, n}$ for brevity. Since $u_n = u_n \chi_E + u_n \chi_{E^c}$, (2.7) implies $u_n \chi_E \in L^{q_1}(\Omega)$ and $u_n \chi_{E^c} \in L^{q_2}(\Omega)$ and thus, by definition (2.2), one has $\|u_n\| \leq \|u_n\|_{L^{q_1}(E)} + \|u_n\|_{L^{q_2}(E^c)} < \varepsilon^{1/q_1} + \varepsilon^{1/q_2}$ for all $n > n_\varepsilon$, with ε arbitrary. $\blacksquare$

Proposition 2.8. *The convergence in $L(\Omega)$ implies pointwise convergence (μ-a.e. and up to a subsequence).*

Proof. Let $\{u_n\} \subseteq L(\Omega)$ be such that $\|u_n\|_1 \to 0$ as $n \to \infty$. Then for every $n > 0$ there exists $(u_{1,n}, u_{2,n}) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ such that $u_n = u_{1,n} + u_{2,n}$ and $\|(u_{1,n}, u_{2,n})\|_1 \leq \|u_n\|_1 + 1/n$. Hence $u_{1,n} \to 0$ in $L^{q_1}(\Omega)$ and $u_{2,n} \to 0$ in $L^{q_2}(\Omega)$, which implies that, up to a subsequence, $u_{1,n}, u_{2,n} \to 0$ μ-a.e. in Ω. Therefore $u_n = u_{1,n} + u_{2,n} \to 0$ μ-a.e. in Ω. $\blacksquare$

We now prove the most important result of this section, an isometrical identification between $(L(\Omega), \|\cdot\|^*)$ and the dual space of $(L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega), \|\cdot\|_{L^{q_1'} \cap L^{q_2'}})$, where

$$
q_i' = \frac{q_i}{q_i - 1} \quad \text{and} \quad \|\varphi\|_{L^{q_1'} \cap L^{q_2'}} := \|\varphi\|_{L^{q_1'}(\Omega)} + \|\varphi\|_{L^{q_2'}(\Omega)}. \tag{2.8}
$$

This will also ensure that $L(\Omega)$ is a reflexive Banach space (see Corollary 2.11 below). For future reference, let us give first the following lemma.

9

Lemma 2.9. *For any $u \in L(\Omega)$ and $\varphi \in L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$ one has*

$$\int_\Omega |u\varphi| \, d\mu \leq \|u\|^* \|\varphi\|_{L^{q_1'} \cap L^{q_2'}}$$

and

$$\int_\Omega |u\varphi| \, d\mu \leq \|u\| \max \left\{ \|\varphi\|_{L^{q_1'}(\Omega)}, \|\varphi\|_{L^{q_2'}(\Omega)} \right\}.$$

Proof. Let $u \in L(\Omega)$ and let $u_1 \in L^{q_1}(\Omega)$ and $u_2 \in L^{q_2}(\Omega)$ be such that $u_1 + u_2 = u$. Then, by Hölder inequality, $\forall \varphi \in L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$ one has

$$\int_\Omega |u\varphi| \, d\mu \leq \int_\Omega |u_1\varphi| \, d\mu + \int_\Omega |u_2\varphi| \, d\mu \leq \|u_1\|_{L^{q_1}(\Omega)} \|\varphi\|_{L^{q_1'}(\Omega)} + \|u_2\|_{L^{q_2}(\Omega)} \|\varphi\|_{L^{q_2'}(\Omega)},$$

which easily yields the result. ∎

Theorem 2.10. *For any $u \in L(\Omega)$ and $\varphi \in L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$,*

$$J(u)\varphi := \int_\Omega u\varphi \, d\mu$$

defines a linear continuous functional $J(u) : L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega) \to \mathbb{R}$. Moreover the linear operator $J : L(\Omega) \to (L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega))'$ is bijective and one has

$$\|u\|^* = \sup_{0 \neq \varphi \in L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)} \frac{\int_\Omega u\varphi \, d\mu}{\|\varphi\|_{L^{q_1'}(\Omega)} + \|\varphi\|_{L^{q_2'}(\Omega)}} \qquad \text{for every } u \in L(\Omega). \qquad (2.9)$$

Proof. Denote $L := L(\Omega)$ and $L^p := L^p(\Omega)$ for brevity.

We begin with some preliminary remarks about the dual space $(L^{q_1'} \cap L^{q_2'})'$ of $L^{q_1'} \cap L^{q_2'}$ equipped with the norm (2.8). First we observe that there is a natural linear isometry between $L^{q_1'} \cap L^{q_2'}$ and the closed subspace

$$\Delta := \left\{ (\varphi, \psi) \in L^{q_1'} \times L^{q_2'} : \varphi = \psi \right\}$$

of the Banach space $L^{q_1'} \times L^{q_2'}$ equipped with the norm $\|(\varphi, \psi)\|_{L^{q_1'} \times L^{q_2'}} := \|\varphi\|_{L^{q_1'}} + \|\psi\|_{L^{q_2'}}$. Hence $(L^{q_1'} \cap L^{q_2'})'$ isometrically identifies with the dual space Δ', so that for any $g \in (L^{q_1'} \cap L^{q_2'})'$ there exists a unique $G \in \Delta'$ such that $\|G\|_{\Delta'} = \|g\|_{(L^{q_1'} \cap L^{q_2'})'}$ and $G(\varphi, \varphi) = g(\varphi)$ for all $\varphi \in L^{q_1'} \cap L^{q_2'}$. Then, by the Hahn-Banach theorem, there exists $\widetilde{G} \in (L^{q_1'} \times L^{q_2'})'$ such that $\|\widetilde{G}\|_{(L^{q_1'} \times L^{q_2'})'} = \|g\|_{(L^{q_1'} \cap L^{q_2'})'}$ and $\widetilde{G}(\varphi, \varphi) = g(\varphi)$ for all $\varphi \in L^{q_1'} \cap L^{q_2'}$. Now, since $\widetilde{G}(\varphi, \psi) = \widetilde{G}(\varphi, 0) + \widetilde{G}(0, \psi)$ with $\widetilde{G}(\cdot, 0) \in (L^{q_1'})'$ and $\widetilde{G}(0, \cdot) \in (L^{q_2'})'$, by the Riesz representation theorem there exist $v_1 \in L^{q_1}$ and $v_2 \in L^{q_2}$ such that

$$\forall (\varphi, \psi) \in L^{q_1'} \times L^{q_2'}, \qquad \widetilde{G}(\varphi, \psi) = \int_\Omega v_1\varphi \, d\mu + \int_\Omega v_2\psi \, d\mu.$$

This gives in particular

$$\forall \varphi \in L^{q_1'} \cap L^{q_2'} \qquad g\left(\varphi\right) = \int_\Omega \left(v_1 + v_2\right) \varphi \, d\mu \,. \tag{2.10}$$

Moreover if $v_1 \neq 0$ then

$$\left\|\widetilde{G}\right\|_{(L^{q_1'} \times L^{q_2'})'} = \sup_{(\varphi, \psi) \neq (0,0)} \frac{\int_\Omega v_1 \varphi \, d\mu + \int_\Omega v_2 \psi \, d\mu}{\|\varphi\|_{L^{q_1'}} + \|\psi\|_{L^{q_2'}}} \geq \frac{\int_\Omega |v_1|^{q_1} \, d\mu}{\|v_1\|_{L^{q_1}}^{q_1/q_1'}} = \|v_1\|_{L^{q_1}}$$

where, for the inequality, we have taken $\psi = 0$ and $\varphi = |v_1|^{q_1 - 2} v_1 \in L^{q_1'}$. Similarly one obtains $\left\|\widetilde{G}\right\|_{(L^{q_1'} \times L^{q_2'})'} \geq \|v_2\|_{L^{q_2}}$ if $v_2 \neq 0$. Thus we conclude

$$\|g\|_{(L^{q_1'} \cap L^{q_2'})'} = \left\|\widetilde{G}\right\|_{(L^{q_1'} \times L^{q_2'})'} \geq \max\left\{\|v_1\|_{L^{q_1}}, \|v_2\|_{L^{q_2}}\right\} \geq \|v_1 + v_2\|^*, \tag{2.11}$$

which trivially holds even when $v_1 = v_2 = 0$.

Now let $u \in L$ and let $u_1 \in L^{q_1}$ and $u_2 \in L^{q_2}$ be such that $u_1 + u_2 = u$. For every $\varphi \in L^{q_1'} \cap L^{q_2'}$ one has

$$\left|\int_\Omega u\varphi \, d\mu\right| \leq \int_\Omega |u_1 \varphi \, d\mu| + \int_\Omega |u_2 \varphi \, d\mu| \leq \|u_1\|_{L^{q_1}} \|\varphi\|_{L^{q_1'}} + \|u_2\|_{L^{q_2}} \|\varphi\|_{L^{q_2'}}$$
$$\leq \max\left\{\|u_1\|_{L^{q_1}}, \|u_2\|_{L^{q_2}}\right\} \|\varphi\|_{L^{q_1'} \cap L^{q_2'}} \,.$$

Hence $J\left(u\right) \in (L^{q_1'} \times L^{q_2'})'$ and

$$\|J\left(u\right)\|_{(L^{q_1'} \cap L^{q_2'})'} \leq \|u\|^*, \tag{2.12}$$

which also implies that the linear operator $J : L \to (L^{q_1'} \cap L^{q_2'})'$ is continuous. The injectivity of J is plain, as $\int_\Omega u\varphi \, d\mu = 0$ for all $\varphi \in L^{q_1'} \cap L^{q_2'}$ implies $u = 0$ μ-a.e. in Ω. Moreover J is surjective by the previous preliminary discussion, since (2.10) ensures that for any $g \in (L^{q_1'} \cap L^{q_2'})'$ there exist $v_1 \in L^{q_1}$ and $v_2 \in L^{q_2}$ such that $g = J\left(v_1 + v_2\right)$. Finally, if $u \in L$ is fixed, then (2.12) and (2.11) (in which we take $g = J\left(u\right)$) yield

$$\|u\|^* \geq \|J\left(u\right)\|_{(L^{q_1'} \cap L^{q_2'})'} \geq \|v_1 + v_2\|^* = \|u\|^*$$

where the last equality holds thanks to the injectivity of J, because (2.10) means $J\left(u\right) = J\left(v_1 + v_2\right)$. ∎

Corollary 2.11. $L\left(\Omega\right)$ *is a reflexive Banach space.*

Proof. It follows from the completeness and reflexivity of $L^{q_1'}\left(\Omega\right) \cap L^{q_2'}\left(\Omega\right)$, by Theorem 2.10. ∎

Corollary 2.12. *Let $u \in \mathcal{M}\left(\Omega\right)$ and $v \in L\left(\Omega\right)$. Then $|u| \leq v$ implies $\|u\|^* \leq \|v\|^*$.*

Proof. Recall that $|u| \leq v \in L(\Omega)$ implies $|u| \in L(\Omega)$ by Proposition 2.3. Since $0 \leq |u| \leq v$, one has

$$\sup_{\varphi \neq 0} \frac{\int_\Omega |u|\, \varphi}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} = \sup_{\varphi \geq 0} \frac{\int_\Omega |u|\, \varphi}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} \leq \sup_{\varphi \geq 0} \frac{\int_\Omega v\varphi}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} = \sup_{\varphi \neq 0} \frac{\int_\Omega v\varphi}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}}$$

where φ varies in $L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$. Then $\|u\|^* = \|\,|u|\,\|^* \leq \|v\|^*$ by Proposition 2.4 and Theorem 2.10. $\blacksquare$

2.3. Some inequalities and continuous embeddings

Recall from Proposition 2.1 that $L(\Omega) \cap L^\infty(E) \subseteq L^{q_2}(E)$ for any measurable $E \subseteq \Omega$, and $L(\Omega) \subseteq L^{q_1}(E)$ if $\mu(E) < +\infty$.

Proposition 2.13. *Let $u \in L(\Omega)$ and let $E \subseteq \Omega$ be a measurable set such that $\mu(E) < +\infty$ and $u \in L^\infty(E^c)$. Then*

$$\|u\|^* \leq \max\left\{ \|u\|_{L^{q_1}(E)}, \|u\|_{L^{q_2}(E^c)} \right\}, \tag{2.13}$$

$$\|u\| \leq \|u\|_{L^{q_1}(E)} + \|u\|_{L^{q_2}(E^c)}. \tag{2.14}$$

Proof. From Proposition 2.1 we know that $u\chi_E \in L^{q_1}(\Omega)$ and $u\chi_{E^c} \in L^{q_2}(\Omega)$. Hence the claim follows by definitions (2.2) and (2.4), since $u = u\chi_E + u\chi_{E^c}$. $\blacksquare$

Proposition 2.1 can be complemented by the following result.

Proposition 2.14. *Let $E \subseteq \Omega$ be a measurable set.*

i) *If $\mu(E) < +\infty$, then for every $u \in L(\Omega)$ one has*

$$\|u\|_{L^{q_1}(E)} \leq \left(1 + \mu(E)^{1/q_1 - 1/q_2} \right) \|u\|^*, \tag{2.15}$$

$$\|u\|_{L^{q_1}(E)} \leq \max\left\{ 1, \mu(E)^{1/q_1 - 1/q_2} \right\} \|u\|. \tag{2.16}$$

ii) *For every $u \in L(\Omega) \cap L^\infty(E)$ one has*

$$\|u\|_{L^{q_2}(E)}^{q_2/q_1} \leq \left(\|u\|_{L^\infty(E)}^{q_2/q_1 - 1} + \|u\|_{L^{q_2}(E)}^{q_2/q_1 - 1} \right) \|u\|^*, \tag{2.17}$$

$$\|u\|_{L^{q_2}(E)}^{q_2/q_1} \leq \max\left\{ \|u\|_{L^\infty(E)}^{q_2/q_1 - 1}, \|u\|_{L^{q_2}(E)}^{q_2/q_1 - 1} \right\} \|u\|. \tag{2.18}$$

Proof. First we prove (2.17)-(2.18), which are is obvious if $u = 0$. Accordinlgy, assume $u \in L(\Omega) \cap L^\infty(E)$, $u \neq 0$, and define

$$\varphi_u := \frac{|u|^{q_2-2}\, u}{\|u\|_{L^{q_2}(E)}^{q_2-1}} \chi_E.$$

Then
$$\int_\Omega |\varphi_u|^{q_2'}\, d\mu = \frac{1}{\|u\|_{L^{q_2}(E)}^{q_2}} \int_E |u|^{q_2}\, d\mu = 1$$

and

$$
\begin{aligned}
\int_\Omega |\varphi_u|^{q_1'}\, d\mu &= \frac{1}{\|u\|_{L^{q_2}(E)}^{(q_2-1)q_1'}} \int_E |u|^{(q_2-1)q_1'}\, d\mu \\
&\le \frac{\|u\|_{L^\infty(E)}^{(q_2-1)q_1'-q_2}}{\|u\|_{L^{q_2}(E)}^{(q_2-1)q_1'}} \int_E |u|^{q_2}\, d\mu = \frac{\|u\|_{L^\infty(E)}^{(q_2-1)q_1'-q_2}}{\|u\|_{L^{q_2}(E)}^{(q_2-1)q_1'-q_2}},
\end{aligned}
$$

since $q_1 < q_2$ implies $(q_2-1)q_1' > q_2$. So $\varphi_u \in L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$ and thus, since

$$\int_\Omega u\varphi_u\, d\mu = \frac{1}{\|u\|_{L^{q_2}(E)}^{q_2-1}} \int_E |u|^{q_2}\, d\mu = \|u\|_{L^{q_2}(E)},$$

Lemma 2.9 gives

$$\|u\|_{L^{q_2}(E)} \le \|u\|^* \left(\|\varphi_u\|_{L^{q_1'}(\Omega)} + \|\varphi_u\|_{L^{q_2'}(\Omega)} \right) \le \|u\|^* \left(\frac{\|u\|_{L^\infty(E)}^{q_2-1-q_2/q_1'}}{\|u\|_{L^{q_2}(E)}^{q_2-1-q_2/q_1'}} + 1 \right)$$

and

$$\|u\|_{L^{q_2}(E)} \le \|u\| \max\left\{ \|\varphi_u\|_{L^{q_1'}(\Omega)}, \|\varphi_u\|_{L^{q_2'}(\Omega)} \right\} \le \|u\| \max\left\{ \frac{\|u\|_{L^\infty(E)}^{q_2-1-q_2/q_1'}}{\|u\|_{L^{q_2}(E)}^{q_2-1-q_2/q_1'}}, 1 \right\},$$

which yield the results, since $q_2 - 1 - q_2/q_1' = q_2/q_1 - 1$ and

$$\max\left\{ \frac{\|u\|_{L^\infty(E)}^{q_2/q_1-1}}{\|u\|_{L^{q_2}(E)}^{q_2/q_1-1}}, 1 \right\} = \frac{\max\left\{ \|u\|_{L^\infty(E)}^{q_2/q_1-1}, \|u\|_{L^{q_2}(E)}^{q_2/q_1-1} \right\}}{\|u\|_{L^{q_2}(E)}^{q_2/q_1-1}}.$$

Now we let $u \in L(\Omega)$, assume that $\mu(E) < +\infty$ and let $u_1 \in L^{q_1}(\Omega)$ and $u_2 \in L^{q_2}(\Omega)$ be such that $u = u_1 + u_2$. By Hölder inequality we get

$$\|u\|_{L^{q_1}(E)} \le \|u_1\|_{L^{q_1}(E)} + \|u_2\|_{L^{q_1}(E)} \le \|u_1\|_{L^{q_1}(E)} + \mu(E)^{1/q_1-1/q_2} \|u_2\|_{L^{q_2}(E)},$$

whence

$$
\begin{aligned}
\|u\|_{L^{q_1}(E)} &\le \left(1 + \mu(E)^{1/q_1-1/q_2} \right) \max\left\{ \|u_1\|_{L^{q_1}(E)}, \|u_2\|_{L^{q_2}(E)} \right\}, \\
\|u\|_{L^{q_1}(E)} &\le \max\left\{ 1, \mu(E)^{1/q_1-1/q_2} \right\} \left(\|u_1\|_{L^{q_1}(E)} + \|u_2\|_{L^{q_2}(E)} \right).
\end{aligned}
$$

Then (2.15)-(2.16) ensue by passing to the infima. ∎

Recall that $L(\Omega)$ and $L^{q_1}(\Omega)$ are the same set if $\mu(\Omega) < +\infty$.

13

Corollary 2.15. *If $\mu(\Omega) < +\infty$, the norms of $L(\Omega)$ and $L^{q_1}(\Omega)$ are equivalent.*

Proof. One has $\|u\|^* \leq \|u\|_{L^{q_1}(\Omega)} \leq (1 + \mu(\Omega)^{1/q_1 - 1/q_2}) \|u\|^*$ by (2.13) and (2.15). ∎

Recall from Corollary 2.2 that $L(\Omega) \cap L^\infty(\Omega)$ and $L^{q_2}(\Omega) \cap L^\infty(\Omega)$ are the same set.

Corollary 2.16. *On the subspace $L^{q_2}(\Omega) \cap L^\infty(\Omega)$ of $L(\Omega)$, the norms $\|u\| + \|u\|_{L^\infty(\Omega)}$ and $\|u\|_{L^{q_2}(\Omega)} + \|u\|_{L^\infty(\Omega)}$ are equivalent.*

Proof. If $u \in L^{q_2}(\Omega) \cap L^\infty(\Omega)$ then (2.14), with $E = \varnothing$, gives $\|u\| \leq \|u\|_{L^{q_2}(\Omega)}$. On the other hand, if $\{u_n\} \subseteq L^{q_2}(\Omega) \cap L^\infty(\Omega)$ is such that $\|u_n\| + \|u_n\|_{L^\infty(\Omega)} \to 0$, then (2.18) yields

$$\|u_n\|_{L^{q_2}(\Omega)}^{q_2/q_1} \leq \left(1 + \|u_n\|_{L^{q_2}(\Omega)}^{q_2/q_1 - 1}\right) o(1) \quad \text{as } n \to \infty,$$

which implies $\|u_n\|_{L^{q_2}(\Omega)} \to 0$. ∎

The next proposition collects some embedding properties of $L(\Omega)$, some of which are consequences of the above inequalites. Another relevant embedding result will be proved in Section 4 (Theorem 4.1) for $\Omega = \mathbb{R}^N$.

Proposition 2.17. *The following continuous embeddings hold:*

  i) *$L(\Omega) \hookrightarrow L(E)$ for any measurable $E \subseteq \Omega$, namely, for every $u \in L(\Omega)$ one has $\chi_E u \in L(E)$ and $\|\chi_E u\|_{L(E)} \leq \|u\|_{L(\Omega)}$;*

  ii) *$L(\Omega) \hookrightarrow L^{q_1}(E)$ for any measurable $E \subseteq \Omega$ such that $\mu(E) < +\infty$;*

  iii) *$L^q(\Omega) \hookrightarrow L(\Omega)$ for any $q \in [q_1, q_2]$.*

Proof. The first and second continuous embeddings straighforwardly follow from definition (2.2) and Proposition 2.14.i) respectively. So we take $q \in [q_1, q_2]$ and show that $L^q(\Omega) \hookrightarrow L(\Omega)$. Let $u \in L^q(\Omega)$ and recall the definition (2.1) of Λ_u. Then $q_1 \leq q$ and

$$\mu(\Lambda_u) = \int_{\Lambda_u} d\mu \leq \int_{\Lambda_u} |u|^q \, d\mu \leq \|u\|_{L^q(\Omega)}^q < +\infty$$

imply $u \in L^{q_1}(\Lambda_u)$ and, by Hölder inequality,

$$\int_{\Lambda_u} |u|^{q_1} \, d\mu \leq \mu(\Lambda_u)^{1 - q_1/q} \left(\int_{\Lambda_u} |u|^q \, d\mu\right)^{q_1/q} \leq \|u\|_{L^q(\Omega)}^{q(1 - q_1/q)} \|u\|_{L^q(\Omega)}^{q_1} = \|u\|_{L^q(\Omega)}^q.$$

On the other hand, $q \leq q_2$ and $|u| \leq 1$ imply $|u|^{q_2} \leq |u|^q$, so that $u \in L^{q_2}(\Lambda_u^c)$ and

$$\int_{\Lambda_u^c} |u|^{q_2} \, d\mu \leq \int_{\Lambda_u^c} |u|^q \, d\mu \leq \|u\|_{L^q(\Omega)}^q.$$

Thus $u \in L(\Omega)$ since $u = u\chi_{\Lambda_u} + u\chi_{\Lambda_u^c}$, and Proposition 2.13 gives

$$\|u\| \leq \|u\|_{L^{q_1}(\Lambda_u)} + \|u\|_{L^{q_2}(\Lambda_u^c)} \leq \|u\|_{L^r(\Omega)}^{q/q_1} + \|u\|_{L^r(\Omega)}^{q/q_2}.$$

This implies that $\|u_n\| \to 0$ if $\|u_n\|_{L^q(\Omega)} \to 0$ and the proof is thus complete. ∎

Remark 1. On $L^q(\Omega)$, $q_1 \leq q \leq q_2$, the norms (2.2)-(2.4) and the standard $L^q(\Omega)$ norm are not equivalent in general. For this, we have two different counterexamples, according as $q_1 \leq q < q_2$ or $q = q_2$, both for $\Omega = \mathbb{R}^N$ endowed with the Lebesgue measure. In the first case, $q \in [q_1, q_2)$, define the sequence

$$u_n(x) := \begin{cases} 1/n & \text{if } |x| \leq n^{q_2/N} \\ 0 & \text{otherwise.} \end{cases}$$

Then one has

$$\|u_n\|_{L^q(\mathbb{R}^N)}^q = \frac{1}{n^q} \int_{|x| \leq n^{q_2/N}} dx = (\text{const.}) \, n^{q_2-q} \to +\infty,$$

whereas (2.14) (with $E = \varnothing$) gives

$$\|u_n\| \leq \left(\int_{|x| \leq n^{q_2/N}} \frac{1}{n^{q_2}} dx \right)^{1/q_2} = \frac{1}{n} \left(\int_{|x| \leq n^{q_2/N}} dx \right)^{1/q_2} = (\text{const.}).$$

Similarly, if $q = q_2$, the sequence

$$u_n(x) := \begin{cases} n & \text{if } |x| \leq n^{-q_1/N} \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

is bounded in $L(\Omega)$ (by (2.14) again, with $E = B_{n^{-q_1/N}}$) and such that $\|u_n\|_{L^{q_2}(\mathbb{R}^N)}^{q_2} \to +\infty$.

The next result is a corollary of Propositions 2.13 and 2.14.

Corollary 2.18. Let $u \in L(\Omega)$ and recall the definition (2.1) of Λ_u. One has

$$\max\left\{ \frac{\|u\|_{L^{q_1}(\Lambda_u)}}{1+\mu(\Lambda_u)^{1/q_1-1/q_2}}, \frac{1}{2}\|u\|_{L^{q_2}(\Lambda_u^c)} - \frac{1}{2} \right\} \leq \|u\|^* \leq \max\left\{ \|u\|_{L^{q_1}(\Lambda_u)}, \|u\|_{L^{q_2}(\Lambda_u^c)} \right\} \tag{2.20}$$

and

$$\max\left\{ \frac{\|u\|_{L^{q_1}(\Lambda_u)}}{\max\{1,\mu(\Lambda_u)^{1/q_1-1/q_2}\}}, \|u\|_{L^{q_2}(\Lambda_u^c)} - l \right\} \leq \|u\| \leq \|u\|_{L^{q_1}(\Lambda_u)} + \|u\|_{L^{q_2}(\Lambda_u^c)}, \tag{2.21}$$

where $l := (q_2 - q_1)(q_1/q_2)^{q_2/(q_2-q_1)}/q_1$.

Proof. Since $\|u\|_{L^\infty(\Lambda_u^c)} \leq 1$ and $\mu(\Lambda_u) < +\infty$ (recall Proposition 2.3), the right hand inequalities of (2.20)-(2.21) and part of the left hand ones directly follow from Propositions 2.13 and 2.14 respectively. Then, setting $t := \|u\|_{L^{q_2}(\Lambda_u^c)}$, from (2.18) we get

$$\frac{t^{q_2/q_1}}{\max\{1, t^{q_2/q_1-1}\}} \leq \|u\|,$$

15

so that the remaining part of (2.21) follows from the inequality

$$\frac{t^{q_2/q_1}}{\max\left\{1, t^{q_2/q_1-1}\right\}} - t \geq -l$$

which holds for every $t \geq 0$. Finally we show that

$$\|u\|_{L^{q_2}(\Lambda_u^c)} \leq \max\left(2\|u\|^*, 1\right),$$

which completes the proof of (2.20). To this end, we set $t := \|u\|_{L^{q_2}(\Lambda_u^c)}^{q_2/q_1-1}$ and use (2.17) to deduce

$$\frac{t}{1+t}\|u\|_{L^{q_2}(\Lambda_u^c)} \leq \|u\|^*.$$

Then $\|u\|_{L^{q_2}(\Lambda_u^c)} > 1$ implies $t > 1$ (recall that $q_2/q_1 > 1$) and thus we get $\|u\|_{L^{q_2}(\Lambda_u^c)} \leq 2\|u\|^*$, since $t > (1+t)/2$. ∎

Remark 2. *Note that, according to the above proof of Corollary 2.18, the inequality $\|u\|_{L^{q_2}(\Lambda_u^c)} - 1 \leq 2\|u\|^*$ of (2.20) actually holds in the stronger form*

$$\|u\|_{L^{q_2}(\Lambda_u^c)} \leq \max\left(2\|u\|^*, 1\right).$$

On the other hand, inequalites (2.20)-(2.21) cannot be improved in the following sense: there is no constant $C > 0$ such that

$$\|u\|_{L^{q_2}(\Lambda_u^c)} \leq C\|u\|^* \quad \text{for all } u \in L(\Omega). \tag{2.22}$$

Indeed, arguing by contradiction, for any $u \in L(\Omega) \cap L^\infty(\Omega)$, $u \neq 0$, we set $\tilde{u} := u/\|u\|_{L^\infty(\Omega)}$ (so that $\Lambda_{\tilde{u}} = \varnothing$) and by (2.22) we obtain

$$C\frac{\|u\|^*}{\|u\|_{L^\infty(\Omega)}} = C\|\tilde{u}\|^* \geq \|\tilde{u}\|_{L^{q_2}(\Lambda_u^c)} = \|\tilde{u}\|_{L^{q_2}(\Omega)} = \frac{\|u\|_{L^{q_2}(\Omega)}}{\|u\|_{L^\infty(\Omega)}},$$

that is,

$$\|u\|_{L^{q_2}(\Omega)} \leq C\|u\|^*. \tag{2.23}$$

But the sequence $\{u_n\} \subseteq L(\Omega) \cap L^\infty(\Omega)$ defined in (2.19) does not satisfy (2.23) and thus a contradiction ensues.

We end this section with a characterization of the boundedness of a sequence in $L(\Omega)$.

Proposition 2.19. *Let $\{u_n\}$ be a sequence in $L(\Omega)$ and denote $\Lambda_n := \Lambda_{u_n}$. Then $\{u_n\}$ is bounded in $L(\Omega)$ if and only if $\{\|u_n\|_{L^{q_1}(\Lambda_n)}\}$, $\{\|u_n\|_{L^{q_2}(\Lambda_n^c)}\}$ and $\{\mu(\Lambda_n)\}$ are bounded.*

Proof. The "if part" directly follows from Corollary 2.18, which readily gives also that $\{\|u_n\|_{L^{q_2}(\Lambda_n^c)}\}$ is bounded if $\{u_n\}$ is bounded in $L(\Omega)$. We now use an argument from [15]: since $|u_n| > 1$ on Λ_n, from (2.20) we get

$$\|u_n\|^* \geq \frac{\|u_n\|_{L^{q_1}(\Lambda_u)}}{1 + \mu(\Lambda_u)^{1/q_1 - 1/q_2}} \geq \frac{\mu(\Lambda_n)^{1/q_1}}{1 + \mu(\Lambda_n)^{1/q_1 - 1/q_2}} \to \infty \quad \text{as } \mu(\Lambda_n) \to \infty$$

and therefore the boundedness of $\{\|u_n\|^*\}$ implies the one of $\{\mu(\Lambda_n)\}$, and then of $\{\|u_n\|_{L^{q_1}(\Lambda_n)}\}$. ∎

2.4. The Orlicz structure of $L(\Omega)$

Define

$$\phi(t) := \min\{t^{q_1 - 1}, t^{q_2 - 1}\} \quad \text{for all } t \geq 0$$

and set

$$\Phi(t) := \int_0^{|t|} \phi(s)\, ds \quad \text{for all } t \in \mathbb{R},$$

that is,

$$\Phi(t) = \begin{cases} \frac{1}{q_2} |t|^{q_2} & \text{if } |t| \leq 1 \\ \frac{1}{q_1} |t|^{q_1} + \frac{1}{q_2} - \frac{1}{q_1} & \text{if } |t| > 1. \end{cases} \tag{2.24}$$

Then $\Phi : \mathbb{R} \to [0, +\infty)$ is a *nice Young function*, i.e., an even, convex and continuous function such that

$$\lim_{t \to 0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \Phi(t) = 0 \Leftrightarrow t = 0,$$

and we can consider the Orlicz class

$$L^\Phi(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega \Phi(u)\, d\mu < +\infty \right\}.$$

We will show that $L^\Phi(\Omega)$ is exactly $L(\Omega)$. Note that Φ satisfies the so-called *global Δ_2 condition*, that is, there exists $\eta > 0$ such that

$$\Phi(2t) \leq \eta \Phi(t) \quad \text{for all } t \geq 0,$$

so that $L^\Phi(\Omega)$ is a vector space (see [33, Theorem 3.2]).

Proposition 2.20. *One has $L^\Phi(\Omega) = L(\Omega)$.*

Proof. By (2.24), for any $u \in \mathcal{M}(\Omega)$ we have

$$\int_\Omega \Phi(u)\, d\mu = \frac{1}{q_1} \int_{\Lambda_u} (|t|^{q_1} - 1)\, d\mu + \frac{1}{q_2} \mu(\Lambda_u) + \frac{1}{q_2} \int_{\Lambda_u^c} |t|^{q_2}\, d\mu$$

(where Λ_u is defined in (2.1)), so that $u \in L^\Phi(\Omega)$ if and only if $\mu(\Lambda_u) < +\infty$ and $u \in L^{q_1}(\Lambda_u) \cap L^{q_2}(\Lambda_u^c)$. This is equivalent to $u \in L(\Omega)$, by Proposition 2.3. ∎

17

As to the *complementary function*

$$\Psi(t) := \sup_{s \geq 0} \left(|t| \, s - \Phi(s)\right) \quad \text{for all } t \in \mathbb{R}$$

and the corresponding Orlicz class

$$L^{\Psi}(\Omega) = \left\{ \varphi \in \mathcal{M}(\Omega) : \int_{\Omega} \Psi(\varphi) \, d\mu < +\infty \right\},$$

an easy computation shows that Ψ is actually given by

$$\Psi(t) = \max_{s \geq 0} \left(|t| \, s - \Phi(s)\right) = \begin{cases} \frac{1}{q_2'} |t|^{q_2'} & \text{if } |t| \leq 1 \\ \frac{1}{q_1'} \left(|t|^{q_1'} - 1\right) + \frac{1}{q_2'} & \text{if } |t| > 1 \end{cases} \tag{2.25}$$

(where $q_i' = q_i/(q_i - 1)$ as usual), i.e.,

$$\Psi(t) = \int_0^{|t|} \max\{s^{q_1'-1}, s^{q_2'-1}\} \, ds \quad \text{for all } t \in \mathbb{R},$$

so that the same argument of the proof of Proposition 2.20 yields that

$$L^{\Psi}(\Omega) = L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega).$$

We now show that the Orlicz norm

$$\|u\|_{\Phi} := \sup \left\{ \int_{\Omega} |u \, \varphi| \, d\mu : \varphi \in L^{\Psi}(\Omega), \int_{\Omega} \Psi(\varphi) \, d\mu \leq 1 \right\} \tag{2.26}$$

gives rise on $L(\Omega)$ to the same Banach structure we have considered so far. Recall the definition (2.8) of $\|\cdot\|_{L^{q_1'} \cap L^{q_2'}}$.

Lemma 2.21. *There exists $\delta_0 > 0$ such that for every $\varphi \in L^{\Psi}(\Omega) = L^{q_1'}(\Omega) \cap L^{q_2'}(\Omega)$ one has*

$$\|\varphi\|_{L^{q_1'} \cap L^{q_2'}} \leq \delta_0 \implies \int_{\Omega} \Psi(\varphi) \, d\mu \leq 1. \tag{2.27}$$

Proof. Since

$$\int_{\Lambda_\varphi^c} |\varphi|^{q_2'} \, d\mu \leq \int_{\Omega} |\varphi|^{q_2'} \, d\mu \leq \|\varphi\|_{L^{q_1'} \cap L^{q_2'}}^{q_2'}$$

and

$$\mu(\Lambda_\varphi) \leq \int_{\Lambda_\varphi} |\varphi|^{q_1'} \, d\mu \leq \int_{\Omega} |\varphi|^{q_1'} \, d\mu \leq \|\varphi\|_{L^{q_1'} \cap L^{q_2'}}^{q_1'}$$

(recall definition (2.1)), from (2.25) we get

$$\begin{aligned}
\int_{\Omega} \Psi(\varphi) \, d\mu &= \frac{1}{q_1'} \int_{\Lambda_\varphi} |\varphi|^{q_1'} \, d\mu + \left(\frac{1}{q_2'} - \frac{1}{q_1'} \right) \mu(\Lambda_\varphi) + \frac{1}{q_2'} \int_{\Lambda_\varphi^c} |\varphi|^{q_2'} \, d\mu \\
&\leq \frac{1}{q_2'} \left(\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}^{q_1'} + \|\varphi\|_{L^{q_1'} \cap L^{q_2'}}^{q_2'} \right)
\end{aligned}$$

(recall that $q_2' \leq q_1'$) and the result ensues. $\blacksquare$

18

Proposition 2.22. *The Orlicz norm (2.26) is equivalent to (2.4).*

Proof. Let $u \in L^{\Phi}(\Omega) = L(\Omega)$. For any $\varphi \in L^{\Psi}(\Omega)$, $\varphi \neq 0$, the mapping

$$\widetilde{\varphi} := \delta_0 \frac{\varphi}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} \in L^{\Psi}(\Omega),$$

where δ_0 is given by Lemma 2.21, satisfies $\int_{\Omega} \Psi(\widetilde{\varphi}) \, d\mu \leq 1$ by (2.27), so that we get

$$\|u\|_{\Phi} \geq \int_{\Omega} |u \, \widetilde{\varphi}| \, d\mu \geq \left| \int_{\Omega} u \, \widetilde{\varphi} \, d\mu \right| = \delta_0 \frac{\left| \int_{\Omega} u \, \varphi \, d\mu \right|}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}}$$

by (2.26). Hence (2.9) yields

$$\|u\|^* = \sup_{0 \neq \varphi \in L^{\Psi}(\Omega)} \frac{\int_{\Omega} u\varphi \, d\mu}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} = \sup_{0 \neq \varphi \in L^{\Psi}(\Omega)} \frac{\left| \int_{\Omega} u\varphi \, d\mu \right|}{\|\varphi\|_{L^{q_1'} \cap L^{q_2'}}} \leq \frac{1}{\delta_0} \|u\|_{\Phi}.$$

On the other hand, since $q_2' \leq q_1'$, for every $\varphi \in L^{\Psi}(\Omega)$ we have

$$\int_{\Lambda_{\varphi}} |\varphi|^{q_1'} \, d\mu \geq \int_{\Lambda_{\varphi}} |\varphi|^{q_2'} \, d\mu \quad \text{and} \quad \int_{\Lambda_{\varphi}^c} |\varphi|^{q_2'} \, d\mu \geq \int_{\Lambda_{\varphi}^c} |\varphi|^{q_1'} \, d\mu$$

(recall definition (2.1)), so that, using (2.25), we get

$$\int_{\Omega} \Psi(\varphi) \, d\mu = \frac{1}{q_1'} \int_{\Lambda_{\varphi}} |\varphi|^{q_1'} \, d\mu + \left(\frac{1}{q_2'} - \frac{1}{q_1'} \right) \mu(\Lambda_{\varphi}) + \frac{1}{q_2'} \int_{\Lambda_{\varphi}^c} |\varphi|^{q_2'} \, d\mu$$

$$\geq \frac{1}{q_1'} \left(\int_{\Lambda_{\varphi}} |\varphi|^{q_1'} \, d\mu + \int_{\Lambda_{\varphi}^c} |\varphi|^{q_2'} \, d\mu \right) \geq \frac{1}{q_1'} \max \left\{ \int_{\Omega} |\varphi|^{q_1'} \, d\mu, \int_{\Omega} |\varphi|^{q_2'} \, d\mu \right\}.$$

Hence, by Lemma 2.9, $\int_{\Omega} \Psi(\varphi) \, d\mu \leq 1$ implies

$$\int_{\Omega} |u \, \varphi| \, d\mu \leq \left(\|\varphi\|_{L^{q_1'}(\Omega)} + \|\varphi\|_{L^{q_2'}(\Omega)} \right) \|u\|^* \leq \left((q_1')^{1/q_1'} + (q_1')^{1/q_2'} \right) \|u\|^*$$

and we conclude

$$\|u\|_{\Phi} \leq \left((q_1')^{1/q_1'} + (q_1')^{1/q_2'} \right) \|u\|^*$$

by (2.26). This completes the proof. ∎

3. The Nemytskiĭ operator on $L(\Omega)$

As in the previous section, we fix $1 < q_1 \leq q_2 < \infty$ and let $(\Omega, \mathcal{A}, \mu)$ be a nonempty σ-finite measure space, on which we also consider here a second measure λ, possibly not different from μ, such that μ and λ are absolutely continuous with respect to each other, that is,

$$d\mu = \omega(x) \, d\lambda \quad \text{for some measurable function } \omega : \Omega \to (0, +\infty).$$

Accordingly, we use the expanded notation $L^p(\Omega, d\lambda)$ for the Lebesgue spaces with respect to the measure λ, while we still briefly denote

$$L^p := L^p(\Omega, d\mu), \quad L := L^{q_1} + L^{q_2} = L^{q_1}(\Omega, d\mu) + L^{q_2}(\Omega, d\mu) =: L(\Omega, d\mu).$$

Notice that a proposition holds μ-a.e. if and only if it holds λ-a.e., since μ and λ have the same null measure sets.

Theorem 3.1. *Let $1 \leq q < \infty$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$\int_\Omega |f(x, u(x))|^q \, d\lambda < +\infty \quad \text{for all } u \in L(\Omega, d\mu). \tag{3.1}$$

Then the operator

$$\mathcal{N} : L(\Omega, d\mu) \to L^q(\Omega, d\lambda) \tag{3.2}$$

$$\mathcal{N}(u)(x) := f(x, u(x)) \tag{3.3}$$

is continuous.

The proof of Theorem 3.1 will be achieved through several lemmas.

Lemma 3.2. *Let $\{u_n\}$ be such that $u_n \to 0$ in L. Then, up to a subsequence, there exist $\{u'_n\} \subseteq L^{q_1}$ and $\{u''_n\} \subseteq L^{q_2}$ such that*

$$u_n = u'_n + u''_n, \quad \sum_{n=1}^\infty \int_\Omega |u'_n|^{q_1} \, d\mu < +\infty, \quad \sum_{n=1}^\infty \int_\Omega |u''_n|^{q_2} \, d\mu < +\infty. \tag{3.4}$$

Proof. Let $\{\varepsilon_n\}$ be an arbitrary sequence of real numbers such that

$$\sum_{n=1}^\infty \varepsilon_n < +\infty. \tag{3.5}$$

Since $u_n \to 0$ in L, $\forall n$ there exists $k_n \in \mathbb{N}$ such that $\|u_{k_n}\| < \min\{\varepsilon_n^{1/q_1}, \varepsilon_n^{1/q_2}\}$, and thus, by definition (2.2) of $\|\cdot\|$, there exist $u'_{k_n} \in L^{q_1}$ and $u''_{k_n} \in L^{q_2}$ such that

$$u_{k_n} = u'_{k_n} + u''_{k_n} \quad \text{and} \quad \left\|u'_{k_n}\right\|_{L^{q_1}} + \left\|u''_{k_n}\right\|_{L^{q_2}} < \min\{\varepsilon_n^{1/q_1}, \varepsilon_n^{1/q_2}\}.$$

Together with (3.5), this gives the result. ∎

Lemma 3.3. *Let $1 \leq q < \infty$ and let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $f_0(\cdot, 0) = 0$ and $f_0(\cdot, u(\cdot)) \in L^q(\Omega, d\lambda)$ for all $u \in L(\Omega, d\mu)$. Then the operator*

$$\mathcal{N}_0 : L(\Omega, d\mu) \to L^q(\Omega, d\lambda)$$

$$\mathcal{N}_0(u)(x) := f_0(x, u(x)) \tag{3.6}$$

is continuous at 0.

Proof. First, we notice that $\mathcal{N}_0$ is well defined and $\mathcal{N}_0(0) = 0$. Then, arguing by contradiction, we assume that $\mathcal{N}_0$ is not continuous at 0, that is, there exist $\delta > 0$ and a sequence $\{u_n\} \subseteq L$ such that $u_n \to 0$ in L and

$$\|\mathcal{N}_0(u_n)\|_{L^q(\Omega, d\lambda)} > \delta \quad \text{for all } n,$$

that is,

$$\int_\Omega \frac{|f_0(x, u_n(x))|^q}{\omega(x)} d\mu = \int_\Omega |f_0(x, u_n(x))|^q \, d\lambda > \delta^q \quad \text{for all } n. \tag{3.7}$$

Passing in case to a subsequence, let $\{u_n'\} \subseteq L^{q_1}$ and $\{u_n''\} \subseteq L^{q_2}$ be such that (3.4) holds according to Lemma 3.2.

We claim that $\forall k \in \mathbb{N}$ there exist $D_k \subseteq \Omega$ and $n_k \in \mathbb{N}$ such that

$$\mu(D_k) \quad < \quad +\infty \tag{3.8}$$

$$D_k \cap D_{k'} \quad = \quad \varnothing \quad \text{if } k \neq k' \tag{3.9}$$

$$\int_{D_k} \frac{|f_0(x, u_{n_k}(x))|^q}{\omega(x)} d\mu \quad > \quad \frac{\delta^q}{2}. \tag{3.10}$$

Let us proceed by induction on k. For $k = 0$, we set $n_0 = 0$, so that (3.7) and the σ-finiteness of μ imply that $\exists D_0 \subseteq \Omega$ such that $\mu(D_0) < +\infty$ and

$$\int_{D_0} \frac{|f_0(x, u_{n_0}(x))|^q}{\omega(x)} d\mu > \delta^q > \frac{\delta^q}{2}.$$

Then suppose that we have $D_0, ..., D_k \subseteq \Omega$ and $n_0, ..., n_k \in \mathbb{N}$ satisfying (3.8)-(3.10), and define the set

$$D := \bigcup_{j=0}^k D_j.$$

Since $\mu(D) < +\infty$, by Proposition 2.17 we know that $u_n \to 0$ in $L^{q_1}(D, d\mu)$ and we can apply the classical theorem on Nemytskiĭ operators (see [37, Theorem 19.1]): the operator

$$\widetilde{\mathcal{N}}_0 : L^{q_1}(D, d\mu) \to L^q(D, d\mu)$$

$$\widetilde{\mathcal{N}}_0(u)(x) := \omega(x)^{-1/q} f_0(x, u(x))$$

(associated to the Caratheodory function $\widetilde{f}_0(x, t) := \omega(x)^{-1/q} f_0(x, t)$) is continuous and such that $\widetilde{\mathcal{N}}_0(0) = 0$, and thus there exists $n_{k+1} \in \mathbb{N}$, $n_{k+1} > n_k$, such that

$$\int_D \frac{\left|f_0\left(x, u_{n_{k+1}}(x)\right)\right|^q}{\omega(x)} d\mu < \frac{\delta^q}{2}. \tag{3.11}$$

On the other hand, again by (3.7) and the σ-finiteness of μ, there exists $D' \subseteq \Omega$ such that $\mu(D') < +\infty$ and

$$\int_{D'} \frac{\left|f_0\left(x, u_{n_{k+1}}(x)\right)\right|^q}{\omega(x)} d\mu > \delta^q. \tag{3.12}$$

Then, setting

$$D_{k+1} = D' \setminus D,$$

from (3.11)-(3.12) we deduce

$$\int_{D_{k+1}} \frac{\left|f_0\left(x, u_{n_{k+1}}(x)\right)\right|^q}{\omega(x)} d\mu > \frac{\delta^q}{2}$$

and the claim is proved.

Now, using (3.9), we can define

$$u' := \sum_{k=0}^{\infty} u'_{n_k} \chi_{D_k}$$

and, by (3.4), we get

$$\int_{\Omega} |u'|^{q_1} d\mu = \sum_{k=0}^{\infty} \int_{D_k} \left|u'_{n_k}\right|^{q_1} d\mu \leq \sum_{k=0}^{\infty} \int_{\Omega} \left|u'_{n_k}\right|^{q_1} d\mu < +\infty,$$

so that $u' \in L^{q_1}$. Analogously, one defines $u'' \in L^{q_2}$ by setting

$$u'' := \sum_{k=0}^{\infty} u''_{n_k} \chi_{D_k}.$$

Therefore $u := u' + u''$ belongs to L and satisfies

$$u = \sum_{k=0}^{\infty} \left(u'_{n_k} + u''_{n_k}\right) \chi_{D_k} = \sum_{k=0}^{\infty} u_{n_k} \chi_{D_k},$$

which, by $f_0(\cdot, 0) = 0$ and (3.10), implies

$$\int_{\Omega} |f_0(x, u(x))|^q \, d\lambda = \int_{\Omega} \frac{|f_0(x, u(x))|^q}{\omega(x)} d\mu = \sum_{k=0}^{\infty} \int_{D_k} \frac{|f_0(x, u_{n_k}(x))|^q}{\omega(x)} d\mu = +\infty.$$

So we get a contradiction with $f_0(\cdot, u(\cdot)) \in L^q(\Omega, d\lambda)$. ∎

Proof of Theorem 3.1. Let $u_0 \in L$ and define

$$f_0(x, t) := f(x, t + u_0(x)) - f(x, u_0(x))$$

(for almost every $x \in \Omega$ and every $t \in \mathbb{R}$). Then the operator $\mathcal{N}_0 : L \to L^q(\Omega, d\lambda)$ defined by (3.6) is continuous at 0 thanks to Lemma 3.3, and therefore $\mathcal{N}$ is continuous at u_0 since $\mathcal{N}(u) - \mathcal{N}(u_0) = \mathcal{N}_0(u - u_0)$. ∎

We now introduce the following condition:

($\boldsymbol{\mu}$) *for every* $v \in L^1(\Omega, d\mu)$ *and* $n \geq 1$ *there exists a partition* $\{\Omega_i\}_{1 \leq i \leq n}$ *of* Ω *such that*

$$\int_{\Omega_i} |v|\, d\mu = \frac{1}{n} \int_{\Omega} |v|\, d\mu \quad \text{for all } i = 1, ..., n.$$

We observe that ($\boldsymbol{\mu}$) holds for example if $\Omega \subseteq \mathbb{R}^N$ and $d\mu = \omega(x)\, dx$ for some $\omega \in L^1_{\text{loc}}(\Omega, dx)$; indeed, in this case, the mapping the mapping $\varphi : \rho \mapsto \int_{\Omega \cap B_\rho} |v|\, d\mu$ is continuous, monotone and such that $\varphi([0, +\infty)) = [0, \alpha]$ where $\alpha := \int_{\Omega} |v|\, d\mu$, so that, setting $\rho_i := \varphi^{-1}(\alpha i / n)$ and $\Omega_i := \Omega \cap (B_{\rho_i} \setminus B_{\rho_{i-1}})$ for $i = 1, ..., n$, one has

$$\int_{\Omega_i} |v|\, d\mu = \int_{\Omega \cap B_{\rho_i}} |v|\, d\mu - \int_{\Omega \cap B_{\rho_{i-1}}} |v|\, d\mu = \frac{\alpha i}{n} - \frac{\alpha(i-1)}{n} = \frac{\alpha}{n}.$$

Theorem 3.4. *If* ($\boldsymbol{\mu}$) *holds then, under the same assumptions of Theorem 3.1, the operator (3.2)-(3.3) is bounded (i.e., it maps bounded sets into bounded sets).*

Proof. Define

$$f_0(x, t) := f(x, t) - f(x, 0) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

so that the operator $\mathcal{N}_0 : L \to L^q(\Omega, d\lambda)$ defined by (3.6) is continuous at 0 thanks to Lemma 3.3. Note that

$$\mathcal{N}_0(u) = \mathcal{N}(u) - \mathcal{N}(0) \tag{3.13}$$

and thus $\mathcal{N}_0(0) = 0$. Hence there exists $R > 0$ such that $\forall u \in L$ one has

$$\|u\| \leq R \implies \|\mathcal{N}_0(u)\|_{L^q(\Omega, d\lambda)} \leq 1. \tag{3.14}$$

Now let $u \in L$ and let $n \in \mathbb{N}$ be such that

$$n^{1/q_2} \leq \left\| \frac{2u}{R} \right\| \leq (n+1)^{1/q_2}. \tag{3.15}$$

By definition (2.2) of $\|\cdot\|$, there exist $u' \in L^{q_1}$ and $u'' \in L^{q_2}$ such that

$$\frac{2u}{R} = \frac{2u'}{R} + \frac{2u''}{R}, \tag{3.16}$$

$$n^{1/q_2} \leq \left\| \frac{2u'}{R} \right\|_{L^{q_1}} + \left\| \frac{2u''}{R} \right\|_{L^{q_2}} < (n+1)^{1/q_2},$$

whence we get

$$\left\| \frac{2u'}{R} \right\|_{L^{q_1}} < (n+1)^{1/q_2} \leq (n+1)^{1/q_1} \quad \text{and} \quad \left\| \frac{2u''}{R} \right\|_{L^{q_2}} < (n+1)^{1/q_2},$$

that is,

$$\int_{\Omega} |u'|^{q_1}\, d\mu < (n+1) \left(\frac{R}{2} \right)^{q_1} \quad \text{and} \quad \int_{\Omega} |u''|^{q_2}\, d\mu < (n+1) \left(\frac{R}{2} \right)^{q_2}.$$

23

Hence, by assumption $(\boldsymbol{\mu})$, there exist two partitions $\{A_i\}_{1 \leq i \leq n+1}$ and $\{B_i\}_{1 \leq i \leq n+1}$ of Ω such that for all $i = 1, ..., n+1$ one has

$$\int_{A_i} |u'|^{q_1} \, d\mu < \left(\frac{R}{2}\right)^{q_1} \quad \text{and} \quad \int_{B_i} |u''|^{q_2} \, d\mu < \left(\frac{R}{2}\right)^{q_2}.$$

Define a new partition $\{C_{ij}\}$ of Ω by setting

$$C_{ij} := A_i \cap B_j \quad \text{for } i, j = 1, ..., n+1.$$

Then $u' \in L^{q_1}(C_{ij}, d\mu)$, $u'' \in L^{q_2}(C_{ij}, d\mu)$ and $\|u'\|_{L^{q_1}(C_{ij}, d\mu)}, \|u''\|_{L^{q_2}(C_{ij}, d\mu)} < R/2$, so that the mapping defined by

$$u_{ij}(x) := \begin{cases} u(x) & \text{if } x \in C_{ij} \\ 0 & \text{otherwise} \end{cases}$$

belongs to L by Proposition 2.3 and satisfies

$$\|u_{ij}\| \leq \|u'\|_{L^{q_1}(C_{ij}, d\mu)} + \|u''\|_{L^{q_2}(C_{ij}, d\mu)} < R,$$

since (3.16) implies $u_{ij} = u' \chi_{C_{ij}} + u'' \chi_{C_{ij}}$. Hence (3.14) gives $\|\mathcal{N}_0(u_{ij})\|_{L^q(\Omega, d\lambda)} \leq 1$, and thus we get

$$
\begin{aligned}
\|\mathcal{N}_0(u)\|_{L^q(\Omega, d\lambda)}^q &= \int_{\Omega} |f_0(x, u)|^q \, d\lambda = \sum_{i,j=1}^{n+1} \int_{C_{ij}} |f_0(x, u)|^q \, d\lambda \\
&= \sum_{i,j=1}^{n+1} \int_{\Omega} |f_0(x, u_{ij})|^q \, d\lambda = \sum_{i,j=1}^{n+1} \|\mathcal{N}_0(u_{ij})\|_{L^q(\Omega, d\lambda)}^q \\
&\leq (n+1)^2.
\end{aligned}
$$

Therefore, by (3.13) and the first inequality of (3.15), we deduce

$$\|\mathcal{N}(u) - \mathcal{N}(0)\|_{L^q(\Omega, d\lambda)}^q = \|\mathcal{N}_0(u)\|_{L^q(\Omega, d\lambda)}^q \leq \left(\left(\frac{2}{R}\right)^{q_2} \|u\|^{q_2} + 1\right)^2.$$

This yields the result and the proof is thus complete. $\blacksquare$

A growth condition on f ensuring (3.1) can be easily obtained by Proposition 2.3.

Proposition 3.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function and let $\alpha, \beta > 0$ be such that*

$$\frac{\alpha}{\beta} \leq \frac{q_1}{q_2} \quad \text{and} \quad \beta \leq q_2.$$

Assume that there exist $q \in [q_2/\beta, q_1/\alpha]$ and $g \in L^q(\Omega, d\mu)$ such that for almost every $x \in \Omega$ and every $t \in \mathbb{R}$ one has

$$|f(x, t)| \leq \left(M \min\{|t|^\alpha, |t|^\beta\} + g(x)\right) \omega(x)^{1/q} \tag{3.17}$$

where $M > 0$ is a constant. Then the Nemytskiĭ operator $\mathcal{N} : L(\Omega, d\mu) \to L^q(\Omega, d\lambda)$ given by (3.3) is well defined and continuous, and it is bounded if $(\boldsymbol{\mu})$ holds.

Proof. By Theorems 3.1 and 3.4, it is sufficient to show that $f(\cdot, u(\cdot)) \in L^q(\Omega, d\lambda)$ for all $u \in L$. So let $u \in L$ and recall the definition (2.1) of Λ_u. Since $\alpha \le \beta$, we have

$$|f(\cdot, u(\cdot))| \le \left(M |u|^\alpha \chi_{\Lambda_u} + M |u|^\beta \chi_{\Lambda_u^c} + g \right) \omega^{1/q} \quad \text{almost everywhere on } \Omega,$$

so that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\int_\Omega |f(x, u)|^q \, d\lambda &\le \int_\Omega \left(M |u|^\alpha \chi_{\Lambda_u} + M |u|^\beta \chi_{\Lambda_u^c} + g \right)^q \omega(x) \, d\lambda \\
&\le C \left(\int_{\Lambda_u} |u|^{\alpha q} \, d\mu + \int_{\Lambda_u^c} |u|^{\beta q} \, d\mu + \int_\Omega g^q d\mu \right) \\
&\le C \left(\int_{\Lambda_u} |u|^{q_1} \, d\mu + \int_{\Lambda_u^c} |u|^{q_2} \, d\mu + \|g\|_{L^q}^q \right),
\end{aligned}
$$

where we have used the fact that $\alpha q \le q_1$ and $\beta q \ge q_2$. The conclusion then follows from Proposition 2.3.iii. ∎

Remark 3. *If $\mu = \lambda$, then Proposition 3.5 ensures that the operator $\mathcal{N} : L \to L^q$ given by (3.3) is continuous (and bounded if ($\boldsymbol{\mu}$) holds) for every $q \in [q_2/\beta, q_1/\alpha]$, provided that (3.17) holds with $\omega(x) \equiv 1$ and $g \in L^{q_1/\alpha} \cap L^{q_2/\beta}$. Taking into account the characterization of Proposition 2.20, such a result was partially given in [26, Theorem 2.3], where it is shown that a necessary and sufficient condition in order that $\mathcal{N}$ acts from L^Φ into L^q is that there exist $\tilde{M} > 0$ and $\tilde{g} \in L^1$ such that*

$$|f(x, t)|^q \le \tilde{M} \Phi(t) + \tilde{g}(x) \quad \text{for almost every } x \in \Omega \text{ and every } t \in \mathbb{R}. \qquad (3.18)$$

Indeed, under the assumptions of Proposition 3.5, it is easy to check that there exists a constant $C > 0$ such that $\min\{|t|^\alpha, |t|^\beta\}^q \le C \Phi(t)$ for all $t \in \mathbb{R}$, so that (3.18) holds provided that (3.17) holds.

The following corollary concerns the case, of particular interest in the applications, in which the Nemytskiĭ operator works between $L = L(\Omega, d\mu)$ and its dual space L'. Recall from Theorem 2.10 that L' identifies with $L^{q_1'} \cap L^{q_2'} = L^{q_1'}(\Omega, d\mu) \cap L^{q_2'}(\Omega, d\mu)$.

Corollary 3.6. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function and assume that there exist $M > 0$ and $g \in L^{q_1'} \cap L^{q_2'}$ such that for almost every $x \in \Omega$ and every $t \in \mathbb{R}$ one has*

$$|f(x, t)| \le M \min\{|t|^{q_1 - 1}, |t|^{q_2 - 1}\} + g(x).$$

Then the Nemytskiĭ operator $\mathcal{N} : L \to L^{q_1'} \cap L^{q_2'}$ given by (3.3) is well defined and continuous, and it is bounded if ($\boldsymbol{\mu}$) holds.

Proof. Recalling the intersection norm (2.8), the continuity and the boundedness of $\mathcal{N} : L(\Omega) \to L^{q_1'} \cap L^{q_2'}$ is equivalent to the ones of $\mathcal{N}$ from L into both $L^{q_1'}$ and $L^{q_2'}$. On the other hand, f satisfies (3.17) with $\alpha = q_1 - 1$, $\beta = q_2 - 1$, $\omega(x) \equiv 1$ and $g \in L^{q_1'} \cap L^{q_2'}$. Hence the result follows from applying Proposition 3.5 with $q = q_1', q_2'$. ∎

Another consequence of Theorem 3.1 (and in particular of Corollary 3.6) is the next differentiability result, which will be exploited in Section 5.

Proposition 3.7. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function and set*

$$F(x,t) := \int_0^t f(x,s)\, ds \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Assume that there exist $M > 0$ and $g \in L^{q_1'} \cap L^{q_2'}$ such that for almost every $x \in \Omega$ and every $t \in \mathbb{R}$ one has

$$|f(x,t)| \leq \left(M \min\{|t|^{q_1-1}, |t|^{q_2-1}\} + g(x) \right) \omega(x). \tag{3.19}$$

Then the operator

$$\mathcal{F} : L(\Omega, d\mu) \to L^1(\Omega, d\lambda) \tag{3.20}$$

$$\mathcal{F}(u)(x) := F(x, u(x))$$

is well defined and of class C^1, with Fréchet derivative $\mathcal{F}'(u)$ at any $u \in L(\Omega, d\mu)$ given by

$$\langle \mathcal{F}'(u), h \rangle (x) = f(x, u(x)) h(x) \qquad \text{for all } h \in L(\Omega, d\mu).$$

Proof. From (3.19) it follows that $\exists \tilde{M} > 0$ such that

$$|F(x,t)| \leq \left(\tilde{M} \min\{|t|^{q_1}, |t|^{q_2}\} + |t|\, g(x) \right) \omega(x)$$

(for almost every $x \in \Omega$ and every $t \in \mathbb{R}$), so that $u \in L$ implies

$$|\mathcal{F}(u)| \leq \tilde{M} \min\{|u|^{q_1}, |u|^{q_2}\} \omega + |u|\, g\omega \in L^1(\Omega, d\lambda)$$

since

$$\int_\Omega \min\{|u|^{q_1}, |u|^{q_2}\} \omega\, d\lambda = \int_\Omega \min\{|u|^{q_1}, |u|^{q_2}\}\, d\mu < +\infty,$$

$$\int_\Omega |u|\, g\omega\, d\lambda = \int_\Omega |u|\, g\, d\mu < +\infty$$

by Proposition 3.5 and Lemma 2.9 respectively. Hence the operator (3.20) is well defined.

Now let $u, h \in L$ and let $\{t_n\} \subseteq (-1, 1)$ be any sequence such that $t_n \to 0$. By the mean value theorem, we infer that for almost every $x \in \Omega$ and every n there exists $\theta_n = \theta_n(x) \in [0, 1]$ such that

$$
\begin{aligned}
|F(x, u + t_n h) - F(x, u)| &= |f(x, u + \theta_n t_n h)|\, |t_n|\, |h| \\
&\leq \left(M \min\{|u + \theta_n t_n h|^{q_1-1}, |u + \theta_n t_n h|^{q_2-1}\} + g \right) |t_n|\, |h|\, \omega \\
&\leq \left(M \min\{(|u| + |h|)^{q_1-1}, (|u| + |h|)^{q_2-1}\} + g \right) |t_n|\, |h|\, \omega,
\end{aligned}
$$

26

so that almost everywhere on Ω one has

$$\frac{\mathcal{F}\left(u+t_n h\right)-\mathcal{F}\left(u\right)}{t_n} \to f\left(x,u\right) h \quad \text{as } n \to \infty,$$

$$\left|\frac{\mathcal{F}\left(u+t_n h\right)-\mathcal{F}\left(u\right)}{t_n}\right| \leq M \min\{(|u|+|h|)^{q_1-1},(|u|+|h|)^{q_2-1}\}\,|h|\,\omega + g\,|h|\,\omega,$$

where

$$\min\{(|u|+|h|)^{q_1-1},(|u|+|h|)^{q_2-1}\}\,|h|\,\omega,\ g\,|h|\,\omega \in L^1\left(\Omega, d\lambda\right)$$

by Lemma 2.9, since $|u|+|h| \in L$ (recall Proposition 2.3.iv) and Corollary 3.6 gives $\min\{(|u|+|h|)^{q_1-1},(|u|+|h|)^{q_2-1}\} \in L^{q_1'} \cap L^{q_2'}$. Hence

$$\frac{\mathcal{F}\left(u+t_n h\right)-\mathcal{F}\left(u\right)}{t_n} \to f\left(x,u\right) h \quad \text{in } L^1\left(\Omega, d\lambda\right) \tag{3.21}$$

by dominated convergence. On the other hand, the Caratheodory function defined by

$$\widetilde{f}\left(x,t\right) := \frac{f\left(x,t\right)}{\omega\left(x\right)} \quad \text{for all } \left(x,t\right) \in \Omega \times \mathbb{R}$$

satisfies

$$|\widetilde{f}\left(x,t\right)| = \frac{|f\left(x,t\right)|}{\omega\left(x\right)} \leq M \min\{|t|^{q_1-1},|t|^{q_2-1}\} + g\left(x\right)$$

(for almost every $x \in \Omega$ and every $t \in \mathbb{R}$), so that, by Corollary 3.6, the operator $u \mapsto \widetilde{f}\left(x,u\right)$ acts from L into $L^{q_1'} \cap L^{q_2'}$ and it is continuous. Hence the linear operator

$$\mathcal{F}'\left(u\right) : h \in L \mapsto f\left(x,u\right) h \in L^1\left(\Omega, d\lambda\right)$$

is continuous by Lemma 2.9 and thus, by (3.21), it is the Gâteaux derivative of $\mathcal{F}$ at u. Moreover, denoting by $\mathcal{L}$ the space of linear and continuous operators from L into $L^1\left(\Omega, d\lambda\right)$, for every $u,v \in L$ one has

$$\|\mathcal{F}'\left(u\right)-\mathcal{F}'\left(v\right)\|_{\mathcal{L}} = \sup_{\|h\|=1} \int_\Omega \left|\widetilde{f}\left(x,u\right)-\widetilde{f}\left(x,v\right)\right| |h|\, d\mu \leq \|\widetilde{f}\left(x,u\right)-\widetilde{f}\left(x,v\right)\|_{L^{q_1'} \cap L^{q_2'}}$$

by Lemma 2.9 again, so that the mapping $u \in L \mapsto \mathcal{F}'\left(u\right) \in \mathcal{L}$ is continuous and thus $\mathcal{F}'\left(u\right)$ is the Fréchet derivative of $\mathcal{F}$ at u. $\blacksquare$

4. A compactness result

As a particular case of the previous sections, here we consider the space $L(\mathbb{R}^N, d\mu)$ where μ is a σ-finite Borel measure such that μ and the Lebesgue measure of $\mathbb{R}^N$ are absolutely continuous with respect to each other, that is,

$$d\mu = \omega\left(x\right) dx \quad \text{for some measurable function } \omega : \mathbb{R}^N \to \left(0, +\infty\right).$$

Accordingly, we only omit the indication of the Lebesgue measure, briefly writing

$$L^p(\Omega) = L^p(\Omega, dx) \quad \text{for any } \Omega \subseteq \mathbb{R}^N,$$

while, in order to exclude possible misunderstandings, whenever the measure μ is concerned we shall use expanded notations, that is,

$$L\left(\mathbb{R}^N, \omega(x)\,dx\right) = L\left(\mathbb{R}^N, d\mu\right) = L^{q_1}\left(\mathbb{R}^N, d\mu\right) + L^{q_2}\left(\mathbb{R}^N, d\mu\right),$$

just avoiding the mentioning of the exponents q_1, q_2 in the sum space notation.

Assuming $1 < p < N$, we will prove a compactness result involving the radial subspace

$$D^{1,p}_{\mathrm{rad}}\left(\mathbb{R}^N\right) := \left\{ u \in D^{1,p}\left(\mathbb{R}^N\right) : u(x) = u(|x|) \right\}$$

of the Sobolev space

$$D^{1,p}\left(\mathbb{R}^N\right) = \left\{ u \in L^{p^*}\left(\mathbb{R}^N\right) : |\nabla u| \in L^p\left(\mathbb{R}^N\right) \right\},$$

where $p^* := pN/(N-p)$. Recall that $D^{1,p}(\mathbb{R}^N)$ is actually the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,p}} := \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p},$$

as well as $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is the closure in $D^{1,p}(\mathbb{R}^N)$ of the radial subspace $C^\infty_{\mathrm{c,rad}}(\mathbb{R}^N)$ of $C^\infty_c(\mathbb{R}^N)$. Moreover, the continuous embedding

$$D^{1,p}\left(\mathbb{R}^N\right) \hookrightarrow L^{p^*}\left(\mathbb{R}^N\right) \tag{4.1}$$

holds, thanks to Sobolev inequality.

Theorem 4.1. *Assume $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and that there exist $\theta_1 > -(1 + N/p')$ and $\theta_2 \in \mathbb{R}$ such that*

$$\lim_{r \to 0^+} \operatorname*{esssup}_{0 < |x| < r} \frac{\omega(x)}{|x|^{\theta_1}} < +\infty \quad \text{and} \quad \lim_{r \to +\infty} \operatorname*{esssup}_{|x| > r} \frac{\omega(x)}{|x|^{\theta_2}} < +\infty. \tag{4.2}$$

Then $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L(\mathbb{R}^N, \omega(x)\,dx)$ for every $q_1, q_2 > 1$ such that

$$q_1 < p^* + \frac{\theta_1 p}{N - p} \quad \text{and} \quad p^* + \frac{\theta_2 p}{N - p} < q_2.$$

The proof of Theorem 4.1 relies on the following lemmas, where the first one is a generalization of a pointwise estimate which is well known for $p = 2$. Some easy consequences of Theorem 4.1 will be pointed out at the end of the section.

Lemma 4.2. *There exists a constant $C_{N,p} > 0$ (only depending on N, p) such that $\forall u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ one has*

$$|u(x)| \leq C_{N,p} \|u\|_{D^{1,p}} \frac{1}{|x|^{\frac{N-p}{p}}} \quad \text{almost everywhere on } \mathbb{R}^N.$$

Proof. See for example [36, Lemma 1]. ∎

For future convenience, in the next two lemmas we give more general results than the ones needed in the proof of Theorem 4.1 (where we will apply with $h = u$).

Lemma 4.3. *Let $\theta > -\left(1 + N/p'\right)$ and $1 < q < p^* + \theta p/\left(N - p\right)$. Then there exists $C_0 = C_0\left(N, p, \theta, q\right) > 0$ such that for every $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $h \in D^{1,p}(\mathbb{R}^N)$ and $r > 0$ one has*

$$\int_{B_r} |x|^\theta \, |u|^{q-1} \, |h| \, dx \leq C_0 r^{\left(p^* + \frac{\theta p}{N-p} - q\right)\frac{N-p}{p}} \|u\|^{q-1}_{D^{1,p}} \|h\|_{D^{1,p}}.$$

Proof. We denote by C any positive constant only depending on N, p, θ, q. Take $\delta = \delta\left(N, p, q\right) > 0$ such that $q - p^* < \delta < q - 1$, in such a way that $(p^* - 1)/(q - 1 - \delta) > 1$. Then by Hölder inequality, Sobolev embedding (4.1) and Lemma 4.2, for every $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $h \in D^{1,p}(\mathbb{R}^N)$ and $r > 0$ we have

$$
\begin{aligned}
\int_{B_r} |x|^\theta \, |u|^{q-1} \, |h| \, dx \quad \leq \quad & \left(\int_{B_r} |x|^{\frac{\theta p^*}{p^*-1}} |u|^{\frac{(q-1)p^*}{p^*-1}} \, dx \right)^{(p^*-1)/p^*} \left(\int_{B_r} |h|^{p^*} \, dx \right)^{1/p^*} \\
\leq \quad & \left(\int_{B_r} |x|^{\frac{\theta p^*}{p^*-1}} |u|^{\frac{\delta p^*}{p^*-1}} |u|^{\frac{(q-1-\delta)p^*}{p^*-1}} \, dx \right)^{(p^*-1)/p^*} \|h\|_{D^{1,p}} \\
\leq \quad & C \|u\|^{\delta}_{D^{1,p}} \left(\int_{B_r} |x|^{(\theta - \frac{N-p}{p}\delta)\frac{p^*}{p^*-1}} |u|^{(q-1-\delta)p^{*\prime}} \, dx \right)^{1/p^{*\prime}} \|h\|_{D^{1,p}}
\end{aligned}
$$

Applying Hölder inequality again (with conjugate exponents $t = (p^* - 1)/(q - 1 - \delta)$ and $t' = (p^* - 1)/(p^* - q + \delta)$) we get

$$
\begin{aligned}
& \int_{B_r} |x|^\theta \, |u|^{q-1} \, |h| \, dx \\
\leq \quad & C \left(\int_{B_r} |x|^{\left(\theta - \frac{\delta N}{p^*}\right)\frac{p^*}{p^*-q+\delta}} \, dx \right)^{\frac{p^*-q+\delta}{p^*}} \left(\int_{B_r} |u|^{p^*} \, dx \right)^{\frac{q-1-\delta}{p^*}} \|u\|^{\delta}_{D^{1,p}} \|h\|_{D^{1,p}} \\
\leq \quad & C \left(\int_0^r \rho^{\left(\theta - \frac{\delta N}{p^*}\right)\frac{p^*}{p^*-q+\delta} + N - 1} d\rho \right)^{(p^*-q+\delta)/p^*} \|u\|^{q-1}_{D^{1,p}} \|h\|_{D^{1,p}} \quad\quad (4.3)
\end{aligned}
$$

where

$$\left(\theta - \frac{\delta N}{p^*}\right) \frac{p^*}{p^* - q + \delta} + N = \frac{N}{p^* - q + \delta} \left(p^* + \frac{\theta p}{N - p} - q\right) > 0$$

by assumption. Hence we conclude

$$\int_{B_r} |x|^\theta \, |u|^{q-1} \, |h| \, dx \leq C \|u\|^q_{D^{1,p}} \left(r^{\frac{N}{p^*-q+\delta}\left(p^* + \frac{\theta p}{N-p} - q\right)} \right)^{(p^*-q+\delta)/p^*},$$

which yields the result. ∎

29

Lemma 4.4. Let $\theta \in \mathbb{R}$ and let $q > 1$ be such that $q > p^* + \theta p / (N - p)$. Then there exists $C_\infty = C_\infty(N, p, \theta, q) > 0$ such that for every $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $h \in D^{1,p}(\mathbb{R}^N)$ and $R > 0$ one has

$$\int_{B_R^c} |x|^\theta |u|^{q-1} |h| \, dx \leq \frac{C_\infty}{R^{\left(q - p^* - \frac{\theta p}{N-p}\right)\frac{N-p}{p}}} \|u\|_{D^{1,p}}^{q-1} \|h\|_{D^{1,p}}.$$

Proof. We denote by C any positive constant only depending on N, p, θ, q. Taking $\delta = \delta(N, p, q) > 0$ as in the previous lemma, the same computation giving (4.3) yields that for every $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $h \in D^{1,p}(\mathbb{R}^N)$ and $R > 0$ one has

$$\int_{B_R^c} |x|^\theta |u|^{q-1} |h| \, dx \leq C \left(\int_R^{+\infty} \rho^{\left(\theta - \frac{\delta N}{p^*}\right)\frac{p^*}{p^* - q + \delta} + N - 1} d\rho \right)^{(p^* - q + \delta)/p^*} \|u\|_{D^{1,p}}^{q-1} \|h\|_{D^{1,p}},$$

where

$$\left(\theta - \frac{\delta N}{p^*} \right) \frac{p^*}{p^* - q + \delta} + N = \frac{N}{p^* - q + \delta} \left(p^* + \frac{\theta p}{N - p} - q \right) < 0$$

by assumption. Hence we obtain

$$\int_{B_R^c} |x|^\theta |u|^{q-1} |h| \, dx \leq C \left(R^{\left(\theta - \frac{\delta N}{p^*}\right)\frac{p^*}{p^* - q + \delta} + N} \right)^{(p^* - q + \delta)/p^*} \|u\|_{D^{1,p}}^{q-1} \|h\|_{D^{1,p}},$$

and the result then ensues. ∎

Proof of Theorem 4.1. Let $u_n \rightharpoonup 0$ in $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, whence $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$, and let $\varepsilon > 0$. By assumption (4.2), there exist $C_1, C_2 > 0$ such that $\omega(x) \leq C_1 |x|^{\theta_1}$ for almost every $|x|$ small enough and $\omega(x) \leq C_2 |x|^{\theta_2}$ for almost every $|x|$ large enough, so that, by Lemmas 4.3 and 4.4 (applied with $u = h = u_n$), we can fix $r_\varepsilon, R_\varepsilon > 0$ such that

$$\int_{B_{r_\varepsilon}} |u_n|^{q_1} \omega(x) \, dx + \int_{B_{R_\varepsilon}^c} |u_n|^{q_2} \omega(x) \, dx$$

$$\leq \quad C_1 \int_{B_{r_\varepsilon}} |x|^{\theta_1} |u_n|^{q_1} \, dx + C_2 \int_{B_{R_\varepsilon}^c} |x|^{\theta_2} |u_n|^{q_2} \, dx$$

$$\leq \quad C_1 C_0 r_\varepsilon^{\left(p^* + \frac{\theta p}{N-p} - q_1\right)\frac{N-p}{p}} \|u_n\|_{D^{1,p}}^{q_1} + \frac{C_2 C_\infty}{R_\varepsilon^{\left(q_2 - p^* - \frac{\theta p}{N-p}\right)\frac{N-p}{p}}} \|u_n\|_{D^{1,p}}^{q_2}$$

$$\leq \quad C \left(r_\varepsilon^{\left(p^* + \frac{\theta p}{N-p} - q_1\right)\frac{N-p}{p}} + \frac{1}{R_\varepsilon^{\left(q_2 - p^* - \frac{\theta p}{N-p}\right)\frac{N-p}{p}}} \right) < \frac{\varepsilon}{2}$$

for some suitable constant $C > 0$ and for all n. Then, if $q_1 \leq p$, from the compactness of the embedding $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ we deduce that

$$\int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^{q_1} \omega(x) \, dx \leq \|\omega\|_{L^\infty(B_{R_\varepsilon} \setminus B_{r_\varepsilon})} \int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^{q_1} \, dx \to 0 \quad \text{as } n \to \infty.$$

30

On the other hand, if $q_1 > p$, we use Lemma 4.2 to deduce that there exists a constant $C_\varepsilon > 0$ such that

$$
\begin{aligned}
\int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^{q_1} \, \omega(x) \, dx
&\leq \|\omega\|_{L^\infty(B_{R_\varepsilon} \setminus B_{r_\varepsilon})} \int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^{q_1 - p} \, |u_n|^p \, dx \\
&\leq \|\omega\|_{L^\infty(B_{R_\varepsilon} \setminus B_{r_\varepsilon})} \frac{C_{N,p}^{q_1 - p} \|u_n\|_{D^{1,p}(\mathbb{R}^N)}^{q_1 - p}}{r_\varepsilon^{\frac{N-p}{p}(q_1 - p)}} \int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^p \, dx \\
&\leq C_\varepsilon \int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^p \, dx \to 0
\end{aligned}
$$

as $n \to \infty$. Therefore condition (2.7) holds with $E_{\varepsilon,n} = B_{R_\varepsilon}$ and the conclusion follows from Proposition 2.7. ■

The following results are straightforward consequences of Theorem 4.1. In particular, the first one contains a compactness lemma due to Benci-Fortunato (see [15, Lemma 3]), corresponding to the case $p = 2$ and ω constant.

Corollary 4.5. *If $\omega \in L^\infty(\mathbb{R}^N)$, then the space $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L(\mathbb{R}^N, \omega(x) \, dx)$ for every $1 < q_1 < p^* < q_2$.*

Proof. If $\omega \in L^\infty(\mathbb{R}^N)$ then condition (4.2) holds with $\theta_1 = \theta_2 = 0$ (which are also the best exponents for such a condition) and the result follows from Theorem 4.1. ■

Corollary 4.6. *If $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and (4.2) holds for some $\theta_1 > -(1 + N/p')$ and $\theta_2 < \theta_1$, then $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N, \omega(x) dx)$ for every $q > 1$ such that*

$$
p^* + \frac{\theta_2 p}{N - p} < q < p^* + \frac{\theta_1 p}{N - p}.
$$

Proof. Since $\theta_2 < \theta_1$ implies $p^* + \theta_2 p/(N - p) < p^* + \theta_1 p/(N - p)$, we can apply Theorem 4.1 with $q_1 = q_2 = q$ and the result ensues, because $L(\mathbb{R}^N, \omega(x) \, dx) = L^q(\mathbb{R}^N, \omega(x) dx)$. ■

5. Application to quasilinear equations

Assume $1 < p < N$ and let $V : (0, +\infty) \to [0, +\infty]$ and $f : (0, +\infty) \times \mathbb{R} \to [0, +\infty)$ be, respectively, a measurable and a Carathéodory function satisfying (**V**) and (**f**).

We define the weighted Sobolev spaces

$$
\begin{aligned}
W &= W^{1,p}\left(\mathbb{R}^N, V\right) := \left\{ u \in D^{1,p}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(|x|) |u|^p \, dx < +\infty \right\}, \\
W_{\mathrm{rad}} &= W^{1,p}_{\mathrm{rad}}\left(\mathbb{R}^N\right) := \overline{C^\infty_{\mathrm{c,rad}}\left(\mathbb{R}^N\right) \cap W^{1,p}\left(\mathbb{R}^N, V\right)}^{\, W^{1,p}(\mathbb{R}^N, V)}
\end{aligned}
$$

equipped with the norm given by

$$
\|u\|_W^p := \int_{\mathbb{R}^N} \left(|\nabla u|^p + V(|x|) |u|^p\right) dx,
$$

with respect to which they are uniformly convex Banach spaces. Note that assumption (**V**) ensures that both W and W_{rad} are nonempty. Moreover, since W-convergence implies almost everywhere convergence (up to a subsequence), all the mappings in W_{rad} are spherically symmetric.

We will apply some results from the previous sections (in particular Theorem 4.1, Proposition 3.7 and Corollary 2.18) with $\Omega = \mathbb{R}^N$ and $d\mu = \omega\left(x\right)dx$, where

$$\omega\left(x\right) = \max\{|x|^{\theta_1}, |x|^{\theta_2}\}$$

according to assumption (**f**). Note that W_{rad} is continuously embedded into $D_{\mathrm{rad}}^{1,p}(\mathbb{R}^N)$ and thus the embedding

$$W_{\mathrm{rad}} \hookrightarrow L(\mathbb{R}^N, \omega\left(x\right)dx) := L^{q_1}(\mathbb{R}^N, \omega\left(x\right)dx) + L^{q_2}(\mathbb{R}^N, \omega\left(x\right)dx)$$

(where q_1, q_2 are given by (**f**) again) is compact by Theorem 4.1.

Thanks to the continuity of the embedding $W_{\mathrm{rad}} \hookrightarrow L(\mathbb{R}^N, \omega\left(x\right)dx)$, assumption (**f**) and Proposition 3.7 ensure that the functional $I : W_{\mathrm{rad}} \to \mathbb{R}$ given by

$$I\left(u\right) := \frac{1}{p}\left\|u\right\|_W^p - \int_{\mathbb{R}^N} F\left(|x|, u\right)dx$$

(where $F\left(r, t\right) := \int_0^t f\left(r, s\right)ds$) is well defined and of class C^1, with Fréchet derivative $I'\left(u\right) \in W_{\mathrm{rad}}'$ at any $u \in W_{\mathrm{rad}}$ given by

$$\langle I'\left(u\right), h\rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla h + V\left(|x|\right)|u|^{p-2}uh\right)dx - \int_{\mathbb{R}^N} f\left(|x|, u\right)h\,dx \quad (5.1)$$

for all $h \in W_{\mathrm{rad}}$. Hence the critical points $u \in W_{\mathrm{rad}}$ of I satisfy (1.3) for all $h \in W_{\mathrm{rad}}$. The next lemma shows that W_{rad} actually is, in some sense, a natural constraint for finding weak solutions of equation (1.2). Observe that the classical Palais' principle of symmetric criticality [31] does not apply in this case, because we do not know whether I is differentiable, not even well defined, on the whole space W or not.

Lemma 5.1. *Every critical point of $I : W_{\mathrm{rad}} \to \mathbb{R}$ is a weak solution to equation (1.2).*

Proof. We show that if $u \in W_{\mathrm{rad}}$ satisfies (1.3) for all $h \in W_{\mathrm{rad}}$, then (1.3) holds also true for all $h \in W$. Let $u \in W_{\mathrm{rad}}$. By (**f**) and Lemmas 4.3 and 4.4, $\forall h \in W$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |f\left(|x|, u\right)h|\,dx &\leq M \int_{\mathbb{R}^N} \min\{|u|^{q_1-1}, |u|^{q_2-1}\}|h|\,\omega\left(x\right)dx \\
&\leq M \int_{B_1} |x|^{\theta_1}|u|^{q_1-1}|h|\,dx + M \int_{B_1^c} |x|^{\theta_2}|u|^{q_2-1}|h|\,dx \\
&\leq M\left(C_0\|u\|_W^{q_1-1} + C_\infty\|u\|_W^{q_2-1}\right)\|h\|_W,
\end{aligned}
$$

so that the linear functional defined by

$$\langle T\left(u\right), h\rangle := \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla h + V\left(|x|\right)|u|^{p-2}uh\right)dx - \int_{\mathbb{R}^N} f\left(|x|, u\right)h\,dx$$

32

is well defined and continuous on W. Hence, by uniform convexity, there exists a unique $\tilde{u} \in W$ such that $T(u)\tilde{u} = \|\tilde{u}\|_W^2 = \|T(u)\|_{W'}^2$. Then, by means of obvious changes of variable, one checks that for every $h \in W$ we have

$$\langle T(u), h(S\cdot)\rangle = \langle T(u), h\rangle \quad \text{and} \quad \|h(S\cdot)\|_W = \|h\|_W \quad \text{for all } S \in O(N),$$

whence, applying with $h = \tilde{u}$, one deduces $\tilde{u}(S\cdot) = \tilde{u}$ by uniqueness. This means $\tilde{u} \in W_{\text{rad}}$, so that, if $\langle T(u), h\rangle = 0$ for all $h \in W_{\text{rad}}$, one has $\langle T(u), \tilde{u}\rangle = 0$, that is, $\|T(u)\|_{W'} = 0$. ∎

By virtue of Lemma 5.1, the proof of Theorems 1.1 and 1.2 reduces to finding critical points of the functional I, which exhibits a right amount of compactness, according to the following lemma.

Lemma 5.2. *The functional $I : W_{\text{rad}} \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subseteq W_{\text{rad}}$ be a sequence such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in W'_{rad}. One has to show that $\{u_n\}$ contains a W-converging subsequence. Exploiting the condition $\gamma F(r, t) \leq f(r, t)t$ with $\gamma > p$ of assumption (**f**), a standard argument shows that $\{u_n\}$ is bounded in W_{rad}. Then Theorem 4.1 applies, yielding the existence of $u \in W_{\text{rad}}$ such that (up to a subsequence)

$$u_n \rightharpoonup u \quad \text{in } W_{\text{rad}}$$
$$u_n \to u \quad \text{in } L(\mathbb{R}^N, \omega(x)\,dx).$$

Now set

$$I_1(u) := \frac{1}{p}\|u\|_W^p \quad \text{and} \quad I_2(u) := I_1(u) - I(u)$$

for brevity. Then, by (5.1) and Proposition 3.7, we get

$$\|u_n\|_W^p = \langle I'(u_n), u_n\rangle + \langle I_2'(u_n), u_n\rangle = \langle I_2'(u), u\rangle + o(1)_{n\to\infty},$$

so that $\lim_{n\to\infty}\|u_n\|_W$ exists and one has $\|u\|_W^p \leq \lim_{n\to\infty}\|u_n\|_W^p$ by weak lower semicontinuity. Moreover the convexity of $I_1 : W_{\text{rad}} \to \mathbb{R}$ implies

$$I_1(u) - I_1(u_n) \geq \langle I_1'(u_n), u - u_n\rangle = \langle I'(u_n), u - u_n\rangle + \langle I_2'(u_n), u - u_n\rangle = o(1)_{n\to\infty}$$

and thus

$$\frac{1}{p}\|u\|_W^p = I_1(u) \geq \lim_{n\to\infty} I_1(u_n) = \frac{1}{p}\lim_{n\to\infty}\|u_n\|_W^p.$$

So $\|u_n\|_W \to \|u\|_W$ and one concludes that $u_n \to u$ in W_{rad} by uniform convexity. ∎

Proof of Theorem 1.1. As we are interested in nonnegative solutions, it is not restrictive to assume

$$f(r, t) = 0 \quad \text{for all } r > 0 \text{ and } t < 0 \tag{5.2}$$

We want to apply the well known Mountain-Pass Theorem [2]. To this end, observe that, by (**f**) and Lemmas 4.3 and 4.4 (applied with $h = u$), for every $u \in W_{\mathrm{rad}}$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} F\left(|x|, u\right) dx &\leq c_1 \int_{\mathbb{R}^N} \min\{|u|^{q_1}, |u|^{q_2}\} \omega\left(x\right) dx \\
&\leq \int_{B_1} |u|^{q_1} |x|^{\theta_1} dx + \int_{B_1^c} |u|^{q_2} |x|^{\theta_2} dx \leq c_2 \|u\|_W^{q_1} + c_3 \|u\|_W^{q_2}
\end{aligned}
$$

(where c_1, c_2, c_3 denote some positive constants), so that

$$
I\left(u\right) \geq \frac{1}{p} \|u\|_W^p - c_2 \|u\|_W^{q_1} - c_3 \|u\|_W^{q_2}. \tag{5.3}
$$

Since $q_2 > q_1 > p$, this proves that I has a mountain pass geometry near the origin, i.e., there exist $\delta, \rho > 0$ such that for all $u \in W_{\mathrm{rad}}$ with $\|u\|_W = \rho$ one has $I\left(u\right) \geq \delta$. On the other hand, there exists $\bar{u} \in W_{\mathrm{rad}}$ such that $\|\bar{u}\|_W > \rho$ and $I\left(\bar{u}\right) < 0$. Indeed, by assumption (1.7) and condition $\gamma F\left(r, t\right) \leq f\left(r, t\right) t$ of (**f**), one easily deduces

$$
F\left(r, t\right) \geq \frac{F\left(r, t_*\right)}{t_*^\gamma} t^\gamma \quad \text{for almost every } r > 0 \text{ and all } t \geq t_*
$$

so that, $\forall \lambda > 1$ and $\forall u \in W_{\mathrm{rad}}$ nonnegative such that the set $\{x \in \mathbb{R}^N : u\left(x\right) \geq t_*\}$ has positive Lebesgue measure, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} F\left(|x|, \lambda u\right) dx &\geq \int_{\{\lambda u \geq t_*\}} F\left(|x|, \lambda u\right) dx \geq \frac{\lambda^\gamma}{t_*^\gamma} \int_{\{\lambda u \geq t_*\}} F\left(|x|, t_*\right) u^\gamma dx \\
&\geq \frac{\lambda^\gamma}{t_*^\gamma} \int_{\{u \geq t_*\}} F\left(|x|, t_*\right) u^\gamma dx \geq \lambda^\gamma \int_{\{u \geq t_*\}} F\left(|x|, t_*\right) dx > 0
\end{aligned}
$$

(recall that $F \geq 0$ and $F\left(|\cdot|, t_*\right) > 0$ almost everywhere), which gives

$$
I\left(\lambda u\right) \leq \frac{\lambda^p}{p} \|u\|_W^p - \lambda^\gamma \int_{\{u \geq t_*\}} F\left(|x|, t_*\right) dx \to -\infty \quad \text{as } \lambda \to +\infty
$$

since $\gamma > p$. As a conclusion, I exhibits a full mountain-pass geometry and, by Lemma 5.2, the Mountain-Pass Theorem provides the existence of a nontrivial critical point for I, which is a weak solution to equation (1.2) by Lemma 5.1. Finally, by (5.2), a standard argument shows that any $u \in W$ satisfying (1.3) for all $h \in W$ has to be nonnegative. ∎

Proof of Theorem 1.2. By the oddness assumption (1.8), one has $I\left(u\right) = I\left(-u\right)$ for all $u \in W_{\mathrm{rad}}$ and thus we can apply the Symmetric Mountain-Pass Theorem (see for example [35, Theorem 6.5]). To this end, taking into account (5.3) and Lemma 5.2, we need only to show that I satisfies the following geometrical condition: for any finite dimensional subspace $Y \neq \{0\}$ of W_{rad} there exists $R > 0$ such that for all $u \in Y$ with $\|u\|_W \geq R$ one has $I\left(u\right) \leq 0$. In fact it is sufficient to prove

that any sequence $\{u_n\} \subseteq Y$ with $\|u_n\|_W \to +\infty$ admits a subsequence such that $I(u_n) \leq 0$. Recall the definition (2.1) (where $\Omega = \mathbb{R}^N$) of Λ_{u_n} and briefly denote $L^{q_1}(\Lambda_{u_n}) := L^{q_1}(\Lambda_{u_n}, \omega(x)\,dx)$, $L^{q_2}(\Lambda_{u_n}^c) := L^{q_2}(\Lambda_{u_n}^c, \omega(x)\,dx)$. Since all norms are equivalent on Y, one has

$$\|u_n\|_{L^{q_1}(\Lambda_{u_n})} + \|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)} \geq \|u_n\| \geq c_0 \|u_n\|_W \to +\infty \qquad (5.4)$$

for some constant $c_0 > 0$, where the right hand inequality of (2.21) has been used. Hence, up to a subsequence, at least one of the sequences $\{\|u_n\|_{L^{q_1}(\Lambda_{u_n})}\}$, $\{\|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)}\}$ diverges. We now use assumption (1.9) to obtain

$$\int_{\mathbb{R}^N} F(|x|, u_n)\, dx \geq m \int_{\Lambda_{u_n}} |u_n|^{q_1} \omega(x)\, dx + m \int_{\Lambda_{u_n}^c} |u_n|^{q_2} \omega(x)\, dx.$$

Thus, using inequalities (5.4), there exists a constant $c > 0$ such that

$$\begin{aligned}
I(u_n) &\leq c\left(\|u_n\|_{L^{q_1}(\Lambda_{u_n})}^p + \|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)}^p\right) - m\left(\|u_n\|_{L^{q_1}(\Lambda_{u_n})}^{q_1} + \|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)}^{q_2}\right) \\
&= c\|u_n\|_{L^{q_1}(\Lambda_{u_n})}^p - m\|u_n\|_{L^{q_1}(\Lambda_{u_n})}^{q_1} + c\|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)}^p - m\|u_n\|_{L^{q_2}(\Lambda_{u_n}^c)}^{q_2},
\end{aligned}$$

so that $I(u_n) \to -\infty$, since $q_2 > q_1 > p$. Therefore the Symmetric Mountain-Pass Theorem implies the existence of an unbounded sequence of critical values for I, to which there corresponds a sequence of nontrivial critical points and thus a sequence of weak solutions to equation (1.2), thanks to Lemma 5.1. ∎

References

[1] A.Ambrosetti, V.Felli, A.Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. **7** (2005), 117-144.

[2] A.Ambrosetti, P.H.Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[3] A.Ambrosetti, Z.Q.Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials*, Diff. Int. Equations **18** (2005), 1321-1332.

[4] A.Azzollini, *A multiplicity result for a semilinear Maxwell type equation*, Topol. Methods Nonlinear Anal. **31** (2008), 83–110.

[5] A.Azzollini, L.Pisani, A.Pomponio, *Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system*, Proc. R. Soc. Edinb., in press.

[6] A.Azzollini, A.Pomponio, *On a "zero mass" nonlinear Schrödinger equation*, Adv. Nonlinear Stud. **7** (2007), 599-627.

[7] A.Azzollini, A.Pomponio, *Compactness results and applications to some "zero mass" elliptic problems*, Nonlinear Anal. **69** (2008), 3559-3576.

[8] M.Badiale, V.Benci, S.Rolando, *A nonlinear elliptic equation with singular potential and applications to nonlinear field equations*, J. Eur. Math. Soc. **9** (2007), 355-381.

[9] M.Badiale, V.Benci, S.Rolando, *Three dimensional vortices in the nonlinear wave equation*, Bollettino U.M.I., serie IX, **2** (2009), 105-134.

[10] M.Badiale, M.Guida, S.Rolando, *Elliptic equations with decaying cylindrical potentials and power-type nonlinearities*, Adv. Differential Equations **12** (2007), 1321-1362.

[11] M.Badiale, S.Rolando, *Elliptic problems with singular potentials and double-power nonlinearity*, Mediterr. J. Math. **2** (2005), 417-436.

[12] M.Badiale, S.Rolando, *A note on nonlinear elliptic problems with singular potentials*, Rend. Lincei Mat. Appl. **16** (2006), 1-13.

[13] M.Badiale, S.Rolando, *Nonlinear elliptic equations with subhomogeneous potentials*, Nonlinear Anal. **72** (2010), 602-617.

[14] V.Benci, D.Fortunato, *A strongly degenerate elliptic equation arising from the semilinear Maxwell equations*, C.R. Acad. Sci. Paris, série I **339** (2004), 839-842.

[15] V.Benci, D.Fortunato, *Towards a unified field theory for classical electrodynamics*, Arch. Rational Mech. Anal. **173** (2004), 379-414.

[16] V.Benci, C.R.Grisanti, A.M.Micheletti, *Existence and non existence of the ground state solution for the nonlinear Schrödinger equation with $V(\infty) = 0$*, Topol. Methods Nonlinear Anal. **26** (2005), 203-220.

[17] V.Benci, C.R.Grisanti, A.M.Micheletti, *Existence of solutions for the nonlinear Schrödinger equation with $V(\infty) = 0$*, Progress in Nonlinear Differential Equations and Their Applications, vol. 66, Birkhäuser, 2005.

[18] V.Benci, A.M.Micheletti, *Solutions in exterior domains of null mass nonlinear field equations*, Adv. Nonlinear Stud. **6** (2006), 171-198.

[19] H.Berestycki, P.L.Lions, *Nonlinear Scalar Field Equations, I - Existence of a Ground State*, Arch. Rational Mech. Anal. **82** (1983), 313-345.

[20] D.Bonheure, J.Van Schaftingen, *Bound states solutions for a class of nonlinear Schrödinger equations*, Rev. Mat. Iberoamericana **24** (2008), 297-351.

[21] H.Brezis, L.Dupaigne, A.Tesei, *On a semilinear elliptic equation with inverse-square potential*, Selecta Math. New Series **11** (2005), 1-7.

[22] J.Byeon, Z.Q.Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. **165** (2002), 295-316.

[23] J.Byeon, Z.Q.Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations. II*, Calc. Var. Partial Differential Equations **18** (2003), 207-219.

[24] J.Byeon, Z.Q.Wang, *Spherical semiclassical states of a critical frequency for Schrödinger equations with decaying potentials*, J. Eur. Math. Soc. **8** (2006), 217-228.

[25] M.Conti, S.Crotti, D.Pardo, *On the existence of positive solutions for a class of singular elliptic equations*, Adv. Differential Equations **3** (1998), 111-132.

[26] L.Drewnowski, W.Orlicz, *A note on modular spaces. XI*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **16** (1968), 877-882.

[27] M.Ghimenti, A.M.Micheletti, *Solutions for a nonhomogeneous nonlinear Schroedinger equation with double-power nonlinearity*, Diff. Int. Equations **10** (2007), 1131-1152.

[28] M.A.Krasnosel''skii, Y.B.Rutickii, *Convex functions and Orlicz spaces*, P. Noordhoff Ltd, Groningen, 1961.

[29] L.Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics 5, Univ. Estadual de Campinas, Campinas, 1989.

[30] R.Musina, *Existence and multiplicity results for a weighted p-Laplace equation involving Hardy potentials and critical nonlinearities*, Rend. Lincei Mat. Appl. **20** (2009), 1-17.

[31] R.S.Palais, *The principle of symmetric criticality*, Commun. Math. Phys. **69** (1979), 19-30.

[32] L.Pisani, *Remarks on the sum of Lebesgue spaces*, unpublished note, 2004.

[33] M.M.Rao, Z.D.Ren, *Theory of Orlicz spaces*, Marcel Dekker, New York, Basel, Hong Kong, 1991.

[34] S.Rolando, *Nonlinear elliptic equations with singular symmetric potentials*, PhD Thesis, Dipartimento di Matematica, Università degli Studi di Torino, 2006. (http://www2.dm.unito.it/paginepersonali/rolando)

[35] M.Struwe, *Variational methods*, Springer-Verlag, Berlin-Heidelberg, 1990.

[36] J.Su, Z.Q.Wang, M.Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 201-219.

[37] M.M.Vainberg, *Variational methods for the study of nonlinear operators*, Holden-Day, San Francisco, London, Amsterdam, 1964.