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Planar Dirac-type systems: the eigenvalue problem and a global bifurcation result*†

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Abstract

We deal with a boundary value problem on the half-line for planar systems of Dirac type. We first study the eigenvalue problem in the linear case and define an index for nontrivial solutions. We then give a global bifurcation result for nonlinear problems.

1 Introduction

In this paper we study a nonlinear Dirac-type system in $\mathbf{R}^2$ of the form

$$Jz' + P(x)z = \lambda z + S(x,z)z, \quad x \in [1,+\infty), \quad \lambda \in \mathbf{R}, \quad z = (u,v) \in \mathbf{R}^2, \tag{1.1}$$

where J is the standard symplectic matrix and $P(x), S(x,z)$ are continuous symmetric matrices.

The linear system $Jz' + P(x)z = \lambda z$ is a particular case of the so-called Dirac systems (see [18] for a complete description of such systems), which arise in a natural way after separation of variables in the physical Dirac operator (cfr. [7], [18]). In the classical Dirac system the matrix P has the form

$$P(x) = \begin{pmatrix} -1 - aZ/x & k/x \\ k/x & 1 - aZ/x \end{pmatrix},$$

for some constants a, Z and k, and the system is studied in the interval $(0,+\infty)$. This leads to a singular situation from two different points of view. On the one hand, the interval is unbounded; on the other hand, the matrix P has a singularity at the endpoint $x = 0$.

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Regular boundary value problems in a bounded interval have been considered in [3], [17] (see also [9]). In this paper, we will consider only singular systems as (1.1), where the singularity is due to the fact that the interval is unbounded; we will suppose that P is regular at $x = 1$. In a forthcoming paper we will study the case of a singularity also at $x = 1$.

As far as the functional setting is concerned, we will be interested in solutions z of (1.1) satisfying a boundary condition of the form $v(1) = 0$ and belonging to the space $H^1(1, +\infty)$; in particular, the solutions are convergent to zero at infinity. This choice is strictly related to the spectral properties of the linear operator $\tau z = Jz' + P(x)z$ and to the possibility to consider self-adjoint extensions of τ (see Section 2).

The main difficulty in the study of (1.1) lies in the fact that the interval under consideration is unbounded. In case of a bounded interval several techniques can be used to prove the existence of solutions of boundary value problems associated to a system such as the one in (1.1); indeed, the existence of a double sequence of eigenvalues of the linear operator τ, together with the knowledge of the oscillatory properites (in terms of rotation number in the phase-plane) of solutions, has been obtained in [1] using the rotation number approach and the results in [18]. Finally, the compactness of the interval leads in a straightforward way to compactness properties of abstract operators associated to (1.1).

When passing to the case of an unbounded interval many problems arise. First of all, no complete spectral theory is available for τ. Indeed, in most cases an essential spectrum appears (according to the asymptotic behaviour of P at infinity) and the operator τ might have no eigenvalues. We point out that a similar situation characterizes the analogous problem for second order differential operators of Schrödinger type (see [5], [14]); the main difference is that for an operator like $Au = u'' + q(x)u$ the essential spectrum has the form $[Q, +\infty)$, with $Q = \lim_{x\to+\infty} q(x)$, while in our situation the essential spectrum is $(-\infty, \mu^-] \cup [\mu^+, +\infty)$, being $\mu^\pm$ the eigenvalues of the limit matrix of $P(x)$ when $x \to +\infty$. We also remark that the oscillation theory of systems defined in an unbounded interval is not completely developed; our first aim is to study in a deep way this question, trying also to relate it to the eigenvalue problem for the operator τ. To do this, we consider the angular coordinate $\theta(\cdot, \lambda)$ associated to nontrivial solutions of $\tau z = \lambda z$ (which generalizes the well-known angular coordinate for solutions of second order differential equations [18]) and we show that, under suitable assumptions on P, it has a limit when $x \to +\infty$. It is interesting to observe that, contrary to the case of second order equations, in case of planar Dirac-type systems the angular coordinate is not, in general, an increasing function of x. One useful tool for the study of the asymptotic behaviour of θ is the classical Levinson theorem. Indeed, this result provides an asymptotic expansion of the solutions of a first order linear differential system.

By means of the knowledge of the asymptotic behaviour of θ we will be able to define an index for nontrivial solutions of $\tau z = \lambda z$, related to the number of rotations of the solution in the phase-plane. A careful analysis (based on this index) of the properties of θ leads then to the characterization of the eigenvalues of the linear boundary value problem associated to τ. More precisely, in Theorem 3.7 not only we characterize the existence of eigenvalues in terms of the angular coordinate of the corresponding solution, but we describe by means of the index the nodal properties of the eigenfunctions. In particular, according to the properties of θ, we can prove the existence of a finite number or an infinite number of eigenvalues in the spectral gap (cfr. Theorems 3.12 and 3.13).

In the case of a matrix P of the form

$$P(x) = \begin{pmatrix} -1 + V(x) & k/x \\ k/x & 1 + V(x) \end{pmatrix}$$

we prove e.g. that there exists an infinite sequence of eigenvalues of τ converging to 1 when V is a strictly increasing negative potential decaying to zero at infinity as $1/x^\alpha$, with $0 < \alpha \leq 1$. A similar result (under more restrictive conditions on α) has been obtained in case V is singular in zero by Schmid-Tretter in [12]; however, in [12] no information on the nodal properties of the eigenfunctions is provided.

As far as the nonlinear system is concerned, we will use a bifurcation approach, which has been widely applied in studying parameter dependent boundary value problems for ordinary and partial differential equations. As already observed, in the present situation we have a lack of compactness; we overcome this difficulty by applying a suitable abstract bifurcation result, which has been proved by Stuart in [14] in the context of α-contraction mappings and has been applied to the study of second order equations in the half-line. In order to guarantee the applicability of this theorem, we need to recover some compactness from the nonlinear term, requiring the condition

$$||S(x,z)|| \leq \alpha(x)\eta(z), \quad \forall\ x \geq 1,\ z \in \mathbf{R}^2,$$

with $\lim_{x\to+\infty} \alpha(x) = 0$.

The application of the abstract theorem by Stuart gives the existence of connected branches of solutions of (1.1) bifurcating from eigenvalues of odd multiplicity of the linear operator τ; moreover, some global property of the branches is known. Indeed, the continuum bifurcating from a fixed eigenvalue is unbounded or meets the lines $\lambda = \mu^\pm$, where $(-\infty, \mu^-] \cup [\mu^+, +\infty)$ is the essential spectrum of τ, or contains another eigenvalue of τ.

In order to exclude the second alternative we develop a continuity and connectivity argument (as we did in [2]) based on a linearization approach; indeed, we associate to every nontrivial solution (w, μ) of (1.1) the index of (w, μ) as solution of the linear equation

$$Jz' + P(x)z = \mu z + S(x, w(x))z.$$

It turns out that along each branch the index is preserved. In a forthcoming paper we will study the nonlinear eigenvalue problem (1.1) for a fixed value of λ; this requires to analyze in a more precise way the global behaviour of the bifurcating branches, proving that they are bounded in the variable z.

It has to be pointed out that in the literature one can find global bifurcation results and applications to boundary value problems in unbounded domains (cf., among others, [4],[10], [13], [15], [16] and references therein) which generalize in various directions those in [11], [14]. Thanks to the fact that we work in $\mathbf{R}^2$, we have been able to develop a precise qualitative analysis of the solutions to the linear problem associated to (1.1) and, as a consequence, to introduce a topological invariant by elementary methods. Thus, having at our disposal a well-established linear theory, we can enter the framework of [14] and state our global bifurcaton result under rather simple assumptions. For first order systems in $\mathbf{R}^{2n}$, a suitable topological invariant (the Maslov index) is available, but more work is needed in order to obtain a satisfactory knowledge of the linear theory.

In Section 2 we introduce the index of a solution to linear and nonlinear problems on the half-line $[1, +\infty)$ and prove that it is continuous.
In Section 3 we study the linear eigenvalue associated to the differential operator τ and, under suitable assumptions on P, prove the existence of simple eigenvalues. We also describe the index of the corresponding eigenfunctions.
In Section 4 we state and prove our global bifurcation result for (1.1).

In what follows, we will denote by $M_S^{2,2}$ the set of symmetric 2×2 matrices. Moreover, D_0 will be

$$D_0 = \{z = (u, v) \in H^1(1, +\infty) : \ v(1) = 0\}$$

endowed with the usual norm

$$||z||^2 = ||z||^2_{L^2} + ||z'||^2_{L^2}.$$

2 Definition of the index

In this section we will be interested in associating an index, related to the rotation number on the phase-plane, to nontrivial solutions of the nonlinear equation

$$Jz' + P(x)z = \lambda z + S(x, z)z, \quad x \in [1, +\infty), \quad \lambda \in \mathbf{R}, \quad z = (u, v) \in \mathbf{R}^2. \tag{2.1}$$

In what follows, by a solution of (2.1) we mean a function $z \in AC_{loc}(1, +\infty)$ satisfying (2.1) a.e. in $[1, +\infty)$.

Let us first consider the linear equation

$$Jz' + P(x)z = \lambda z, \tag{2.2}$$

where $P : [1, +\infty) \longrightarrow M_S^{2,2}$ is continuous.
We denote by $\mathcal{P}$ the class of continuous maps $P : [1, +\infty) \longrightarrow M_S^{2,2}$ such that

$$\lim_{x \to +\infty} P(x) = \begin{pmatrix} \mu^- & 0 \\ 0 & \mu^+ \end{pmatrix} := P_0, \tag{2.3}$$

for some $\mu^- < \mu^+$, and there exists $q \geq 1$ such that

$$\int_1^{+\infty} |R(x)|^q \, dx < +\infty, \tag{2.4}$$

where $R(x) = P(x) - P_0$, for every $x \geq 1$, and (2.4) means that every entry of the matrix R belongs to $L^q(1, +\infty)$.
In what follows we denote

$$P(x) = \begin{pmatrix} p_1(x) & p_{12}(x) \\ p_{12}(x) & p_2(x) \end{pmatrix}, \quad \forall\, x \geq 1,$$

and $\Lambda = (\mu^-, \mu^+) \subset \mathbf{R}$.

Remark 2.1 *In many important situations the coefficients of the matrix P satisfy conditions of the form*

$$p_1(x) = \mu^- + O(1/x^a), \quad p_2(x) = \mu^+ + O(1/x^b), \quad p_{12}(x) = O(1/x^c),$$

for $x \to +\infty$. It is immediate to see that in this case condition (2.4) is fulfilled for every $a > 0$, $b > 0$ and $c > 0$.

As we will see in Section 3, we will be interested in solutions $z = (u, v)$ of (2.2) (or of (2.1)) satisfying the condition

$$v(1) = 0. \tag{2.5}$$

It is possible to associate an index to nontrivial solutions of (2.2)-(2.5); to do this, we observe that assumption (2.3) implies that for every $\lambda \in \Lambda$ there exists $x_\lambda \geq 1$ such that

$$p_1(x) < \lambda < p_2(x), \quad \forall\, x \geq x_\lambda. \tag{2.6}$$

To define x_λ univocally, we set

$$x_\lambda = \max\left(\sup\{x: \ p_1(x) = \lambda\}, \sup\{x: \ p_2(x) = \lambda\}\right), \tag{2.7}$$

when one of the sets

$$\{x: \ p_i(x) = \lambda\}, \ i = 1, 2,$$

is non-empty; otherwise, we set $x_\lambda = 1$. Moreover, we assume that the function $\lambda \mapsto x_\lambda$ is continuous in Λ (remark that this condition is trivially satisfied when p_1 and p_2 are monotone).

Now, let us observe that (2.2) can be written as

$$z' = B_\lambda z + Q(x) z, \tag{2.8}$$

where

$$B_\lambda = \begin{pmatrix} 0 & \mu^+ - \lambda \\ \lambda - \mu^- & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p_{12}(x) & p_2(x) - \mu^+ \\ \mu^- - p_1(x) & -p_{12}(x) \end{pmatrix}, \quad \forall\, x \geq 1.$$

Note that, if $\lambda \in \Lambda$, then there exists an invertible matrix T_λ such that

$$T_\lambda^{-1} B_\lambda T_\lambda = B_\lambda^D, \tag{2.9}$$

where

$$B_\lambda^D = \begin{pmatrix} -\sqrt{\Delta} & 0 \\ 0 & \sqrt{\Delta} \end{pmatrix}, \quad \Delta = (\mu^+ - \lambda)(\lambda - \mu^-) > 0.$$

Now, let us denote by $\{e_1, e_2\}$ the standard basis of $\mathbf{R}^2$; the following result gives an asymptotic estimate for the solutions of (2.8):

Proposition 2.2 *For every $\lambda \in \Lambda$ system (2.2) has two linearly independent solutions z_1 and z_2 satisfying*

$$\begin{aligned} z_1(x) &= (T_\lambda e_1 + o(1)) e^{-\sqrt{\Delta}(x-1) + \int_1^x q_{\lambda,1}(t)\, dt}, \ x \to +\infty \\ z_2(x) &= (T_\lambda e_2 + o(1)) e^{\sqrt{\Delta}(x-1) + \int_1^x q_{\lambda,2}(t)\, dt}, \ x \to +\infty, \end{aligned} \tag{2.10}$$

for some functions $q_{\lambda,i} \in L^q(1, +\infty)$ depending on the matrix Q_λ $(1 = 1, 2)$.

Proof. For $q \in [1,2]$ it is sufficient to apply Theorem 1.8.1 or 1.8.2 in [6]. To this aim, let us recall that the quoted Theorems are valid when B_λ has eigenvalues with distinct real parts and the remainder term Q satisfies

$$\int_1^{+\infty} |Q(x)|^q \, dx < +\infty. \tag{2.11}$$

The condition on B_λ is trivially verified, while (2.11) follows from the definition of Q and assumption (2.4).

In the case $q > 2$ the result is a consequence of the application of Theorem 1.5.2 in [6] to the linear system obtained from (2.8) by means of the change of variables $z = T_\lambda w$. ■

Now, we shall study in a more careful way the asymptotic behaviour of z_1 and z_2 for $x \to +\infty$; for every $x \geq 1$ and for $i = 1,2$, let us define

$$\beta_i(x) = (-1)^i \sqrt{\Delta}(x-1) + \int_1^x q_{\lambda,i}(t)\, dt. \tag{2.12}$$

Let q' be the conjugate exponent of q; for every $x \geq 1$ and for $i = 1,2$ from assumption (2.4) we infer

$$\begin{aligned} |q_{\lambda,i}(x)| &\leq \int_1^x |Q_\lambda(t)|\, dt \leq \int_1^x |T_\lambda^{-1}|\; |Q(t)|\; |T_\lambda|\, dt \leq \\ &\leq C_\lambda \int_1^x |Q(t)|\, dt \leq C_\lambda \left(\int_1^x |Q(t)|^q\, dt\right)^{1/q} \left(\int_1^x 1\, dt\right)^{1/q'} \leq C'_{\lambda,q}(x-1)^{1/q'}, \end{aligned} \tag{2.13}$$

for some constant $C'_{\lambda,q} > 0$. From (2.13) and (2.12), observing that $1/q' < 1$, we obtain that

$$\lim_{x\to+\infty} \beta_1(x) = -\infty, \qquad \lim_{x\to+\infty} \beta_2(x) = +\infty. \tag{2.14}$$

We are now in position to prove the following:

Lemma 2.3 *Assume that $\lambda \in \Lambda$ and let $z = (u,v)$ be a nontrivial solution of* (2.2)*. Then*

$$\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} v(x) = 0 \tag{2.15}$$

or

$$\lim_{x\to+\infty} |u(x)| = \lim_{x\to+\infty} |v(x)| = +\infty. \tag{2.16}$$

Proof. Let z be a nontrivial solution of (2.2); then, there exist real constants c_1 and c_2 such that

$$z(x) = c_1 z_1(x) + c_2 z_2(x), \quad \forall\, x \geq 1.$$

If $c_2 = 0$, then $z = c_1 z_1$ and (2.15) follows from (2.10) and (2.14).
In the case when $c_2 \neq 0$ then z satisfies (2.16); indeed, it is sufficient to observe that (2.10) and (2.14) imply

$$|u(x)| = |c_1 z_{1,1}(x) + c_2 z_{2,1}(x)| \sim |c_2 z_{2,1}(x)| = |c_2(\sqrt{\Delta} + o(1))|\; e^{\beta_2(x)},$$

for $x \to +\infty$. An analogous estimate holds for v. ■

Now, for every nontrivial solution (u, v, λ) of (2.2) let us introduce the polar coordinates $(\rho, \theta) = (\rho(x, \lambda), \theta(x, \lambda))$ according to

$$\begin{cases} u = \rho \cos \theta \\ v = \rho \sin \theta. \end{cases}$$

Observe that θ is defined mod. 2π; if we fix

$$\theta(1, \lambda) = 0, \quad \forall\, \lambda \in \Lambda,$$

according to the boundary condition $v(1) = 0$, then θ is uniquely determined and is a continuous function of x and λ. Moreover, we have

$$uv = 0 \quad \Leftrightarrow \quad \theta = 0, \text{ mod. } \pi/2; \tag{2.17}$$

since the equation in (2.2) can be written as

$$\begin{cases} u' = (p_2(x) - \lambda)v + p_{12}(x)u \\ v' = (\lambda - p_1(x))u - p_{12}(x)v, \end{cases} \tag{2.18}$$

it then follows that θ satisfies the differential equation

$$\theta'(x) = (\lambda - p_1(x)) \cos^2 \theta(x) - 2p_{12}(x) \cos \theta(x) \sin \theta(x) + (\lambda - p_2(x)) \sin^2 \theta(x), \tag{2.19}$$

for every $x \geq 1$.

Using the asymptotic properties of u and v proved before, we are able to show that θ has limit as $x \to +\infty$; indeed, the following result holds true:

Proposition 2.4 *For every $\lambda \in \Lambda$ the function $\theta(\cdot, \lambda)$ has limit at infinity and we have*

$$\lim_{x \to +\infty} \theta(x, \lambda) = \pi - \arctan \sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}} \quad (\text{mod } \pi) \tag{2.20}$$

or

$$\lim_{x \to +\infty} \theta(x, \lambda) = \arctan \sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}} \quad (\text{mod } \pi). \tag{2.21}$$

Proof. The proof follows the same lines of the proof of Proposition 3.1 in [12]. Let $z = (u, v)$ be a nontrivial solution of (2.2) and let c_1 and c_2 be such that

$$z(x) = c_1 z_1(z) + c_2 z_2(x), \quad \forall\, x \geq 1.$$

We prove the result in the case $c_2 = 0$, leading to (2.20); when $c_2 \neq 0$ then an analogous argument shows that (2.21) holds true.
Assume then $z = c_1 z_1$; from (2.10) we infer, as before, that

$$u(x) = c_1 e^{\beta_1(x)} u_1(x)$$

$$v(x) = c_1 e^{\beta_1(x)} v_1(x),$$

with $u_1(x) \neq 0$ for x sufficiently large. Hence for large x we can write (mod. π)

$$\lim_{x\to+\infty} \theta(x,\lambda) = \lim_{x\to+\infty} \arctan \frac{v(x)}{u(x)} = \lim_{x\to+\infty} \arctan \frac{v_1(x)}{u_1(x)} =$$

$$= \arctan \frac{-\sqrt{\Delta}}{\mu^+ - \lambda} = \pi - \arctan \sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}}.$$

■

Remark 2.5 *Note that the case when* (2.20) *holds corresponds to the situation in which* (2.15) *is fulfilled; similarly,* (2.21) *occurs when z satisfies* (2.16).

From Proposition 2.4 we deduce that for every $\lambda \in \Lambda$ there exist $x^*_\lambda \geq x_\lambda$ and $k_\lambda \in \mathbf{Z}$ such that

$$k_\lambda \pi \leq \theta(x,\lambda) < (k_\lambda + 1)\pi, \quad \forall\, x \geq x^*_\lambda. \tag{2.22}$$

We are now in position to define the index associated to a nontrivial solution of (2.2):

Definition 2.6 *Assume that $P \in \mathcal{P}$ and let $(z,\lambda) = (u,v,\lambda)$ be a nontrivial solution to* (2.2), (2.5). *We define*

$$i(z,\lambda) = \left[\frac{\theta(x^*_\lambda,\lambda)}{\pi}\right],$$

where x^*_λ is given in (2.22) *and $[\cdot]$ means the integer part.*

Remark 2.7 *For every fixed $\lambda \in \Lambda$, the function $\theta(\cdot,\lambda)$ counts the rotations of the solution vector (u,v) in the phase-plane; in particular, $[\theta(x^*_\lambda,\lambda)/\pi]$ is the number of rotations of the solution in the time interval $[0,x^*_\lambda]$.*
From (2.22) *we deduce that*

$$i(z,\lambda)\pi \leq \theta(x,\lambda) < (i(z,\lambda)+1)\pi, \quad \forall\, x \geq x^*_\lambda.$$

Therefore, $i(z,\lambda)$ takes into account the total number of rotations of the solution z in $[1,+\infty)$.

For the study of the continuity properties of the index, we will use the following result:

Lemma 2.8 *Assume that for some $\lambda \in \Lambda$ problem* (2.2)-(2.5) *has a nontrivial solution z belonging to $H^1(1,+\infty)$. Then*

$$i(z,\lambda) = \left[\frac{\theta(x_\lambda,\lambda)}{\pi}\right].$$

PROOF. Let $(z,\lambda) = (u,v,\lambda) \in H^1(1,+\infty)$ be a nontrivial solution of (2.2) and assume by contradiction that

$$\left[\frac{\theta(x_\lambda,\lambda)}{\pi}\right] \neq \left[\frac{\theta(x_\lambda^*,\lambda)}{\pi}\right]. \tag{2.23}$$

This condition implies that there exists $X_\lambda > x_\lambda$ such that $v(X_\lambda) = 0$; from (2.19) and (2.6) we deduce that $\theta'(X_\lambda) > 0$. Moreover, using the fact that $\theta'(x) < 0$ when $\theta(x) = \pi/2$ (mod. π), we infer that $u(x) \neq 0$, for every $x > X_\lambda$; as a consequence we obtain that

$$0 \leq \theta(x,\lambda) < \frac{\pi}{2}, \quad \forall\, x \geq X_\lambda \quad (\text{mod.}\pi).$$

From this relation we deduce that (2.21) and (2.16) hold true, contradicting the fact that $z \in H^1(1,+\infty)$. ■

Remark 2.9 *From the proof of Lemma 2.8 we also deduce that*

$$i(z,\lambda) = \left[\frac{\theta(x,\lambda)}{\pi}\right], \quad \forall\, x \geq x_\lambda$$

when the assumptions of the same Lemma are satisfied.

Now, let us pass to the study of the nonlinear equation (2.1). We denote by $\mathcal{S}$ the set of continuous functions $S : [1,+\infty) \times \mathbf{R}^2 \longrightarrow M_S^{2,2}$ satisfying the conditions

1. there exist $\alpha \in L^\infty(1,+\infty)$, $\eta_1, \eta_2, \eta_{12} : \mathbf{R}^2 \longrightarrow \mathbf{R}$, continuous and with $\eta_1(0) = \eta_2(0) = \eta_{12}(0) = 0$, and $p \geq 1$ for which

$$\begin{aligned} &|s_i(x,z)| \leq \alpha(x)\eta_i(z), \quad \forall\, x \geq 1, \quad z \in \mathbf{R}^2,\ i = 1,2, \\ &|s_{12}(x,z)| \leq \alpha(x)\eta_{12}(z), \quad \forall\, x \geq 1, \quad z \in \mathbf{R}^2; \end{aligned} \tag{2.24}$$

2. for every compact $K \subset \mathbf{R}^2$ there exists $A_K > 0$ such that

$$||S(x,z) - S(x,z')|| \leq A_K ||z - z'||, \quad \forall\, x \geq 1, \quad z, z' \in K. \tag{2.25}$$

Definition 2.10 *Assume that $P \in \mathcal{P}$ and $S \in \mathcal{S}$ and let (w,μ) be a solution of* (2.1),(2.5).
If $(w,\mu) \neq (0,\mu)$, then the index of (w,μ) as a solution of (2.1), (2.5) *is defined as the index $i(w,\mu)$ of (w,μ) as a solution of*

$$\begin{cases} Jz' + P(x)z = \mu z + S(x,w(x))z, \\ v(1) = 0. \end{cases} \tag{2.26}$$

If $(w,\mu) = (0,\mu)$ and the linear problem

$$\begin{cases} Jz' + P(x)z = \mu z, \\ v(1) = 0 \end{cases} \tag{2.27}$$

has a nontrivial solution z_μ belonging to $H^1(1,+\infty)$, then the index of (w,μ) as a solution of (2.1),(2.5) *is defined as the index $i(z_\mu,\mu)$ of (z_μ,μ) as a solution of* (2.27).

In the following result we prove a continuity property of the above defined index.

Proposition 2.11 *Let (z^*, λ^*) be a nontrivial solution of (2.1), (2.5) and let $z^* \in H^1(1, +\infty)$. Then, there exists $\epsilon > 0$ such that for every nontrivial solution (z^+, λ^+) of (2.1)-(2.5) with $z^+ \in H^1(1, +\infty)$, $||z^* - z^+|| \le \epsilon$ and $|\lambda^* - \lambda^+| \le \epsilon$, we have*

$$i(z^+, \lambda^+) = i(z^*, \lambda^*).$$

Proof. First of all, let us recall that the index of a nontrivial solution (w, μ) is defined by means of the linearized equation

$$Jz' + (P(x) - S(x, w(x))z = \mu z.$$

If $w \in H^1(1, +\infty)$, the matrix $P_w(x) = P(x) - S(x, w(x))$ satisfies assumption (2.3). Moreover, we observe that assumption (2.24) implies that an hypothesis analogous to (2.4) is fulfilled by $P_w - P_0$.

Let now (z^*, λ^*) be a nontrivial solution of (2.1)-(2.5) with $z^* \in H^1(1, +\infty)$ and $\lambda \in \Lambda$. Let

$$\epsilon_0 = \min\left(\frac{\mu^+ - \lambda^*}{4}, \frac{\lambda^* - \mu^-}{4}\right). \tag{2.28}$$

From the continuity of η_1 and η_2 we deduce that there exists $\delta_0' > 0$ such that

$$z \in \mathbf{R}^2, \ ||z|| < \delta_0' \quad \Rightarrow \quad \eta_i(z) < \frac{\epsilon_0}{||\alpha||_\infty}, \ i = 1, 2.$$

Now, from the fact that $z^* \in H^1(1, +\infty)$ we infer that there exist $\delta_0'' > 0$ and $x_0 > 1$ such that

$$z^+ \in H^1(1, +\infty), \ ||z^+ - z^*|| \le \delta_0'' \quad \Rightarrow \quad ||z^+(x)|| \le \delta_0', \ \forall \ x \ge x_0.$$

As a consequence, from (2.24) we obtain that $z^+ \in H^1(1, +\infty)$ and $||z^+ - z^*|| \le \delta_0''$ imply

$$|s_i(x, z^+(x))| \le \alpha(x)\eta_i(z(x)) < \epsilon_0, \quad \forall \ x \ge x_0. \tag{2.29}$$

Now, given $P \in \mathcal{P}$, consider the numbers

$$x_{\lambda^* + (\mu^+ - \lambda^*)/2}, \ x_{\lambda^* - (\lambda^* - \mu^-)/2},$$

defined according to (2.7), and let

$$X = \max\left(x_0, x_{\lambda^* + (\mu^+ - \lambda^*)/2}, x_{\lambda^* - (\lambda^* - \mu^-)/2}\right).$$

Claim 1. Let us consider the Cauchy problem

$$\begin{cases} \theta'(x) = (\lambda - p_1(x)) \cos^2 \theta(x) - 2p_{12}(x) \cos \theta(x) \sin \theta(x) + (\lambda - p_2(x)) \sin^2 \theta(x) \\ \theta(1) = 0 \end{cases} \tag{2.30}$$

and let us study the dependence of the solution θ from the matrix coefficients p_1, p_2, p_{12} and from λ. From an usual continuous dependence argument, we deduce that there exists $\delta \in (0, 1)$ such that

$$|\lambda^* - \lambda^+| < \delta, \quad ||P^*(x) - P^*(x)|| \le \delta, \ \forall \ x \in [1, X] \tag{2.31}$$

implies

$$|\theta^*(x) - \theta^+(x)| < \frac{\pi}{2}, \quad \forall\, x \in [1, X], \tag{2.32}$$

where θ^* and θ^+ denote the solutions of (2.30) with $(P, \lambda) = (P^*, \lambda^*)$ and $(P, \lambda) = (P^+, \lambda^+)$, respectively.

Claim 2. Let (z^*, λ^*) and (z^+, λ^+) be nontrivial solutions of (2.1)-(2.5) with z^*, $z^+ \in H^1(1, +\infty)$, $||z^* - z^+|| \leq \epsilon$ and $|\lambda^+ - \lambda^*| \leq \epsilon < 1$, for some $\epsilon > 0$. From the Sobolev embedding $H^1(1, +\infty) \hookrightarrow L^\infty(1, +\infty)$ we deduce that there exists a constant C (not depending on z^* and z^+) such that

$$||z^*(x) - z^+(x)|| \leq C||z^* - z^+|| \leq C\epsilon, \quad \forall\, x \geq 1. \tag{2.33}$$

Let us denote by D^* the L^∞-norm of z^*; then, (2.33) implies that

$$||z^+(x)|| \leq C + D^*, \quad \forall\, x \geq 1.$$

Now, let $P^*(x) = P(x) - S(x, z^*(x))$ and $P^+(x) = P(x) - S(x, z^+(x))$; from (2.25), applied to the set $K = \{z \in \mathbf{R}^2 : \ ||z|| \leq C + D^*\}$, we infer

$$||P^*(x) - P^+(x)|| \leq ||S(x, z^*(x)) - S(x, z^+(x))|| \leq A_K \ ||z^*(x) - z^+(x)|| \leq A_K \ C\epsilon, \tag{2.34}$$

for every $x \geq 1$.

Now, we can conclude the proof of the result. Given δ as in Claim 1., let us take $\epsilon > 0$ in Claim 2. such that $\epsilon < \min(\delta_0'', \delta)$ and $A_K \ C\epsilon < \delta$; as a consequence, (2.32) holds true and in particular we have

$$|\theta^*(X) - \theta^+(X)| < \frac{\pi}{2}. \tag{2.35}$$

Now, it is easy to see that

$$\begin{aligned} p_1^*(x) &< \lambda^* < p_2^*(x) \\ p_1^+(x) &< \lambda^+ < p_2^+(x), \end{aligned} \tag{2.36}$$

for every $x \geq X$. Indeed, let us prove e.g. the second part of the first inequality in (2.36); from (2.29) and the definitions of X and ϵ_0 we deduce that

$$p_2^*(x) = p_2(x) - s_2(x, z^*(x)) > \lambda^* + \frac{\mu^+ - \lambda^*}{2} - \epsilon_0 > \lambda^* + \frac{\mu^+ - \lambda^*}{4} > \lambda^*.$$

In an analogous way we can show the validity of the other inequalities in (2.36).

Conditions (2.36) imply that the numbers x_{λ^*} and x_{λ^+}, defined according to (2.7) for the matrices P^* and P^+, respectively, satisfy $x_{\lambda^*} \leq X$ and $x_{\lambda^+} \leq X$. Hence, Remark 2.9 guarantees that

$$i(z^*, \lambda^*) = \left[\frac{\theta^*(X)}{\pi}\right], \quad i(z^+, \lambda^+) = \left[\frac{\theta^+(X)}{\pi}\right].$$

This implies that

$$i(z^*, \lambda^*) = i(z^+, \lambda^+),$$

since, from (2.35), we have

$$\left|\frac{\theta^*(X)}{\pi} - \frac{\theta^+(X)}{\pi}\right| < \frac{1}{2}.$$

■

In a very similar way it is possible to prove the following:

Proposition 2.12 *Let $(0,\mu)$ be a nontrivial solution of* (2.1)-(2.5) *and suppose that* (2.27) *has a nontrivial solution belonging to $H^1(1,+\infty)$. Then, there exists $\epsilon > 0$ such that for every nontrivial solution (z^+,λ^+) of* (2.1)-(2.5) *with $z^+ \in H^1(1,+\infty)$, $||z^+|| \le \epsilon$ and $|\lambda^+ - \mu| \le \epsilon$, we have*

$$i(z^+,\lambda^+) = i(0,\mu).$$

3 The eigenvalue problem

In this Section we are dealing with the study of the spectral theory for the linear operator

$$\tau z = Jz' + P(x)z, \;\; x \ge 1, \tag{3.1}$$

where $P \in \mathcal{P}$. Some information on the spectrum of τ follows directly from a standard spectral theory (see e.g. [18]). Indeed the regularity of P in $[1,+\infty)$ guarantees that τ is in the limit circle case in $x = 1$. Moreover, Theorem 6.8 in [18] ensures that τ is in the limit point case at infinity.

Let us consider the operator A_0 defined by

$$D(A_0) = \{z \in L^2(1,+\infty) : \;\; z \in AC(1,+\infty), \;\; \tau z \in L^2(1,+\infty), \;\; v(1) = 0\},$$

$$A_0 z = \tau z, \quad \forall \; z \in D(A_0).$$

From [18, Th. 5.8] we deduce that A_0 is a self-adjoint realization of τ; moreover, using (2.3), it is easy to see that

$$D(A_0) = \{z \in H^1(1,+\infty) : \;\; v(1) = 0\} := D_0.$$

Hence, A_0 is a selfadjoint operator from D_0 to $L^2(1,+\infty)$.

The following result gives some information on the spectral properties of A_0:

Proposition 3.1 *The operator A_0 satisfies the following properties:*

1. $\sigma_{\rm ess}(A_0) = (-\infty,\mu^-] \cup [\mu^+,+\infty)$.

2. $\lambda \in (\mu^-,\mu^+)$ *is an eigenvalue of A_0 if and only if there exists a nontrivial solution $z = (u,v)$ of*

$$\begin{cases} Jz' + P(x)z = \lambda z, \\ v(1) = 0 \end{cases} \tag{3.2}$$

such that $z \in H^1(0,+\infty)$.

For the proof of Proposition 3.1, we observe that Statement 1. directly follows from assumption (2.3) and [18, Th. 16.5-16.6], while Statement 2. is a consequence of the above discussion.

We also remark that, being τ in the limit point case at infinity, every eigenvalue of A_0 is necessarily simple.

The aim of this Section is to study the problem of the existence of eigenvalues of A_0 in Λ; it is possible to give some characterization of the eigenvalues by means of the angular function θ associated to solutions of (3.2) introduced in Section 2. Indeed, we are able to prove the following preliminary results:

Proposition 3.2 *A real number $\lambda \in \Lambda$ is an eigenvalue of A_0 if and only if*

$$\lim_{x\to+\infty} \theta(x,\lambda) = \pi - \arctan\sqrt{\frac{\lambda-\mu^-}{\mu^+-\lambda}} \pmod{\pi}. \tag{3.3}$$

PROOF. 1. Assume that λ is an eigenvalue of A_0; hence there exists a nontrivial solution $z \in H^1(1,+\infty)$ of (3.2). From Proposition 2.3 we deduce that (2.15) holds true; according to Remark 2.5 this implies that (2.20) is valid, i.e. (3.3) is fulfilled.

2. Assume now that (3.3) holds true; an argument similar to the previous one proves that (2.15) is satisfied. Hence

$$\lim_{x\to+\infty} z(x) = 0.$$

Moreover, (2.10) implies that z goes to zero exponentially; this is sufficient to conclude that $z \in H^1(1,+\infty)$, i.e. λ is an eigenvalue of A_0. ■

Now, let λ be an eigenvalue of A_0 and let z_λ be an eigenfunction associated to λ; in what follows and in Section 4 we will use the notation

$$i(\lambda) = i(z_\lambda, \lambda),$$

where $i(z_\lambda,\lambda)$ is defined in Definition 2.6. By means of Proposition 3.2 we are able to study $i(\lambda)$ as a function of λ. Indeed, for every $\lambda, \lambda' \in \Lambda$, let $N(\lambda,\lambda')$ denote (assuming $\lambda < \lambda'$) the number of eigenvalues of A_0 in $(\lambda,\lambda']$; from [18, Th. 16.4] we deduce that

$$n(\lambda,\lambda') - 2 \le N(\lambda,\lambda') \le n(\lambda,\lambda') + 2, \tag{3.4}$$

where

$$n(\lambda,\lambda') = \lim_{x\to+\infty} \frac{\theta(x,\lambda') - \theta(x,\lambda)}{\pi}.$$

From Proposition 3.2 we know that if λ is eigenvalue, then there exists $k_\lambda \in \mathbf{N}$ such that

$$\lim_{x\to+\infty} \theta(x,\lambda) = k_\lambda\pi + \pi - \arctan\sqrt{\frac{\lambda-\mu^-}{\mu^+-\lambda}}.$$

Hence, if λ and λ' are eigenvalues of A_0 (3.4) becomes

$$k_{\lambda'} - k_\lambda + \sigma_{\lambda,\lambda'} - 2 \le N(\lambda,\lambda') \le k_{\lambda'} - k_\lambda + \sigma_{\lambda,\lambda'} + 2, \tag{3.5}$$

for some $\sigma_{\lambda,\lambda'} \in (-1/2, 0)$. By means of (3.5) it is easy to prove the following two results:

Proposition 3.3 *For every $k \in \mathbf{N}$ there exist at most two different eigenvalues λ and λ' of A_0 such that*

$$i(\lambda) = i(\lambda') = k.$$

Proposition 3.4 *Suppose that there exist two eigenvalues $\lambda < \lambda'$ of A_0 such that*

$$i(\lambda) = i(\lambda') = k^*,$$

and denote by $\lambda_1, \lambda_2, \ldots$ the eigenvalues of A_0 greater than λ'. Then, we have

$$i(\lambda_k) = k^* + k, \quad \forall\ k \geq 1.$$

From Proposition 3.3 and Proposition 3.4 we deduce the following:

Proposition 3.5 *Let $\{\lambda_k\}_{k \in K}$ be the set of eigenvalues of A_0 in Λ, for some $K \subset \mathbf{N}$. Then, there exist at most two indeces k_1 and $k_2 \in K$ such that*

$$i(\lambda_{k_1}) = i(\lambda_{k_2})$$

and

$$i(\lambda_j) \neq i(\lambda_m), \quad \forall\ j \neq m,\ j, m \in K \setminus \{k_1, k_2\}.$$

A more useful characterization of the eigenvalues of A_0 can be given by studying in a very careful way the qualitative behaviour of θ. To this aim, let us consider the Cauchy problem

$$\begin{cases} \theta'(x) = (\lambda - p_1(x)) \cos^2 \theta(x) - 2p_{12}(x) \cos \theta(x) \sin \theta(x) + (\lambda - p_2(x)) \sin^2 \theta(x) \\ \\ \theta(x_\lambda) = \alpha, \end{cases} \tag{3.6}$$

where $\alpha \in [k\pi, (k+1)\pi)$, for some $k \in \mathbf{N}$. The differential equation in (3.6) is π-periodic in θ, so the results we are going to prove are independent of k.

Recall that (2.6) implies that for every $x > x_\lambda$ we have

$$\begin{aligned} &\theta'(x) > 0 \quad \text{if } \theta(x) = 0 \ (\text{mod } \pi), \\ &\theta'(x) < 0 \quad \text{if } \theta(x) = \pi/2 \ (\text{mod } \pi). \end{aligned} \tag{3.7}$$

Let us observe that Proposition 2.4 shows that for every λ and for every α the function θ has limit at infinity; moreover, according to Proposition 3.2 we will be interested in those values α such that this limit is $\pi - \arctan \sqrt{(\lambda - \mu^-)/(\mu^+ - \lambda)}$ (mod. π). From (3.7) it is easy to see that this can occur only if $\alpha \in (k\pi + \pi/2, k\pi + \pi)$.

We can now prove the following result:

Lemma 3.6 *For every $\lambda \in \Lambda$ and for every $k \in \mathbf{N}$, there exists a unique $\alpha^*_\lambda \in [k\pi, (k+1)\pi)$ such that the solution $\theta(\cdot, \lambda)$ of (3.6) with $\alpha = \alpha^*_\lambda$ satisfies*

$$\lim_{x\to+\infty} \theta(x,\lambda) = (k+1)\pi - \arctan\sqrt{\frac{\lambda-\mu^-}{\mu^+-\lambda}}.$$

Proof. Let us introduce the sets

$$\begin{aligned}
\Sigma^+ = \quad & \{\alpha \in (k\pi + \pi/2, k\pi + \pi) : \\
& \lim\nolimits_{x\to+\infty} \theta(x,\lambda) = (k+1)\pi + \arctan\sqrt{(\lambda-\mu^-)/(\mu^+-\lambda)}\},\\
\Sigma^0 = \quad & \{\alpha \in ((k\pi + \pi/2, k\pi + \pi) : \\
& \lim\nolimits_{x\to+\infty} \theta(x,\lambda) = (k+1)\pi - \arctan\sqrt{(\lambda-\mu^-)/(\mu^+-\lambda)}\},\\
\Sigma^- = \quad & \{\alpha \in (k\pi + \pi/2, k\pi + \pi) : \\
& \lim\nolimits_{x\to+\infty} \theta(x,\lambda) = k\pi + \arctan\sqrt{(\lambda-\mu^-)/(\mu^+-\lambda)}\}.
\end{aligned}$$

From the uniqueness of solutions to initial value problems associated to the equation in (3.6) we deduce that the sets Σ^+, Σ^0 and Σ^- are intervals and

$$\sigma < \delta < \tau, \quad \forall\, \sigma \in \Sigma^-,\ \forall\, \delta \in \Sigma^0,\ \forall\, \tau \in \Sigma^+.$$

The result is proved if we are able to show that Σ^0 reduces to a single point. To this aim, let $\sigma^+_\lambda = \inf \Sigma^+$ and $\sigma^-_\lambda = \sup \Sigma^-$ and notice that $\sigma^-_\lambda \le \sigma^+_\lambda$.

Claim 1. $\sigma^+_\lambda \notin \Sigma^+$, $\sigma^-_\lambda \notin \Sigma^-$.

Indeed, suppose by contradiction that $\sigma^+_\lambda \in \Sigma^+$ and let θ^+_λ be the solution of (3.6) with $\alpha = \sigma^+_\lambda$; then, there exists $x^+_\lambda > x_\lambda$ such that

$$\begin{aligned}
&\sigma^+_\lambda \le \theta^+_\lambda(x) < k\pi + \pi, \quad \forall\, x_\lambda \le x < x^+_\lambda,\\
&\theta^+_\lambda(x) > k\pi + \pi, \quad \forall\, x > x^+_\lambda.
\end{aligned}$$

Let us denote by θ^*_λ the solution of the Cauchy problem

$$\begin{cases}
\theta'(x) = (\lambda - p_1(x))\cos^2\theta(x) - 2p_{12}(x)\cos\theta(x)\sin\theta(x) + (\lambda - p_2(x))\sin^2\theta(x)\\
\theta(x^*_\lambda) = k\pi + \pi,
\end{cases}$$

for a fixed $x^*_\lambda > x^+_\lambda$; let also be $\delta^*_\lambda = \theta^*_\lambda(x_\lambda)$. From the uniqueness of initial value problems associated to the equation in (3.6) we deduce that $\delta^*_\lambda < \sigma^+_\lambda$; moreover, it is easy to see that

$$\lim_{x\to+\infty} \theta^*_\lambda(x) = k\pi + \pi + \arctan\sqrt{\frac{\lambda-\mu^-}{\mu^+-\lambda}},$$

i.e. $\delta_\lambda^* \in \Sigma^+$. This contradicts the fact that σ_λ^+ is the minimum of Σ^+.

In an analogous way we prove that $\sigma_\lambda^- \notin \Sigma^-$; indeed, suppose that $\sigma_\lambda^- \in \Sigma^-$ and let θ_λ^- be the solution of (3.6) with $\alpha = \sigma_\lambda^-$; there exists $x_\lambda^- > x_\lambda$ such that

$$\theta_\lambda^-(x) > k\pi + \pi + \pi/2, \quad \forall\, x_\lambda \leq x < x_\lambda^-,$$

$$\theta_\lambda^-(x) < k\pi + \pi + \pi/2, \quad \forall\, x > x_\lambda^-.$$

Let us denote by θ_λ^* the solution of the Cauchy problem

$$\begin{cases} \theta'(x) = (\lambda - p_1(x))\cos^2\theta(x) - 2p_{12}(x)\cos\theta(x)\sin\theta(x) + (\lambda - p_2(x))\sin^2\theta(x) \\ \\ \theta(x_\lambda^*) = k\pi + \pi/2, \end{cases}$$

for some $x_\lambda^* > x_\lambda^-$; let also be $\delta_\lambda^* = \theta_\lambda^*(x_\lambda)$. From the uniqueness of initial value problems associated to the equation in (3.6) we deduce that $\delta_\lambda^* > \sigma_\lambda^+$; moreover, it is easy to see that

$$\lim_{x\to+\infty} \theta_\lambda^*(x) = k\pi + \arctan\sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}},$$

i.e. $\delta_\lambda^* \in \Sigma^-$. This contradicts the fact that σ_λ^- is the maximum of Σ^+.

Claim 2. Let

$$f(x,\theta) = (\lambda - p_1(x))\cos^2\theta - 2p_{12}(x)\cos\theta\sin\theta + (\lambda - p_2(x))\sin^2\theta,$$

for every $x \geq x_\lambda$ and $\theta \in (k\pi + \pi/2, k\pi + \pi)$ and define

$$E^+ = \left\{(x,\theta) \in [x_\lambda, +\infty) \times (k\pi + \pi/2, k\pi + \pi) : \ \frac{\partial f}{\partial \theta}(x,\theta) > 0\right\}.$$

Then, a straighforward computation shows that there exist two functions $u_\pm : [x_\lambda, +\infty) \longrightarrow \mathbf{R}$ with $u_\pm(x) < 0$ for every $x \geq x_\lambda$ and such that

$$E^+ = \left\{(x,\theta) \in [x_\lambda, +\infty) \times (k\pi + \pi/2, k\pi + \pi) : \ p_{12}(x) > 0, \ k\pi + \frac{\pi}{2} < \theta < k\pi + \pi + \arctan u_-(x)\right\} \cup$$

$$\cup \{(x,\theta) \in [x_\lambda, +\infty) \times (k\pi + \pi/2, k\pi + \pi) : \ p_{12}(x) < 0, \ k\pi + \pi + \arctan u_+(x) < \theta < k\pi + \pi\} \cup$$

$$\cup \{(x,\theta) \in [x_\lambda, +\infty) \times (k\pi + \pi/2, k\pi + \pi) : \ p_{12}(x) = 0\}$$

and

$$\lim_{x\to+\infty} u_-(x) = 0, \qquad \lim_{x\to+\infty} u_+(x) = -\infty. \tag{3.8}$$

Claim 3. $\sigma_\lambda^+ = \sigma_\lambda^-$.

Suppose that $\sigma_\lambda^+ > \sigma_\lambda^-$. Then, both σ_λ^+ and σ_λ^- do not belong neither to Σ^+ neither to Σ^-; as a consequence, if θ_λ^+ and θ_λ^- are the solutions of (3.6) with $\alpha = \sigma_\lambda^+$ and $\alpha = \sigma_\lambda^-$, respectively, we have

$$\lim_{x\to+\infty} \theta_\lambda^+(x) = \lim_{x\to+\infty} \theta_\lambda^-(x) = k\pi + \pi - \arctan\sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}}. \tag{3.9}$$

Moreover, from the uniqueness of solutions to initial value problems associated to the equation in (3.6) we deduce that

$$\theta_\lambda^-(x) < \theta_\lambda^+(x), \quad \forall\, x \geq x_\lambda. \tag{3.10}$$

Now, from (3.9) and (3.8) we deduce that there exists $x_\lambda^* \geq x_\lambda$ such that

$$(x, \theta_\lambda^\pm(x)) \in E^+, \quad \forall\, x \geq x_\lambda^*,$$

where E^+ as in Claim 2. Hence, we have

$$f(x, \theta_\lambda^-(x)) < f(x, \theta_\lambda^+(x)), \quad \forall\, x \geq x_\lambda^*.$$

By integrating the equation in (3.6) we then obtain

$$\theta_\lambda^+(x) - \theta_\lambda^-(x) = \theta_\lambda^+(x_\lambda^*) - \theta_\lambda^-(x_\lambda^*) + \int_{x_\lambda^*}^{x} [f(t, \theta_\lambda^+(t)) - f(t, \theta_\lambda^-(t))]\, dt \geq \theta_\lambda^+(x_\lambda^*) - \theta_\lambda^-(x_\lambda^*) > 0,$$

for every $x > x_\lambda^*$. Passing to the limit for $x \to +\infty$, we reach a contradiction with (3.9).

From Claim 1 and Claim 3 we immediately deduce that the set Σ^0 reduces to $\sigma_\lambda^+ = \sigma_\lambda^-$ and this proves the result. ■

From Proposition 3.2 and Lemma 3.6 we deduce the validity of the following fundamental result:

Theorem 3.7 *A real number $\lambda \in \Lambda$ is an eigenvalue of A_0 if and only if*

$$\theta(x_\lambda, \lambda) = \alpha_\lambda^* \pmod{\pi}.$$

In this case, we have

$$i(\lambda) = \left[\frac{\theta(x_\lambda, \lambda)}{\pi}\right].$$

In view of Theorem 3.7, the existence of eigenvalues of A_0 is related to the study of the intersection between the graphs of the functions α^* and ϕ, where

$$\phi(\lambda) = \theta(x_\lambda, \lambda), \quad \forall\, \lambda \in \Lambda.$$

In order to study the number of intersections of the graphs of these functions, we will use the intermediate values Theorem; the continuity of ϕ follows directly from the continuity of θ and x_λ. As far as the function α^* is concerned, we can prove the following:

Proposition 3.8 *The function $\alpha^* : \Lambda \longrightarrow \mathbf{R}$ is continuous.*

Proof. 1. We first show that for every $\epsilon > 0$ and for every fixed $\lambda \in \Lambda$ there exists $\delta > 0$ such that

$$|\lambda - \lambda'| < \delta \quad \Longrightarrow \quad \alpha_{\lambda'}^* < \alpha_\lambda^* + \epsilon. \tag{3.11}$$

To this aim, let us fix $\epsilon > 0$ such that

$$\frac{\epsilon}{2} < \arctan \sqrt{\frac{\lambda - \mu^-}{\mu^+ - \lambda}} \tag{3.12}$$

and let $\theta_\epsilon(\cdot,\lambda)$ be the solution of

$$\begin{cases} \theta'(x) = (\lambda - p_1(x))\cos^2\theta(x) - 2p_{12}(x)\cos\theta(x)\sin\theta(x) + (\lambda - p_2(x))\sin^2\theta(x) \\ \theta(x_\lambda) = \alpha^*_\lambda + \epsilon \in \Sigma^+. \end{cases} \tag{3.13}$$

Since

$$\lim_{x\to+\infty} \theta_\epsilon(x,\lambda) = (k+1)\pi + \arctan\sqrt{\frac{\lambda-\mu^-}{\mu^+-\lambda}} > (k+1)\pi + \frac{\epsilon}{2},$$

there exists $x_\epsilon > x_\lambda$ such that

$$\theta_\epsilon(x_\epsilon,\lambda) = (k+1)\pi + \frac{\epsilon}{4}. \tag{3.14}$$

Moreover, from the continuity of the map $\lambda \mapsto x_\lambda$ we infer that there exists $\delta' > 0$ such that

$$|\lambda-\lambda'| < \delta' \quad \Longrightarrow \quad |x_\lambda - x_{\lambda'}| < \frac{x_\epsilon - x_\lambda}{4} \quad \Longrightarrow \quad x_{\lambda'} < x_\epsilon. \tag{3.15}$$

Now, let $\theta_\epsilon(\cdot,\lambda')$ be the solution of

$$\begin{cases} \theta'(x) = (\lambda' - p_1(x))\cos^2\theta(x) - 2p_{12}(x)\cos\theta(x)\sin\theta(x) + (\lambda' - p_2(x))\sin^2\theta(x) \\ \theta(x_\lambda) = \alpha^*_\lambda + \epsilon \in \Sigma^+ \end{cases} \tag{3.16}$$

and observe that there exists a constant $K > 0$ (independent from λ') such that

$$|\theta'_\epsilon(x,\lambda')| \le K, \quad \forall\, x \ge 1. \tag{3.17}$$

From an elementary continuous dependence argument we deduce that there exists $\delta'' < \delta'$ such that

$$|\lambda-\lambda'| < \delta'' \quad \Longrightarrow \quad |\theta_\epsilon(x,\lambda) - \theta_\epsilon(x,\lambda')| < \frac{\epsilon}{8}, \quad \forall\, x \in [x_\lambda, x_\epsilon], \tag{3.18}$$

and hence, by (3.14),

$$\theta_\epsilon(x_\epsilon,\lambda') \ge \theta_\epsilon(x_\epsilon,\lambda) - \frac{\epsilon}{8} = (k+1)\pi + \frac{\epsilon}{8} > (k+1)\pi.$$

As a consequence, using (3.15) and (3.7), we obtain

$$|\lambda-\lambda'| < \delta'' \quad \Longrightarrow \quad \theta_\epsilon(x,\lambda') > (k+1)\pi, \quad \forall\, x \ge x_\epsilon,$$

and then $\theta_\epsilon(x_{\lambda'},\lambda') \in \Sigma^+$. Finally, the continuity of $\lambda \mapsto x_\lambda$ and (3.17) imply that there exists $\delta''' > 0$ such that

$$|\lambda-\lambda'| < \delta''' \quad \Longrightarrow \quad |\theta_\epsilon(x_{\lambda'},\lambda') - \theta_\epsilon(x_\lambda,\lambda')| \le K|x_{\lambda'} - x_\lambda| < \frac{\epsilon}{2}. \tag{3.19}$$

Define $\delta = \min(\delta'',\delta''')$ and let $|\lambda'-\lambda| < \delta$; we then have

$$\alpha^*_{\lambda'} \le \theta_\epsilon(x_{\lambda'},\lambda') < \theta_\epsilon(x_\lambda,\lambda') + \frac{\epsilon}{2} = \alpha^*_\lambda + \frac{3}{2}\epsilon,$$

which proves (3.11).

2. In a analogous way, by considering initial conditions of the form $\theta(x_\lambda) = \alpha^*_\lambda - \epsilon$ in (3.13) and (3.16), it is possible to show that for every $\epsilon > 0$ and for every fixed $\lambda \in \Lambda$ there exists $\delta^* > 0$ such that

$$|\lambda - \lambda'| < \delta^* \quad \Longrightarrow \quad \alpha^*_{\lambda'} > \alpha^*_\lambda - \epsilon \tag{3.20}$$

and this completes the proof of the Proposition. ■

We are now able to prove the auxiliary result:

Lemma 3.9 *Assume that there exist $k \in \mathbf{N}$ and $[\delta_k^-, \delta_k^+] \subset (\mu^-, \mu^+)$ such that*

$$\begin{aligned} & k\pi + \frac{\pi}{2} \le \phi(\lambda) \le (k+1)\pi + \frac{\pi}{2}, \quad \forall\ \lambda \in [\delta_k^-, \delta_k^+], \\ & \phi(\delta_k^-) = k\pi + \frac{\pi}{2} \\ & \phi(\delta_k^+) = (k+1)\pi + \frac{\pi}{2}. \end{aligned} \tag{3.21}$$

Then, the set

$$I_k = \{\lambda \in [\delta_k^-, \delta_k^+] :\ \phi(\lambda) = k\pi + \alpha^*(\lambda)\}$$

is non empty and contains at most two points. Moreover, every $\lambda \in I_k$ is an eigenvalue of A_0 and

$$i(\lambda) = k, \quad \forall\ \lambda \in I_k.$$

Proof. Let $\alpha^*_k(\lambda) = k\pi + \alpha^*(\lambda)$, for every $\lambda \in \Lambda$. From the definitions of Σ^0 and α^*_λ we know that

$$k\pi + \frac{\pi}{2} < \alpha^*_k(\lambda) < k\pi + \pi, \quad \forall\ \lambda \in \Lambda.$$

From this relation and the assumptions on ϕ, by an elementary intermediate values argument, it is immediate to conclude that I_k is non empty. The fact that every element of I_k is an eigenvalue whose index is k follows directly from Theorem 3.7.

Finally, the fact that I_k contains at most two points is a consequence of Proposition 3.3. ■

By the previous Lemma, we deduce that the existence of eigenvalues is related to the intersections of the image of the continuous function ϕ with the intervals $[k\pi + \pi/2, (k+1)\pi + \pi/2)$, with $k \in \mathbf{N}$. The study of such intersections is easy when the behaviour of ϕ at the endpoints of Λ is known; this requires more assumptions on the matrix P. Indeed, let us assume that

$$p_1(x) < \mu^-, \quad \forall\ x \ge 1 \tag{3.22}$$

and

$$\lim_{\lambda \to \mu^+} x_\lambda = +\infty, \tag{3.23}$$

where x_λ is defined in (2.7); observe that (3.23) is trivially satisfied when $p_2 < \mu^+$ is monotone. Finally, we denote by $\mathcal{P}^*$ the set of matrices $P \in \mathcal{P}$ such that (3.22) and (3.23) are satisfied.

Let us first study the asymptotic behaviour of ϕ when $\lambda \to \mu^-$; we point out that assumption (3.22) guarantees that the number x_{μ^-}, according to (2.7), is well defined. Hence, letting

$$\lambda^- = \max\left(\mu^-, \inf_{x\geq 1} p_2(x)\right),$$

we can define

$$k^- = \left[\frac{\phi(\lambda^-)}{\pi}\right].$$

Then we have $\phi(\lambda^-) \geq k^-\pi$; moreover, it is straightforward to prove that $\phi'(\lambda) > 0$ when $\phi(\lambda) = 0$, mod. π, implying that $\phi(\lambda) > k^-\pi$, for every $\lambda \in \Lambda$.

The study of ϕ for $\lambda \to \mu^+$ is more delicate; the behaviour of ϕ strongly depends on the behaviour of $\theta(\cdot, \mu^+)$. In the following results we consider two different situations.

Lemma 3.10 *Assume that*

$$\lim_{x\to+\infty} \theta(x, \mu^+) = +\infty. \tag{3.24}$$

Then, for every $k \in \mathbf{N}$ *there exists* $\mu_k < \mu^+$ *such that*

$$\phi(\lambda) > k\pi, \quad \forall\ \lambda \in (\mu_k, \mu^+).$$

Proof. Let us fix $k \in \mathbf{N}$; from (3.24) we deduce that there exists $X_k > 1$ such that

$$\theta(X_k, \mu^+) > k\pi. \tag{3.25}$$

From (3.23) we infer that there exists $\mu'_k \in \Lambda$ such that

$$x_\lambda > X_k, \quad \forall\ \lambda > \mu'_k.$$

Moreover, for every $\epsilon > 0$ there exists μ''_k such that

$$|\theta(X_k, \lambda) - \theta(X_k, \mu^+)| < \epsilon, \quad \forall\ \lambda \in (\mu''_k, \mu^+). \tag{3.26}$$

By taking $\epsilon > 0$ small enough, from (3.25) and (3.26) we can ensure that

$$\theta(X_k, \lambda) > k\pi, \quad \forall\ \lambda \in (\mu''_k, \mu^+).$$

Recalling that $\theta'(x, \lambda) > 0$ when $\theta(x, \lambda) = 0$, mod. π, we deduce that

$$\phi(\lambda) = \theta(x_\lambda, \lambda) > k\pi, \quad \forall\ \lambda \in (\mu_k, \mu^+),$$

where $\mu_k = \max(\mu'_k, \mu''_k)$. ■

Lemma 3.11 *Assume that*

$$\lim_{x\to+\infty} \theta(x, \mu^+) = \theta^+ \in \mathbf{R}. \tag{3.27}$$

Then, there exists $\Phi > 0$ *such that*

$$\phi(\lambda) < \Phi, \quad \forall\ \lambda \in (\mu^-, \mu^+).$$

Proof. Observe that from (3.27), by a continuity argument, we can deduce that there exists $\Phi \in \mathbf{R}$ such that

$$\theta(x, \mu^+) < \Phi, \quad \forall\, x \geq 1.$$

Now, we recall that for every $x \geq 1$ the function $\theta(x, \cdot)$ is strictly increasing (cfr. [18, Cor. 16.2]); hence, we obtain

$$\theta(x, \lambda) < \theta(x, \mu^+) < \Phi, \quad \forall\, x \geq 1,\ \lambda < \mu^+.$$

This is sufficient to conclude. ■

We are now in position to state our main results on the existence of eigenvalues of A_0; they are a consequence of Lemmas 3.9, 3.10 and 3.11.

Theorem 3.12 *Assume $P \in \mathcal{P}^*$ and that condition* (3.24) *holds true. Then, for every $k > k^-$ there exists at least one eigenvalue $\lambda_k \in \Lambda$ of A_0 such that*

$$i(\lambda_k) = k.$$

Theorem 3.13 *Assume $P \in \mathcal{P}^*$ and that condition* (3.27) *holds true. Then, there exists at most a finite number of eigenvalues of A_0 in Λ.*

Remark 3.14 *Conditions* (3.24) *and* (3.27) *are, in some sense, assumptions on the oscillatory behaviour of the linear equation* (2.2)-(2.5) *when $\lambda = \mu^+$. Hence, Theorem 3.12 and Theorem 3.13 show that the number of eigenvalues of A_0 in a left neighbourhood of μ^+ is related to the number of rotations of solutions of* (2.2)-(2.5) *for $\lambda = \mu^+$; when this number of rotation is infinite, an infinite sequence of eigenvalues accumulating to μ^+ does exist. When the solutions of* (2.2)-(2.5) *for $\lambda = \mu^+$ have a finite number of rotations in the phase-plane, then only finitely many eigenvalues of A_0 fall in a left neighbourhood of μ^+.*

The fact that the number of eigenvalues not belonging to the essential spectrum depends on the oscillatory behaviour of the solutions of the equation for a value of λ corresponding to the bottom of the essential spectrum is well-known in the case of second order differential operator (see e.g. [5, Th. 53-Th. 55]). Our result extends to the case of first order systems this classical theory.

We also point out that an analogous result on the accumulation of eigenvalues to μ^- can be obtained by replacing (3.22) *by $p_2(x) > \mu^+$, for every $x \geq 1$.*

We conclude this section by studying the applicability of Theorem 3.12 and 3.13 to the case of

$$P(x) = \begin{pmatrix} -1 + V(x) & k/x \\ k/x & 1 + V(x) \end{pmatrix}, \tag{3.28}$$

where $k \in \mathbf{N}$ and $V \in C(1, +\infty)$. Dirac operators of this form (which are the classical Dirac operators when $V(x) = c/x$) have been considered e.g. in [12]; in this paper, the authors study the eigenvalue problem on $(0, +\infty)$, which makes the operator τ singular also at $x = 0$.

Let us observe that if V is an increasing negative function, tending to zero at infinity, then the matrix P given in (3.28) belongs to the class $\mathcal{P}^*$. Moreover, the essential spectrum of the operator A_0 associated to the matrix P is $(-\infty,-1]\cup[1,+\infty)$. Now, consider the differential equation (2.19) satisfied by $\theta(\cdot,\lambda)$, which can be written as

$$\theta'(x,\lambda) =< Q_{\lambda,P}(x)[\cos\theta,\sin\theta],[\cos\theta,\sin\theta] >,$$

where $Q_{\lambda,P}(x)$ denotes the quadratic form associated to the matrix $\lambda - P(x)$. It is easy to see that the matrix $P(x)$ has the eigenvalues

$$\mu(x) = V(x) \pm \frac{1}{x}\sqrt{x^2+k^2}.$$

Therefore, we have

$$Q_{1,P}(x)[\cos\theta,\sin\theta],[\cos\theta,\sin\theta] \geq 1 - V(x) - \frac{1}{x}\sqrt{x^2+k^2}.$$

As a consequence, we obtain

$$\theta'(x,1) \geq 1 - V(x) - \frac{1}{x}\sqrt{x^2+k^2},$$

which implies

$$\theta(x,1) \geq \int_1^x \left(1 - V(t) - \frac{1}{t}\sqrt{t^2+k^2}\right) dt, \quad \forall\, x \geq 1.$$

From this relation, it is easy to prove the following result:

Proposition 3.15 *Assume that $V \in C(1,+\infty)$ is a strictly increasing negative potential such that*

$$V(x) \sim \frac{c}{x^\alpha}, \quad x \to +\infty,$$

with $\alpha \in (0,1]$. Then, the selfadjoint extension $A_0 : D_0 \longrightarrow L^2(1,+\infty)$ of the operator τ with P as in (3.28) *has a sequence of eigenvalues in $(-1,1)$ converging to* 1.

4 Global bifurcation for nonlinear problems

In this section we are interested in proving a global bifurcation result for (2.1). To do this, we will use a slight variant of a bifurcation theorem due to Stuart [14] and a continuity-connectivity argument already introduced by the authors in [2].

Let us first state the abstract bifurcation result we will apply; consider a real Hilbert space B and let $A_0 : D(A_0) \longrightarrow B$ be an unbounded self-adjoint operator in B with

$$\sigma_{ess}(A_0) = (-\infty,\mu^-] \cup [\mu^+,+\infty),$$

for some $\mu^- < \mu^+$. Let H denote the real Hilbert space obtained from the domain of A_0 equipped with the graph topology and let us consider the nonlinear problem

$$A_0 u + M(u) = \lambda u, \;\; (u,\lambda) \in H \times \mathbf{R}, \tag{4.1}$$

where $M : H \times \mathbf{R} \longrightarrow B$ is a continuous and compact map such that

$$M(u) = o(||u||), \quad u \to 0. \tag{4.2}$$

Let Σ denote the set of nontrivial solutions of (4.1) in $H \times \mathbf{R}$ and let $\Sigma' = \Sigma \cup \{(0, \lambda) \in H \times \mathbf{R} :$ λ is an eigenvalue of $A_0\}$.

Then, we have the following:

Theorem 4.1 *Under the previous assumptions, let $\mu \in (\mu^-, \mu^+)$ be an eigenvalue of A_0 of odd multiplicity. Moreover, let C_μ denote the component of Σ' containing $(0, \mu)$.*

Then, C_μ has one of the following properties:

(1) C_μ is unbounded in $H \times \mathbf{R}$.

(2) $\sup\{\lambda : (u, \lambda) \in C_\mu\} \geq \mu^+$ or $\inf\{\lambda : (u, \lambda) \in C_\mu\} \leq \mu^-$.

(3) C_μ contains an element $(0, \mu^*) \in \Sigma'$ with $\mu^* \neq \mu$.

Theorem 4.1 is a straighforward variant of Theorem 1.2 in [14], where the author considers the case of a linear operator A_0 satisfying $\sigma_{ess}(A_0) = [Q, +\infty)$, for some Q (as it happens for the one-dimensional Schrödinger operators considered in [14]).

The proof of Theorem 4.1 follows exactly the same lines of the proof of [14, Th. 1.2] and hence it is omitted.

For the applications, we will be interested in excluding alternative (3) in Theorem 4.1, concluding that the bifurcating branch is unbounded or meets the essential spectrum of the linear part of the equation. To do this, we can use, as in [2], a standard continuity and connectivity argument and obtain the following:

Theorem 4.2 *Under the assumptions of Theorem 4.1, suppose that there exists a continuous functional $\phi : \Sigma' \longrightarrow \mathbf{N}$ such that*

$$\phi(0, \mu) \neq \phi(0, \mu'), \quad \forall\ \mu \neq \mu',\ \textit{eigenvalues of } A_0. \tag{4.3}$$

Then, the continuum C_μ given in Theorem 4.1 satisfies alternative (1) or (2). Moreover we have

$$\phi(u, \lambda) = \phi(0, \mu), \quad \forall\ (u, \lambda) \in C_\mu.$$

Now, let us go back to (2.1) and suppose that $P \in \mathcal{P}$ and $S \in \mathcal{S}$; this equation fits into the framework of the abstract bifurcation theorem with A_0 as in Section 2, $H = D_0$, $B = L^2(1, +\infty)$ and M being the Nemitskii operator associated to S, given by

$$M(u)(x) = S(x, u(x))u(x), \quad \forall\ x \geq 1,$$

for every $u \in D_0$. Arguing as in [14], standard computations show the validity of the following:

Proposition 4.3 *Assume that $S \in \mathcal{S}$ and that*

$$\lim_{x \to +\infty} \alpha(x) = 0, \tag{4.4}$$

where α is given in (2.24). *Then $M : D_0 \longrightarrow L^2(1, +\infty)$ is a continuous compact map and satifies* (4.2).

In view of Proposition 4.3, we can apply Theorem 4.1 to obtain the existence of a continuum bifurcating from every eigenvalue of A_0 (recall that all the eigenvalues of A_0 are simple). Moreover, any bifurcating branch satisfies one of the alternatives (1), (2) or (3).

We shall now exhibit the continuous functional suitable for the application of Theorem 4.2. Indeed, for every solution $(\lambda, z) \in \Lambda \times D_0$ of (2.1), we define

$$\phi(\lambda, z) = i(z, \lambda),$$

where i is the rotation index given in Definition 2.10 in Section 2. The continuity of ϕ directly follows from Proposition 2.11 and Proposition 2.12.

As far as condition (4.3) is concerned, we observe that in the well-known case of a second order equation (where ϕ is the number of zeros of the solution) this can be proved by some oscillatory arguments. In our situation, we have shown in Section 3 that it is not possible to guarantee the validity of (4.3) for every eigenvalue of A_0 (cfr. Proposition 3.5); more precisely, we have proved that this is true for the set $\{\lambda_k\}_{k \in K^*}$ of eigenvalues of A_0, where $K^* = K \setminus \{k_1, k_2\}$.

By applying Theorem 4.2 to these eigenvalues we obtain the following result:

Theorem 4.4 *Assume that $P \in \mathcal{P}$, $S \in \mathcal{S}$ and* (4.4) *hold true. Then, for every $k \in K^*$ there exists a continuum C_k of nontrivial solutions of* (2.1)-(2.5) *in $D_0 \times \mathbf{R}$ bifurcating from $(0, \lambda_k)$ and such that*

(1) C_k is unbounded in $D_0 \times \mathbf{R}$, or

(2) $\sup\{\lambda : (u, \lambda) \in C_k\} \geq \mu^+$ *or* $\inf\{\lambda : (u, \lambda) \in C_k\} \leq \mu^-$.

Moreover, we have

$$i(z, \lambda) = i(\lambda_k), \quad \forall\ (z, \lambda) \in C_k.$$

We observe that when K is infinite, as for instance in the case considered in Proposition 3.15, Theorem 4.4 gives the existence of infinitely many branches of solutions to (2.1)-(2.5). Analogously, only a finite number of bifurcating branches can be obtained when K is a finite set.

Remark 4.5 *As for the continuum C_{k_1} bifurcating from $(0, \lambda_{k_1})$, whose existence is guaranteed by Theorem 4.1, we can ensure that one of the following alternatives holds true:*

(1) C_{k_1} is unbounded in $D_0 \times \mathbf{R}$.

(2) $\sup\{\lambda : (u, \lambda) \in C_{k_1}\} \geq \mu^+$ *or* $\inf\{\lambda : (u, \lambda) \in C_{k_1}\} \leq \mu^-$.

(3) C_{k_1} contains $(0, \lambda_{k_2})$.

An analogous remark holds for C_{k_2}.

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