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FREE COOLING AND HIGH-ENERGY TAILS OF GRANULAR GASES WITH VARIABLE RESTITUTION COEFFICIENT*

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Abstract. We prove the so-called generalized Haff's law yielding the optimal algebraic cooling rate of the temperature of a granular gas described by the homogeneous Boltzmann equation for inelastic interactions with nonconstant restitution coefficient. Our analysis is carried through a careful study of the infinite system of moments of the solution to the Boltzmann equation for granular gases and precise L^p estimates in the self-similar variables. In the process, we generalize several results on the Boltzmann collision operator obtained recently for homogeneous granular gases with constant restitution coefficient to a broader class of physical restitution coefficients that depend on the collision impact velocity. This generalization leads to the so-called L^1-exponential tails theorem for this model.

Key words. Boltzmann equation, inelastic hard spheres, granular gas, cooling rate, Haff's law, tail behavior

AMS subject classifications. 76P05, 76P05, 47G10, 82B40, 35Q70, 35Q82

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1. Introduction.

1.1. General setting. Rapid granular flows can be successfully described by the Boltzmann equation conveniently modified to account for the energy dissipation due to the inelasticity of collisions. For such a description, one usually considers the collective dynamics of inelastic hard spheres interacting through binary collisions [12, 27, 30]. The loss of mechanical energy due to collisions is characterized by the so-called normal restitution coefficient which quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Namely, if v and $v_\star$ denote the velocities of two particles before they collide, their respective velocities v' and $v'_\star$ after collisions are such that

$$(u' \cdot \widehat{n}) = -(u \cdot \widehat{n})\, e, \tag{1.1}$$

where the restitution coefficient e is such that $0 \leqslant e \leqslant 1$ and $\widehat{n} \in \mathbb{S}^2$ determines the impact direction; i.e., $\widehat{n}$ stands for the unit vector that points from the v-particle center to the $v_\star$-particle center at the instant of impact. Hereafter

$$u = v - v_\star, \qquad u' = v' - v'_\star,$$

denote, respectively, the relative velocity before and after collision. The major part of the investigation, at the physical as well as the mathematical levels, has been devoted to the particular case of a constant normal restitution. However, as described in the

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monograph [12], it appears that a more relevant description of granular gases should deal with a variable restitution coefficient $e(\cdot)$ depending on the impact velocity; i.e.,

$$e := e(|u \cdot \widehat{n}|).$$

The most common model is the one corresponding to viscoelastic hard spheres for which the restitution coefficient has been derived by Schwager and Pöschel in [27]. For this peculiar model, $e(\cdot)$ admits the following representation as an infinite expansion series:

$$e(|u \cdot \widehat{n}|) = 1 + \sum_{k=1}^{\infty} (-1)^k a_k |u \cdot \widehat{n}|^{k/5}, \qquad u \in \mathbb{R}^3, \quad \widehat{n} \in \mathbb{S}^2, \tag{1.2}$$

where $a_k \geqslant 0$ for any $k \in \mathbb{N}$. We refer the reader to [12, 27] for the physical considerations leading to the above expression (see also Appendix A for several properties of $e(\cdot)$ in the case of viscoelastic hard spheres). This is the principal example we have in mind for most of the results in the paper, though, as we shall see, our approach will cover more general cases including the one of constant restitution coefficient.

In a kinetic framework, behavior of the granular flows is described, in the spatially homogeneous situation we shall consider here, by the so-called velocity distribution $f(v,t)$ which represents the probability density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geqslant 0$. The time-evolution of the one-particle distribution function $f(t,v)$, $v \in \mathbb{R}^3$, $t > 0$, satisfies the following:

$$\partial_t f(t,v) = \mathcal{Q}_e(f,f)(t,v), \qquad f(t=0,v) = f_0(v), \tag{1.3}$$

where $\mathcal{Q}_e(f,f)$ is the inelastic Boltzmann collision operator, expressing the effect of binary collisions of particles. The collision operator $\mathcal{Q}_e$ shares a common structure with the classical Boltzmann operator for elastic collision [16, 29] but is conveniently modified in order to take into account the inelastic character of the collision mechanism. In particular, $\mathcal{Q}_e$ depends in a very strong and explicit way on the restitution coefficient e. Of course, for $e \equiv 1$, one recovers the classical Boltzmann operator. We postpone to section 2.1 the precise expression of $\mathcal{Q}_e$. Due to the dissipation of kinetic energy during collisions, in the absence of external forces, the granular temperature

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2 \, \mathrm{d}v$$

is continuously decreasing and is expected to go to zero as time goes to infinity, expressing the *cooling of the granular gases.*

Determining the precise rate of decay to zero for the granular temperature is the main goal of the present work. The asymptotic behavior for the granular temperature was first explained in [18] by Haff at the beginning of the '80s for the case of constant restitution coefficient; thus, it has become standard to refer to this behavior simply as *Haff's law.*

The mathematical study of Boltzmann models for granular flows was first restricted to the so-called inelastic Maxwell molecules where the collision rate is independent of the relative velocity [5, 6, 9, 10, 13, 15]. Later, the mathematical investigation of hard-sphere interactions was initiated in [17] for diffusively heated gases and continued in a series of papers [22, 23] where the first rigorous proof of Haff's law was presented in the case of constant restitution coefficient. Additional relevant work in the existence and stability of the homogeneous cooling state can be found in [24, 25]. We refer to [30] for a mathematical overview of the relevant questions addressed by the kinetic theory of granular gases and a complete bibliography on the topic.

From the mathematical viewpoint the literature on granular gases with *variable restitution coefficient* is rather limited. However, the Cauchy problem for the homogeneous inelastic Boltzmann equation has been studied in great detail and full generality in [22], including the class of restitution coefficients that we are dealing with in this paper. For the inhomogeneous inelastic Boltzmann equation the literature is more scarce; in this respect we mention the work by one of the authors [1] that treats the Cauchy problem in the case of near-vacuum data. It is worthwhile mentioning that the scarcity of results regarding existence of solutions for the inhomogeneous case is explained by the lack of entropy estimates for the inelastic Boltzmann equation; thus, well-known theories like the DiPerna–Lions *renormalized solutions* are no longer available. More complex behaviors that involve boundaries, for instance, clusters and Maxwell demons, are well beyond of the present techniques. Notice, however, that staying somehow at a formal level and resorting to hydrodynamic closures at the Euler or Navier–Stokes accuracy, it is possible to recover an algebraic decay rate of the granular temperature for variable restitution coefficient in some peculiar quasi-elastic regime in spatially inhomogeneous situations [28, 7] (we also refer to [14] for a macroscopic description of granular gases with constant resitution coefficient and a justification of Haff's law in this case).

1.2. Main results and methodology. Physical considerations and careful dimensional analysis led Haff [18] to predict that, for *constant restitution coefficient*, the temperature $\mathcal{E}(t)$ of a granular gas should cool down at a quadratic rate as follows:

$$\mathcal{E}(t) = O\left(t^{-2}\right) \text{ as } t \to \infty.$$

Similar considerations led Schwager and Pöschel [27] to conclude that, for the restitution coefficient associated with the viscoelastic hard spheres (1.2), the decay should be slower than the one predicted by Haff, namely, at an algebraic rate proportional to $t^{-5/3}$. These considerations are precisely described in the main result of this paper, where the key intuitive fact is that the decay rate of $\mathcal{E}(t)$ is completely determined by the behavior of the restitution coefficient $e(|u\cdot\widehat{n}|)$ for small impact velocity (see Assumption 3.1(1)). Precisely, our result is valid for restitution coefficient such that there exist some constants $\alpha > 0$ and $\gamma \geqslant 0$ such that

$$e(|u\cdot\widehat{n}|) \simeq 1 - \alpha|u\cdot\widehat{n}|^{\gamma} \quad \text{for} \quad |u\cdot\widehat{n}| \simeq 0,$$

and it reads as follows.

THEOREM 1.1. *For any initial distribution velocity $f_0 \geqslant 0$ satisfying the conditions given by* (2.8) *with $f_0 \in L^{p_0}(\mathbb{R}^3)$ for some $1 < p_0 < \infty$, the solution $f(t,v)$ to the associated Boltzmann equation* (2.7) *satisfies the generalized Haff's law for variable restitution coefficient $e(\cdot)$ fulfilling Assumptions* 3.1 *and* 4.10*,*

$$c(1+t)^{-\frac{2}{1+\gamma}} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-\frac{2}{1+\gamma}}, \qquad t \geqslant 0, \tag{1.4}$$

where $\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2\,\mathrm{d}v$ and c and C are positive constants.

We recover with Theorem 1.1 the optimal decay for constant restitution coefficient ($\gamma = 0$) given in [24] and the one predicted for viscoelastic hard spheres ($\gamma = 1/5$) in [27]. The method of the proof has similarities to that of the constant restitution coefficient [24] but is technically more challenging.

The main tools to prove Theorem 1.1 are the following:

- The study of the moments of solutions to the Boltzmann equation uses a generalization of Povzner's lemma developed in [8, 11].
- Precise L^p estimates, in the same spirit of [24], of the solution to the Boltzmann equation for $p > 1$ are used.
- For the previous item, the analysis is understood in the easiest way using *rescaled solutions* to (1.3) of the form

  $$f(t,v) = V(t)^3 g(\tau(t), V(t)v),$$

  where $\tau(\cdot)$ and $V(\cdot)$ are fixed time-scaling functions to be crafted depending upon the restitution coefficient. In the *self-similar variables* (τ, w) the function $g(\tau, w)$ is a solution of an evolution problem of the type

  $$\partial_\tau g(\tau,w) + \xi(\tau)\nabla_w \cdot (wg(\tau,w)) = \mathcal{Q}_{\widetilde{e}(\tau)}(g,g) \tag{1.5}$$

  for some $\xi(\tau)$ depending on the time scale τ. The collision operator $\mathcal{Q}_{\widetilde{e}(\tau)}(g,g)$ is associated with a time-dependent restitution coefficient $\widetilde{e}(\tau)$ (see section 2.3 for details). In this respect we note that one notable difference with respect to the case of a constant restitution coefficient treated in [23] is that the rescaled collision operator depends on the (rescaled) time τ, leading to a *nonautonomous problem* for g. This is the main reason why the construction of self-similar profile g independent of τ obtained in [23] (homogeneous cooling state) is not valid for nonconstant restitution coefficient.

Let us explain in more details our method of proof.

1. We start proving in sections 2 and 3 an upper bound for the decay of the energy. This shows that, for restitution coefficients satisfying 3.1, the cooling of the temperature is *at least algebraic*. More precisely, under suitable assumptions on the restitution coefficient $e(\cdot)$, we exhibit a convex and increasing mapping $\mathbf{\Psi}_e$ such that

   $$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -\mathbf{\Psi}_e(\mathcal{E}(t)) \qquad \forall t \geqslant 0,$$

   which leads to an upper bound for $\mathcal{E}(t)$ of the type

   $$\mathcal{E}(t) \leqslant C(1+t)^{-\frac{2}{1+\gamma}} \qquad \forall t \geqslant 0$$

   for some positive constant $C > 0$.
2. The lower bound for the free cooling is much more difficult to establish and consists in proving that the cooling rate found above is optimal; i.e., there exists $c > 0$ such that

   $$\mathcal{E}(t) \geqslant c(1+t)^{-\frac{2}{1+\gamma}} \quad \forall t \geqslant 0. \tag{1.6}$$

   A careful study of the moments of the solution to (1.3) shows that it suffices to prove a similar algebraic lower bound with some arbitrary rate; i.e., (1.6) will hold if there exists $\lambda > 0$ and $c > 0$ such that

   $$\mathcal{E}(t) \geqslant c(1+t)^{-\lambda} \qquad \forall t \geqslant 0.$$

   These two points are proved in the last part of section 3.

3. To prove that the lower bound with some unprescribed rate λ holds, we use, as in [23], precise L^p estimates ($p > 1$) for solutions to (1.3) in self-similar variables. We craft correct time-scaling functions $\tau(t)$ and $V(t)$ such that (1.6) is equivalent to $\mathbf{\Theta}(\tau(t)) \geqslant c$ (here $\mathbf{\Theta}(\cdot)$ denotes the second moment of g). Once this scale is fixed, the function $g(\tau, w)$ satisfies the rescaled Boltzmann equation (1.5) with $\xi(\tau) \to 0$ as $\tau \to \infty$. This is a major difference from the constant restitution coefficient case where $\xi(\tau) \equiv 1$. This technical difficulty is overcome by proving that the L^p-norm of $g(\tau)$ behaves at most polynomially with respect to τ. The details can be found in section 5.

The derivation of precise L^p estimates for the solution $g(\tau, w)$ to (1.5) requires a careful study of the collision operator $\mathcal{Q}_e$ and its regularity properties. We present in section 4 a full discussion of the regularity and integrability properties of the gain part of the collision operator $\mathcal{Q}^+_{B,e}$ associated to a general collision kernel $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfying Grad's cut-off assumption (see section 2 for the definition). This section is divided in four subsections starting with the Carleman representation of the gain operator $\mathcal{Q}^+_{B,e}$. It is well known [19, 20, 21, 26, 31, 23] that such a representation is essential for the study of regularizing properties of the gain operator $\mathcal{Q}^+_{B,e}$ when smooth assumptions are imposed on the kernel $B(u,\sigma)$. Our contribution in sections 4.3 and 4.4 is to extend the existing theory to the inelastic case with variable restitution coefficient. Since the estimates of section 4 will be applied for solutions written in self-similar variables, we make sure that such estimates are *independent* of the restitution coefficient. This allows us to overcome the technical problem of the time dependence of the gain operator in the self-similar variables. Additional convolution-like inequalities [3, 26] are derived in subsection 4.2 assuming minimal regularity of the angular kernel $b(\cdot)$.

The final part of this work is devoted to the proof of propagation of exponential L^1-tails where the full power of Povzner's lemma is exploited. Much of the argument, with a minor adaptation, is taken from [11]. This important result is presented in the final section for convenience and not because the machinery of sections 4 and 5 is needed to prove it.

THEOREM 1.2 (L^1-exponential tails theorem). *Let $B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma)$ be the collision kernel with $b(\cdot)$ satisfying* (2.6) *and $b(\cdot) \in L^q(\mathbb{S}^2)$ for some $q \geqslant 1$. Assume that the variable restitution coefficient $e(\cdot)$ satisfies Assumption* 3.1. *Furthermore, assume that f_0 satisfies* (2.8), *and that there exists $r_0 > 0$ such that*

$$\int_{\mathbb{R}^3} f_0(v) \exp\left(r_0|v|\right) \, \mathrm{d}v < \infty.$$

Then there exists some $r \leqslant r_0$ such that

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} f(t,v) \exp\left(rV(t)|v|\right) \, \mathrm{d}w < \infty. \tag{1.7}$$

The function $V(t)$ is the appropriate scaling, depending solely on the restitution coefficient, given in (3.17).

1.3. Notations. Let us introduce the notations we shall use in the following. Throughout the paper we shall use the notation $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. We denote, for any $\eta \in \mathbb{R}$, the Banach space

$$L^1_\eta = \left\{ f : \mathbb{R}^3 \to \mathbb{R} \text{ measurable}\,; \;\; \|f\|_{L^1_\eta} := \int_{\mathbb{R}^3} |f(v)| \, \langle v\rangle^\eta \, \mathrm{d}v < +\infty \right\}.$$

More generally, we define the weighted Lebesgue space $L^p_\eta(\mathbb{R}^3)$ ($p \in [1,+\infty)$, $\eta \in \mathbb{R}$) by the norm

$$\|f\|_{L^p_\eta(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3} |f(v)|^p \, \langle v\rangle^{p\eta} \, \mathrm{d}v\right]^{1/p}, \qquad 1 \leqslant p < \infty,$$

while $\|f\|_{L^\infty_\eta(\mathbb{R}^3)} = \sup_{v\in\mathbb{R}^3} (|f(v)|\langle v\rangle^\eta)$ for $p = \infty$.

For any $k \in \mathbb{N}$, we denote by $H^k = H^k(\mathbb{R}^3)$ the usual Sobolev space defined by the norm

$$\|f\|_{H^k} = \left[\sum_{|j|\leqslant k} \|\partial^j_v f\|^p_{L^2}\right]^{1/p},$$

where ∂^j_v denotes the partial derivative associated with the multi-index $j \in \mathbb{N}^N$. Moreover, this definition can be extended to H^s for any $s \geqslant 0$ by using the Fourier transform $\mathcal{F}$. The binomial coefficients for noninteger $p \geqslant 0$ and $k \in \mathbb{N}$ are defined as

$$\begin{pmatrix} p \\ k \end{pmatrix} = \frac{p(p-1)\dots(p-k+1)}{k!}, \qquad k \geqslant 1, \quad \begin{pmatrix} p \\ 0 \end{pmatrix} = 1.$$

2. Preliminaries.

2.1. The kinetic model. We assume the granular particles to be perfectly smooth hard spheres of mass $m = 1$ performing inelastic collisions. Recall that, as explained in the introduction, the inelasticity of the collision mechanism is characterized by a single parameter, namely, the coefficient of normal restitution $0 \leqslant e \leqslant 1$ which we assume to be *nonconstant*. More precisely, let $(v, v_\star)$ denote the velocities of two particles before they collide. Their respective velocities after collisions v' and $v'_\star$ are given, by virtue of (1.1) and the conservation of momentum, by

$$v' = v - \frac{1+e}{2}\,(u\cdot\widehat{n})\widehat{n}, \qquad v'_\star = v_\star + \frac{1+e}{2}\,(u\cdot\widehat{n})\widehat{n}, \tag{2.1}$$

where the symbol u stands for the relative velocity $u = v - v_\star$ and $\widehat{n}$ is the impact direction. From the physical viewpoint, a common approximation consists in choosing e as a suitable function of the impact velocity; i.e., $e := e(|u\cdot\widehat{n}|)$. The main assumptions on the function $e(\cdot)$ are listed in the following (see [1]).

Assumption 2.1. Assume the following hold.

(1) The mapping $r \in \mathbb{R}_+ \mapsto e(r) \in (0,1]$ is absolutely continuous.

(2) The mapping $r \in \mathbb{R}_+ \to \vartheta(r) := r\,e(r)$ is strictly increasing.

Further assumptions on the function $e(\cdot)$ shall be needed later on. Given assumption (2), the Jacobian of the transformation (2.1) can be computed as

$$J := \left|\frac{\partial(v', v'_\star)}{\partial(v, v_\star)}\right| = e(|u\cdot\widehat{n}|) + |u\cdot\widehat{n}|\frac{\mathrm{d}e}{\mathrm{d}r}(|u\cdot\widehat{n}|) = \frac{\mathrm{d}\vartheta}{\mathrm{d}r}(|u\cdot\widehat{n}|) > 0.$$

In practical situations, the restitution coefficient $e(\cdot)$ is usually chosen among the following three examples.

Example 2.2 (constant restitution coefficient). The most documented example in the literature is the one in which

$$e(r) = e_0 \in (0,1] \qquad \text{for any } r \geqslant 0.$$

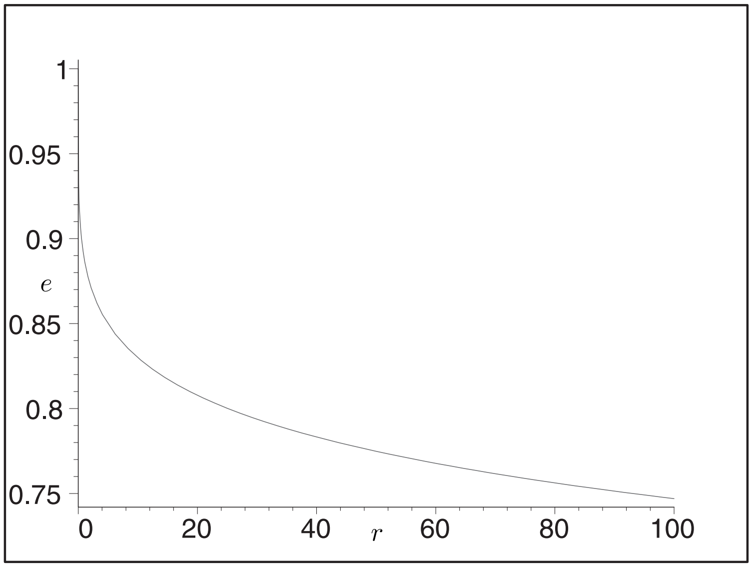

FIG. 1. *Restitution coefficient for viscoelastic hard spheres given by* (2.3) *with* $a = 0.12$

Example 2.3 (monotone decreasing). A second example of interest is the one in which the restitution coefficient $e(\cdot)$ is a monotone decreasing function:

$$e(r) = \frac{1}{1 + ar^{\eta}} \qquad \forall r \geqslant 0, \tag{2.2}$$

where $a > 0$ and $\eta > 0$ are two given constants.

Example 2.4 (viscoelastic hard spheres). This is the most physically relevant model treated in this work. For such a model, the properties of the restitution coefficient have been derived in [12, 27] where representation (1.2) is given. It also accepts the implicit representation

$$e(r) + ar^{1/5} e(r)^{3/5} = 1, \tag{2.3}$$

where $a > 0$ is a suitable positive constant depending on the material viscosity (see Figure 1).

In the following, it shall be more convenient to use the following equivalent parametrization of the postcollisional velocities. For distinct velocities v and $v_\star$, let $\widehat{u} = \frac{u}{|u|}$ be the relative velocity unit vector. The change of variables,

$$\sigma = \widehat{u} - 2\,(\widehat{u} \cdot \widehat{n})\widehat{n} \in \mathbb{S}^2,$$

provides an alternative parametrization of the unit sphere $\mathbb{S}^2$ for which the impact velocity reads

$$|u \cdot \widehat{n}| = |u|\,|\widehat{u} \cdot \widehat{n}| = |u|\sqrt{\frac{1 - \widehat{u} \cdot \sigma}{2}},$$

Then the postcollisional velocities $(v', v'_\star)$ given in (2.1) are transformed to

$$v' = v - \beta\frac{u - |u|\sigma}{2}, \qquad v'_\star = v_\star + \beta\frac{u - |u|\sigma}{2}, \tag{2.4}$$

where

$$\beta = \beta\left(|u|\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\right) = \frac{1+e}{2} \in \left(\tfrac{1}{2}, 1\right].$$

In this representation, the *weak formulation* of the Boltzmann collision operator $\mathcal{Q}_{B,e}$ given a collision kernel $B(u,\sigma)$ reads

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_\star)\mathcal{A}_{B,e}[\psi](v,v_\star)\;\mathrm{d}v_\star\,\mathrm{d}v \tag{2.5}$$

for any suitable test function $\psi = \psi(v)$. Here

$$\mathcal{A}_{B,e}[\psi](v,v_\star) = \int_{\mathbb{S}^2}\Big(\psi(v') + \psi(v'_\star) - \psi(v) - \psi(v_\star)\Big)B(u,\sigma)\,\mathrm{d}\sigma,$$

with $v', v'_\star$ defined in (2.4). We assume that the collision kernel $B(u,\sigma)$ takes the form

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma),$$

where $\Phi(\cdot)$ is a suitable nonnegative function known as *potential*, while the *angular kernel* $b(\cdot)$ is usually assumed to belong to $L^1(-1,1)$. For any fixed vector $\widehat{u}$, the angular kernel defines a measure on the sphere through the mapping $\sigma \in \mathbb{S}^2 \mapsto b(\widehat{u}\cdot\sigma) \in [0,\infty]$ that we assume to satisfy the renormalized *Grad's cut-off* hypothesis:

$$\|b\|_{L^1(\mathbb{S}^2)} = 2\pi\,\|b\|_{L^1(-1,1)} = 1. \tag{2.6}$$

The most relevant model in our case is hard spheres that correspond to $\Phi(|u|) = |u|$ and $b(\widehat{u}\cdot\sigma) = \frac{1}{4\pi}$. We shall also consider the *generalized hard-sphere collision kernel* for which $\Phi(|u|) = |u|$ and the angular kernel is satisfying (2.6) without being necessarily constant. For the particular model of hard-sphere interactions, we simply denote the collision operator $\mathcal{Q}_{B,e}$ by $\mathcal{Q}_e$.

2.2. On the Cauchy problem. We consider the following homogeneous Boltzmann equation:

$$\begin{cases} \partial_t f(t,v) &= \mathcal{Q}_{B,e}(f,f)(t,v), \qquad t>0,\ v\in\mathbb{R}^3, \\ f(0,v) &= f_0(v), \qquad v\in\mathbb{R}^3, \end{cases} \tag{2.7}$$

where the initial datum f_0 is a *nonnegative* velocity function such that

$$\int_{\mathbb{R}^3} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}^3} f_0(v)v\,\mathrm{d}v = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} f_0(v)|v|^3\,\mathrm{d}v < \infty. \tag{2.8}$$

There is no loss of generality in assuming the two conditions in (2.8) due to scaling and translational arguments. We say that a nonnegative $f = f(t,v)$ is a solution to (2.7) if $f \in \mathcal{C}([0,\infty), L^1_2(\mathbb{R}^3))$ and

$$\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3}\Big(f(t,v)\partial_t\psi(t,v) + \mathcal{Q}_{B,e}(f,f)(t,v)\psi(t,v)\Big)\,\mathrm{d}v = \int_{\mathbb{R}^3} f_0(v)\psi(0,v)\,\mathrm{d}v$$

holds for any compactly supported $\psi \in \mathcal{C}^1([0,\infty)\times\mathbb{R}^3)$. Under Assumption 2.1, Assumptions H1 and H2 of [22] are fulfilled. (With the terminology of [22], we are dealing with a noncoupled collision rate and, more precisely, with the so-called generalized viscoelastic model; see [22, p. 661].) In particular, [22, Theorem 1.2] applies directly and allows us to state the following.

THEOREM 2.5 (Mischler, Mouhot, and Rodriguez Ricard [22]). *For any nonnegative velocity function f_0 satisfying* (2.8)*, there is a unique solution $f = f(t, v)$ to* (2.7)*. Moreover,*

$$\int_{\mathbb{R}^3} f(t,v)\,\mathrm{d}v = 1, \qquad \int_{\mathbb{R}^3} f(t,v)v\,\mathrm{d}v = 0 \qquad \forall t \geqslant 0. \tag{2.9}$$

2.3. Self-similar variables. Let us discuss precisely the rescaling using self-similar variables. Let $f(t, v)$ be the solution to (2.7) associated to some initial datum f_0 satisfying (2.8) and collision kernel

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma),$$

with $b(\cdot)$ satisfying (2.6). The rescaled solution $g = g(\tau, w)$ is defined such that

$$f(t,v) = V(t)^3 g(\tau(t), V(t)v), \tag{2.10}$$

where $\tau(\cdot)$ and $V(\cdot)$ are time-scaling functions to be determined solely on the behavior of the restitution coefficient in the low-impact velocity region. Since these are scaling functions they are increasing and satisfy $\tau(0) = 0$ and $V(0) = 1$. One has

$$1 = \int_{\mathbb{R}^3} f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^3} g(\tau(t),w)\,\mathrm{d}w \qquad \forall t \geqslant 0,$$

and $g(0, w) = f_0(w)$. Furthermore, some elementary calculations show that the function $g(\tau, w)$ satisfies

$$V(t)^{-2}\mathcal{Q}_e(f,f)(t,v) = \dot{\tau}(t)V(t)\partial_\tau g(\tau,w) + \dot{V}(t)\nabla_w\cdot(wg(\tau,w))\Big|_{\substack{w=V(t)v\\ \tau=\tau(t)}}, \tag{2.11}$$

where the overdot symbol denotes the derivative with respect to t. Moreover, the expression of the collision operator in the self-similar variables is

$$V(t)^{-2}\mathcal{Q}_{B,e}(f,f)\left(t,\frac{v}{V(t)}\right) = \mathcal{Q}_{B_\tau,\widetilde{e}_\tau}(g,g)(\tau(t),v),$$

where the rescaled collision kernel B_τ is given by

$$B_{\tau(t)}(u,\sigma) := V(t)\Phi\left(\frac{|u|}{V(t)}\right)b(\widehat{u}\cdot\sigma).$$

The rescaled restitution coefficient $\widetilde{e}_\tau$ has been defined by

$$\widetilde{e}_\tau\ :\ (r,t)\longmapsto \widetilde{e}_{\tau(t)}(r) := e\left(\frac{r}{V(t)}\right) \quad \text{for} \quad r\geqslant 0, \quad t\geqslant 0.$$

Since the mapping $t \in \mathbb{R}^+ \longmapsto \tau(t) \in \mathbb{R}^+$ is injective with inverse ζ, one can rewrite (2.11) in terms of τ only. Thus, $g(\tau, w)$ is a solution to the following rescaled Boltzmann equation:

$$\lambda(\tau)\partial_\tau g(\tau,w) + \xi(\tau)\nabla_w\cdot(wg(\tau,w)) = \mathcal{Q}_{B_\tau,\widetilde{e}_\tau}(g,g)(\tau,w) \qquad \forall \tau > 0, \tag{2.12}$$

with

$$\lambda(\cdot) = \dot{\tau}(\zeta(\cdot))V(\zeta(\cdot)) \quad \text{and} \quad \xi(\cdot) = \dot{V}(\zeta(\cdot)),$$

and model parameters

$$B_\tau(u,\sigma) = V(\zeta(\tau))\Phi\left(\frac{|u|}{V(\zeta(\tau))}\right)b(\widehat{u}\cdot\sigma) \quad \text{and} \quad \widetilde{e}_\tau(r) = e\left(\frac{r}{V(\zeta(\tau))}\right). \tag{2.13}$$

Notice that, for generalized hard-sphere interactions (i.e., whenever $\Phi(|u|) = |u|$), one has $B_\tau = B$. For true hard-sphere interaction (i.e., $b(\cdot) = \frac{1}{4\pi}$), one simply denotes the rescaled collision operator by $\mathcal{Q}_{\widetilde{e}_\tau}$. In addition, observe that the rescaled operator depends on time, and therefore, g is a solution to a nonautonomous problem.

2.4. Povzner-type inequalities. We extend in this section the results of [11] and [24] to the case of variable restitution coefficient satisfying 2.1. We consider a collision kernel of the form

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma),$$

with angular kernel $b(\cdot)$ satisfying the renormalized Grad's cut-off assumption (2.6). Let f be a *nonnegative* function satisfying (2.9) and $\psi(v) = \Psi(|v|^2)$ be a given test function with Ψ nondecreasing and convex. Then (2.5) leads to

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)\mathcal{A}_{B,e}[\psi](v,v_\star)\,\,\mathrm{d}v_\star\,\mathrm{d}v,$$

with

$$\mathcal{A}_{B,e}[\psi](v,v_\star) = \Phi(|u|)\left(A^+_{B,e}[\Psi](v,v_\star) - A^-_{B,e}[\Psi](v,v_\star)\right),$$

where

$$A^+_{B,e}[\Psi](v,v_\star) = \int_{\mathbb{S}^2}\left(\Psi(|v'|^2) + \Psi(|v'_\star|^2)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

Using (2.6) we also have

$$A^-_{B,e}[\Psi](v,v_\star) = \int_{\mathbb{S}^2}\left(\psi(v) + \psi(v_\star)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma = \left(\Psi(|v|^2) + \Psi(|v_\star|^2)\right).$$

Following [11], we define the velocity of the center of mass $U = \frac{v+v_\star}{2}$ so that

$$v' = U + \frac{|u|}{2}\omega, \qquad v'_\star = U - \frac{|u|}{2}\omega, \qquad \text{with} \quad \omega = (1-\beta)\widehat{u} + \beta\sigma.$$

Recall that for any vector $x \in \mathbb{R}^3$, we set $\widehat{x} = \frac{x}{|x|}$. When e, or equivalently β, is constant, the strategy of [11] consists, roughly speaking, of performing a suitable change of unknown $\sigma \to \widehat{\omega}$ to carefully estimate $A^+_{B,e}[\psi]$. For variable β, such strategy does not apply directly. Instead, observe that $|\omega| \leqslant 1$ and, since Ψ is increasing, one has

$$\begin{aligned}\Psi(|v'|^2) + \Psi(|v'_\star|^2) &\leqslant \Psi\left(|U|^2 + \frac{|u|^2}{4} + |u||U|\widehat{U}\cdot\omega\right) + \Psi\left(|U|^2 + \frac{|u|^2}{4} - |u||U|\widehat{U}\cdot\omega\right)\\ &= \Psi\left(E\frac{1+\xi\,\widehat{U}\cdot\omega}{2}\right) + \Psi\left(E\frac{1-\xi\,\widehat{U}\cdot\omega}{2}\right),\end{aligned}$$

where we have set $E := |v|^2 + |v_\star|^2 = 2|U|^2 + \frac{|u|^2}{2}$ and $\xi = 2\frac{|U|\,|u|}{E}$. Since $\Psi(\cdot)$ is convex the mapping

$$\Psi_0(t) = \Psi(x+ty) + \Psi(x-ty)$$

is even and nondecreasing for $t \geqslant 0$ and $x, y \in \mathbb{R}$ (see [11]). Therefore, using that $\xi \leqslant 1$ one gets

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{1+\widehat{U}\cdot\omega}{2}\right) + \Psi\left(E\frac{1-\widehat{U}\cdot\omega}{2}\right). \tag{2.14}$$

In the case that $\widehat{U}\cdot\sigma \geqslant 0$ it follows that

$$\left|\widehat{U}\cdot\omega\right| = \left|(1-\beta)\widehat{U}\cdot\widehat{u} + \beta\widehat{U}\cdot\sigma\right| \leqslant (1-\beta) + \beta\widehat{U}\cdot\sigma;$$

thus, using the fact that $\Psi_0(t)$ is even and nondecreasing for $t \geqslant 0$, we conclude from (2.14) that

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta-\beta\widehat{U}\cdot\sigma}{2}\right).$$

When $\widehat{U}\cdot\sigma \leqslant 0$, a similar argument shows that

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{2-\beta-\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta+\beta\widehat{U}\cdot\sigma}{2}\right).$$

Hence, setting $\tilde{b}(s) = b(s) + b(-s)$ and using these last two estimates with the change of variables $\sigma \to -\sigma$, we get

$$\begin{aligned} A^+_{B,e}[\Psi](v,v_\star) &\leqslant \int_{\{\widehat{U}\cdot\sigma\geqslant 0\}} \left[\Psi\left(E\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta-\beta\widehat{U}\cdot\sigma}{2}\right)\right]\tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma \\ &\leqslant \int_{\{\widehat{U}\cdot\sigma\geqslant 0\}} \left[\Psi\left(E\frac{3+\widehat{U}\cdot\sigma}{4}\right) + \Psi\left(E\frac{1-\widehat{U}\cdot\sigma}{4}\right)\right]\tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{2.15}$$

where the second inequality can be shown by writing

$$\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2} = \frac{1}{2} + \left(\frac{1}{2} - \frac{\beta}{2}\left(1-\widehat{U}\cdot\sigma\right)\right) \quad \text{and}$$
$$\frac{\beta-\beta\widehat{U}\cdot\sigma}{2} = \frac{1}{2} - \left(\frac{1}{2} - \frac{\beta}{2}\left(1-\widehat{U}\cdot\sigma\right)\right).$$

The term in parentheses is maximized when $\beta = 1/2$; thus, the monotonicity of Ψ_0 implies the result.

Next, we particularize the previous estimates to the important case $\Psi(x) = x^p$. This choice will lead to the study of the moments of solutions.

LEMMA 2.6. *Let $q \geqslant 1$ be such that $b \in L^q(\mathbb{S}^2)$. Then, for any restitution coefficient $e(\cdot)$ satisfying Assumption* 2.1 *and any real $p \geqslant 1$, there exists an explicit constant $\kappa_p > 0$ such that*

$$\begin{aligned} \Phi(|u|)^{-1}\mathcal{A}_{B,e}[|\cdot|^p](v,v_\star) \leqslant &-(1-\kappa_p)\left(|v|^{2p} + |v_\star|^{2p}\right) \\ &+ \kappa_p\left[\left(|v|^2+|v_\star|^2\right)^p - |v|^{2p} - |v_\star|^{2p}\right]. \end{aligned} \tag{2.16}$$

This constant κ_p has the following properties.

(1) $\kappa_1 \leqslant 1$.

(2) *For $p \geqslant 1$ the map $p \mapsto \kappa_p$ is strictly decreasing. In particular, $\kappa_p < 1$ for $p > 1$.*

(3) $\kappa_p = O(1/p^{1/q'})$ *for large p, where $1/q + 1/q' = 1$.*

(4) *For $q = 1$, one still has $\kappa_p \searrow 0$ as $p \to \infty$.*

Proof. Let $\Psi_p(x) = x^p$. From (2.15), one sees that

$$A^+_{B,e}[\Psi_p](v, v_\star) \leqslant \kappa_p\, E^p,$$

where we recall that $E = |v|^2 + |v_\star|^2$, and we set

$$\kappa_p = \sup_{\widehat{U},\widehat{u}} \int_{\widehat{U}\cdot\sigma\geqslant 0} \left[\Psi_p\left(\frac{3+\widehat{U}\cdot\sigma}{4}\right) + \Psi_p\left(\frac{1-\widehat{U}\cdot\sigma}{4}\right)\right] \tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma. \tag{2.17}$$

It is clear that the above inequality yields (2.16). Let us prove that κ_p satisfies the aforementioned conditions. First, we use the Hölder inequality to obtain

$$\kappa_p \leqslant 4\pi\, \|b\|_{L^q(\mathbb{S}^2)} \left(\int_{-1}^{1} \left[\Psi_p\left(\frac{3+s}{4}\right) + \Psi_p\left(\frac{1-s}{4}\right)\right]^{q'} \mathrm{d}s\right)^{1/q'} < \frac{16\pi\, \|b\|_{L^q(\mathbb{S}^2)}}{(q'p+1)^{1/q'}}.$$

This proves that κ_p is finite and also yields item (3) for $q > 1$. For items (1) and (2) observe that the integral in the right-hand side (2.15) is continuous in the vectors $\widehat{U},\ \widehat{u} \in \mathbb{S}^2$. This can be shown by changing the integral to polar coordinates. Thus, the supremum in these arguments is achieved. Therefore, there exist $\widehat{U}_0,\ \widehat{u}_0 \in \mathbb{S}^2$ (depending on the angular kernel b) such that

$$\kappa_p = \int_{\{\widehat{U}_0\cdot\sigma\geqslant 0\}} \left[\Psi_p\left(\frac{3+\widehat{U}_0\cdot\sigma}{4}\right) + \Psi_p\left(\frac{1-\widehat{U}_0\cdot\sigma}{4}\right)\right] \tilde{b}(\widehat{u}_0\cdot\sigma)\,\mathrm{d}\sigma.$$

A simple computation with this estimate shows that $\kappa_1 = \|b\|_{L^1(\mathbb{S}^2)} = 1$. Moreover, the integrand is almost everywhere strictly decreasing as p increases, and this proves (2). Finally, let $p \to \infty$ in this expression and use dominated convergence to conclude (4) for the case $q = 1$. $\square$

The above lemma is analogous to [11, Corollary 1] for variable restitution coefficient $e(\cdot)$, and it proves that the subsequent results of [11] extend readily to variable restitution coefficient. In particular, [11, Lemma 3] reads[1] as follows.

PROPOSITION 2.7. *Let f be a nonnegative function satisfying* (2.9). *For any $p \geqslant 1$, we set*

$$m_p = \int_{\mathbb{R}^3} f(v)|v|^{2p}\,\mathrm{d}v.$$

Assume that the collision kernel $B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma)$ is such that $b(\cdot)$ satisfies (2.6) *with $b(\cdot) \in L^q(\mathbb{S}^2)$ for some $q \geqslant 1$. For any restitution coefficient $e(\cdot)$ satisfying Assumption* 2.1 *and any real $p \geqslant 1$, one has*

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(v)|v|^{2p}\,\mathrm{d}v \leqslant -(1-\kappa_p)m_{p+1/2} + \kappa_p\, S_p, \tag{2.18}$$

[1] Notice that, though stated for hard-sphere interactions only, [11, Lemma 3] applies to our situation thanks to Lemma 2.6 above and [11, Lemma 1].

where

$$S_p = \sum_{k=1}^{[p+1/2]} \left(\begin{array}{c} p \\ k \end{array} \right) \left(m_{k+1/2}\, m_{p-k} + m_k\, m_{p-k+1/2}\right),$$

with $[\frac{p+1}{2}]$ *denoting the integer part of* $\frac{p+1}{2}$ *and* κ_p *being the constant of Lemma* 2.6.

Inequality (2.18) was introduced in [11] because the term S_p involves only moments of order $p-1/2$. Thus, the above estimate has important consequences on the propagation of moments for the solution to (2.7) (see section 3 for more discussion).

3. Free cooling of granular gases: Generalized Haff's law. We investigate in this section the so-called generalized *Haff's law for granular gases with variable restitution coefficient*. More precisely, we aim to derive the exact rate of decay of the temperature $\mathcal{E}(t)$ of the solution to (2.7). In this section, we *exclusively* study the generalized hard-sphere collision kernel,

$$B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma),$$

where $b(\cdot)$ satisfies (2.6), but generalization to the so-called variable hard-sphere interactions (i.e., $\Phi(|u|) = |u|^s$ for $s \geqslant 0$) is easy to handle. Let f_0 be a nonnegative velocity distribution satisfying (2.8), and let $f(t,v)$ be the associated solution to the Cauchy problem (2.7). We denote its temperature by $\mathcal{E}(t)$:

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2\,\mathrm{d}v.$$

The conditions (2.8) imply that $\sup_{t\geqslant 0}\mathcal{E}(t) < \infty$. Indeed, the evolution of $\mathcal{E}(t)$ is governed by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = \int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(t,v)|v|^2\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)|u|\\ \times \int_{\mathbb{S}^2}\Big(|v'|^2+|v'_\star|^2-|v|^2-|v_\star|^2\Big)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\;\,\mathrm{d}v_\star\,\mathrm{d}v,\end{aligned}$$

where we applied (2.5) with $\psi(v) = |v|^2$. One checks readily that

$$|v'|^2+|v'_\star|^2-|v|^2-|v_\star|^2 = -|u|^2\frac{1-\widehat{u}\cdot\sigma}{4}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)|u|^3\,\mathrm{d}v\,\mathrm{d}v_\star\\ \times\int_{\mathbb{S}^2}\frac{1-\widehat{u}\cdot\sigma}{4}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\end{aligned}$$

We compute this last integral over $\mathbb{S}^2$ (for fixed v and $v_\star$) using polar coordinates to get the following:

$$\begin{aligned}|u|^3\int_{\mathbb{S}^2}\frac{1-\widehat{u}\cdot\sigma}{8}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\\ = 2\pi|u|^3\int_0^1\left(1-e^2(|u|y)\right)b(1-2y^2)y^3\,\mathrm{d}y = \mathbf{\Psi}_e(|u|^2),\end{aligned}$$

where we have defined

$$\boldsymbol{\Psi}_e(r) := 2\pi r^{3/2}\int_0^1 \left(1-e(\sqrt{r}z)^2\right) b\left(1-2z^2\right) z^3\,\mathrm{d}z \qquad \forall r\geqslant 0. \tag{3.1}$$

In other words, the evolution of the temperature $\mathcal{E}(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)\boldsymbol{\Psi}_e(|u|^2)\,\mathrm{d}v\,\mathrm{d}v_\star \leqslant 0, \qquad t\geqslant 0.$$

In addition to Assumption 2.1, we assume in the rest of the paper that the restitution coefficient $e(\cdot)$ satisfies the following.

Assumption 3.1. Assume that the mapping $r\mapsto e(r)\in(0,1]$ satisfies Assumption 2.1 and that

(1) there exist $\alpha>0$ and $\gamma\geqslant 0$ such that

$$e(r)\simeq 1-\alpha\, r^{\gamma} \quad \text{for} \quad r\simeq 0,$$

(2) $\liminf_{r\to\infty} e(r) = e_0 < 1$,
(3) $b(\cdot)\in L^q(\mathbb{S}^2)$ for some $q\geqslant 1$, and
(4) the function $r>0\longmapsto \boldsymbol{\Psi}_e(r)$ defined in (3.1) is strictly increasing and convex over $(0,+\infty)$.

Remark 3.2. For hard-sphere interactions, $b(\widehat{u}\cdot\sigma)=\frac{1}{4\pi}$; thus, $\boldsymbol{\Psi}_e$ reduces to

$$\boldsymbol{\Psi}_e(r) = \frac{1}{2\sqrt{r}}\int_0^{\sqrt{r}} \left(1-e(y)^2\right) y^3\,\mathrm{d}y, \qquad r>0.$$

We prove in Appendix A that Assumption 3.1 is satisfied for the viscoelastic hard spheres of Example 2.4 with $\gamma=1/5$. More generally, in the case of hard-sphere interactions, Assumption 3.1(4) is fulfilled if $e(\cdot)$ is continuously decreasing (see Lemma A.1 in Appendix A). For constant restitution coefficient $e(r)=e_0$, these assumptions are trivially satisfied.

3.1. Upper bound for $\mathcal{E}(t)$. We first prove the first half of Haff's law; namely, the temperature $\mathcal{E}(t)$ has at least algebraic decay.

PROPOSITION 3.3. *Let f_0 be a nonnegative velocity distribution satisfying* (2.8), *and let $f(t,v)$ be the associated solution to the Cauchy problem* (2.7) *where the variable restitution coefficient satisfies Assumption* 3.1*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)\leqslant -\boldsymbol{\Psi}_e(\mathcal{E}(t)) \qquad \forall t\geqslant 0.$$

Moreover, there exists $C>0$ such that

$$\mathcal{E}(t)\leqslant C\,(1+t)^{-\frac{2}{1+\gamma}} \qquad \forall t\geqslant 0. \tag{3.2}$$

Proof. Recall that the evolution of the temperature is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)\boldsymbol{\Psi}_e(|u|^2)\,\mathrm{d}v\,\mathrm{d}v_\star, \qquad t\geqslant 0, \tag{3.3}$$

where $u=v-v_\star$. Since $\boldsymbol{\Psi}_e(|\cdot|^2)$ is convex according to Assumption 3.1(4) and $f(t,v_\star)\,\mathrm{d}v_\star$ is a probability measure over $\mathbb{R}^3$, Jensen's inequality implies

$$\int_{\mathbb{R}^3} f(t,v_\star)\boldsymbol{\Psi}_e(|u|^2)\,\mathrm{d}v_\star \geqslant \boldsymbol{\Psi}_e\left(\left|v-\int_{\mathbb{R}^3} v_\star f(t,v_\star)\,\mathrm{d}v_\star\right|^2\right) = \boldsymbol{\Psi}_e(|v|^2),$$

where we used (2.9). Applying Jensen's inequality again, we obtain

$$\int_{\mathbb{R}^3} f(t,v)\boldsymbol{\Psi}_e(|v|^2)\,\mathrm{d}v \geqslant \boldsymbol{\Psi}_e\left(\int_{\mathbb{R}^3} f(t,v)|v|^2\,\mathrm{d}v\right),$$

and therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -\boldsymbol{\Psi}_e(\mathcal{E}(t)) \qquad \forall t \geqslant 0.$$

Note that $\boldsymbol{\Psi}_e(\cdot)$ is strictly increasing with $\lim_{x\to 0}\boldsymbol{\Psi}_e(x) = 0$; this ensures that

$$\lim_{t\to\infty}\mathcal{E}(t) = 0.$$

Moreover, according to Assumption 3.1(1), it is clear from (3.1) that

$$\boldsymbol{\Psi}_e(x) \simeq C_\gamma x^{\frac{3+\gamma}{2}} \quad \text{ for } \quad x \simeq 0,$$

where the constant can be taken as $C_\gamma = 2\pi\alpha\int_0^1 y^{3+\gamma}b(1-2y^2)\,\mathrm{d}y < \infty$. Since $\mathcal{E}(t) \to 0$, there exists $t_0 > 0$ such that $\boldsymbol{\Psi}_e(\mathcal{E}(t)) \geqslant \frac{1}{2}C_\gamma\mathcal{E}(t)^{\frac{3+\gamma}{2}}$ $\forall\, t \geqslant t_0$ which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -\frac{C_\gamma}{2}\mathcal{E}(t)^{\frac{3+\gamma}{2}} \qquad \forall t \geqslant t_0.$$

This proves (3.2) and, hence, Proposition 3.3. ∎

Example 3.4. In the case of constant restitution coefficient $e(r) = e_0 \in (0,1)$ for any $r \geqslant 0$, for hard-sphere interactions, one has

$$\boldsymbol{\Psi}_e(x) = \frac{1-e_0^2}{8}x^{3/2}.$$

Thus, one recovers from (3.2) the decay of the temperature established from physical considerations (dimension analysis) in [18] and proved in [23]; namely, $\mathcal{E}(t) \leqslant C(1+t)^{-2}$ for large t.

Example 3.5. For the restitution coefficient $e(\cdot)$ associated to viscoelastic hard spheres (see Example 2.4), one has $\gamma = 1/5$; thus, the above estimate (3.2) leads to a decay of the temperature faster than $(1+t)^{-5/3}$ which is the one obtained in [27] (see also [12]) from physical considerations and dimensional analysis.

Notice that, since $\mathcal{E}(t) \to 0$ as $t \to \infty$, it is possible to resume the arguments of [22, Proposition 5.1] to prove that the solution $f(t,v)$ to (2.7) converges to a Dirac mass as t goes to infinity; namely,

$$f(t,v) \underset{t\to\infty}{\longrightarrow} \delta_{v=0} \quad \text{ weakly } * \text{ in } \quad M^1(\mathbb{R}^3),$$

where $M^1(\mathbb{R}^3)$ denotes the space of normalized probability measures on $\mathbb{R}^3$. We shall not investigate further on the question of longtime asymptotic behavior of the distribution $f(t,v)$ but rather try to capture the very precise rate of convergence of the temperature to zero.

Using the Povzner-like estimate of section 2.4, it is possible, from the decay in $\mathcal{E}(t)$, to deduce the decay of any moments of f. Indeed, for any $t \geqslant 0$ and any $p \geqslant 1$, we define the *p-moment* of f as

$$m_p(t) := \int_{\mathbb{R}^3} f(t,v)|v|^{2p}\,\mathrm{d}v. \tag{3.4}$$

COROLLARY 3.6. *Let f_0 be a nonnegative velocity distribution satisfying* (2.8)*, and let $f(t,v)$ be the associated solution to the Cauchy problem* (2.7) *where the variable restitution coefficient satisfies Assumption* 3.1*. For any $p \geqslant 1$, there exists $K_p > 0$ such that*

$$m_p(t) \leqslant K_p\,(1+t)^{-\frac{2p}{1+\gamma}} \qquad \forall t \geqslant 0. \tag{3.5}$$

Proof. Set $u(t) = (1+t)^{-\frac{2}{1+\gamma}}$. We prove that, for any $p \geqslant 1$, there exists $K_p > 0$ such that $m_p(t) \leqslant K_p u^p(t)$ for any $t \geqslant 0$. Observe that using classical interpolation, it suffices to prove this for any p such that $2p \in \mathbb{N}$. We argue by induction. It is clear from Proposition 3.3 that estimate (3.5) holds for $p = 1$. Let $p > 1$, with $2p \in \mathbb{N}$, be fixed, and assume that for any integer $1 \leqslant j \leqslant p - 1/2$ there exists $K_j > 0$ such that $m_j(t) \leqslant K_j u^j(t)$ holds. According to Proposition 2.7,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_p(t) = \int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(t,v)|v|^{2p}\,\mathrm{d}v \leqslant -(1-\kappa_p) m_{p+1/2}(t) + \kappa_p\, S_p(t), \tag{3.6}$$

where

$$S_p(t) = \sum_{k=1}^{[\frac{p+1}{2}]} \begin{pmatrix} p \\ k \end{pmatrix} \big(m_{k+1/2}(t)\; m_{p-k}(t) + m_k(t)\; m_{p-k+1/2}(t) \big) \quad \forall t \geqslant 0.$$

For $p \geqslant 2$, the above expression $S_p(t)$ involves moments of order less than $p - 1/2$. The case $p = 3/2$ is treated independently.

Step 1 *($p = 3/2$)*. In this case (3.6) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{3/2}(t) \leqslant -(1-\kappa_{3/2}) m_2(t) + m_{3/2}(t) m_{1/2}(t) + \mathcal{E}^2(t) \qquad \forall t \geqslant 0. \tag{3.7}$$

Let K be a positive number to be chosen later, and define

$$U_{3/2}(t) := m_{3/2}(t) - K u(t)^{3/2}.$$

Using (3.7) one has

$$\frac{\mathrm{d}U_{3/2}}{\mathrm{d}t}(t) \;\leqslant\; -(1 - \kappa_{3/2}) m_2(t) + m_{3/2}(t) m_{1/2}(t) + \mathcal{E}^2(t) + \frac{3K}{1+\gamma}(1+t)^{-\frac{4+\gamma}{1+\gamma}}.$$

From Holder's inequality,

$$m_{3/2}(t) \leqslant \sqrt{\mathcal{E}(t)}\sqrt{m_2(t)} \quad \text{and} \quad m_{1/2}(t) \leqslant \sqrt{\mathcal{E}(t)} \qquad \forall t \geqslant 0; \tag{3.8}$$

hence,

$$\frac{\mathrm{d}U_{3/2}}{\mathrm{d}t}(t) \leqslant -(1-\kappa_{3/2}) \frac{m_{3/2}^2(t)}{\mathcal{E}(t)} + \sqrt{\mathcal{E}(t)} m_{3/2}(t) + \mathcal{E}^2(t) + \frac{3K}{1+\gamma}(1+t)^{-\frac{4+\gamma}{1+\gamma}}.$$

Since $\mathcal{E}(t) \leqslant C(1+t)^{-\frac{2}{1+\gamma}}$, there exist $a, b, c > 0$ such that

$$\begin{aligned} \frac{\mathrm{d}U_{3/2}}{\mathrm{d}t}(t) \leqslant{}& -a\, m_{3/2}^2(t)(1+t)^{\frac{2}{1+\gamma}} + b\,(1+t)^{-\frac{4}{1+\gamma}} \\ & + c\,(1+t)^{-\frac{1}{1+\gamma}} m_{3/2}(t) + \frac{3}{1+\gamma} K (1+t)^{-\frac{4+\gamma}{1+\gamma}} \qquad \forall t > 0. \end{aligned} \tag{3.9}$$

Inequality (3.9) implies the result for the case $p = 3/2$, provided K is large enough. Indeed, choose K so that $m_{3/2}(0) < Ku^{3/2}(0) = K$. Then, by time-continuity of the moments, the result follows at least for some finite time. Assume that there exists a time $t_\star > 0$ such that $m_{3/2}(t_\star) = Ku^{\frac{3}{2}}(t_\star) = K(1+t_\star)^{-\frac{3}{1+\gamma}}$; then (3.9) implies

$$\frac{\mathrm{d}U_{3/2}}{\mathrm{d}t}(t_\star) \leqslant \left(-aK^2 + b + cK + \frac{3}{1+\gamma}K\right)(1+t_\star)^{-\frac{4}{1+\gamma}} < 0$$

whenever K is large enough. Thus, (3.5) holds for $p = 3/2$ choosing $K_{3/2} := K$.

Step 2 *($p \geqslant 2$).* The induction hypothesis implies that there exists a constant $C_p > 0$ such that

$$S_p(t) \leqslant C_p\, u(t)^{p+1/2} \qquad \forall t \geqslant 0,$$

where C_p can be taken as

$$C_p = \sum_{k=1}^{\left[\frac{p+1}{2}\right]} \left(\begin{array}{c} p \\ k \end{array}\right) \left(K_{k+1/2}\, K_{p-k} + K_k\, K_{p-k+1/2}\right).$$

Furthermore, according to Jensen's inequality $m_{p+1/2}(t) \geqslant m_p^{1+1/2p}(t)$ for any $t \geqslant 0$. Thus, from (3.6), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_p(t) \leqslant -(1-\kappa_p) m_p^{1+1/2p}(t) + \kappa_p\, C_p\, u(t)^{p+1/2} \qquad \forall t \geqslant 0.$$

Arguing as in Step 1 for some $K > 0$ to be chosen later, we define

$$U_p(t) := m_p(t) - Ku(t)^p.$$

In this way,

$$\frac{\mathrm{d}}{\mathrm{d}t} U_p(t) \leqslant -(1-\kappa_p) m_p^{1+\frac{1}{2p}}(t) + \kappa_p\, C_p\, u(t)^{p+\frac{1}{2}} + \frac{2pK}{1+\gamma}(1+t)^{-\frac{2p+1}{1+\gamma}} \qquad \forall t \geqslant 0.$$

Then, if K is such that $U_p(0) < 0$, the result holds at least for some finite time. For any $t_\star > 0$ such that $U_p(t_\star) = 0$, one notices then that

$$\frac{\mathrm{d}}{\mathrm{d}t} U_p(t_\star) \leqslant \left(-(1-\kappa_p) K^{1+\frac{1}{2p}} + \kappa_p\, C_p + \frac{2pK}{1+\gamma}\right)(1+t_\star)^{-\frac{2p+1}{1+\gamma}} < 0,$$

provided K is large enough. This proves (3.5) for any $p \geqslant 1$. ∎

3.2. Lower bound for $\mathcal{E}(t)$: Preliminary considerations. The next goal is to complete the proof of Haff's law by showing that the cooling rate (3.2) is optimal under Assumption 3.1. Thus, we have to show that there exists $C > 0$ such that

$$\mathcal{E}(t) \geqslant C(1+t)^{-\frac{2}{1+\gamma}} \qquad \forall t \geqslant 0.$$

First, we prove the following result that simplifies our endeavor.

THEOREM 3.7. *Assume a nonconstant ($\gamma > 0$) restitution coefficient $e(\cdot)$ satisfying Assumption* 3.1. *If there exist $C_0 > 0$ and $\lambda > 0$ such that*

$$\mathcal{E}(t) \geqslant C_0\,(1+t)^{-\lambda} \qquad \forall t \geqslant 0, \tag{3.10}$$

then there exists $C_p > 0$ such that

$$m_p(t) \leqslant C_p\, \mathcal{E}^p(t) \qquad \text{for any } t \geqslant 0 \ \text{ and } \ p \geqslant 1. \tag{3.11}$$

As a consequence, there exists $C > 0$ such that

$$\mathcal{E}(t) \geqslant C\,(1+t)^{-\frac{2}{1+\gamma}} \qquad \forall t \geqslant 0. \tag{3.12}$$

Proof. According to Assumption 3.1(1),

$$\mathbf{\Psi}_e(x) \simeq C_\gamma x^{\frac{3+\gamma}{2}} \ \text{ for } \ x \simeq 0.$$

In addition, Assumption 3.1(2) implies that there exists $C_b > 0$ such that

$$\mathbf{\Psi}_e(x) \simeq C_b x^{3/2} \ \text{ for large } x,$$

where the constant can be taken as $C_b = 2\pi(1-e_0^2)\int_0^1 b(1-2z^2)z^3\,\mathrm{d}z$. Thus, there exists another constant $C > 0$ such that

$$\mathbf{\Psi}_e(x) \leqslant C x^{\frac{3+\gamma}{2}} \qquad \forall x > 0. \tag{3.13}$$

Therefore, there exists $C > 0$ such that, for any $\varepsilon > 0$ and any $p > \frac{3+\gamma}{2}$,

$$\mathbf{\Psi}_e(x) \leqslant C x^{\frac{3+\gamma}{2}} \leqslant C\left(\varepsilon^{\frac{\gamma}{2}} x^{\frac{3}{2}} + \frac{x^p}{\varepsilon^{p-\frac{3+\gamma}{2}}}\right) \qquad \forall x > 0.$$

Then from (3.3) one deduces that for any $\varepsilon > 0$ and $p > \frac{3+\gamma}{2}$,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant C\Bigg(\varepsilon^{\frac{\gamma}{2}}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)|u|^3\,\mathrm{d}v\,\mathrm{d}v_\star \\ + \frac{1}{\varepsilon^{p-\frac{3+\gamma}{2}}}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,v)f(t,v_\star)|u|^{2p}\,\mathrm{d}v\,\mathrm{d}v_\star\Bigg).$$

In other words, for any $\varepsilon > 0$ and any $p > \frac{3+\gamma}{2}$, there is some $C > 0$ such that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &\leqslant C\left(\varepsilon^{\frac{\gamma}{2}} m_{3/2}(t) + \frac{1}{\varepsilon^{p-\frac{3+\gamma}{2}}} m_p(t)\right) \\ &\leqslant C\left(\varepsilon^{\frac{\gamma}{2}} m_{3/2}(t) + \frac{C_p}{\varepsilon^{p-\frac{3+\gamma}{2}}}(1+t)^{-\frac{2p}{1+\gamma}}\right) \qquad \forall t \geqslant 0, \end{aligned}$$

where we have used Corollary 3.6 for the second inequality. In particular, using (3.10) and the fact that $\mathcal{E}(t)$ is a nonincreasing function, one can choose p sufficiently large so that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant C\left(\varepsilon^{\frac{\gamma}{2}} m_{3/2}(t) + \frac{\tilde{C}_p}{\varepsilon^{p-\frac{3+\gamma}{2}}}\mathcal{E}(t)^{\frac{3}{2}}\right)$$

for some positive constant $\tilde{C}_p$. In other words, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant \delta m_{3/2}(t) + C_\delta \mathcal{E}(t)^{3/2} \qquad \forall t \geqslant 0. \tag{3.14}$$

With this preliminary observation, the proof of (3.11) is a direct adaptation of that of Corollary 3.6. Here again, by simple interpolation, it is enough to prove the result for any p such that $2p \in \mathbb{N}$ and argue using induction. The result is clearly true for $p = 1$ with $C_1 = 1$. For $p = 3/2$, let $K > 0$ be a constant chosen later and define

$$u_{3/2}(t) = m_{3/2}(t) - K\mathcal{E}(t)^{3/2}.$$

Thus, from (3.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{3/2}(t) \leqslant -(1-\kappa_{3/2})m_2(t) + m_{3/2}(t)m_{1/2}(t) + \mathcal{E}^2(t) - \frac{3}{2}K\sqrt{\mathcal{E}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t).$$

Using (3.8), one deduces from (3.14) that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{3/2}(t) \leqslant -(1-\kappa_{3/2})\frac{m_{3/2}^2(t)}{\mathcal{E}(t)} + \left(1 + \frac{3}{2}K\delta\right) m_{3/2}(t)\sqrt{\mathcal{E}(t)} + \left(1 + \frac{3}{2}KC_\delta\right)\mathcal{E}^2(t).$$

Fix $\delta = \frac{1-\kappa_{3/2}}{3}$ and choose $K > 0$ such that $u_{3/2}(0) < 0$. If $t_\star > 0$ is such that $u_{3/2}(t_\star) = 0$, then the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{3/2}(t_\star) \leqslant \left(-\frac{1-\kappa_{3/2}}{2}K^2 + \left(K + 1 + \frac{3}{2}KC_\delta\right)\right)\mathcal{E}(t_\star)^2 < 0,$$

provided K is sufficiently large. This proves (3.11) for $p = 3/2$ with $C_{3/2} := K$. The case $p \geqslant 2$ follows in the same lines as the proof of Corollary 3.6 interchanging the roles of $\mathcal{E}(t)$ and $u(t)$.

To conclude the proof, observe that according to (3.13) and (3.3), there exists $C > 0$ such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant C m_{\frac{3+\gamma}{2}}(t) \qquad \forall t \geqslant 0.$$

Then, applying (3.11) with $p = \frac{3+\gamma}{2}$, one deduces that there is $C_\gamma > 0$ such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant C_\gamma \mathcal{E}(t)^{\frac{3+\gamma}{2}} \qquad \forall t \geqslant 0.$$

A simple integration of this inequality yields (3.12). $\square$

Remark 3.8. For constant restitution coefficient $e = e_0$, since $\gamma = 0$, (3.14) does not hold anymore. However, for some $C_e > 0$, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant C_e m_{3/2}(t) \qquad \forall t \geqslant 0. \tag{3.15}$$

Assuming that $e_0 \simeq 1$ (quasi-elastic regime), the constant C_e is small; thus, the argument above can be reproduced to prove that the conclusion of Theorem 3.7 still holds. Recall that for $\gamma = 0$, the second part of Haff's law (3.12) has been proved in [23, Theorem 1.2] with the additional requirement that $f_0 \in L^p(\mathbb{R}^3)$ for some $1 < p < \infty$. It appears that, by (3.15), in the quasi-elastic regime $e_0 \simeq 1$, Haff's law holds true for constant restitution coefficient under the sole assumption (2.8) on the initial datum.

In order to prove that (3.10) is satisfied for some $\lambda > 0$, we will need precise L^p estimates, following the spirit of [23], for the rescaled function g given in section 2.3. The idea to craft the correct time-scaling functions $\tau(\cdot)$ and $V(\cdot)$ is choosing them such that the corresponding temperature of g is bounded away from zero. Indeed, for any $\tau > 0$, define

$$\mathbf{\Theta}(\tau) := \int_{\mathbb{R}^3} g(\tau, w)|w|^2 \, \mathrm{d}w.$$

Since

$$\mathcal{E}(t) = V(t)^{-2}\mathbf{\Theta}(\tau(t)) \qquad \forall t \geqslant 0, \tag{3.16}$$

we choose

$$V(t) = (1+t)^{\frac{1}{\gamma+1}} \qquad \forall t \geqslant 0. \tag{3.17}$$

In this way, (3.12) is equivalent to $\mathbf{\Theta}(\tau(t)) \geqslant C$ for any $t \geqslant 0$. Notice that (3.2) immediately translates into

$$\sup_{t>0} \mathbf{\Theta}(\tau(t)) < \infty. \tag{3.18}$$

Moreover, for simplicity we pick $\tau(t)$ such that $\dot{\tau}(t)V(t) = 1$; therefore, for $\gamma > 0$,

$$\tau(t) = \int_0^t \frac{\mathrm{d}s}{V(s)} = \frac{\gamma+1}{\gamma}\left((1+t)^{\frac{\gamma}{1+\gamma}} - 1\right), \tag{3.19}$$

which is an acceptable time-scaling function. Thus, the rescaled solution $g(\tau, w)$ satisfies (2.12) with $\lambda(\tau) = 1$ with

$$\xi(\tau) = \frac{1}{\gamma\tau + (1+\gamma)} \quad \text{and} \quad \widetilde{e}_\tau(r) = e\left(r\left(1 + \frac{\gamma}{\gamma+1}\tau\right)^{-1/\gamma}\right). \tag{3.20}$$

If $\gamma = 0$, the restitution coefficient is constant [23]; in particular, $\widetilde{e}_\tau = e$, and the rescaling reads $V(t) = 1 + t$ and $\tau(t) = \ln(1+t)$. In such a case, $\xi(\tau) \equiv 1$.

The next sections are devoted to the proof of the second part of Haff's law. With the choice of the scaling, it is equivalent to prove that

$$\inf_{\tau>0} \mathbf{\Theta}(\tau) > 0.$$

However, it appears difficult to prove directly such a lower bound, and we shall instead prove a lower bound of the type

$$\mathbf{\Theta}(\tau) \geqslant C_1 \, (1+\tau)^{-\kappa_1} \qquad \forall \tau > 0$$

for some positive constants $C_1, \kappa_1 > 0$ and use Theorem 3.7 to get the conclusion (see Proposition 5.1 and Theorem 5.2 below). To do so, one has to perform a careful study of the properties of the collision operator $\mathcal{Q}_e$ in Sobolev or L^p spaces $p > 1$.

4. Regularity properties of the collision operator. In this section the regularity properties studied originally for the elastic case in [19, 20, 21, 26, 31] and later for the constant restitution coefficient in [23] are generalized to cover variable restitution coefficients depending on the impact velocity. The path that we follow closely follows [26].

4.1. The Carleman representation. We establish here a technical representation of the gain term $\mathcal{Q}^+_{B,e}$ which is reminiscent of the classical Carleman representation in the elastic case. More precisely, let $B(u,\sigma)$ be a collision kernel of the form

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma),$$

where $\Phi(\cdot) \geqslant 0$, and $b(\cdot) \geqslant 0$ satisfies (2.6). For any $\psi = \psi(v)$, define the following linear operators:

$$\mathcal{S}_\pm(\psi)(u) = \int_{\mathbb{S}^2} \psi(u^\pm)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma \qquad \forall u \in \mathbb{R}^3, \tag{4.1}$$

where the symbols u^- and u^+ are defined by

$$u^- := \beta\left(|u\,|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\,\frac{u-|u|\,\sigma}{2} \quad \text{and} \quad u^+ := u - u^-.$$

LEMMA 4.1. *For any bounded and measurable functions ψ and φ,*

$$\int_{\mathbb{R}^3} \varphi(u)\mathcal{S}_-(\psi)(u)\Phi(|u|)\,\mathrm{d}u = \int_{\mathbb{R}^3} \psi(x)\Gamma_B(\varphi)(x)\,\mathrm{d}x,$$

where the linear operator Γ_B is given by

$$\Gamma_B(\varphi)(x) = \int_{\omega^\perp} \mathcal{B}(z+\alpha(r)\omega, \alpha(r))\varphi(\alpha(r)\omega + z)\,\mathrm{d}\pi_z, \qquad x = r\omega,\ r \geqslant 0,\ \omega \in \mathbb{S}^2. \tag{4.2}$$

Here $\mathrm{d}\pi_z$ is the Lebesgue measure in the hyperplane $\omega^\perp$ perpendicular to ω, and $\alpha(\cdot)$ is the inverse of the mapping $s \mapsto s\beta(s)$. Moreover,

$$\mathcal{B}(z,\varrho) = \frac{8\Phi(|z|)}{|z|(\varrho\beta(\varrho))^2} b\left(1 - 2\frac{\varrho^2}{|z|^2}\right)\frac{\varrho}{1+\vartheta'(\varrho)}, \qquad \varrho \geqslant 0, \quad z \in \mathbb{R}^3, \tag{4.3}$$

with $\vartheta(\cdot)$ defined in Assumption 2.1(2) *and $\vartheta'(\cdot)$ denoting its derivative.*

Proof. For simplicity assume that $\Phi \equiv 1$. Define

$$I := \int_{\mathbb{R}^3} \varphi(u)\mathcal{S}_-(\psi)(u)\,\mathrm{d}u = \int_{\mathbb{R}^3} \varphi(u)\,\mathrm{d}u \int_{\mathbb{S}^2} \psi(u^-)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

For fixed $u \in \mathbb{R}^3$, we perform the integration over $\mathbb{S}^2$ using the formula

$$\int_{\mathbb{S}^2} F\left(\frac{u-|u|\sigma}{2}\right)\mathrm{d}\sigma = \frac{4}{|u|}\int_{\mathbb{R}^3} \delta(|x|^2 - x\cdot u)F(x)\,\mathrm{d}x$$

valid for any given function F. Then

$$I = 4\int_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(u)|u|^{-1}\delta(|x|^2 - x\cdot u)\psi\big(x\beta(|x|)\big)b\left(1 - 2\frac{|x|^2}{|u|^2}\right)\mathrm{d}x\,\mathrm{d}u.$$

Now setting $u = z + x$, we get

$$I = 4\int_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(x+z)|x+z|^{-1}\delta(x\cdot z)\psi\big(x\beta(|x|)\big)b\left(1 - 2\frac{|x|^2}{|x+z|^2}\right)\mathrm{d}z\,\mathrm{d}x.$$

Keeping x fixed, we remove the Dirac mass using the identity

$$\int_{\mathbb{R}^3} F(z)\delta(x\cdot z)\,\mathrm{d}z = \frac{1}{|x|}\int_{x^\perp} F(z)\,\mathrm{d}\pi_z,$$

which leads to

$$I = 4\int_{\mathbb{R}^3}\psi\big(x\beta(|x|)\big)\frac{\mathrm{d}x}{|x|}\int_{x^\perp}\frac{\varphi(x+z)}{|x+z|}b\left(1-2\frac{|x|^2}{|x+z|^2}\right)\,\mathrm{d}\pi_z.$$

We compute the x-integral using polar coordinates $x=\varrho\,\omega$ and the change of variables $r=\varrho\,\beta(\varrho)$. Recall that $\alpha(r)$ is the inverse of such mapping; furthermore, notice that $\mathrm{d}r=\frac{1}{2}(1+\vartheta'(\varrho))\,\mathrm{d}\varrho$. This yields

$$I = 8\int_0^\infty\frac{\alpha(r)\,\mathrm{d}r}{1+\vartheta'(\alpha(r))}\int_{\mathbb{S}^2}\psi(r\omega)\,\mathrm{d}\omega\int_{\omega^\perp}\frac{\varphi(z+\alpha(r)\omega)}{|z+\alpha(r)\omega|}b\left(1-2\frac{\alpha(r)^2}{|z+\alpha(r)\omega|^2}\right)\,\mathrm{d}\pi_z.$$

Turning back to Cartesian coordinates $x=r\omega$, we obtain the desired expression,

$$I=\int_{\mathbb{R}^3}\psi(x)\Gamma_B(\varphi)(x)\,\mathrm{d}x,$$

with Γ_B given by (4.2). ∎

The above result leads to a Carleman-like expression for $\mathcal{Q}^+_{B,e}$.

Corollary 4.2 (Carleman representation). *Let $e(\cdot)$ be a restitution coefficient satisfying Assumption* 2.1*, and let*

$$B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$$

be a collision kernel satisfying (2.6)*. Then, for any velocity distribution functions f and g, one has*

$$\mathcal{Q}^+_{B,e}(f,g)(v)=\int_{\mathbb{R}^3}f(z)\left[(t_z\circ\Gamma_B\circ t_z)\,g\right](v)\,\mathrm{d}z,$$

where $[t_v\psi](x)=\psi(v-x)$ for any $v,x\in\mathbb{R}^3$ and test function ψ.

Proof. The proof readily follows from Lemma 4.1, and the identity,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}f(v)g(v-u)\Phi(|u|)\mathcal{S}_-(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u \tag{4.4}$$

is valid for any test function ψ. ∎

4.2. Convolution-like estimates for $\mathcal{Q}^+_{B,e}$. General convolution-like estimates are obtained in [3, Theorem 1] for nonconstant restitution coefficient. Such estimates are given in L^p_η with $\eta\geqslant 0$, and, for the applications we have in mind, we need to extend some of them to $\eta\leqslant 0$. This can be done using the method developed in [26] (see also [17]) together with the estimates of [3].[2]

Theorem 4.3. *Assume that the collision kernel $B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfies* (2.6) *and $\Phi(\cdot)\in L^\infty_{-k}$ for some $k\in\mathbb{R}$. In addition, assume that $e(\cdot)$ fulfills Assumption* 2.1*. Then, for any $1\leqslant p\leqslant\infty$ and $\eta\in\mathbb{R}$, there exists $\mathbf{C}_{\eta,p,k}(B)>0$ such that*

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta}\leqslant\mathbf{C}_{\eta,p,k}(B)\,\|f\|_{L^1_{|\eta+k|+|\eta|}}\,\|g\|_{L^p_{\eta+k}},$$

[2]Notice that the constants $\gamma(\eta,p,b)$ and $\widetilde{\gamma}(\eta,p,b)$ given by (4.6) and (4.8) are not finite for arbitrary angular kernel b. It is implicitly assumed that the theorem applies to the range of parameters leading to finite constants (see also Remark 4.4).

where the constant $\mathbf{C}_{\eta,p,k}(B)$ *is given by*

$$\mathbf{C}_{\eta,p,k}(B) = c_{k,\eta,p}\ \gamma(\eta,p,b)\,\|\Phi\|_{L^\infty_{-k}}\,, \tag{4.5}$$

with a constant $c_{k,\eta,p} > 0$ *depending only on* $k, \eta,$ *and* p. *Furthermore, the dependence on the angular kernel is given by*

$$\gamma(\eta,p,b) = \int_{-1}^{1} \left(\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s, \tag{4.6}$$

where $1/p + 1/p' = 1$, *and* η_+ *is the positive part of* η: $\eta_+ = \max(\eta, 0)$. *Similarly, there exists* $\widetilde{\mathbf{C}}_{\eta,p,k}(B) > 0$ *such that*

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta} \leqslant \widetilde{\mathbf{C}}_{\eta,p,k}(B)\,\|g\|_{L^1_{|\eta+k|+|\eta|}}\,\|f\|_{L^p_{\eta+k}},$$

where the constant $\widetilde{\mathbf{C}}_{\eta,p,k}(B)$ *is given by*

$$\widetilde{\mathbf{C}}_{\eta,p,k}(B) = \widetilde{c}_{k,\eta,p}\ \widetilde{\gamma}(\eta,p,b)\,\|\Phi\|_{L^\infty_{-k}} \tag{4.7}$$

for some constant $\widetilde{c}_{k,\eta,p} > 0$ *depending only on* $k, \eta,$ *and* p. *The dependence on the angular kernel is given by*

$$\widetilde{\gamma}(\eta,p,b) = \int_{-1}^{1} \left(\frac{1+s}{2} + (1-\beta_0)^2\,\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s, \tag{4.8}$$

where $1/p + 1/p' = 1$ *and* $\beta_0 = \beta(0) = \frac{1+e(0)}{2}$.

Proof. Fix $1 \leqslant p \leqslant \infty$ and $\eta \in \mathbb{R}$, and use the convention $1/p' + 1/p = 1$. By duality,

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta} = \sup\left\{\left|\int_{\mathbb{R}^3} \mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v\right| \ ;\ \|\psi\|_{L^{p'}_{-\eta}} \leqslant 1\right\}.$$

Using (4.4),

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v-u)\mathcal{T}_-(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u,$$

with

$$\mathcal{T}_-(\psi)(u) = \Phi(|u|)\mathcal{S}_-(\psi)(u), \quad \text{and} \quad t_v\psi(x) = \psi(v-x),$$

with $\mathcal{S}_-$ defined in (4.1). With the notation of [3], one recognizes that $\mathcal{S}_-(h) = \mathcal{P}(h,1)$, thus, applying [3, Theorem 5] with $q = \infty$ and $\alpha = -\eta$,

$$\|\mathcal{S}_-(h)\|_{L^{p'}_{-\eta}} \leqslant \gamma(\eta,p,b)\|h\|_{L^{p'}_{-\eta}},$$

with $\gamma(\eta,p,b)$ given by (4.6). Notice that, with respect to [3], we used the weight $\langle v\rangle^\eta$ instead of $|v|^\eta$; this is the reason to introduce η_+ in our definition of $\gamma(\eta,p,b)$. As a consequence,

$$\|\mathcal{T}_-(h)\|_{L^{p'}_{-\eta-k}} \leqslant \gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|h\|_{L^{p'}_{-\eta}}. \tag{4.9}$$

Now,

$$\left|\int_{\mathbb{R}^3} \mathcal{Q}^+_{B,e}(f,g)\psi \,\mathrm{d}v\right| \leqslant \int_{\mathbb{R}^3} |f(v)| \,\mathrm{d}v \left(\int_{\mathbb{R}^3} |g(u)| \left[(t_v \circ \mathcal{T}_- \circ t_v)\psi\right](u)\,\mathrm{d}u\right)$$
$$\leqslant \|g\|_{L^p_{\eta+k}} \int_{\mathbb{R}^3} |f(v)| \,\|(t_v \circ \mathcal{T}_- \circ t_v)\psi\|_{L^{p'}_{-k-\eta}} \,\mathrm{d}v.$$

Using the inequality $\|t_v h\|_{L^{p'}_s} \leqslant 2^{|s|/2}\langle v\rangle^{|s|}\|h\|_{L^{p'}_s}$ for any $s \in \mathbb{R}$ and v,

$$\left|\int_{\mathbb{R}^3} \mathcal{Q}^+_{B,e}(f,g)\psi \,\mathrm{d}v\right| \leqslant 2^{|\eta+k|/2}\|g\|_{L^p_{\eta+k}} \int_{\mathbb{R}^3} |f(v)|\langle v\rangle^{|\eta+k|} \,\|(\mathcal{T}_- \circ t_v)\psi\|_{L^{p'}_{-k-\eta}} \,\mathrm{d}v$$
$$\leqslant 2^{|\eta+k|/2}\gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|g\|_{L^p_{\eta+k}} \int_{\mathbb{R}^3} |f(v)|\langle v\rangle^{|\eta+k|} \,\|t_v\psi\|_{L^{p'}_{-\eta}} \,\mathrm{d}v$$
$$\leqslant 2^{|\eta+k|+|\eta|/2}\gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|g\|_{L^p_{\eta+k}} \int_{\mathbb{R}^3} |f(v)|\langle v\rangle^{|\eta+k|+|\eta|} \,\|\psi\|_{L^{p'}_{-\eta}} \,\mathrm{d}v,$$

which proves the first part of the theorem. To prove the second part, observe that

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v-u)g(v)\mathcal{T}_+(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u,$$

where $\mathcal{T}_+(\psi)(u) = \Phi(|u|)\mathcal{S}_+(\psi)(u)$ and $\mathcal{S}_+$ is defined as in (4.1). Using the notation of [3], we identify $\mathcal{S}_+(h) = \mathcal{P}(1,h)$. Thus, applying [3, Theorem 5] with $p = \infty$ and $\alpha = -\eta$,

$$\|\mathcal{S}_-(h)\|_{L^{p'}_{-\eta}} \leqslant \widetilde{\gamma}(\eta,p,b)\|h\|_{L^{p'}_{-\eta}},$$

where $\widetilde{\gamma}(\eta,p,b)$ is given by (4.8). One concludes as above, interchanging the roles of f and g. ∎

Remark 4.4. The careful reader will notice that the constants given in the theorem are independent of $e(\cdot)$ except for $\tilde{\gamma}(\eta,p,b)$ which depends only on the value $e(0)$.

Clearly, the constants $\gamma(\eta,p,b)$ and $\widetilde{\gamma}(\eta,p,b)$ are not finite for arbitrary $b(\cdot)$ because of the possible singularity at $s = \pm 1$. For instance, for hard-sphere interactions, i.e., $b \equiv \frac{1}{4\pi}$, one has

$$\gamma(\eta,p,b) < \infty \iff \widetilde{\gamma}(\eta,p,b) < \infty \iff 1 \leqslant p < \frac{3+\eta_+}{1+\eta_+}.$$

However, if one assumes, as in [23, Theorem 2.1], that the angular kernel $b(\cdot)$ vanishes in the vicinity of $s = 1$, then $\gamma(\eta,p,b) < \infty$ for any $1 \leqslant p \leqslant \infty$ and $\eta \in \mathbb{R}$. This is an additional difficulty of the inelastic regime that is overcome in the elastic case using symmetry, i.e., defining b in half the domain. In the inelastic regime such difficulty can be handled when dealing with *quadratic estimates* (i.e., whenever $f = g$ in the above). Precisely, one has the following corollary.

COROLLARY 4.5. *Assume that the collision kernel $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfies (2.6) and $\Phi(\cdot) \in L^\infty_{-k}$ for some $k \in \mathbb{R}$. In addition, assume that $e(\cdot)$ fulfills Assumption 2.1. Then, for any $1 \leqslant p \leqslant \infty$ and $\eta \in \mathbb{R}$, there exists some positive constant $C(k,\eta,p) > 0$ (independent of B) such that*

$$\left\|\mathcal{Q}^+_{B,e}(f,f)\right\|_{L^p_\eta} \leqslant C(k,\eta,p)\,\|\Phi\|_{L^\infty_{-k}}\,\|b\|_{L^1(\mathbb{S}^2)}\,\|f\|_{L^1_{|\eta+k|+|\eta|}}\,\|f\|_{L^p_{\eta+k}}$$

for any $f \in L^1_{|\eta+k|+|\eta|} \cap L^p_{\eta+k}$.

Proof. Recall that $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$. Thus, one can write

$$B(u,\sigma) = B_0(u,\sigma) + B_1(u,\sigma) = \Phi(|u|)b_0(\widehat{u}\cdot\sigma) + \Phi(|u|)b_1(\widehat{u}\cdot\sigma),$$

where

$$b_0(s) = b(s)\chi_{[-1,0]} \quad \text{and} \quad b_1(s) = b(s)\chi_{[0,1]}.$$

Then, with the notations of Theorem 4.3, it is clear that

$$\gamma(\eta,p,b_0) = \int_{-1}^{0} \left(\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s \leqslant 2^{\frac{3+\eta_+}{2p'}} \|b\|_{L^1(\mathbb{S}^2)} < \infty,$$

while

$$\widetilde{\gamma}(\eta,p,b_1) = \int_0^1 \left(\frac{1+s}{2} + (1-\beta_0)^2\,\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s \leqslant c_{\eta,p}\|b\|_{L^1(\mathbb{S}^2)} < \infty$$

for some explicit numerical constant $c_{\eta,p}$ (depending on $e(\cdot)$ through β_0). Then writing $\mathcal{Q}^+_{B,e}(f,f) = \mathcal{Q}^+_{B_0,e}(f,f) + \mathcal{Q}^+_{B_1,e}(f,f)$, we get from Theorem 4.3 that

$$\left\|\mathcal{Q}^+_{B,e}(f,f)\right\|_{L^p_\eta} \leqslant \left(\mathbf{C}_{\eta,p,k}(B_0) + \widetilde{\mathbf{C}}_{\eta,p,k}(B_1)\right) \|f\|_{L^1_{|\eta+k|+|\eta|}} \|f\|_{L^p_{\eta+k}},$$

which is the desired conclusion. □

4.3. Sobolev regularity for smooth collision kernel. For this section we assume $\Phi(\cdot)$ and $b(\cdot)$ are smooth and compactly supported as follows:

$$\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^3 \setminus \{0\}), \qquad b \in \mathcal{C}_0^\infty(-1,1). \tag{4.10}$$

Denote by $\mathcal{Q}_{B,e}$ the associated collision operator defined by (2.5).

Lemma 4.6. *Assume that* $e(\cdot)$ *satisfies Assumption* 2.1 *with* $e(\cdot) \in \mathcal{C}^m(0,\infty)$ *for some integer* $m \in \mathbb{N}$. *Then, for the collision kernel satisfying* (4.10), *for any* $0 \leqslant s \leqslant m$, *there exists* $C = C(s,B,e)$ *such that*

$$\|\Gamma_B(f)\|_{H^{s+1}} \leqslant C(s,B,e)\,\|f\|_{H^s} \qquad \forall f \in H^s,$$

where Γ_B *is the operator defined in Lemma* 4.1. *The constant* $C(s,B,e)$ *depends only on* s, *on the collision kernel* B, *and on the restitution coefficient* $e(\cdot)$. *More precisely,* $C(s,B,e)$ *depends on* $e(\cdot)$ *through the* L^∞ *norm of the derivatives* $D^k e(\cdot)$ *(*$k = 1,\dots,m$*) over some compact interval bounded away from zero depending only on* B.

We postpone the proof of Lemma 4.6 and first prove its important consequence.

Theorem 4.7. *Let* $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ *be a collision kernel satisfying* (4.10), *and let* $e(\cdot)$ *satisfy Assumption* 2.1. *In addition, assume that* $e(\cdot) \in \mathcal{C}^m(0,\infty)$ *for some integer* $m \in \mathbb{N}$. *Then, for any* $0 \leqslant s \leqslant m$,

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{H^{s+1}} \leqslant C(s,B,e)\,\|g\|_{H^s}\,\|f\|_{L^1},$$

with constant $C(s,B,e)$ *given in Lemma* 4.6.

Proof. Let $\mathcal{F}[\mathcal{Q}^+_{B,e}(f,g)](\xi)$ denote the Fourier transform of $\mathcal{Q}^+_{B,e}(f,g)$. According to Corollary 4.2,

$$\mathcal{F}\left[\mathcal{Q}^+_{B,e}(f,g)\right](\xi) = \int_{\mathbb{R}^3} f(v)\mathcal{F}\left[(t_v \circ \Gamma_B \circ t_v)\, g\right](\xi)\,\mathrm{d}v.$$

To simplify notation, set $G(v,\xi) = \mathcal{F}\left[(t_v \circ \Gamma_B \circ t_v)\, g\right](\xi)$. Thus,

$$\begin{aligned}
\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|^2_{H^{s+1}} &= \int_{\mathbb{R}^3} \left|\mathcal{F}\left[\mathcal{Q}^+_{B,e}(f,g)\right](\xi)\right|^2 \langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^3} \langle\xi\rangle^{2(s+1)} \left|\int_{\mathbb{R}^3} f(v)G(v,\xi)\,\mathrm{d}v\right|^2 \mathrm{d}\xi \qquad (4.11)\\
&\leqslant \|f\|_{L^1} \int_{\mathbb{R}^3\times\mathbb{R}^3} |f(v)|\,|G(v,\xi)|^2 \langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi\,\mathrm{d}v.
\end{aligned}$$

Since $G(v,\xi) = \mathcal{F}\left[(t_v \circ \Gamma_B \circ t_v)\, g\right](\xi)$,

$$\int_{\mathbb{R}^3} |G(v,\xi)|^2 \langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi = \left\|(t_v \circ \Gamma_B \circ t_v)\, g\right\|^2_{H^{s+1}} \leqslant C(s,B,e)^2\ \|g\|^2_{H^s}.$$

For this inequality we used Lemma 4.6 and the fact that the translation operator t_v has norm one in any Sobolev space. Hence, estimate (4.11) yields the desired estimate. $\square$

Proof of Lemma 4.6. The proof of the regularity property of Γ_B can be obtained following the lines of the corresponding one for the elastic Boltzmann operator [26]. Indeed, note that

$$\begin{aligned}
\widetilde{\Gamma_B}(f)(r,\omega) :&= \Gamma_B(f)(\alpha^{-1}(r),\omega) = \Gamma_B(f)(r\beta(r),\omega)\\
&= \int_{\omega^\perp} \mathcal{B}(z+r\omega, r)\varphi(r\omega+z)\,\mathrm{d}\pi_z.
\end{aligned}$$

The condition (4.10) on the collision kernel implies that there exists $\delta > 0$ such that $b(x) = 0$ for $|x \pm 1| \leqslant \delta$ and $\{|z|\,;\, z \in \mathrm{Supp}(\Phi)\} \subset (a, M)$ for some positive constants $0 < a < M$. Then, by virtue of (4.3), $\mathcal{B}(z+r\omega, r) = 0$ for any $r > 0$, $\omega \in \mathbb{S}^2$, and $z \in \omega^\perp$ provided that $|z|^2 > \frac{2-\delta}{\delta}r^2$. For $|z|^2 \leqslant \frac{2-\delta}{\delta}r^2$, one has $|z+r\omega|^2 \leqslant 2r^2/\delta$; thus, $\mathcal{B}(z+r\omega, r) = 0$ if $r < \sqrt{\delta a^2/2}$. Putting these together, we conclude that

$$\mathcal{B}(z+r\omega, r) = 0 \qquad \forall r \notin I := \left(\sqrt{\delta a^2/2}, M\right),\ \omega \in \mathbb{S}^2,\ \text{and any } z \perp \omega. \tag{4.12}$$

In particular, $\widetilde{\Gamma_B}(f)(r,\omega) = 0$ for any $r \notin I$ independently of f. Define

$$\mathcal{B}_0(z,\varrho) := \frac{1+\vartheta'(\varrho)}{\varrho}\beta^2(\varrho)\mathcal{B}(z,\varrho) = \frac{\Phi(|z|)b\left(1 - 2\frac{\varrho^2}{|z|^2}\right)}{|z|\varrho^2},$$

and denote $\widetilde{\Gamma_0}(f)$ the associated operator

$$\widetilde{\Gamma_0}(f)(r,\omega) := \int_{\omega^\perp} \mathcal{B}_0(z+r\omega, r)\varphi(r\omega+z)\,\mathrm{d}\pi_z.$$

Then $\mathcal{B}_0$ does not depend on the restitution coefficient $e(\cdot)$, and $\widetilde{\Gamma_0}$ is exactly of the form of the operator T studied in [26, Theorem 3.1]. Therefore, arguing as in [26], for any $s \geqslant 0$, there is an explicit constant $C_0 = C_0(s, \Phi, b)$ such that

$$\left\|\widetilde{\Gamma_0}(f)\right\|_{H^{s+1}} \leqslant C_0(s, \Phi, b)\, \|f\|_{H^s} \qquad \forall f \in H^s. \tag{4.13}$$

Setting

$$G_e(\varrho) = \frac{\varrho}{(1 + \vartheta'(\varrho))\, \beta^2(\varrho)} \qquad \forall \varrho \geqslant 0, \tag{4.14}$$

one observes that G_e is a $\mathcal{C}^m$ function over I whose derivatives $D^k G_e$ are bounded over I for any $k \leqslant m$ and

$$\widetilde{\Gamma_B}(f)(r, \omega) = G_e(r)\chi_I(r)\widetilde{\Gamma_0}(f)(r, \omega).$$

Here χ_I is the characteristic function of $I = (\sqrt{\delta a^2/2}, M)$ (see (4.12)). Therefore, for any $0 \leqslant s \leqslant m$, there exists some constant $C = C_0(s, b, e)$ such that

$$\left\|\widetilde{\Gamma_B}(f)\right\|_{H^{s+1}} \leqslant C_0(s, B, e)\, \|f\|_{H^s} \qquad \forall f \in H^s, \tag{4.15}$$

where the constant $C_0(s, B, e)$ can be chosen as

$$C_0(s, B, e) = C_0(s, \Phi, b) \max_{k=0,\dots,s} \|D^k G_e\|_{L^\infty(I)}. \tag{4.16}$$

From estimate (4.15) we deduce Lemma 4.6 with the following argument. Assume first $s = k \geqslant 1$ is an integer. Using polar coordinates

$$\|\Gamma_B(f)\|^2_{H^k} = \sum_{|j| \leqslant k} \int_0^\infty F_j(\varrho)\varrho^2\, \mathrm{d}\varrho \int_{\mathbb{S}^2} |\partial_v^j \widetilde{\Gamma_B}(f)(\varrho, \omega)|^2\, \mathrm{d}\omega,$$

where, for any $|j| \leqslant k$, the function $F_j(\varrho)$ can be written as

$$F_j(\varrho) = P_j(\vartheta^{(1)}(\varrho), \dots, \vartheta^{(j)}(\varrho))(1 + \vartheta^{(1)}(\varrho))^{-n_j}. \tag{4.17}$$

Here $P_j(y_1, \dots, y_j)$ is a suitable polynomial, $n_j \in \mathbb{N}$, and $\vartheta^{(p)}$ denotes the pth derivative of $\vartheta(\cdot)$. Since $\vartheta \in \mathcal{C}^m(0, \infty)$ and I is a compact interval away from zero, one has $\sup_{\varrho \in I} F_j(\varrho) = C_k < \infty$ for any $|j| \leqslant k$. Thus,

$$\|\Gamma_B(f)\|_{H^k} \leqslant C_k \|\widetilde{\Gamma_B}(f)\|_{H^k}, \tag{4.18}$$

where C_k is an explicit constant involving the L^∞ norm of the first kth order derivatives of $\alpha(\cdot)$ on I. This proves that the conclusion of Lemma 4.6 holds true for any integer $s \leqslant m$, and we deduce the general case using interpolation. ◻

Remark 4.8. It is important, for our subsequent analysis, to obtain a precise expression for the constant $C(s, B, e)$. For instance, in the case in which $e(\cdot) \in \mathcal{C}^1(0, \infty)$, one obtains that

$$C(1, B, e) \leqslant C_0(1, B, e) \sup_{\varrho \in I} F_1(\varrho),$$

where F_1 is of the form (4.17), with I defined in (4.12). Note that $C_0(1, B, e)$ and $G_e(\varrho)$ are given by (4.16) and (4.14), respectively. In particular, under Assumption 2.1, $G_e(\varrho) \leqslant 4\varrho$ for large ϱ and $G_e(\varrho) \simeq \varrho/2$ for $\varrho \simeq 0$.

Arguing as in [26, Corollary 3.2], we translate the gain of regularity obtained in Theorem 4.7 in gain of integrability.

COROLLARY 4.9. *Let $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ be a collision kernel satisfying* (4.10)*, and let $e(\cdot) \in \mathcal{C}^1(0,\infty)$ satisfy Assumption* 2.1*. Then, for any $1 < p < \infty$,*

$$\left\| \mathcal{Q}^+_{B,e}(f,g) \right\|_{L^p} \leqslant C(p,B,e) \left(\|g\|_{L^q} \|f\|_{L^1} + \|g\|_{L^1} \|f\|_{L^q} \right),$$

where the constant $C(p,B,e)$ depends on B and e through the constant $C(1,B,e)$ of Theorem 4.7*. The exponent $q < p$ is given by*

$$q = \begin{cases} \dfrac{5p}{3+2p} & \text{if} \quad p \in (1,6], \\ \dfrac{p}{3} & \text{if} \quad p \in [6,\infty). \end{cases} \tag{4.19}$$

4.4. Regularity and integrability for hard spheres. We consider in this section the case of hard-sphere collision kernel

$$B(u,\sigma) = \frac{|u|}{4\pi}.$$

Such a collision kernel does not enjoy the regularity properties assumed in the previous section. This does not present a problem since the dependence of the constant on the collision kernel B permits us to adapt the method developed in [26] for the elastic case. We need some supplementary assumptions on the restitution coefficient $e(\cdot)$.

Assumption 4.10. In addition to Assumption 2.1, suppose that $e(\cdot) \in \mathcal{C}^1(0,\infty)$ and that there exists $k \in \mathbb{R}$ such that

$$e'(r) = O(r^k) \qquad \text{when} \quad r \to \infty,$$

where $e'(\cdot)$ denotes the derivative of $e(\cdot)$.

The above assumption implies $\vartheta'(\varrho) = O(\varrho^{k+1})$ for large ϱ and $\vartheta'(\varrho) \simeq 1$ when $\varrho \simeq 0$. Recall that $\vartheta'(\cdot)$ is the derivative of $\vartheta(r) = re(r)$.

THEOREM 4.11. *Assume that $e(\cdot)$ satisfies Assumption* 4.10*. For any $1 < p < \infty$, there exist $\kappa > 0$, $\theta \in (0,1)$, and a constant $C_e > 0$ depending only on p and the restitution coefficient $e(\cdot)$ such that, for any $\delta > 0$,*

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_e(f,f)\, f^{p-1}\, \mathrm{d}v \leqslant C_e \delta^{-\kappa} \|f\|_{L^1}^{1+p\theta} \|f\|_{L^p}^{p(1-\theta)} + \delta \|f\|_{L^1_2} \|f\|^p_{L^p_{1/p}}.$$

Proof. We follow the same lines presented in [26] and subsequently used in [23, 24]. We present the argument for convenience. Fix $p \geqslant 1$, and let $\Theta : \mathbb{R} \to \mathbb{R}^+$ be an even $\mathcal{C}^\infty$ function with compact support in $(-1,1)$ and $\int_{-1}^1 \Theta(s)\,\mathrm{d}s = 1$. In the same way, consider a radial $\mathcal{C}^\infty$ function $\Xi : \mathbb{R}^3 \to \mathbb{R}$ with support in the ball $B(0,1)$ and $\int_{\mathbb{R}^3} \Xi(v)\,\mathrm{d}v = 1$. Define the mollifications $\Xi_n(v) := n^3 \Xi(nv)$ and $\Theta_m(s) := m\Theta(ms)$ for $m, n \geqslant 1$. Thus, $\Phi_{S_n} = \Xi_n * (|\cdot|\chi_{A_n})$ and $b_{S_m} = \Theta_m * (\frac{1}{4\pi}\chi_{[-1+\frac{2}{m},1-\frac{2}{m}]})$ are smooth mollifications of the collision kernel. Here we have defined the set

$$A_n = \left\{ v \in \mathbb{R}^3\,;\, |v| \in \left[\frac{2}{n}, n\right] \right\}, \qquad n \geqslant 1.$$

Consider the smooth collision kernel

$$B_{S_{m,n}}(|u|, \widehat{u}\cdot\sigma) = \Phi_{S_n}(|u|)\, b_{S_m}(\widehat{u}\cdot\sigma),$$

and observe that

$$\operatorname{supp}(\Phi_{S_n}) \subseteq \left\{\frac{1}{n} \leqslant |v| \leqslant n+1\right\} \quad \text{and} \quad \operatorname{supp}(b_{S_m}) \subseteq \left[-1+\frac{1}{m}, 1-\frac{1}{m}\right].$$

Define naturally

$$\begin{aligned} B_{SR_{m,n}}(|u|, \widehat{u}\cdot\sigma) &:= \Phi_{S_n}(|u|)\, b_{R_m}(\widehat{u}\cdot\sigma), \\ B_{RS_{m,n}}(|u|, \widehat{u}\cdot\sigma) &:= \Phi_{R_n}(|u|)\, b_{S_m}(\widehat{u}\cdot\sigma), \quad \text{and} \\ B_{RR_{m,n}}(|u|, \widehat{u}\cdot\sigma) &:= \Phi_{R_n}(|u|)\, b_{R_m}(\widehat{u}\cdot\sigma). \end{aligned}$$

Here $\Phi_{R_n}(|u|) = |u| - \Phi_{S_n}(|u|)$ and $b_{R_m}(\widehat{u}\cdot\sigma) = \frac{1}{4\pi} - b_{S_m}(\widehat{u}\cdot\sigma)$ are the remainder parts. Thus, one splits $\mathcal{Q}_e^+$ in four parts using obvious notation as follows:

$$\mathcal{Q}_e^+ = \mathcal{Q}^+_{B_{S_{m,n}},e} + \mathcal{Q}^+_{B_{SR_{m,n}},e} + \mathcal{Q}^+_{B_{RS_{m,n}},e} + \mathcal{Q}^+_{B_{RR_{m,n}},e}.$$

Since $B_{S_{m,n}}(|u|, \widehat{u}\cdot\sigma)$ fulfills (4.10), one deduces from Corollary 4.9 that there is a constant $C(m,n)$ such that

$$\left\|\mathcal{Q}^+_{B_{S_{m,n}},e}(f,f)\right\|_{L^p} \leqslant C(m,n)\|f\|_{L^q}\,\|f\|_{L^1}$$

for $q < p$ given by (4.19). A simple application of Hölder's inequality yields

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{B_{S_{m,n}},e}(f,f)\, f^{p-1}\,\mathrm{d}v \leqslant C(m,n)\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|^{p-1}_{L^p}. \tag{4.20}$$

Recall from Corollary 4.9 that $C(m,n)$ depends on m and n through the constant $C(1, B_{S_{m,n}}, e)$ in Theorem 4.7. Moreover, according to Remark 4.8, one sees that

$$C(1, B_{S_{m,n}}, e) \leqslant C_0(1, \Phi_{S_n}, b_{S_m}) \max_{k=0,1} \|D^k G_e\|_{L^\infty(I)} \sup_{\varrho\in I} F_1(\varrho),$$

where $C_0(s,\Phi,b)$ is the constant appearing in (4.13), $G_e(\cdot)$ is given by (4.14), and F_1 is of the form (4.17). The interval $I = I_{m,n}$ is defined in (4.12), with $\delta = 1/m$, $M = n+1$, and $a = 1/n$,

$$I = \left(\sqrt{\frac{1}{2mn^2}}, n+1\right).$$

That $C_0(1, \Phi_{S_n}, b_{S_m})$ depends on m and n in a polynomial way follows as in [26]. Moreover, from the properties of G_e given in Remark 4.8 and the fact that $F_1(\varrho)$ is a rational function in $\vartheta'(\varrho)$, one deduces from Assumption 4.10 and the above expression of I that there exist $a, b > 0$ such that

$$C(m,n) = O(m^a\, n^b) \ \text{ as } m,n \to \infty. \tag{4.21}$$

Now, applying Corollary 4.5 with $k = 1$ and $\eta = -1/p'$, we get

$$\left\|\mathcal{Q}^+_{B_{SR_{m,n}},e}(f,f)\right\|_{L^p_\eta} + \left\|\mathcal{Q}^+_{B_{RR_{m,n}},e}(f,f)\right\|_{L^p_\eta} \leqslant \varepsilon(m)\|f\|_{L^1_1}\,\|f\|_{L^p_{1/p}},$$

where $\varepsilon(m) \leqslant c\|b_m\|_{L^1(\mathbb{S}^2)}$ for some positive constant $c > 0$ (independent of n since $\|\Phi_n\|_{L^\infty_{-1}} \leqslant \|\Phi\|_{L^\infty_{-1}}$ for any $n \geqslant 0$). In particular, one can choose a regularizing function Θ such that there exists some $r > 0$ such that

$$\varepsilon(m) = O(m^{-r}) \text{ as } m \to \infty. \tag{4.22}$$

Using the above estimate with $\eta = -1/p'$, we get

$$\int_{\mathbb{R}^3} \left[\mathcal{Q}^+_{B_{SR_{m,n}},e}(f,f) + \mathcal{Q}^+_{B_{RR_{m,n}},e}(f,f)\right] f^{p-1}\,\mathrm{d}v \leqslant \varepsilon(m)\|f\|_{L^1_1}\|f\|^p_{L^p_{1/p}}. \tag{4.23}$$

It remains only to estimate

$$\mathrm{I} := \int_{\mathbb{R}^3} \mathcal{Q}^+_{B_{RS_{m,n}},e}(f,f)\, f^{p-1}\,\mathrm{d}v.$$

One notes that

$$\Phi_{R_n}(|v-v_\star|) \leqslant Cn^{-1}\left(|v|^2+|v_\star|^2\right) \qquad \forall v, v_\star \in \mathbb{R}^3$$

for some $C>0$. Thus,

$$\mathrm{I} \leqslant Cn^{-1}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)\left(|v|^2+|v_\star|^2\right)\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

Define

$$\mathrm{I}_1 := \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)|v|^2\,\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma, \quad \text{and}$$
$$\mathrm{I}_2 := \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)|v_\star|^2\,\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

Observe that I_1 can be written as

$$\mathrm{I}_1 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{Q}^+_{B_m,e}(F,f)(v)\psi(v)\,\mathrm{d}v,$$

where

$$F(v) = |v|^2 f(v), \qquad \psi(v) = f^{p-1}(v) \in L^{p'}(\mathbb{R}^3),$$

with the collision kernel $B_m(|u|,\widehat{u}\cdot\sigma) = b_{S_m}(\widehat{u}\cdot\sigma)$. Applying Theorem 4.3 with $\eta = k = 0$ gives

$$\begin{aligned}\mathrm{I}_1 &\leqslant \left\|\mathcal{Q}^+_{B_m,e}(F,f)\right\|_{L^p}\|\psi\|_{L^{p'}}\\ &\leqslant \mathbf{C}_{0,p,0}(B_m)\,\|F\|_{L^1}\|f\|_{L^p}\|\psi\|_{L^{p'}} \leqslant \mathbf{C}_{0,p,0}(B_m)\|f\|_{L^1_2}\|f\|^p_{L^p},\end{aligned}$$

where $\mathbf{C}_{0,p,0}(B_m)$ is defined by (4.5). Now, with the same notation,

$$\mathrm{I}_2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{Q}^+_{B_m,e}(f,F)(v)\psi(v)\,\mathrm{d}v;$$

therefore, applying Theorem 4.3 with $\eta = 0$ and $k=-2$ yields

$$\mathrm{I}_2 \leqslant \mathbf{C}_{0,p,-2}(B_m)\,\|f\|_{L^1_2}\,\|F\|_{L^p_{-2}}\,\|\psi\|_{L^{p'}} \leqslant \mathbf{C}_{0,p,-2}(B_m)\,\|f\|_{L^1_2}\,\|f\|^p_{L^p}.$$

Combining the two estimates for I_1 and I_2,

$$\mathrm{I} \leqslant \frac{C(m)}{n}\|f\|_{L^1_2}\|f\|^p_{L^p},$$

where $C(m) = \mathbf{C}_{0,p,0}(B_m) + \mathbf{C}_{0,p,-2}(B_m)$. The support of $b_{S_m}(s)$ lies to a positive distance, of order $1/m$, from $s = 1$. Then we use the expression (4.5) to conclude that

$$C(m) \leqslant m^{-\frac{3}{2p'}} \text{ as } m \to \infty. \tag{4.24}$$

Estimates (4.24), (4.20), and (4.23) give

$$\int_{\mathbb{R}^3} \mathcal{Q}_e^+(f,f)\, f^{p-1}\,\mathrm{d}v \leqslant C(m,n)\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|_{L^p}^{p-1} + \varepsilon(m)\|f\|_{L^1_1}\,\|f\|^p_{L^p_{1/p}} + \frac{C(m)}{n}\|f\|_{L^1_2}\,\|f\|^p_{L^p}.$$

Using the polynomial bounds (4.21), (4.22), and (4.24), we get the result as in [23]. ☐

Remark 4.12. Assumption 4.10 allows us to present the explicit dependence of the constants with respect to $\delta > 0$. This dependence will be crucial in the proof of Haff's law in section 5. Note that the constant C_e in Theorem 4.11 depends on the regularity of the restitution coefficient away from zero.

Remark 4.13. Notice also that the above estimates involving L^p-norms for $p < \infty$ degenerate as $p \to \infty$ and do not allow us to derive L^∞ estimates in some direct way. We refer the reader to [4] for further considerations on pointwise estimates.

COROLLARY 4.14. *Assume that $e(\cdot)$ satisfies Assumption* 4.10*. For any $1 < p < \infty$, there exist $\kappa > 0$, $\theta \in (0,1)$, and a constant $C_e > 0$ depending only on p and the restitution coefficient $e(\cdot)$ such that, for any $\delta > 0$,*

$$\int_{\mathbb{R}^3} \mathcal{Q}_e^+(g,g)\, g^{p-1}\langle v\rangle^{\eta p}\,\mathrm{d}v \leqslant C_e\delta^{-\kappa}\,\|g\|^{1+p\theta}_{L^1_\eta}\,\|g\|^{p(1-\theta)}_{L^p_\eta} + \delta\,\|g\|_{L^1_{2+\eta}}\,\|g\|^p_{L^p_{\eta+1/p}} \qquad \forall \eta \geqslant 0.$$

The constant C_e is provided by Theorem 4.11*.*

Proof. Fix $g \geqslant 0$ and $\eta \geqslant 0$, and set $f(v) = g(v)\langle v\rangle^\eta$. Note that $\langle v'\rangle^\eta \leqslant \langle v\rangle^\eta\,\langle v_\star\rangle^\eta$ for any $v, v_\star \in \mathbb{R}^3$; then, using the weak formulation of $\mathcal{Q}_e^+$,

$$\int_{\mathbb{R}^3} \mathcal{Q}_e^+(g,g)\, g^{p-1}\langle v\rangle^{\eta p}\,\mathrm{d}v = \int_{\mathbb{R}^3}\langle v\rangle^\eta\,\mathcal{Q}_e^+(g,g)\,f^{p-1}\,\mathrm{d}v \leqslant \int_{\mathbb{R}^3}\mathcal{Q}_e^+(f,f)f^{p-1}\,\mathrm{d}v.$$

Conclude the proof with Theorem 4.11. ☐

The following result applies to the rescaled solutions $g(\tau, w)$. Its importance lies in that the estimate is uniform in the rescaled time τ.

COROLLARY 4.15. *Assume that $e(\cdot)$ satisfies Assumption* 4.10*. For any $\tau \geqslant 0$, let $\widetilde{e}_\tau$ be the restitution coefficient defined by* (2.13)*, and let $\mathcal{Q}_{\widetilde{e}_\tau}(f,f)$ be the associated collision operator. Assume that $V(\zeta(\tau))$ is continuous and goes to infinity as $\tau \to \infty$. For any $1 < p < \infty$, there exist $\kappa > 0$, $\theta \in (0,1)$, and $K > 0$ all independent of τ such that, for any $\delta > 0$,*

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{\widetilde{e}_\tau}(g,g)\, g^{p-1}\langle w\rangle^{\eta p}\,\mathrm{d}w \leqslant K\delta^{-\kappa}\|g\|^{1+p\theta}_{L^1_\eta}\,\|g\|^{p(1-\theta)}_{L^p_\eta} + \delta\,\|g\|_{L^1_{2+\eta}}\,\|g\|^p_{L^p_{\eta+1/p}} \qquad \forall \eta \geqslant 0.$$

Proof. From Corollary 4.14, for any $\tau \geqslant 0$, there exists $K(\tau) = C_{\widetilde{e}_\tau}$ for which the above inequality holds. It suffices to prove that $K = \sup_{\tau\geqslant 0} K(\tau) < \infty$. Recall that $K(\tau)$ depends on τ through the restitution coefficient $\widetilde{e}_\tau$; more precisely, $C_{\widetilde{e}_\tau}$ depends on the L^∞ norm of the derivatives $D^k\widetilde{e}_\tau(\cdot)$, $k = 0, 1$, over some compact interval of $(0,\infty)$ bounded away from zero (independent of τ). Now, for any $\tau \geqslant 0$,

$$D^k\widetilde{e}_\tau(\cdot) = \mu^{-k}(\tau)(D^k e)\left(\frac{\cdot}{\mu(\tau)}\right),$$

with $\mu(\tau) = V(\zeta(\tau))$. Since $\mu^{-1}(\tau)$ is continuous and goes to zero as τ goes to ∞, one concludes that all the L^∞ norms of $D^k\widetilde{e}_\tau(\cdot)$ remain uniformly bounded with respect to τ. The same holds for $K(\tau)$. ∎

5. Generalized Haff's law continued.

5.1. Proof of Haff's law. In this section we prove the second part of Haff's law establishing the lower bound of the temperature (3.12). Recall that, from Theorem 3.7 it suffices to prove (3.10). As explained in section 3, this is done using suitable L^p estimates in the self-similar variables. In this section, the restitution coefficient fulfills Assumptions 3.1 and 4.10, and the collision kernel is that of hard-sphere interactions. Recall that the rescaled function $g(\tau, w)$ is the solution to the Boltzmann equation in rescaled variables (2.12):

$$\partial_\tau g(\tau, w) + \xi(\tau)\nabla_w \cdot (wg(\tau, w)) = \mathcal{Q}_{\widetilde{e}_\tau}(g, g)(\tau, w) \qquad \tau > 0. \tag{5.1}$$

The restitution coefficient $\widetilde{e}_\tau$ and the time-dependent mapping $\xi(\tau)$ are given by (3.20).

PROPOSITION 5.1. *Assume that $e(\cdot)$ fulfills Assumption 3.1 with $\gamma > 0$ and Assumption 4.10. Let f_0 satisfy (2.8) with $f_0 \in L^1_2 \cap L^p(\mathbb{R}^3)$ for some $1 < p < \infty$. Let $g(\tau, \cdot)$ be the solution to the rescaled equation (5.1) with initial datum $g(0, w) = f_0(w)$. Then there exist $C_0 > 0$ and $\kappa_0 > 0$ such that*

$$\|g(\tau)\|_{L^p} \leqslant C_0(1+\tau)^{\kappa_0} \qquad \forall \tau \geqslant 0. \tag{5.2}$$

Consequently, there exist $C_1 > 0$ and $\kappa_1 > 0$ such that

$$\mathbf{\Theta}(\tau) := \int_{\mathbb{R}^3} g(\tau, w)|w|^2 \,\mathrm{d}w \geqslant C_1(1+\tau)^{-\kappa_1} \qquad \forall \tau \geqslant 0. \tag{5.3}$$

Proof. The proof relies on Corollary 4.15. Multiply (5.1) by g^{p-1} and integrate over $\mathbb{R}^3$ to obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}\,\|g(\tau)\|^p_{L^p}}{\mathrm{d}\tau} + 3\left(1-\frac{1}{p}\right)\xi(\tau)\,\|g(\tau)\|^p_{L^p} \\ = \int_{\mathbb{R}^3} \mathcal{Q}^+_{\widetilde{e}_\tau}(g,g)g^{p-1}\,\mathrm{d}w - \int_{\mathbb{R}^3} \mathcal{Q}^-(g,g)g^{p-1}\,\mathrm{d}w.\end{aligned} \tag{5.4}$$

From Jensen's inequality, one has

$$\int_{\mathbb{R}^3} \mathcal{Q}^-(g,g)g^{p-1}\,\mathrm{d}w \geqslant \int_{\mathbb{R}^3} g^p(\tau,w)|w|\,\mathrm{d}w \qquad \forall \tau \geqslant 0. \tag{5.5}$$

According to Corollary 4.15, there exist $\kappa > 0$, $\theta \in (0,1)$, and a constant $K > 0$ that does not depend on τ such that

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{\widetilde{e}_\tau}(g,g)\,g^{p-1}\,\mathrm{d}w \leqslant K\delta^{-\kappa}\|g(\tau)\|^{1+p\theta}_{L^1}\,\|g(\tau)\|^{p(1-\theta)}_{L^p} + \delta\,\|g(\tau)\|_{L^1_2}\,\|g(\tau)\|^p_{L^p_{1/p}} \quad \forall \delta > 0.$$

From conservation of mass, $\|g(\tau)\|_{L^1} \equiv 1$; furthermore, $M_2 := \sup_{\tau\geqslant 0}\|g(\tau)\|_{L^1_2} < \infty$ from (3.18). Thus, using (5.4) and (5.5),

$$\frac{\mathrm{d}\,\|g(\tau)\|^p_{L^p}}{\mathrm{d}\tau} \leqslant p\,K\delta^{-\kappa}\,\|g(\tau)\|^{p(1-\theta)}_{L^p} + p\,M_2\,\delta\,\|g(\tau)\|^p_{L^p_{1/p}} - \mu(\tau)\|g(\tau)\|^p_{L^p_{1/p}}, \tag{5.6}$$

where $\mu(\tau)=\min(p,3(p-1)\xi(\tau))$. Since $\xi(\tau)\to 0$ as $\tau\to\infty$ for $\gamma>0$, there exists $\tau_0>0$ such that

$$\mu(\tau)=3(p-1)\xi(\tau)=\frac{3(p-1)}{\gamma\tau+1+\gamma}\qquad\text{for any } \tau\geqslant\tau_0.$$

Choosing $\delta=\mu(\tau)/(pM_2)$ in (5.6), we get

$$\frac{\mathrm{d}\,\|g(\tau)\|_{L^p}^p}{\mathrm{d}\tau}\leqslant p\,K\,(pM_2)^{\kappa}\mu(\tau)^{-\kappa}\|g(\tau)\|_{L^p}^{p(1-\theta)}\leqslant C(\gamma\tau+1+\gamma)^{\kappa}\|g(\tau)\|_{L^p}^{p(1-\theta)}\quad\forall\tau\geqslant\tau_0$$

for some positive constant $C>0$. Integrating the above estimate, we conclude the existence of some constant $C_0>0$ such that

$$\|g(\tau)\|_{L^p}^p\leqslant C_0\,(\gamma\tau+1+\gamma)^{\frac{\kappa+1}{\theta}}\qquad\forall\tau\geqslant\tau_0,$$

and (5.2) readily follows.

Regarding estimate (5.3), note that for any $R>0$,

$$\begin{aligned}\boldsymbol{\Theta}(\tau)&=\int_{|w|\leqslant R}g(\tau,w)|w|^2\,\mathrm{d}w+\int_{|w|>R}g(\tau,w)|w|^2\,\mathrm{d}w\\&\geqslant R^2\int_{|w|>R}g(\tau,w)\,\mathrm{d}w\geqslant R^2\left(1-\int_{|w|\leqslant R}g(\tau,w)|w|\,\mathrm{d}w\right)\qquad\forall\tau\geqslant 0.\end{aligned}$$

From Hölder's inequality,

$$\int_{|w|\leqslant R}g(\tau,w)|w|\,\mathrm{d}w\leqslant\left(\frac{4}{3}\pi R^3\right)^{1/p'}\|g(\tau)\|_{L^p}\ \text{ with the convention }\ \frac{1}{p}+\frac{1}{p'}=1.$$

Therefore, using (5.2), there exists a positive constant $C>0$ independent of R such that

$$\boldsymbol{\Theta}(\tau)\geqslant R^2\left(1-C\,R^{3/p'}(1+\tau)^{\kappa_0}\right)\qquad\forall R>0,\qquad\forall\tau\geqslant 0.$$

Pick $R=R(\tau)>0$ such that $C\,R^{3/p'}(1+\tau)^{\kappa_0}=1/2$; then

$$\boldsymbol{\Theta}(\tau)\geqslant\frac{1}{2}R^2(\tau)=\frac{1}{2}\left(\frac{1}{2C(1+\tau)^{\kappa_0}}\right)^{p'/3}\qquad\forall\tau\geqslant 0,$$

which gives (5.3) with $\kappa_1=p'\kappa_0/3$. ∎

The generalized Haff's law is a consequence of Theorem 3.7 and Proposition 5.1.

THEOREM 5.2. *Let $f_0\geqslant 0$ satisfy the conditions given by* (2.8) *with $f_0\in L^{p_0}(\mathbb{R}^3)$ for some $1<p_0<\infty$. In addition, assume that $e(\cdot)$ fulfills Assumptions* 3.1 *and* 4.10. *Then the solution $f(t,v)$ to the associated Boltzmann equation* (2.7) *satisfies the generalized Haff's law,*

$$c(1+t)^{-\frac{2}{1+\gamma}}\leqslant\mathcal{E}(t)\leqslant C(1+t)^{-\frac{2}{1+\gamma}},\qquad t\geqslant 0,\tag{5.7}$$

where c and C are positive constants independent of time $t\geqslant 0$.

Proof. The upper bound in (5.7) has already been obtained in Proposition 3.3. The proof of the lower bound is a straightforward consequence of Theorem 3.7 and Proposition 5.1. Indeed, recall that for $\gamma > 0$,

$$\mathcal{E}(t) = V^{-2}(t)\mathbf{\Theta}(\tau(t)),$$

where $V(t) = (1+t)^{\frac{1}{1+\gamma}}$ and $\tau(t)$ is given by (3.19). Since $\mathbf{\Theta}(\cdot)$ decays at least algebraically (5.3), one recognizes that there exists some constant $a > 0$ such that $\mathcal{E}(t) \geqslant a\,(1+t)^{-\mu}$, with $\mu = \frac{2+\gamma\kappa_1}{1+\gamma}$ and with κ_1 being the rate in (5.3). The result follows from Theorem 3.7. The proof for $\gamma = 0$ is identical. □

Remark 5.3. Recall that, in self-similar variables, (5.7) reads

$$c \leqslant \mathbf{\Theta}(\tau) \leqslant C \qquad \forall \tau > 0.$$

In particular, as explained in Theorem 3.7, the algebraic lower bound in (5.3) improves as

$$\mathbf{\Theta}_{\min} := \inf_{\tau > 0} \mathbf{\Theta}(\tau) > 0. \tag{5.8}$$

Remark 5.4. As we pointed out (see Remark 3.8), for constant restitution coefficient e_0, the proof of Haff's law for the quasi-elastic regime $e_0 \simeq 1$ follows without the need of the L^p requirement ($p > 1$); this suggests that such requirement might be avoided in the general case. This, of course, would greatly simplify the technicalities of the proof and, more importantly, would clearly separate the L^1 and L^p theories ($p > 1$) for the inelastic Boltzmann equation. The proper treatment of this issue remains an open problem.

Example 5.5. For constant restitution coefficient $\gamma = 0$, we recover the classical Haff's law of [18] proved recently in [23]:

$$c(1+t)^{-2} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-2}, \qquad t \geqslant 0.$$

Example 5.6. For viscoelastic hard spheres given in Example 2.4, one has $\gamma = 1/5$. Thus, Theorem 5.2 provides the first rigorous justification of the following cooling rate conjectured in [12, 27]:

$$c(1+t)^{-5/3} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-5/3}, \qquad t \geqslant 0.$$

Remark 5.7. Theorem 5.2 shows that the decay of the temperature is governed by the behavior of the restitution coefficient $e(r)$ for small impact. The cooling of the gases is slower for larger γ.

From the explicit rate of cooling of the temperature, one deduces the algebraic decay of any moments of the solution to (2.7). Under the assumptions of the above Theorem 5.2, the p-moment $m_p(t)$ defined in (3.4) satisfies

$$c_p(1+t)^{-\frac{2p}{1+\gamma}} \leqslant \mathcal{E}(t)^p \leqslant m_p(t) \leqslant \tilde{C}_p\,\mathcal{E}(t)^p \leqslant C_p(1+t)^{-\frac{2p}{1+\gamma}}, \qquad t \geqslant 0. \tag{5.9}$$

The positive constants c_p, C_p, and $\tilde{C}_p$ depend on p, $m_p(0)$, $\mathcal{E}(0)$, and $e(\cdot)$. The lower bound is a direct consequence of Jensen's inequality and (1.4), while the upper bound was established in Theorem 3.7.

5.2. Application: Propagation of Lebesgue norms. We complement Proposition 5.1 by proving the propagation of L^p-norms for the solution $g(\tau,w)$ satisfying the rescaled equation (5.1). Thus, the method introduced in the elastic case [26] and later used in [23] for constant restitution coefficient is extended to the case of a variable restitution coefficient satisfying Assumptions 3.1 and 4.10.

LEMMA 5.8. *Assume that the initial $f_0 \geqslant 0$ satisfies the conditions given by* (2.8) *with $f_0 \in L^p(\mathbb{R}^3)$ for some $1 < p < \infty$, and let $g(\tau,\cdot)$ be the solution to the rescaled equation* (5.1) *with initial datum $g(0,w) = f_0(w)$. Then there exists a constant $\nu_0 > 0$ such that*

$$\int_{\mathbb{R}^3} g(\tau,w_\star)|w-w_\star|\,\mathrm{d}w_\star \geqslant \max\{\nu_0,|w|\} \geqslant \frac{\nu_0}{2}\langle w\rangle \qquad \forall w\in\mathbb{R}^3, \quad \tau>0.$$

In particular,

$$\int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}_e^-(g,g)\,\mathrm{d}w \geqslant \frac{\nu_0}{2}\int_{\mathbb{R}^3} g^p(\tau,w)(1+|w|^2)^{1/2}\,\mathrm{d}w = \frac{\nu_0}{2}\,\|g(\tau)\|^p_{L^p_{1/p}}.$$

Proof. The proof is a simple consequence of (5.8). Indeed, since $f_0 \in L^1_3$, the propagation of p-moments in the rescaled variables implies $\sup_{t\geqslant 0}\|g(\tau)\|_{L^1_3} < \infty$. Then, for $R>0$ large enough,

$$\begin{aligned}\int_{\{|w|\leqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w &= \int_{\mathbb{R}^3} g(\tau,w)|w|^2\,\mathrm{d}w - \int_{\{|w|\geqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w\\ &\geqslant \boldsymbol{\Theta}_{\min} - \frac{1}{R}\sup_{\{\tau\geqslant 0\}}\|g(\tau)\|_{L^1_3} \geqslant \frac{\boldsymbol{\Theta}_{\min}}{2} > 0.\end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^3} g(\tau,w)|w|\,\mathrm{d}w \geqslant \frac{1}{R}\int_{\{|w|\leqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w \geqslant \frac{\boldsymbol{\Theta}_{\min}}{2R} =: \nu_0 > 0.$$

Using this observation and Jensen's inequality, we obtain the result. ∎

THEOREM 5.9. *Assume the variable restitution coefficient $e(\cdot)$ satisfies Assumptions* 3.1 *and* 4.10 *for some* positive $\gamma > 0$. *Assume that $f_0 \geqslant 0$ satisfies* (2.8) *with $f_0 \in L^1_{2(1+\eta)} \cap L^p_\eta(\mathbb{R}^3)$ for some $1<p<\infty$ and $\eta \geqslant 0$. Then the rescaled solution $g(\tau,\cdot)$ to* (5.1) *with initial datum $g(0,w)=f_0(w)$ satisfies*

$$\sup_{\tau\geqslant 0}\|g(\tau)\|_{L^p_\eta} < \infty.$$

In particular,

$$\sup_{t\geqslant 0}\left\{V(t)^{-3/p'}\,\|f(t)\|_{L^p}\right\} = \sup_{\tau\geqslant 0}\|g(\tau)\|_{L^p} < \infty.$$

Recall that $V(t) = (1+t)^{\frac{1}{1+\gamma}}$.

Proof. Multiplying (5.1) by $g^{p-1}(\tau,w)\langle w\rangle^{\eta p}$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}\,\|g(\tau)\|^p_{L^p_\eta}}{\mathrm{d}\tau} + 3\left(1-\frac{1}{p}\right)\xi(\tau)\,\|g\|^p_{L^p_\eta} &= \int_{\mathbb{R}^3}\mathcal{Q}^+_{\widetilde{e}_\tau}(g,g)g^{p-1}\langle w\rangle^{\eta p}\,\mathrm{d}w\\ &- \int_{\mathbb{R}^3}\mathcal{Q}^-(g,g)g^{p-1}\langle w\rangle^{\eta p}\,\mathrm{d}w + \eta\xi(\tau)\int_{\mathbb{R}^3} g^p(\tau,w)|w|^2\langle w\rangle^{\eta p-2}\,\mathrm{d}w.\end{aligned}$$

Using Lemma 5.8, one has

$$\int_{\mathbb{R}^3} \mathcal{Q}^-(g,g)g^{p-1}\langle w\rangle^{\eta p}\,\mathrm{d}w \geqslant \frac{\nu_0}{2}\|g(\tau)\|^p_{L^p_{\eta+1/p}}.$$

Moreover, $C_\eta = \sup_{\tau\geqslant 0}\|g(\tau)\|_{L^1_{2+\eta}} < \infty$ by virtue of the propagation of moments in self-similar variables (5.9). Applying Corollary 4.15 with $\delta = \frac{\nu_0}{4C}$,

$$\begin{aligned}(5.10)\quad \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\tau}\|g(\tau)\|^p_{L^p_\eta} &+ \frac{\nu_0}{4}\|g(\tau)\|^p_{L^p_{\eta+1/p}} \\ &\leqslant K\|g(\tau)\|^{p(1-\theta)}_{L^p_\eta} + \xi(\tau)\left(\eta - \frac{3}{p'}\right)\|g(\tau)\|^p_{L^p_\eta} \quad \forall \tau > 0\end{aligned}$$

for some uniform constant K. Since $\gamma > 0$, the mapping $\xi(\tau)$ decreases toward zero; thus, (5.10) leads to the result. □

Remark 5.10. We refer to [23, Theorem 1.3] for a proof of the case $\gamma = 0$.

6. High-energy tails for the self-similar solution. We finalize this work studying the high-energy tails of $f(t,v)$ of the solution to (1.3). For models with variable restitution coefficient, the high-energy tail is dynamic since gas changes its behavior during the cooling process. This is noted by a dynamic rate in the tail. Here again, we shall deal with the following generalized hard-sphere collision kernel:

$$B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma),$$

where $b(\cdot)$ satisfies (2.6). We argue in the self-similar variables; thus, it is convenient to define the rescaled p-moments

$$\mathbf{m}_p(\tau) = \int_{\mathbb{R}^3} g(\tau,w)\,|w|^{2p}\,\mathrm{d}w, \quad p\geqslant 0.$$

Notice that (5.9) readily translates into

$$c_p \leqslant \mathbf{m}_p(\tau) \leqslant C_p \quad \text{for} \quad \tau\geqslant 0. \tag{6.1}$$

The following theorem generalizes [23, Proposition 3.1] to the case of a variable restitution coefficient.

THEOREM 6.1 (L^1-exponential tails theorem). *Let $B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma)$ satisfy* (2.6) *with $b\in L^q(\mathbb{S}^2)$ for some $q>1$. Assume that $e(\cdot)$ and f_0 fulfill Assumption* 3.1 *and* (2.8), *respectively. Furthermore, assume that there exists $r_0>0$ such that*

$$\int_{\mathbb{R}^3} f_0(v)\exp(r_0|v|)\,\mathrm{d}v < \infty.$$

Let $g(\tau,w)$ be the rescaled solution defined by (2.10). *Then there exists some $r\leqslant r_0$ such that*

$$\sup_{\tau\geqslant 0}\int_{\mathbb{R}^3} g(\tau,w)\exp(r|w|)\,\mathrm{d}w < \infty. \tag{6.2}$$

Consequently,

$$\sup_{t\geqslant 0}\int_{\mathbb{R}^3} f(t,v)\exp(rV(t)|v|)\,\mathrm{d}w < \infty.$$

Proof. The method of proof is carefully documented in [2, 8, 11]. We sketch the proof dividing the argument in five steps.

Step 1. Note that formally

$$\int_{\mathbb{R}^3} g(\tau,w)\exp\left(r|w|^s\right)\,\mathrm{d}w=\sum_{k=0}^{\infty}\frac{r^k}{k!}\mathbf{m}_{sk/2}(\tau)$$

for any $r>0$ and any $s>0$. Hence, the summability of the integral is described by the behavior of the functions $\frac{\mathbf{m}_{sk/2}(\tau)}{k!}$. This motivates the introduction of the renormalized moments

$$z_p(\tau):=\frac{\mathbf{m}_p(\tau)}{\Gamma(ap+b)},\quad\text{with}\quad a=2/s,$$

where $\Gamma(\cdot)$ denotes the gamma function. We shall prove that the series converges for some $r<r_0$ and with $s=1$ (i.e., $a=2$). To do so, it is enough to prove that, for some $b<1$ and $Q>0$ large enough, one has $z_p(\tau)\leqslant Q^p$ for any $p\geqslant 1$ and any $\tau\geqslant 0$.

Step 2. Recall that, according to Lemma 2.6, the estimates of Proposition 2.7 are independent of the restitution coefficient $e(\cdot)$. In particular, they hold for the time-dependent collision operator $\mathcal{Q}_{\widetilde{e}_\tau}$, providing bounds which are *uniform* with respect to τ. Specifically,

$$\int_{\mathbb{R}^3}\mathcal{Q}_{\widetilde{e}_\tau}(g,g)(\tau,w)|w|^{2p}\,\mathrm{d}w\leqslant-(1-\kappa_p)\mathbf{m}_{p+1/2}(\tau)+\kappa_p\,\mathcal{S}_p(\tau)\qquad\forall\tau\geqslant 0,$$

where κ_p is the constant introduced in Lemma 2.6 and

$$\mathcal{S}_p(\tau)=\sum_{k=1}^{[\frac{p+1}{2}]}\left(\begin{array}{c}p\\k\end{array}\right)\left(\mathbf{m}_{k+1/2}(\tau)\,\mathbf{m}_{p-k}(\tau)+\mathbf{m}_k(\tau)\,\mathbf{m}_{p-k+1/2}(\tau)\right).$$

Step 3. An important simplification, first observed in [11], consists of noticing that the term $\mathcal{S}_p$ satisfies

$$\mathcal{S}_p(\tau)\leqslant A\,\Gamma(ap+a/2+2b)\,\mathcal{Z}_p(\tau)\text{ for }a\geqslant 1,\ b>0,$$

where $A=A(a,b)>0$ does not depend on p and

$$\mathcal{Z}_p(\tau)=\max_{1\leqslant k\leqslant k_p}\left\{z_{k+1/2}(\tau)\,z_{p-k}(\tau),z_k(\tau)\,z_{p-k+1/2}(\tau)\right\}.$$

With such an estimate, the rather involved term $\mathcal{S}_p$ is more tractable.

Step 4. Using the above steps and the evolution problem (5.1) satisfied by the rescaled solution g, we check that

$$\frac{\mathrm{d}\mathbf{m}_p}{\mathrm{d}\tau}(\tau)+(1-\kappa_p)\mathbf{m}_{p+1/2}(\tau)\leqslant\kappa_p\,\Gamma\left(ap+\frac{a}{2}+2b\right)\mathcal{Z}_p(\tau)+2p\,\xi(\tau)\mathbf{m}_p(\tau),$$

where we used the fact that

$$\int_{\mathbb{R}^3}|w|^{2p}\nabla_w\cdot(wg(\tau,w))\,\mathrm{d}w=-2p\,\mathbf{m}_p(\tau).$$

Using the asymptotic formula

$$\lim_{p\to\infty}\frac{\Gamma(p+r)}{\Gamma(p+s)}p^{s-r}=1,$$

the fact that $\xi(\tau) \leqslant 1$ and $\kappa_p \sim 1/p^{1/q'}$ for large p, one concludes that there are constants $c_i > 0$ $(i = 1, 2)$ and $p_0 > 1$ sufficiently large so that

$$\frac{\mathrm{d}z_p}{\mathrm{d}\tau}(\tau) + c_1\, p^{a/2} z_p^{1+1/2p}(\tau) \leqslant c_2\, p^{a/2+b-1/q'}\, \mathcal{Z}_p(\tau) + 2p\, z_p(\tau) \qquad \forall \tau \geqslant 0,\ p \geqslant p_0.$$

We also used that $\mathbf{m}_{p+1/2}(\tau) \geqslant \mathbf{m}_p^{1+1/2p}(\tau)$ for any $\tau \geqslant 0$ thanks to Jensen's inequality.

Final step. We claim that if we choose $a = 2$ and $0 < b < 1/q'$, it is possible to find $Q > 0$ large enough so that $\mathbf{m}_p(\tau) \leqslant Q^p$. Indeed, let p_0 and $Q < \infty$ such that

$$\frac{c_2}{c_1} p_0^{b-1/q'} \leqslant \frac{1}{2}, \quad \text{and} \quad Q \geqslant \left\{ \max_{1 \leqslant k \leqslant p_0} \sup_{\tau \geqslant 0} z_k(\tau), Q_0, \frac{16}{c_1^2}, 1 \right\},$$

where Q_0 is a constant such that $z_p(0) \leqslant Q_0^p$. This constant exists by the exponential integrability assumption on the initial datum. Moreover, since moments of g are uniformly propagated, the existence of such *finite* Q is guaranteed. Arguing by induction and standard comparison of ODEs, one proves that $y_p(\tau) := Q^p$ satisfies for $p \geqslant p_0$

$$\frac{\mathrm{d}y_p}{\mathrm{d}\tau}(\tau) + c_1\, p^{a/2} y_p^{1+1/2p}(\tau) \geqslant c_2\, p^{a/2+b-1/q'}\, \mathcal{Z}_p(\tau) + 2p\, y_p(\tau), \quad y_p(0) \geqslant z_p(0);$$

therefore, $y_p(\tau) \geqslant z_p(\tau)$ for any $p \geqslant p_0$. Since this is trivially true for $p < p_0$, we obtain that

$$\mathbf{m}_p(\tau) \leqslant \Gamma(2p+b) Q^p \qquad \forall p \geqslant 1, \tau \geqslant 0.$$

From Step 1, this is enough to prove the theorem. ∎

Example 6.2. For viscoelastic hard spheres, $V(t) = (1+t)^{5/3}$. Therefore,

$$\int_{\mathbb{R}^3} f_0(v) \exp\left(r_0|v|\right) \mathrm{d}v < \infty \Longrightarrow \sup_{t \geqslant 0} \int_{\mathbb{R}^3} f(t,v) \exp\left(r(1+t)^{5/3}|v|\right) \mathrm{d}v < \infty$$

for some $r < r_0$. In particular, using the terminology of [11], $f(t,v)$ has a (dynamic) exponential tail of order 1.

Appendix A: Viscoelastic hard spheres. In this appendix we prove that Assumption 3.1 is met by the restitution coefficient $e(\cdot)$ associated to the so-called viscoelastic hard spheres as derived in [27] (see also [12, Chapter 4]). In fact, we prove a more general result for the following hard-sphere collision kernel:

$$B(u,\sigma) = \frac{|u|}{4\pi} \qquad \forall u \in \mathbb{R}^3,\ \sigma \in \mathbb{S}^2.$$

Recall that $\mathbf{\Psi}_e$ was defined in (3.1) as

$$\mathbf{\Psi}_e(x) = \frac{1}{2\sqrt{x}} \int_0^{\sqrt{x}} \left(1 - e(z)^2\right) z^3 \,\mathrm{d}z, \qquad x > 0.$$

LEMMA A.1. *Assume that $e(\cdot)$ satisfies Assumption* 2.1 *and that the mapping $r \geqslant 0 \mapsto e(r)$ is decreasing. Then the associated function $\mathbf{\Psi}_e$ defined in* (3.1) *is strictly increasing and convex.*

Proof. Since e is decreasing, $e'(r) \leqslant 0$ for any $r \geqslant 0$. Here $e'(\cdot)$ denotes the derivative of $e(\cdot)$. Define

$$\Phi(x) := \frac{1}{x}\int_0^x \left(1-e^2(z)\right) z^3 \,\mathrm{d}z, \qquad x>0.$$

Note that $\mathbf{\Psi}_e(\cdot)$ is convex if and only if $x\Phi_{xx}(x)-\Phi_x(x) \geqslant 0$ for any $x>0$, where Φ_x and Φ_{xx} denote the first and second derivatives of Φ, respectively. A simple calculation shows that

$$x\Phi_{xx}(x)-\Phi_x(x) = -2x^3 e'(x)e(x) + \frac{3}{x^2}\int_0^x (1-e^2(z))z^3\,\mathrm{d}z \qquad \forall x>0.$$

Since $e'(x) \leqslant 0$ and $e(\cdot)\in(0,1]$, one concludes that $x\Phi_{xx}(x)-\Phi_x(x)\geqslant 0$ for any $x>0$.

Similarly, since $e'(\cdot)\leqslant 0$, the mapping $z\geqslant 0 \mapsto (1-e^2(z))z^3$ is nondecreasing; thus, $\Phi_x(x)>0$ for any $x>0$. This implies that $\mathbf{\Psi}_e(\cdot)$ is strictly increasing over $(0,+\infty)$. ∎

For the viscoelastic hard spheres, as derived in [27], the restitution coefficient e is the solution of the equation

$$e(r)+\alpha\, r^{1/5} e(r)^{3/5} = 1 \qquad \forall r\geqslant 0, \tag{A.1}$$

where $\alpha>0$ is a constant depending on the material viscosity. It was proved in [1, p. 1006] that, on the basis of (A.1), Assumption 2.1 is met. From (A.1), one deduces that

$$\lim_{r\to 0^+} e(r)=1, \quad \text{and} \quad e(r)\simeq 1-\alpha r^{1/5} \quad \text{for} \quad r\simeq 0,$$

which means that Assumption 3.1(1) is met. Furthermore, (A.1) also implies that e is continuously decreasing. According to Lemma A.1, $e(\cdot)$ satisfies Assumption 3.1. Moreover, it is easy to deduce from (A.1) that Assumption 4.10 is satisfied.

*Example A.*2. For monotone decreasing restitution coefficient introduced in Example 2.3, Assumption 3.1 is also met by virtue of the above lemma. In such a case, according to (2.2), the cooling of the temperature $\mathcal{E}(t)$ is

$$\mathcal{E}(t) = O\left((1+t)^{-\frac{2}{1+\eta}}\right) \text{ as } t\to\infty.$$

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