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LOCAL WAVE-FRONT SETS OF BANACH AND FRÉCHET TYPES, AND PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. Let ω, ω_0 be appropriate weight functions and $\mathscr{B}$ be an invariant BF-space. We introduce the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ with respect to the weighted Fourier Banach space $\mathcal{B} = \mathscr{F}\mathscr{B}(\omega)$. We prove that the usual mapping properties for pseudo-differential operators $\mathrm{Op}_t(a)$ with symbols a in $S^{(\omega_0)}_{\rho,0}$ hold for such wave-front sets. In particular we prove $\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}_{\mathcal{B}}(f)$ and $\mathrm{WF}_{\mathcal{B}}(f) \subseteq \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}(a)$. Here $\mathcal{C} = \mathscr{F}\mathscr{B}(\omega/\omega_0)$ and $\mathrm{Char}(a)$ is the set of characteristic points of a.

0. Introduction

In this paper we consider (local) wave-front sets with respect to appropriate Banach and Fréchet spaces. Especially we focus on the case when these spaces agree with Fourier images of translation invariant Banach function spaces (BF-spaces). The family of such wave-front sets contains the wave-front sets of Sobolev type, introduced by Hörmander in [24], the classical wave-front sets (cf. Sections 8.1 and 8.2 in [23]), and wave-front sets of Fourier Lebesgue types, introduced in [29]. Roughly speaking, for any given distribution f and for appropriate Banach (or Frechét) space $\mathcal{B}$ of temperate distributions, the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that no localizations of the distribution at x_0 belongs to $\mathcal{B}$ in the direction ξ_0.

We also establish mapping properties for a quite general class of pseudo-differential operators on such wave-front sets, and show that the micro-local analysis in [29] in background of Fourier Lebesgue spaces can be further generalized. It follows that our approach gives rise to flexible micro-local analysis tools which fit well to the most common approach developed in e.g. [23, 24]. In particular, we prove that usual mapping properties, which are valid for classical wave-front sets (cf. Chapters VIII and XVIII in [23]), also hold for wave-front sets of Fourier BF-types. For example, we show that for an appropriate space $\mathcal{C}$ which is completely determined by $\mathcal{B}$ and the symbol class for a we have

$$\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}_{\mathcal{B}}(f) \subseteq \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}(a). \tag{0.1}$$

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That is, any operator $\operatorname{Op}_t(a)$ shrinks the wave-front sets and opposite embeddings can be obtained by including $\operatorname{Char}(a)$, the set of characteristic points of the operator symbol a.

The symbol classes for the pseudo-differential operators are given by $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, the set of all smooth functions a on $\mathbf{R}^{2d}$ such that $a/\omega_0 \in S^0_{\rho,\delta}(\mathbf{R}^{2d})$. Here $\rho, \delta \in \mathbf{R}$ and ω_0 is an appropriate smooth function on $\mathbf{R}^{2d}$. We note that $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ agrees with the Hörmander class $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ when $\omega_0(x,\xi) = \langle \xi \rangle^r$, where $r \in \mathbf{R}$ and $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$.

The set of characteristic points $\operatorname{Char}(a)$ of $a \in S^{(\omega_0)}_{\rho,\delta}$ is the same as in [29], and depends on the choices of ρ, δ and ω_0 (see Definition 1.8 and Proposition 2.3). We recall that this set is smaller than the set of characteristic points given by [23]. It is empty when a satisfies a local ellipticity condition with respect to ω_0, which is fulfilled for any hypoelliptic partial differential operator with constant coefficients (cf. [29]). As a consequence of (0.1), it follows that such hypoelliptic operators preserve the wave-front sets, as expected (cf. Example 4.9 in [29]).

In view of their definition, the information about regularity of distributions in background of the wave-front sets of Fourier BF-types might be more detailed compared to classical wave-front sets, because the family of Fourier BF-spaces is broad and such spaces can locally be chosen to be "arbitrary close" to C^∞, the set of smooth functions. In this context, the classical wave-front set is exactly the wave-front set with respect to C^∞. For example, the space $\mathscr{F}\mathscr{B}(\omega) = \mathscr{F}L^1_{(\omega)}(\mathbf{R}^d)$, with $\omega(\xi) = \langle \xi \rangle^N$ for some integer $N \geq 0$, is locally close to $C^N(\mathbf{R}^d)$ (cf. the introduction of [29]). Consequently, the wave-front set with respect to $\mathscr{F}L^1_{(\omega)}$ can be used to investigate a sort of regularity which is close to smoothness of order N.

Furthermore, we are able to apply our results on pseudo-differential operators in the context of modulation space theory, when discussing mapping properties of pseudo-differential operators with respect to wave-front sets. The modulation spaces were introduced by Feichtinger in [6], and the theory was developed in [9–11, 14]. The modulation space $M(\omega, \mathscr{B})$, where ω is an appropriate weight function (or time-frequency shift) on the phase space $\mathbf{R}^{2d}$, appears as the set of temperate (ultra-) distributions whose short-time Fourier transform belong to the weighted Banach space $\mathscr{B}(\omega)$. This family of modulation spaces contains the (classical) modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ as well as the space $W^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ related to the Wiener amalgam spaces. In fact, these spaces which occur frequently in the time-frequency community are obtained by choosing $\mathscr{B} = L^{p,q}_1(\mathbf{R}^{2d})$ or $\mathscr{B} = L^{p,q}_2(\mathbf{R}^{2d})$ (see Remark 6.1).

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of the study of continuity of (classical) pseudo-differential operators acting on modulation spaces (cf. [5, 27, 28, 35–37]), and the study of operators of non-classical type, where modulation spaces are used as symbol classes. We refer to [15–19, 21, 22, 27, 32, 33, 38–40, 42] for more facts about pseudo-differential operators in background of modulation space theory.

In the last part of the paper we define wave-front sets with respect to weighted modulation spaces, and prove that they coincide with the wave-front sets of Fourier BF-types.

The paper is organized as follows. In Section 1 we recall the definitions and basic properties for pseudo-differential operators, translation invariant Banach function spaces (BF-spaces) and (weighted) Fourier Banach spaces. Here we also define sets of characteristic points for a broad class of pseudo-differential operators. In Section 2 we prove some properties for the sets of characteristic points, which shows that our definition coincide with the sets of characteristic points defined in [29]. These sets might be smaller than "the classical" characteristic sets in [23] (cf. [29, Example 3.9]).

In Section 3 we define wave-front sets with respect to (weighted) Fourier BF-spaces, and prove some of their main properties. Thereafter, in Section 4 we show how these wave-front sets propagate under the action of pseudo-differential operators. In particular, we prove (0.1), when $\mathcal{B}$ and $\mathcal{C}$ are appropriate spaces and a belongs to $S^{(\omega_0)}_{\rho,0}$ with $\rho > 0$.

In Section 5 we consider wave-front sets obtained from sequences of Fourier BF-spaces. These types of wave-front sets contain the classical ones (with respect to smoothness), and the mapping properties for pseudo-differential operators also hold in this context (cf. Section 18.1 in [23]).

In Section 6 we define wave-front sets with respect to modulation spaces, and prove that they can be identified with wave-front sets of Fourier BF-types. This part can also be considered as a starting point for discussions of global wave-front sets of modulation space types, which are investigated in [3]. Here we remark that a notion of global wave-front sets with respect to smoothness and weighted Sobolev spaces was introduced and investigated in [4, 25].

1. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted. In what follows we let Γ denote an open cone in $\mathbf{R}^d \setminus 0$. An open cone which contains $\xi \in \mathbf{R}^d \setminus 0$ is sometimes denoted by Γ_ξ.

Let $\omega, v \in L^\infty_{loc}(\mathbf{R}^d)$ be positive functions. Then ω is called v-moderate if

$$\omega(x+y) \le C\omega(x)v(y) \tag{1.1}$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderate. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. We say that v is *submultiplicative* when (1.1) holds with $\omega = v$. Throughout, we assume that the submultiplicative weights are even. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation.

We also need to consider classes of weight functions, related to $\mathscr{P}$. More precisely, we let $\mathscr{P}_0(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d) \bigcap C^\infty(\mathbf{R}^d)$ such that $\partial^\alpha \omega/\omega \in L^\infty$ for all multi-indices α. For each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is an equivalent weight $\omega_0 \in \mathscr{P}_0(\mathbf{R}^d)$, that is, $C^{-1}\omega_0 \le \omega \le C\omega_0$ holds for some constant C (cf. [40, Lemma 1.2]).

Assume that $\rho, \delta \in \mathbf{R}$. Then we let $\mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ be the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle \xi \rangle^{\rho|\beta|-\delta|\alpha|} \frac{(\partial_x^\alpha \partial_\xi^\beta \omega)(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}),$$

for every multi-indices α and β. Note that in contrast to $\mathscr{P}_0$, we do not have an equivalence between $\mathscr{P}_{\rho,\delta}$ and $\mathscr{P}$ when $\rho > 0$. On the other hand, if $s \in \mathbf{R}$ and $\rho \in [0,1]$, then $\mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ contains $\omega(x,\xi) = \langle \xi \rangle^s$, which are one of the most important type of weights in the applications.

For any weight ω in $\mathscr{P}(\mathbf{R}^d)$, we let $L^p_{(\omega)}(\mathbf{R}^d)$ be the set of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $f \cdot \omega \in L^p(\mathbf{R}^d)$. We also set $L^p_s(\mathbf{R}^d) = L^p_{(\omega)}$ when $\omega(x) = \langle x \rangle^s$ and $s \in \mathbf{R}$.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi \rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Next we recall the definition of Banach function spaces.

Definition 1.1. Assume that $\mathscr{B}$ is a Banach space of complex-valued measurable functions on $\mathbf{R}^d$ and that $v \in \mathscr{P}(\mathbf{R}^d)$ is submultiplicative. Then $\mathscr{B}$ is called a *(translation) invariant Banach function space (BF-space) on* $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:

(1) $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings).

(2) If $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\|f(\cdot - x)\|_{\mathscr{B}} \le C v(x) \|f\|_{\mathscr{B}}. \tag{1.2}$$

(3) if $f, g \in L^1_{loc}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \le C\|g\|_{\mathscr{B}}.$$

(4) if $f \in \mathscr{B}$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$, then $f * \varphi \in \mathscr{B}$, and

$$\|f * \varphi\|_{\mathscr{B}} \le \|\varphi\|_{L^1_{(v)}}\|f\|_{\mathscr{B}}.$$

Assume that $\mathscr{B}$ is a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|h\|_{L^\infty}. \tag{1.3}$$

The last condition in the definition of BF-spaces is natural in view of Lebesgue spaces. It is also convenient to assume that the convolution map $*$ can be continuously extended to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, such that

$$\|\varphi * f\|_{\mathscr{B}} \le C\|\varphi\|_{L^1_{(v)}}\|f\|_{\mathscr{B}}, \tag{1.4}$$

holds for some constant C, when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. In fact, if $f, g \in \mathscr{S}$, then $f * g \in \mathscr{S} \subseteq \mathscr{B}$ for $\mathscr{B} = L^p$, and Minkowski's inequality gives

$$\|f * g\|_{\mathscr{B}} = \Big\| \int f(\,\cdot\, - y)g(y)\,dy \Big\|_{\mathscr{B}}$$

$$\le \int \|f(\,\cdot - y)\|_{\mathscr{B}}|g(y)|\,dy \le C\int \|f\|_{\mathscr{B}}|g(y)v(y)|\,dy = C\|f\|_{\mathscr{B}}\|g\|_{L^1_{(v)}}.$$

From now on we assume that each translation invariant BF-space $\mathscr{B}$ is such that the convolution map $*$ on $\mathscr{S}(\mathbf{R}^d)$ is *uniquely* extendable to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, and that (1.4) holds when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. We note that $\mathscr{B}$ can be any mixed and weighted Lebesgue space.

Remark 1.2. Assume that $\omega_0, v, v_0 \in \mathscr{P}(\mathbf{R}^d)$ are such that v and v_0 are submultiplicative, ω_0 is v_0-moderate, and assume that $\mathscr{B}$ is a translation-invariant BF-space on $\mathbf{R}^d$ with respect to v. Also let $\mathscr{B}_0$ be the Banach space which consists of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}_0} \equiv \|f\,\omega_0\|_{\mathscr{B}}$ is finite. Then $\mathscr{B}_0$ is a translation invariant BF-space with respect to $v_0 v$.

Remark 1.3. Let $\mathscr{B}$ be an invariant BF-space. Then it is easy to find Sobolev type spaces which are continuously embedded in $\mathscr{B}$. In fact, for each $p \in [1, \infty]$ and integer $N \ge 0$, let $Q^p_N(\mathbf{R}^d)$ be the set of all $f \in L^p(\mathbf{R}^d)$ such that $\|f\|_{Q^p_N} < \infty$, where

$$\|f\|_{Q^p_N} \equiv \sum_{|\alpha+\beta| \le N} \|x^\alpha D^\beta f\|_{L^p}.$$

Then for each p fixed, the topology for $\mathscr{S}(\mathbf{R}^d)$ can be defined by the semi-norms $f \mapsto \|f\|_{Q_N^p}$, for $N = 0, 1, \dots$. A combination of this fact and (1) and (3) in Definition 1.1 now shows that for each $p \in [1, \infty]$ and each translation invariant BF-space $\mathscr{B}$, there is an integer $N \ge 0$ such that $Q_N^p(\mathbf{R}^d) \subseteq \mathscr{B}$. This proves the assertion.

In particular it follows that $\langle \cdot \rangle^{-N} \in \mathscr{B}$, provided $N \ge 0$ is chosen large enough. This gives

$$\|f\|_{\mathscr{B}} = \|\langle \cdot \rangle^{-N}(f\langle \cdot \rangle^N)\|_{\mathscr{B}} \le C_1 \|\langle \cdot \rangle^{-N}\|_{\mathscr{B}} \|f\langle \cdot \rangle^N\|_{L^\infty} = C_2\|f\|_{L_N^\infty},$$

for some costants C_1 and C_2. Hence $L_N^\infty \subseteq \mathscr{B}$ for some $N \ge 0$.

For each translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^d$, and each pair of vector spaces (V_1, V_2) such that $V_1 \oplus V_2 = \mathbf{R}^d$, we define the projection spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\mathscr{B}$ by the formulae

$$\mathscr{B}_1 \equiv \{\, f \in \mathscr{S}'(V_1)\, ;\, f \otimes \varphi \in \mathscr{B} \text{ for every } \varphi \in \mathscr{S}(V_2)\,\} \qquad (1.5)$$

and

$$\mathscr{B}_2 \equiv \{\, f \in \mathscr{S}'(V_2)\, ;\, \varphi \otimes f \in \mathscr{B} \text{ for every } \varphi \in \mathscr{S}(V_1)\,\}. \qquad (1.6)$$

Proposition 1.4. *Assume that $f \in \mathscr{S}'$, $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the same as in* (1.5) *and* (1.6). *Then*

$$\mathscr{B}_1 = \{\, f \in \mathscr{S}'(V_1)\, ;\, f \otimes \varphi \in \mathscr{B} \text{ for some } \varphi \in \mathscr{S}(V_2) \setminus 0\,\} \qquad (1.5)'$$

and

$$\mathscr{B}_2 = \{\, f \in \mathscr{S}'(V_2)\, ;\, \varphi \otimes f \in \mathscr{B} \text{ for some } \varphi \in \mathscr{S}(V_1) \setminus 0\,\}. \qquad (1.6)'$$

In particular, if $\varphi_j \in \mathscr{S}(V_j) \setminus 0$ for $j = 1, 2$ are fixed, then $\mathscr{B}_1$ and $\mathscr{B}_2$ are translation invariant BF-spaces under the norms

$$\|f\|_{\mathscr{B}_1} \equiv \|f \otimes \varphi_1\|_{\mathscr{B}} \quad \text{and} \quad \|f\|_{\mathscr{B}_2} \equiv \|\varphi_2 \otimes f\|_{\mathscr{B}}$$

respectively.

Proof. We only prove (1.6)′. The equality (1.5)′ follows by similar arguments and is left for the reader. We may assume that $V_j = \mathbf{R}^{d_j}$ with $d_1 + d_2 = d$.

Let $\mathscr{B}_0$ be the right-hand side of (1.6)′. Then it is obvious that $\mathscr{B}_2 \subseteq \mathscr{B}_0$. We have to prove the opposite inclusion.

Therefore, assume that $f \in \mathscr{B}_0$, and choose $\varphi_0 \in \mathscr{S}(\mathbf{R}^{d_1}) \setminus 0$ such that $\varphi_0 \otimes f \in \mathscr{B}$. Also let $\varphi \in \mathscr{S}(\mathbf{R}^{d_1})$ be arbitrary. We shall prove that $\varphi \otimes f \in \mathscr{B}$.

First we assume that $\varphi \in C_0^\infty(\mathbf{R}^d)$. Let $Q \subseteq \mathbf{R}^{d_1}$ be an open ball and $c > 0$ be chosen such that $|\varphi_0(x)| > c$ when $x \in Q$. Also let the lattice $\Lambda \subseteq \mathbf{R}^{d_1}$ and $\varphi_1 \in C_0^\infty(Q)$ be such that $0 \le \varphi_1 \le 1$ and

$$\sum_{\{x_j\} \in \Lambda} \varphi_1(\,\cdot\, - x_j) = 1,$$

and let J be a finite set such that $\sum_{j\in J}\varphi_1(\,\cdot\,-x_j)=1$ on $\operatorname{supp}\varphi$. Then $|\varphi_1\varphi(\,\cdot\,+x_j)|\le C|\varphi_0|$, for some constant $C>0$, which gives $(\varphi_1(\,\cdot\,-x_j)\varphi)\otimes f\in\mathscr B$ and

$$\|(\varphi_1(\,\cdot\,-x_j)\varphi)\otimes f\|_{\mathscr B}\le C_1\|\varphi_0(\,\cdot\,-x_j)\otimes f\|_{\mathscr B}\le C_2 v(x_j,0)\|\varphi_0\otimes f\|_{\mathscr B},$$

for some constants C_1 and C_2. From this fact together with the formula

$$\varphi\otimes f=\sum_{j\in J}(\varphi_1(\,\cdot\,-x_j)\varphi)\otimes f,$$

with finite sum on the right-hand side, it follows that $\varphi\otimes f\in\mathscr B$, and

$$\begin{aligned}\|\varphi\otimes f\|_{\mathscr B}&\le\sum\|(\varphi_1(\,\cdot\,-x_j)\varphi)\otimes f\|_{\mathscr B}\\&\le\sum v(x_j,0)\|(\varphi_1\varphi(\,\cdot\,+x_j))\otimes f\|_{\mathscr B}\\&\le C\Big(\sum v(x_j,0)\|\varphi(\,\cdot\,+x_j)\|_{L^\infty(Q)}\Big)\|\varphi_1\otimes f\|_{\mathscr B},\end{aligned}\tag{1.7}$$

where the sums are taken over all $j\in J$. Since $v\in\mathscr P$ and $\varphi\in\mathscr S$, it follows that the sum in the right-hand side of (1.7) is finite. Hence $f\in\mathscr B_2$, and we have proved the assertion in the case $\varphi\in C_0^\infty$. The result now follows for general $\varphi\in\mathscr S$ from (1.7) and the fact that C_0^∞ is dense in $\mathscr S$. The proof is complete. □

Remark 1.5. We note that the last sum in (1.7) is the norm

$$\|\varphi\|_{W_{(v)}}\equiv\sum v(x_j,0)\|\varphi(\,\cdot\,+x_j)\|_{L^\infty(Q)}$$

for the weighted Wiener space

$$W_{(v)}(\mathbf R^d)=\{\,f\in L^\infty_{loc}(\mathbf R^d)\,;\,\|f\|_{W_{(v)}}<\infty\,\}$$

(cf. [15]). The results in Proposition 1.4 can therefore be improved in such way that we may replace $\mathscr S$ by $W_{(v)}$ in (1.5), (1.6), (1.5)$'$ and (1.6)$'$.

Assume that $\mathscr B$ is a translation invariant BF-space on $\mathbf R^d$, and that $\omega\in\mathscr P(\mathbf R^d)$. Then we let $\mathscr F\mathscr B(\omega)$ be the set of all $f\in\mathscr S'(\mathbf R^d)$ such that $\xi\mapsto\widehat f(\xi)\omega(\xi)$ belongs to $\mathscr B$. It follows that $\mathscr F\mathscr B(\omega)$ is a Banach space under the norm

$$\|f\|_{\mathscr F\mathscr B(\omega)}\equiv\|\widehat f\,\omega\|_{\mathscr B}.\tag{1.8}$$

Remark 1.6. In many situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier BF-spaces. More precisely, for each $\omega\in\mathscr P(\mathbf R^{2d})$ and each translation invariant BF-space $\mathscr B$ on $\mathbf R^d$, we let $\mathscr F\mathscr B(\omega)$ be the set of all $f\in\mathscr S'(\mathbf R^d)$ such that

$$\|f\|_{\mathscr F\mathscr B(\omega)}=\|f\|_{\mathscr F\mathscr B(\omega),x}\equiv\|\widehat f\,\omega(x,\,\cdot\,)\|_{\mathscr B}$$

is finite. Since ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$ it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr{F}\mathscr{B}(\omega)} < \infty$ is independent of x, and it follows that $\mathscr{F}\mathscr{B}(\omega)$ is independent of x although $\| \cdot \|_{\mathscr{F}\mathscr{B}(\omega)}$ might depend on x.

Recall that a topological vector space $V \subseteq \mathscr{D}'(X)$ is called *local* if $V \subseteq V_{loc}$. Here $X \subseteq \mathbf{R}^d$ is open, and V_{loc} consists of all $f \in \mathscr{D}'(X)$ such that $\varphi f \in V$ for every $\varphi \in C_0^\infty(X)$. We also let $V_{comp} = V \cap \mathscr{E}'(X)$. This gives $V_{comp} \subseteq V$. For future references we note that if $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, then it follows from (1.4) that $\mathscr{F}\mathscr{B}(\omega)$ is a local space, i. e.

$$\mathscr{F}\mathscr{B}(\omega) \subseteq \mathscr{F}\mathscr{B}(\omega)_{loc} \equiv (\mathscr{F}\mathscr{B}(\omega))_{loc}. \tag{1.9}$$

Next we recall some facts from Chapter XVIII in [23] concerning pseudo-differential operators. Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\, dyd\xi. \tag{1.10}$$

For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{1.11}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and

$$F(x,y) \mapsto F((1-t)x+ty, x-y)$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. We also note that the latter definition of $\operatorname{Op}_t(a)$ agrees with the operator in (1.10) when $a \in \mathscr{S}(\mathbf{R}^{2d})$. If $t = 0$, then $\operatorname{Op}_t(a)$ agrees with the Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x,D)$.

If $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $s,t \in \mathbf{R}$, then there is a unique $b \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$\operatorname{Op}_s(a) = \operatorname{Op}_t(b) \quad \iff \quad b(x,\xi) = e^{i(t-s)\langle D_x, D_\xi\rangle}a(x,\xi). \tag{1.12}$$

(Cf. Section 18.5 in [23].)

Next we discuss symbol classes which we use. Let $r, \rho, \delta \in \mathbf{R}$ be fixed. Then we recall from [23] that $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for each pair of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{r-\rho|\beta|+\delta|\alpha|}.$$

Usually we assume that $0 \le \delta \le \rho \le 1$, $0 < \rho$ and $\delta < 1$.

More generally, assume that $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. Then we recall from the introduction that $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\omega(x,\xi)\langle \xi\rangle^{-\rho|\beta|+\delta|\alpha|}. \tag{1.13}$$

We note that $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d}) = S(\omega, g_{\rho,\delta})$, when $g = g_{\rho,\delta}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{\rho,\delta}\big)_{(y,\eta)}(x,\xi) = \langle \eta\rangle^{2\delta}|x|^2 + \langle \eta\rangle^{-2\rho}|\xi|^2$$

(cf. Section 18.4–18.6 in [23]). Furthermore, $S^{(\omega)}_{\rho,\delta} = S^r_{\rho,\delta}$ when $\omega(x,\xi) = \langle \xi\rangle^r$, as remarked in the introduction.

The following result shows that pseudo-differential operators with symbols in $S^{(\omega)}_{\rho,\delta}$ behave well. We refer to [23] or [29] for the proof.

Proposition 1.7. *Let $\rho,\delta \in [0,1]$ be such that $0 \le \delta \le \rho \le 1$ and $\delta < 1$, and let $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. If $a \in S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, then $\operatorname{Op}_t(a)$ is continuous on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$.*

If $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, then it follows from the definitions that there is a constant $C > 0$ such that

$$|a(x,\xi)| \le C\omega_0(x,\xi).$$

On the other hand, a necessary and sufficient condition for a to be invertible, in the sense that $1/a$ is a symbol in $S^{(1/\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, is that for some constant $c > 0$ we have

$$c\,\omega_0(x,\xi) \le |a(x,\xi)|. \tag{1.14}$$

In the following we discuss more local invertibility conditions for symbols in $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ in terms of sets of characteristic points of the involved symbols. We remark that our definition of such sets is slightly different compared to [23, Definition 18.1.5] in view of Remark 2.4 in the next section.

Definition 1.8. Assume that $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ and that $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$. Then a is called *ψ-invertible* or *ψ-elliptic* with respect to ω_0 at the point $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d\setminus 0)$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and positive constants R and c such that (1.14) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \ge R$.

The point (x_0,ξ_0) is called *ψ-characteristic* for a with respect to ω_0 if a is *not* ψ-invertible with respect to ω_0 at (x_0,ξ_0).

The set of ψ-characteristic points (the ψ-characteristic set), for a symbol $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ with respect to ω_0, is denoted by $\operatorname{Char}(a) = \operatorname{Char}_{(\omega_0)}(a)$.

Remark 1.9. In the case $\omega_0 = 1$ we exclude the phrase "with respect to ω_0" in Definition 1.8. The symbol $a \in S^0_{\rho,\delta}(\mathbf{R}^{2d})$ is *ψ-invertible* at $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d\setminus 0)$ if $(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(a)$ with $\omega_0 = 1$. This means that there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R, c > 0$ such that (1.14) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \ge R$.

We note that $(x_0, \xi_0) \notin \operatorname{Char}_{(\omega_0)}(a)$ means that a is elliptic near x_0 in the direction ξ_0.

It will also be convenient to have the following definition of different types of cutoff functions.

Definition 1.10. Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

(1) A smooth function φ on $\mathbf{R}^d$ is called a *cutoff function* with respect to x_0 and X, if $0 \le \varphi \le 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoff functions with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$;

(2) A smooth function ψ on $\mathbf{R}^d$ is called a *directional cutoff function* with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
- $0 \le \psi \le 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
- $\psi(t\xi) = \psi(\xi)$ when $t \ge 1$ and $|\xi| \ge R$;
- $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \ge R$.

The set of directional cutoff functions with respect to ξ_0 and Γ is denoted by $\mathscr{C}_{\xi_0}^{\mathrm{dir}}(\Gamma)$.

Remark 1.11. We note that if $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}(\Gamma)$ for some $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, then $c \equiv \varphi \otimes \psi$ belongs to $S_{1,0}^0(\mathbf{R}^{2d})$ and is ψ-invertible at (x_0, ξ_0).

2. Pseudo-differential calculus with symbols in $S_{\rho,\delta}^{(\omega)}$

In this section we make a review of basic results for pseudo-differential operators with symbols in classes of the form $S_{\rho,\delta}^{(\omega)}(\mathbf{R}^{2d})$, when $0 \le \delta < \rho \le 1$ and $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. For the standard properties in the pseudo-differential calculus we only state the results and refer to [23] for the proofs.

Some investigations concerns establishment of convenient properties for sets of characteristic points. Such questions are closely related to invertibility of symbols. Similar properties have earlier been proved for such sets (see e. g. [23,29]). However, in order to be more self-contained we include proofs of such properties here.

We start with the following result concerning compositions and invariance properties for pseudo-differential operators. Here we set

$$\sigma_s(x, \xi) = \langle \xi \rangle^s,$$

where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$ as before. We also recall that

$$S^{-\infty}_{\rho,\delta}(\mathbf{R}^{2d}) = S^{-\infty}_{1,0}(\mathbf{R}^{2d}) = S^{-\infty}(\mathbf{R}^{2d})$$

consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for each $N \in \mathbf{R}$ and multi-index α, there is a constant $C_{N,\alpha}$ such that

$$|\partial^\alpha a(x,\xi)| \le C_{N,\alpha}\langle \xi \rangle^{-N}.$$

Proposition 2.1. *Let $0 \le \delta < \rho \le 1$, $\mu = \rho - \delta > 0$ and $\omega, \omega_1, \omega_2 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. Also let $\{m_j\}_{j=0}^\infty$ be a sequence of real numbers such that $m_j \to -\infty$ as $j \to \infty$. Then the following is true:*

(1) *if $a_1 \in S^{(\omega_1)}_{\rho,\delta}(\mathbf{R}^{2d})$ and $a_2 \in S^{(\omega_2)}_{\rho,\delta}(\mathbf{R}^{2d})$, then $\operatorname{Op}(a_1) \circ \operatorname{Op}(a_2) = \operatorname{Op}(c)$, for some $c \in S^{(\omega_1\omega_2)}_{\rho,\delta}(\mathbf{R}^{2d})$. Furthermore,*

$$c(x,\xi) - \sum_{|\alpha|<N} \frac{i^{|\alpha|}(D^\alpha_\xi a_1)(x,\xi)(D^\alpha_x a_2)(x,\xi)}{\alpha!} \in S^{(\omega_1\omega_2\sigma_{-N\mu})}_{\rho,\delta}(\mathbf{R}^{2d}) \quad (2.1)$$

for every $N \ge 0$;

(2) *if $M = \sup_{k\ge 0}(m_k)$, $M_j = \sup_{k \ge j}(m_k)$ and $a_j \in S^{(\omega\sigma_{m_j})}_{\rho,\delta}(\mathbf{R}^{2d})$, then it exists $a \in S^{(\omega\sigma_M)}_{\rho,\delta}(\mathbf{R}^{2d})$ such that*

$$a(x,\xi) - \sum_{j<N} a_j(x,\xi) \in S^{(\omega\sigma_{M_N})}_{\rho,\delta}(\mathbf{R}^{2d}); \quad (2.2)$$

for every $N \ge 0$;

(3) *if $a, b \in \mathscr{S}'(\mathbf{R}^{2d})$ and $s, t \in \mathbf{R}$ are such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$, then $a \in S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, if and only if $b \in S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, and*

$$b(x,\xi) - \sum_{k<N} \frac{(i(t-s)\langle D_x, D_\xi\rangle)^k a(x,\xi)}{k!} \in S^{(\omega\sigma_{-N\mu})}_{\rho,\delta}(\mathbf{R}^{2d}) \quad (2.3)$$

for every $N \ge 0$.

As usual we write

$$a \sim \sum a_j \quad (2.2)'$$

when (2.2) is fulfilled for every $N \ge 0$. In particular it follows from (2.1) and (2.3) that

$$c \sim \sum \frac{i^{|\alpha|}(D^\alpha_\xi a_1)(D^\alpha_x a_2)}{\alpha!} \quad (2.1)'$$

when $\operatorname{Op}(a_1) \circ \operatorname{Op}(a_2) = \operatorname{Op}(c)$, and

$$b \sim \sum \frac{(i(t-s)\langle D_x, D_\xi\rangle)^k a}{k!} \quad (2.3)'$$

when $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$.

In the following proposition we show that the set of characteristic points for a pseudo-differential operator is independent of the choice of pseudo-differential calculus.

Proposition 2.2. *Assume that $s,t \in \mathbf{R}$, $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}$ and that $a,b \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ satisfy $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. Then*

$$\operatorname{Char}_{(\omega_0)}(a) = \operatorname{Char}_{(\omega_0)}(b). \tag{2.4}$$

Proof. Let μ and σ_s be the same as in Proposition 2.1. By Proposition 2.1 (3) we have

$$b = a + h,$$

for some $h \in S^{(\omega_0\sigma_{-\mu})}_{\rho,\delta}$.

Assume that $(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(a)$. By the definitions, there is a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0, $C>0$ and $R>0$ such that

$$|a(x,\xi)| \ge C\omega_0(x,\xi) \quad \text{and} \quad |h(x,\xi)| \le C\omega_0(x,\xi)/2,$$

as $x \in X$, $\xi \in \Gamma$ and $|\xi| \ge R$. This gives

$$|b(x,\xi)| \ge C\omega_0(x,\xi)/2, \quad \text{when} \quad x \in X,\ \xi \in \Gamma,\ |\xi| \ge R,$$

and it follows that $(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(b)$. Hence $\operatorname{Char}_{(\omega_0)}(b) \subseteq \operatorname{Char}_{(\omega_0)}(a)$. By symmetry, the opposite inclusion also holds. Hence $\operatorname{Char}_{(\omega_0)}(a) = \operatorname{Char}_{(\omega_0)}(b)$, and the proof is complete. □

The following proposition shows different aspects of set of characteristic points, and is important when investigating wave-front properties for pseudo-differential operators. In particular it shows that $\operatorname{Op}(a)$ satisfy certain invertibility properties outside the set of characteristic points for a. More precisely, outside $\operatorname{Char}_{(\omega_0)}(a)$, we prove that

$$\operatorname{Op}(b)\operatorname{Op}(a) = \operatorname{Op}(c) + \operatorname{Op}(h), \tag{2.5}$$

for some convenient b, c and h which take the role of inverse, identity symbol and smoothing remainder respectively.

Proposition 2.3. *Let $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, and let $\mu = \rho - \delta$. Then the following conditions are equivalent:*

(1) *$(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(a)$;*
(2) *there is an element $c \in S^0_{\rho,\delta}$ which is ψ-invertible at (x_0,ξ_0), and an element $b \in S^{(1/\omega_0)}_{\rho,\delta}$ such that $ab = c$;*
(3) *there is an element $c \in S^0_{\rho,\delta}$ which is ψ-invertible at (x_0,ξ_0), and elements $h \in S^{-\mu}_{\rho,\delta}$ and $b \in S^{(1/\omega_0)}_{\rho,\delta}$ such that (2.5) holds;*
(4) *for each neighbourhood X of x_0 and conical neighbourhood Γ of ξ_0, there is an element $c = \varphi \otimes \psi$ where $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, and elements $h \in \mathscr{S}$ and $b \in S^{(1/\omega_0)}_{\rho,\delta}$ such that (2.5) holds. Furthermore, the supports of b and h are contained in $X \times \mathbf{R}^d$.*

For the proof we note that μ in Proposition 2.3 is positive, which in turn implies that $\cap_{j\ge 0} S^{(\omega_0\sigma_{-j\mu})}_{\rho,\delta}(\mathbf{R}^{2d})$ agrees with $S^{-\infty}(\mathbf{R}^{2d})$.

Proof. The equivalence between (1) and (2) follows by letting $b(x,\xi) = \varphi(x)\psi(\xi)/a(x,\xi)$ for some appropriate $\varphi\in\mathscr{C}_{x_0}(\mathbf{R}^d)$ and $\psi\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d\setminus 0)$.

(4) $\Rightarrow$ (3) is obvious in view of Remark 1.11. Assume that (3) holds. We shall prove that (1) holds, and since $|b|\le C/\omega_0$, it suffices to prove that

$$|a(x,\xi)b(x,\xi)|\ge 1/2 \tag{2.6}$$

when

$$(x,\xi)\in X\times\Gamma,\ \ |\xi|\ge R \tag{2.7}$$

holds for some conical neighbourhood Γ of ξ_0, some open neighbourhood X of x_0 and some $R>0$.

By Proposition 2.1 (1) it follows that $ab = c+h$ for some $h\in S^{-\mu}_{\rho,\delta}$. By choosing R large enough and Γ sufficiently small conical neighbourhood of ξ_0, it follows that $c(x,\xi)=1$ and $|h(x,\xi)|\le 1/2$ when (2.7) holds. This gives (2.6), and (1) follows.

It remains to prove that (1) implies (4). Therefore assume that (1) holds, and choose an open neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R>0$ such that (1.14) holds when $(x,\xi)\in X\times\Gamma$ and $|\xi|>R$. Also let $\varphi_j\in\mathscr{C}_{x_0}(X)$ and $\psi_j\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ for $j=1,2,3$ be such that $\varphi_j=1$ on $\operatorname{supp}\varphi_{j+1}$, $\psi_j=1$ on $\operatorname{supp}\psi_{j+1}$ when $j=1,2$, and $\psi_j(\xi)=0$ when $|\xi|\le R$. We also set $c_j=\varphi_j\otimes\psi_j$ when $j\le 2$ and $c_j=c_2$ when $j\ge 3$.

If $b_1(x,\xi)=\varphi_1(x)\psi_1(\xi)/a(x,\xi)\in S^{(1/\omega_0)}_{\rho,\delta}$, then the symbol of $\operatorname{Op}(b_1)\operatorname{Op}(a)$ is equal to $c_1 \mod (S^{-\mu}_{\rho,\delta})$. Hence

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c_j)+\operatorname{Op}(h_j) \tag{2.8}$$

holds for $j=1$ and some $h_1\in S^{-\mu}_{\rho,\delta}$.

For $j\ge 2$ we now define $\widetilde b_j\in S^{(1/\omega_0)}_{\rho,\delta}$ by the Neumann serie

$$\operatorname{Op}(\widetilde b_j) = \sum_{k=0}^{j-1}(-1)^k\operatorname{Op}(\widetilde r_k),$$

where $\operatorname{Op}(\widetilde r_k)=\operatorname{Op}(h_1)^k\operatorname{Op}(b_1)\in\operatorname{Op}(S^{(\sigma_{-k\mu}/\omega_0)}_{\rho,\delta})$. Then (2.8) gives

$$\begin{aligned}\operatorname{Op}(\widetilde b_j)\operatorname{Op}(a) &= \sum_{k=0}^{j-1}(-1)^k\operatorname{Op}(h_1)^k\operatorname{Op}(b_1)\operatorname{Op}(a)\\ &= \sum_{k=0}^{j-1}(-1)^k\operatorname{Op}(h_1)^k(\operatorname{Op}(c_1)+\operatorname{Op}(h_1)).\end{aligned}$$

That is

$$\operatorname{Op}(\widetilde{b}_j)\operatorname{Op}(a) = \operatorname{Op}(c_1) + \operatorname{Op}(\widetilde{h}_{1,j}) + \operatorname{Op}(\widetilde{h}_{2,j}), \tag{2.9}$$

where

$$\operatorname{Op}(\widetilde{h}_{1,j}) = (-1)^{j-1}\operatorname{Op}(h_1)^j \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}) \tag{2.10}$$

and

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1}(-1)^k\operatorname{Op}(h_1)^k\operatorname{Op}(1-c_1) \in \operatorname{Op}(S^{-\mu}_{\rho,\delta}).$$

By Proposition 2.1 (1) and asymptotic expansions it follows that

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1}(-1)^k\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(\widetilde{h}_{4,j}), \tag{2.11}$$

for some $\widetilde{h}_{3,j} \in S^{-\mu}_{\rho,\delta}$ which is equal to zero in $\operatorname{supp} c_1$ and $\widetilde{h}_{4,j} \in S^{-j\mu}_{\rho,\delta}$. Now let b_j and r_k be defined by the formulae

$$\operatorname{Op}(b_j) = \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{b}_j) \in \operatorname{Op}(S^{(1/\omega_0)}_{\rho,\delta}),$$

$$\operatorname{Op}(r_k) = \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{r}_k) \in \operatorname{Op}(S^{(\sigma_{-k\mu}/\omega_0)}_{\rho,\delta}).$$

Then

$$\operatorname{Op}(b_j) = \sum_{k=0}^{j-1}(-1)^k\operatorname{Op}(r_k)$$

and (2.9)–(2.11) give

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c_2)\operatorname{Op}(c_1) + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{1,j})$$

$$-\sum_{k=1}^{j-1}(-1)^k\operatorname{Op}(c_2)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{4,j}).$$

Since $c_1 = 1$ and $\widetilde{h}_{3,j} = 0$ on $\operatorname{supp} c_2$, it follows that

$$\operatorname{Op}(c_2)\operatorname{Op}(c_1) = \operatorname{Op}(c_2) \mod \operatorname{Op}(S^{-\infty}),$$

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{1,j}) \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}),$$

$$\sum_{k=1}^{j-1}(-1)^k\operatorname{Op}(c_2)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k \in \operatorname{Op}(S^{-\infty}),$$

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{3,j}) \in \operatorname{Op}(S^{-\infty})$$

and

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{4,j}) \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}).$$

Hence, (2.8) follows for some $h_j \in S^{-j\mu}_{\rho,\delta}$.

By choosing $b_0 \in S^{(1/\omega)}_{\rho,\delta}$ such that

$$b_0 \sim \sum r_k,$$

it follows that $\operatorname{Op}(b_0)\operatorname{Op}(a) = \operatorname{Op}(c_2) + \operatorname{Op}(h_0)$, with

$$h_0 \in S^{-\infty}.$$

The assertion (4) now follows by letting

$$b(x,\xi) = \varphi_3(x)b_0(x,\xi), \quad c(x,\xi) = \varphi_3(x)c_2(x,\xi),$$
$$\text{and} \quad h(x,\xi) = \varphi_3(x)h_0(x,\xi),$$

and using the fact that if $\varphi_3 \in C_0^\infty(\mathbf{R}^d)$ and $h_0 \in S^{-\infty}(\mathbf{R}^{2d})$, then $\varphi_3(x)h_0(x,\xi) \in \mathscr{S}(\mathbf{R}^{2d})$. The proof is complete. □

Remark 2.4. By Proposition 2.3 it follows that Definition 2.3 in [29] is equivalent to Definition 1.8. We also remark that if a is an appropriate symbol, and $\operatorname{Char}'(a)$ the set of characteristic points for a in the sense of [23, Definition 18.1.5], then $\operatorname{Char}_{(\omega_0)}(a) \subseteq \operatorname{Char}'(a)$. Furthermore, strict embedding might occur, especially for symbols to hypoelliptic partial operators with constant coefficients, which are not elliptic (cf. Example 4.9 in [29]).

3. Wave front sets with respect to Fourier BF-spaces

In this section we define wave-front sets with respect to Fourier BF-spaces, and show some basic properties.

Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$. For any $f \in \mathscr{E}'(\mathbf{R}^d)$, let

$$|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = |f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)_x} \equiv \|\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma\|_{\mathscr{B}}. \tag{3.1}$$

Here χ_Γ is the characteristic function on Γ. We note that $\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma \in \mathscr{B}_{loc}$ for every $f \in \mathscr{E}'$. If $\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma \notin \mathscr{B}$, then we set $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = +\infty$. Hence $|\,\cdot\,|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)}$ defines a semi-norm on $\mathscr{E}'$ which might attain the value $+\infty$. Since ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$, it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent semi-norms. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$ and $f \in \mathscr{F}\mathscr{B}(\omega) \cap \mathscr{E}'$, then $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)}$ agrees with $\|f\|_{\mathscr{F}\mathscr{B}(\omega)}$.

For the sake of notational convenience we set

$$|\,\cdot\,|_{\mathcal{B}(\Gamma)} = |\,\cdot\,|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)_x} \tag{3.2}$$

when

$$\mathcal{B} = \mathscr{F}\mathscr{B}(\omega). \tag{3.3}$$

We let $\Theta_{\mathcal{B}}(f) = \Theta_{\mathscr{F}\mathscr{B}(\omega)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some $\Gamma = \Gamma_\xi$. We also let $\Sigma_{\mathcal{B}}(f)$ be the complement of $\Theta_{\mathcal{B}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbf{R}^d$ in (3.1).

Definition 3.1. Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathcal{B}$ be as in (3.3), and let X be an open subset of $\mathbf{R}^d$. The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$ with respect to $\mathcal{B}$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{B}}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

We note that $\mathrm{WF}_{\mathcal{B}}(f)$ in Definition 3.1 is a closed set in $X \times (\mathbf{R}^d \setminus 0)$, since it is obvious that its complement is open. We also note that if $x \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x, \xi)$, then $\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f) = \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega_0)}(f)$, since $\Sigma_{\mathcal{B}}$ is independent of x.

The following theorem shows that wave-front sets with respect to $\mathscr{F}\mathscr{B}(\omega)$ satisfy appropriate micro-local properties. It also shows that such wave-front sets decreases when the local Fourier BF-spaces increases, or when the weight ω decreases.

Theorem 3.2. *Let $X \subseteq \mathbf{R}^d$ be open, $\mathscr{B}_1, \mathscr{B}_2$ be translation invariant BF-spaces, $\varphi \in C^\infty(\mathbf{R}^d)$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ and $f \in \mathscr{D}'(X)$. Also let $\mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j)$ for $j = 1, 2$. If $\mathcal{B}_{1,loc} \subseteq \mathcal{B}_{2,loc}$, then*

$$\mathrm{WF}_{\mathcal{B}_2}(\varphi f) \subseteq \mathrm{WF}_{\mathcal{B}_1}(f).$$

Proof. It suffices to prove

$$\Sigma_{\mathcal{B}_2}(\varphi f) \subseteq \Sigma_{\mathcal{B}_1}(f). \tag{3.4}$$

when $\mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j)$, $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $f \in \mathscr{E}'(\mathbf{R}^d)$. The local properties and Remark 1.2 also imply that it is no restriction to assume that $\omega_1 = \omega_2 = 1$.

Let $\xi_0 \in \Theta_{\mathcal{B}_1}(f)$, and choose open cones Γ_1 and Γ_2 in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has compact support, it follows that $|\widehat{f}(\xi)| \le C\langle \xi \rangle^{N_0}$ for some positive constants C and N_0. The result therefore follows if we prove that for each N, there are constants C_N such that

$$|\varphi f|_{\mathcal{B}_2(\Gamma_2)} \le C_N \Big(|f|_{\mathcal{B}_1(\Gamma_1)} + \sup_{\xi \in \mathbf{R}^d} \big(|\widehat{f}(\xi)| \langle \xi \rangle^{-N} \big) \Big)$$
$$\text{when} \quad \overline{\Gamma}_2 \subseteq \Gamma_1 \quad \text{and} \quad N = 1, 2, \dots . \tag{3.5}$$

By letting $F(\xi) = |\widehat{f}(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)|$, it follows that ψ turns rapidly to zero at infinity and

$$|\varphi f|_{\mathcal{B}_2(\Gamma_2)} = |\varphi f|_{\mathscr{F}\mathscr{B}_2(\Gamma_2)} = \|\mathscr{F}(\varphi f)\chi_{\Gamma_2}\|_{\mathscr{B}_2}$$
$$= C \Big\| \Big(\int_{\mathbf{R}^d} \widehat{\varphi}(\,\cdot\, - \eta) \widehat{f}(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}_2} \le C(J_1 + J_2)$$

for some positive constant C, where

$$J_1 = \Big\| \Big(\int_{\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}_2} \tag{3.6}$$

and

$$J_2 = \Big\| \Big(\int_{\complement\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}_2} \tag{3.7}$$

and χ_{Γ_2} is the characteristic function of Γ_2. First we estimate J_1. By (3) in Definition 1.1 and (1.4), it follows for some constants $C_1, \dots, C_5$ that

$$\begin{aligned} J_1 \le C_1 \Big\| \int_{\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta \Big\|_{\mathscr{B}_2} &= C_1 \|\widehat{\varphi} * (\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{B}_2} \\ &= C_2 \|\varphi \mathscr{F}^{-1}(\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{F}\mathscr{B}_2} \le C_3 \|\varphi \mathscr{F}^{-1}(\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{F}\mathscr{B}_1} \\ &= C_4 \|\widehat{\varphi} * (\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{B}_1} \le C_5 \|\widehat{\varphi}\|_{L^1_{(v)}} \|\chi_{\Gamma_1}\widehat{f}\|_{\mathscr{B}_1} = C|f|_{\mathscr{F}\mathscr{B}_1(\Gamma_1)}, \end{aligned} \tag{3.8}$$

where $C = C_5\|\widehat{\varphi}\|_{L^1_{(v)}} < \infty$, since $\widehat{\varphi}$ turns rapidly to zero at infinity. In the second inequality we have used the fact that $(\mathscr{F}\mathscr{B}_1)_{loc} \subseteq (\mathscr{F}\mathscr{B}_2)_{loc}$.

In order to estimate J_2, we note that the conditions $\xi \in \Gamma_2$, $\eta \notin \Gamma_1$ and the fact that $\overline{\Gamma_2} \subseteq \Gamma_1$ imply that $|\xi - \eta| > c\max(|\xi|, |\eta|)$ for some constant $c > 0$, since this is true when $1 = |\xi| \ge |\eta|$. This implies that for every $s, t \in \mathbf{R}$, there is a constant $C \ge 1$ such that

$$C^{-1}\langle\xi\rangle^s\langle\eta\rangle^t \le \langle\xi - \eta\rangle^{s+t} \le C\langle\xi\rangle^s\langle\eta\rangle^t, \tag{3.9}$$

when $\xi \in \Gamma_2$ and $\eta \notin \Gamma_1$. We also note that if N_1 is large enough, then $\langle\,\cdot\,\rangle^{-N_1} \in \mathscr{B}_2$, because $\mathscr{S}$ is continuously embedded in $\mathscr{B}_2$. Since ψ turns rapidly to zero at infinity, it follows that for each $N_0 > d + N_1$ and $N \in \mathbf{N}$ such that $N > N_0$, and (3.9) gives

$$\begin{aligned} J_2 &\le C_1 \Big\| \Big(\int_{\complement\Gamma_1} \langle\,\cdot\, - \eta\rangle^{-(2N_0+N)} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}_2} \\ &\le C_2 \Big\| \Big(\int_{\complement\Gamma_1} \langle\,\cdot\,\rangle^{-N_0}\langle\eta\rangle^{-N_0}(\langle\eta\rangle^{-N}F(\eta))\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}_2} \\ &\le C_2 \int_{\complement\Gamma_1} \|\langle\,\cdot\,\rangle^{-N_0}\chi_{\Gamma_2}\|_{\mathscr{B}_2}\langle\eta\rangle^{-N_0}(|\langle\eta\rangle^{-N}F(\eta)|)\, d\eta \\ &\le C \sup_{\eta \in \mathbf{R}^d} |\langle\eta\rangle^{-N}F(\eta)|, \end{aligned} \tag{3.10}$$

for some constants C_1 and $C_2 > 0$, where

$$C = C_2\|\langle\,\cdot\,\rangle^{-N_0}\|_{\mathscr{B}_2}\|\langle\,\cdot\,\rangle^{-N_0}\|_{L^1} < \infty.$$

This proves (3.5), and the result follows. □

4. Mapping properties for pseudo-differential operators on wave-front sets

In this section we establish mapping properties for pseudo-differential operators on wave-front sets of Fourier Banach types. More precisely, we prove the following result (cf. (0.1)):

Theorem 4.1. *Let $t \in \mathbf{R}$, $\rho > 0$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$, and $f \in \mathscr{S}'(\mathbf{R}^d)$. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$, $\mathcal{B} = \mathscr{F}\mathscr{B}(\omega)$ and $\mathcal{C} = \mathscr{F}\mathscr{B}(\omega/\omega_0)$. Then*

$$\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}_{\mathcal{B}}(f) \subseteq \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a). \tag{4.1}$$

We shall mainly follow the proof of Theorem 4.1 in [29]. The following restatement of Proposition 4.2 in [29] shows that $(x_0, \xi) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f)$ when $x_0 \notin \operatorname{supp} f$, for any $\xi \in \mathbf{R}^d \setminus 0$.

Proposition 4.2. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $0 \le \delta < \rho \le 1$, and let $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$. Also let $\mathscr{B}$ be a translation invariant BF-space, and let the operator L_a on $\mathscr{S}'(\mathbf{R}^d)$ be defined by the formula*

$$(L_a f)(x) \equiv \varphi_1(x)(\mathrm{Op}(a)(\varphi_2 f))(x), \quad f \in \mathscr{S}'(\mathbf{R}^d), \tag{4.2}$$

where $\varphi_1 \in C_0^\infty(\mathbf{R}^d)$ and $\varphi_2 \in S^0_{0,0}(\mathbf{R}^d)$ are such that

$$\operatorname{supp} \varphi_1 \bigcap \operatorname{supp} \varphi_2 = \emptyset.$$

Then the kernel of L_a belongs to $\mathscr{S}(\mathbf{R}^{2d})$. In particular, the following is true:

1. *$L_a = \mathrm{Op}(a_0)$ for some $a_0 \in \mathscr{S}(\mathbf{R}^{2d})$;*
2. *$\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(L_a f) = \emptyset$.*

Next we consider properties of the wave-front set of $\mathrm{Op}(a)f$ at a fixed point when f is concentrated to that point.

Proposition 4.3. *Let ρ, a, $\mathcal{B}$ and $\mathcal{C}$ be as in Theorem 4.1. Also let $f \in \mathscr{E}'(\mathbf{R}^d)$. Then the following is true:*

1. *if $\Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \setminus 0$ are open cones such that $\overline{\Gamma_2} \subseteq \Gamma_1$, and $|f|_{\mathcal{B}(\Gamma_1)} < \infty$, then $|\mathrm{Op}(a)f|_{\mathcal{C}(\Gamma_2)} < \infty$;*
2. *$\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}_{\mathcal{B}}(f)$.*

We note that $\mathrm{Op}(a)f$ in Proposition 4.3 makes sense as an element in $\mathscr{S}'(\mathbf{R}^d)$, by Proposition 1.7.

Proof. We shall mainly follow the proof of Proposition 4.3 in [29]. We may assume that $\omega(x,\xi) = \omega(\xi)$, $\omega_0(x,\xi) = \omega_0(\xi)$, and that $\operatorname{supp} a \subseteq K \times \mathbf{R}^d$ for some compact set $K \subseteq \mathbf{R}^d$, since the statements only involve local assertions.

Let $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$, and let $\mathscr{F}_1 a$ denote the partial Fourier transform of $a(x,\xi)$ with respect to the x-variable. By straightforward computation, it follows that for every $N \ge 0$, there is a constant C such that

$$|\mathscr{F}(\mathrm{Op}(a)f)(\xi)\omega(\xi)/\omega_0(\xi)| \le C \int_{\mathbf{R}^d} \langle \xi - \eta \rangle^{-N} F(\eta)\, d\eta \tag{4.3}$$

(cf. (4.6) and (4.8) in [29]).

We have to estimate

$$|(\mathrm{Op}(a)f)|_{\mathcal{C}(\Gamma_2)} = \|\mathscr{F}(\mathrm{Op}(a)f)\omega/\omega_0\chi_{\Gamma_2}\|_{\mathscr{B}}.$$

By (4.3) we get

$$\|\mathscr{F}(\mathrm{Op}(a)f)\omega/\omega_0\chi_{\Gamma_2}\|_{\mathscr{B}} \le C\Big\| \Big(\int \langle \cdot - \eta \rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}$$
$$\le C(J_1 + J_2),$$

where C is a constant and

$$J_1 = \Big\| \Big(\int_{\Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}$$

and

$$J_2 = \Big\| \Big(\int_{\complement\Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}.$$

In order to estimate J_1 and J_2 we argue as in the proof of (3.5). More precisely, by (1.4) we get

$$J_1 \le \Big\| \int_{\Gamma_1} \langle\, \cdot\, - \eta \rangle^{-N} F(\eta)\, d\eta \Big\|_{\mathscr{B}} = \|\langle\, \cdot\, \rangle^{-N} * (\chi_{\Gamma_1} F)\|_{\mathscr{B}}$$
$$\le C \|\langle\, \cdot\, \rangle^{-N}\|_{L^1_{(v)}} \|\chi_{\Gamma_1} F\|_{\mathscr{B}} < \infty.$$

Next we estimate J_2. Since $\overline{\Gamma_2} \subseteq \Gamma_1$, we get

$$|\xi - \eta| \ge c \max(|\xi|, |\eta|), \quad \text{when} \quad \xi \in \Gamma_2, \text{ and } \eta \notin \Gamma_1$$

for some constant $c > 0$. (Cf. the proof of Proposition 3.3.)

Since f has compact support, it follows that $F(\eta) \le C\langle \eta \rangle^{N_0}$ for some constants C and N_0. By combining these estimates and (3.9) we obtain

$$J_2 \le \Big\| \Big(\int_{\complement\Gamma_1} F(\eta) \langle\, \cdot\, - \eta \rangle^{-N}\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}$$
$$\le C\Big\| \Big(\int_{\complement\Gamma_1} \langle \eta \rangle^{N_0} \langle\, \cdot\, \rangle^{-N/2} \langle \eta \rangle^{-N/2}\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}$$
$$\le C \|\langle\, \cdot\, \rangle^{-N/2} \chi_{\Gamma_2}\|_{\mathscr{B}} \int_{\complement\Gamma_1} \langle \eta \rangle^{-N/2 + N_0}\, d\eta.$$

Hence, if we choose N sufficiently large, it follows that the right-hand side is finite. This proves (1).

The assertion (2) follows immediately from (1) and the definitions. The proof is complete. □

Proof of Theorem 4.1. By Proposition 2.1 it is no restriction to assume that $t = 0$. The remaining part of the proof is similar to the proof of Theorem 4.1 in [29]. In order to be self contained and to put the previous results in appropriate context, we here present a full proof. We start to prove the first inclusion in (4.1). Assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{B}}(f)$, let $\varphi \in \mathscr{C}_{x_0}(\mathbf{R}^d)$, and set $\psi = 1 - \varphi$ and $a_0(x,\xi) = \varphi(x)a(x,\xi)$. Then it follows from Proposition 4.2 that

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)(\psi f)).$$

Furthermore, by Proposition 4.3 we get

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a_0)(\varphi f)),$$

which implies that

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)(\varphi f)),$$

since $\mathrm{Op}(a)(\varphi f)$ is equal to $\mathrm{Op}(a_0)(\varphi f)$ near x_0. The result is now a consequence of the inclusion

$$\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)(\varphi f)) \bigcup \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)(\psi f)).$$

It remains to prove the last inclusion in (4.1). By Proposition 4.2 it follows that it is no restriction to assume that f has compact support. Assume that

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a),$$

and choose b, c and h as in Proposition 2.3 (4). We shall prove that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{B}}(f)$. Since

$$f = \mathrm{Op}(1-c)f + \mathrm{Op}(b)\,\mathrm{Op}(a)f - \mathrm{Op}(h)f,$$

the result follows if we prove

$$(x_0, \xi_0) \notin \mathsf{S}_1 \bigcup \mathsf{S}_2 \bigcup \mathsf{S}_3,$$

where

$$\mathsf{S}_1 = \mathrm{WF}_{\mathcal{B}}(\mathrm{Op}(1-c)f), \quad \mathsf{S}_2 = \mathrm{WF}_{\mathcal{B}}(\mathrm{Op}(b)\,\mathrm{Op}(a)f),$$
$$\mathsf{S}_3 = \mathrm{WF}_{\mathcal{B}}(\mathrm{Op}(h)f),$$

and $c_0(x,\xi) = \varphi(x)(1 - c(x,\xi))$.

By the first embedding in (4.1) it follows that

$$\mathsf{S}_2 = \mathrm{WF}_{\mathcal{B}}(\mathrm{Op}(b)\,\mathrm{Op}(a)f) \subseteq \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f).$$

Since $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a)f)$, it follows that $(x_0, \xi_0) \notin \mathsf{S}_2$.

Since $h \in \mathscr{S}$, it follows that $\mathrm{Op}(h)f \in \mathscr{S}$, giving that S_3 is empty.

Finally we consider S_1. By the assumptions it follows that c_0 is zero in Γ, and by replacing Γ with a smaller cone, if necessary, we may assume that $c_0 = 0$ in a conical neighborhood of Γ. Hence, if $\Gamma \equiv \Gamma_1$, Γ_2, J_1 and J_2 are the same as in the proof of Proposition 4.3, then it follows from that proof and the fact that $c_0(x,\xi) \in S^0_{\rho,0}$ is compactly supported in the x-variable, that $J_1 < +\infty$, and that for each $N \ge 0$, there are constants C_N and C'_N such that

$$
\begin{aligned}
|\operatorname{Op}(c_0)f|_{\mathcal{C}(\Gamma_2)} &\le C_N(J_1+J_2) \\
&\le C'_N\Big(J_1 + \Big\| \int_{\complement\Gamma_1} \langle\,\cdot\,\rangle^{-N}\langle\eta\rangle^{-N}\,d\eta\,\chi_{\Gamma_2}\Big\|_{\mathscr{B}}\Big). \qquad (4.4)
\end{aligned}
$$

By choosing N large enough, it follows that

$$|\operatorname{Op}(c_0)f|_{\mathcal{C}(\Gamma_2)} < \infty.$$

This proves that $(x_0,\xi_0) \notin \mathsf{S}_1$, and the proof is complete. □

Remark 4.4. We note that the statements in Theorems 4.1 are not true if $\omega_0 = 1$ and the assumption $\rho > 0$ is replaced by $\rho = 0$. (Cf. Remark 4.6 in [29].)

In the following result we apply Theorem 4.1 on elliptic operators.

Theorem 4.5. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $\rho > 0$, $\mathcal{B} = \mathscr{F}\mathscr{B}(\omega)$, $\mathcal{C} = \mathscr{F}\mathscr{B}(\omega/\omega_0)$ and let $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$ be elliptic with respect to ω_0. Also let $\mathscr{B}$ be a translation invariant BF-space. If $f \in \mathscr{S}'(\mathbf{R}^d)$, then*

$$\mathrm{WF}_{\mathcal{C}}(\operatorname{Op}(a)f) = \mathrm{WF}_{\mathcal{B}}(f).$$

5. Wave-front sets of sup and inf types and pseudo-differential operators

In this section we put the micro-local analysis in a more general context compared to previous sections, and define wave-front sets with respect to sequences of Fourier BF-spaces.

Let $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ and $\mathscr{B}_j$ be translation invariant BF-space on $\mathbf{R}^d$ when j belongs to some index set J, and consider the array of spaces, given by

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j\in J}, \quad \text{where} \quad \mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j), \quad j \in J. \qquad (3.3)'$$

If $f \in \mathscr{S}'(\mathbf{R}^d)$, and $(\mathcal{B}_j)$ is given by $(3.3)'$, then we let $\Theta^{\sup}_{(\mathcal{B}_j)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and *each* $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. We also let $\Theta^{\inf}_{(\mathcal{B}_j)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and *some* $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. Finally we let $\Sigma^{\sup}_{(\mathcal{B}_j)}(f)$ and $\Sigma^{\inf}_{(\mathcal{B}_j)}(f)$ be the complements in $\mathbf{R}^d \setminus 0$ of $\Theta^{\sup}_{(\mathcal{B}_j)}(f)$ and $\Theta^{\inf}_{(\mathcal{B}_j)}(f)$ respectively.

Definition 5.1. Let J be an index set, $\mathscr{B}_j$ be translation invariant BF-space on $\mathbf{R}^d$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ when $j \in J$, $(\mathcal{B}_j)$ be as in (3.3)$'$, and let X be an open subset of $\mathbf{R}^d$.

(1) The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\sup}_{(\mathscr{F}\mathscr{B}_j(\omega_j))}(f)$, of *sup-type* with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \backslash 0)$ such that $\xi_0 \in \Sigma^{\sup}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$;

(2) The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\inf}_{(\mathscr{F}\mathscr{B}_j(\omega_j))}(f)$, of *inf-type* with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \backslash 0)$ such that $\xi_0 \in \Sigma^{\inf}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

Remark 5.2. Let $\omega_j(x, \xi) = \langle \xi \rangle^{-j}$ for $j \in J = \mathbf{N}_0$ and $\mathscr{B}_j = L^{q_j}$, where $q_j \in [1, \infty]$. Then it follows that $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)$ in Definition 5.1 is equal to the standard wave-front set $\mathrm{WF}(f)$ in Chapter VIII in [23].

The following result follows immediately from Theorem 4.1 and its proof. We omit the details.

Theorem 4.1$'$. *Let $\rho > 0$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $t \in \mathbf{R}$, $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. Also let $\mathscr{B}_j$ be a translation invariant BF-space on $\mathbf{R}^d$ for every j, and let $\mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j)$ and $\mathcal{C}_j = \mathscr{F}\mathscr{B}_j(\omega_j/\omega_0)$. Then*

$$\mathrm{WF}^{\sup}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f) \subseteq \mathrm{WF}^{\sup}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a), \tag{4.1$'$}$$

and

$$\mathrm{WF}^{\inf}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f) \subseteq \mathrm{WF}^{\inf}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a). \tag{4.1$''$}$$

The following generalization of Theorem 4.5 is an immediate consequence of Theorem 4.1$'$, since $\mathrm{Char}_{(\omega_0)}(a) = \emptyset$, when a is elliptic with respect to ω_0.

Theorem 4.5$'$. *Let $\rho > 0$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $t \in \mathbf{R}$ and let $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$ be elliptic with respect to ω_0. Also let $\mathscr{B}_j$ be a translation invariant BF-space on $\mathbf{R}^d$ for every j, and let $\mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j)$ and $\mathcal{C}_j = \mathscr{F}\mathscr{B}_j(\omega_j/\omega_0)$. If $f \in \mathscr{S}'(\mathbf{R}^d)$, then*

$$\mathrm{WF}^{\inf}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) = \mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f)$$

and

$$\mathrm{WF}^{\sup}_{(\mathcal{C}_j)}(\mathrm{Op}_t(a)f) = \mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f).$$

Remark 5.3. We note that many properties valid for the wave-front sets of Fourier BF-type also hold for wave-front sets in the present section. For example, the conclusion in Remark 4.4 holds for wave-front sets of sup- and inf-types.

Finally we remark that there are some technical generalizations of Theorem 4.1 which involve pseudo-differential operators with symbols in $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ with $0 \le \delta < \rho \le 1$. From these generalizations it follows that

$$\mathrm{WF}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}(f) \subseteq \mathrm{WF}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a),$$

when $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. (Cf. Theorem 5.3′ and Theorem 5.5 in [29].)

6. Wave front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier BF-types. In particular, all micro-local properties for pseudo-differential operators in the previous sections carry over to wave-front sets of modulation space types.

We start with defining general types of modulation spaces. Let (the window) $\phi \in \mathscr{S}'(\mathbf{R}^d) \setminus 0$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\phi f$ is the element in $\mathscr{S}'(\mathbf{R}^{2d})$, defined by the formula

$$(V_\phi f)(x,\xi) \equiv \mathscr{F}(f \cdot \overline{\phi(\cdot - x)})(\xi).$$

We usually assume that $\phi \in \mathscr{S}(\mathbf{R}^d)$, and in this case the short-time Fourier transform $(V_\phi f)$ takes the form

$$(V_\phi f)(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy,$$

when $f \in \mathscr{S}(\mathbf{R}^d)$.

Now let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$. Also let $\phi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. Then the modulation space $M(\omega) = M(\omega,\mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f \cdot \omega \in \mathscr{B}$. We note that $M(\omega,\mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \omega\|_{\mathscr{B}} \tag{6.1}$$

(cf. [9]).

Remark 6.1. Assume that $p,q \in [1,\infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $L^{p,q}_1(\mathbf{R}^{2d})$ and $L^{p,q}_2(\mathbf{R}^{2d})$ be the sets of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L^{p,q}_1} \equiv \Big(\int \Big(\int |F(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \Big(\int \Big(\int |F(x,\xi)|^p\, d\xi \Big)^{q/p} dx \Big)^{1/q} < \infty,$$

respectively (with obvious modifications when $p = \infty$ or $q = \infty$). Then $M(\omega,\mathscr B)$ is equal to the usual modulation space $M^{p,q}_{(\omega)}(\mathbf R^d)$ when $\mathscr B = L^{p,q}_1(\mathbf R^{2d})$. If instead $\mathscr B = L^{p,q}_2(\mathbf R^{2d})$, then $M(\omega,\mathscr B)$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf R^d)$, related to Wiener-amalgam spaces.

In the following proposition we list some important properties for modulation spaces. We refer to [15] for the proof.

Proposition 6.2. *Assume that $\mathscr B$ is a translation invariant BF-space on $\mathbf R^{2d}$ with respect to $v \in \mathscr P(\mathbf R^{2d})$, and that $\omega_0, v_0 \in \mathscr P(\mathbf R^{2d})$ are such that ω is v-moderate. Then the following is true:*

1. *if $\phi \in M^1_{(v_0 v)}(\mathbf R^d)\setminus 0$, then $f \in M(\omega,\mathscr B)$ if and only if $V_\phi f \omega \in \mathscr B$. Furthermore, (6.1) defines a norm on $M(\omega,\mathscr B)$, and different choices of ϕ gives rise to equivalent norms;*
2. *$M^1_{(v_0 v)} \subseteq M(\omega,\mathscr B) \subseteq M^\infty_{(1/(v_0 v))}$.*

The following generalization of Theorem 2.1 in [31] shows that modulation spaces are locally the same as translation invariant Fourier BF-spaces. We recall that if $\varphi \in \mathscr S(\mathbf R^d)\setminus 0$ and $\mathscr B$ is a translation invariant BF-space on $\mathbf R^{2d}$, then it follows from Proposition 1.4 that

$$\mathscr B_0 \equiv \{\, f \in \mathscr S'(\mathbf R^d)\,;\, \varphi \otimes f \in \mathscr B \,\} \tag{6.2}$$

is a translation invariant BF-space on $\mathbf R^d$ which is independent of the choice of φ.

Proposition 6.3. *Let $\varphi \in \mathscr S(\mathbf R^d)\setminus 0$, $\mathscr B$ be a translation invariant BF-space on $\mathbf R^{2d}$, and let $\mathscr B_0$ be as in (6.2). Also let $\omega \in \mathscr P(\mathbf R^{2d})$, and $\omega_0(\xi) = \omega(x_0,\xi)$, for some fixed $x_0 \in \mathbf R^d$. Then*

$$M(\omega,\mathscr B)_{comp} = \mathscr F\mathscr B_0(\omega_0)_{comp} \quad \text{and} \quad M(\omega,\mathscr B)_{loc} = \mathscr F\mathscr B_0(\omega_0)_{loc}.$$

Furthermore, if $K \subseteq \mathbf R^d$ is compact, then

$$C^{-1}\|f\|_{\mathscr F\mathscr B_0(\omega_0)} \le \|f\|_{M(\omega,\mathscr B)} \le C\|f\|_{\mathscr F\mathscr B_0(\omega_0)}, \quad f \in \mathscr E'(K), \tag{6.3}$$

for some constant C.

We need the following lemma for the proof.

Lemma 6.4. *Assume that $f \in \mathscr E'(\mathbf R^d)$. Then the following is true:*

1. *if $\phi \in C_0^\infty(\mathbf R^d)$, then there exists $0 \le \varphi \in C_0^\infty(\mathbf R^d)$ such that*
$$(V_\phi f)(x,\xi) = \varphi(x)(\widehat f * (\mathscr F(\overline{\phi(\cdot - x)})))(\xi)\,; \tag{6.4}$$
2. *if $\varphi \in C_0^\infty(\mathbf R^d)$, then there exists $\phi \in C_0^\infty(\mathbf R^d)$ such that*
$$(\varphi \otimes \widehat f)(x,\xi) = \varphi(x) V_\phi f(x,\xi). \tag{6.5}$$

Proof. (1) From the support properties it follows that there is a compact set $K \subseteq \mathbf{R}^d$ such that $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$. The assertion now follows from Fourier's inversion formula, by choosing $\varphi \in C_0^\infty$ such that $\varphi(x) = (2\pi)^{d/2}$ when $x \in K$.

The assertion (2) follows in a similar way by choosing $\phi \in C_0^\infty$ such that $\phi = 1$ on $\operatorname{supp} f - \operatorname{supp} \varphi$. □

Proof of Proposition 6.3. First we recall that $\mathscr{B}_0$ is independent of the choice of $\varphi \in \mathscr{S} \setminus 0$ in view of Proposition 1.4. Furthermore, we may assume that $\omega = \omega_0 = 1$ in view of Remark 1.2. Assume that $f \in \mathscr{E}'$. First, let $\varphi \in C_0^\infty(\mathbf{R}^d)$. From (2) in Lemma 6.4 it follows that there exists $\phi \in C_0^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{M(\mathscr{B})} = \|V_\phi f\|_{\mathscr{B}} \ge \|\varphi \cdot V_\phi f\|_{\mathscr{B}} = \|\varphi \otimes \widehat{f}\|_{\mathscr{B}} = \|\widehat{f}\|_{\mathscr{B}_0}.$$

This proves that

$$M(\mathscr{B})_{comp} \subseteq (\mathscr{F}\mathscr{B}_0)_{comp} \quad \text{and} \quad M(\mathscr{B})_{loc} \subseteq (\mathscr{F}\mathscr{B}_0)_{loc} \tag{6.6}$$

Now, let $\phi \in C_0^\infty(\mathbf{R}^d)$. Then by (1) in Lemma 6.4 it follows that there exists $0 \le \varphi \in C_0^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{M(\mathscr{B})} = \|V_\phi f\|_{\mathscr{B}} = \|\varphi \otimes (\widehat{f} * (\mathscr{F}(\phi(\cdot - x))))\|_{\mathscr{B}}.$$

Since

$$|\widehat{f} * (\mathscr{F}(\phi(\cdot - x)))(\xi)| \le (|\widehat{f}| * |\widehat{\phi}|)(\xi)$$

it follows, by use of Lemma 6.4 again, that

$$\begin{aligned}\|f\|_{M(\mathscr{B})} = \|V_\phi f\|_{\mathscr{B}} &\le C_1 \|\varphi \otimes (|\widehat{f}| * |\widehat{\phi}|)\|_{\mathscr{B}} \\ &\le C_2 \|\varphi \otimes |\widehat{f}|\|_{\mathscr{B}} \le C_3 \|\varphi \otimes \widehat{f}\|_{\mathscr{B}},\end{aligned}$$

which gives opposite inclusions compared to (6.6), and the assertion follows. The proof is complete. □

Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ be fixed, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, and let χ_Γ be the characteristic function of Γ. For any $f \in \mathscr{S}'(\mathbf{R}^d)$ we set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{M(\omega,\Gamma,\mathscr{B})} = |f|_{M^\phi(\omega,\Gamma,\mathscr{B})} \equiv \|(V_\phi f)\, \omega\, (1 \otimes \chi_\Gamma)\|_{\mathscr{B}} \quad \text{when} \quad \mathcal{B} = M(\omega, \mathscr{B}). \tag{6.7}$$

We note that $|\cdot|_{\mathcal{B}(\Gamma)}$ defines a semi-norm on $\mathscr{S}'$ which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$, then $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M(\omega,\mathscr{B})}$.

The sets $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$, and the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f with respect to $\mathcal{B} = M(\omega, \mathscr{B})$ are now defined in the same way as in Section 3, after replacing the semi-norms of Fourier BF-types in (3.2) with the semi-norms in (6.7).

In Theorem 6.9 below we prove that wave-front sets of Fourier BF-spaces and modulation spaces agree with each others. As a first step we prove that $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ is independent of ϕ in (6.7).

Proposition 6.5. *Let $X \subseteq \mathbf{R}^d$ be open, $f \in \mathscr{D}'(X)$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $\mathcal{B} = M(\omega, \mathscr{B})$. Then $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and $\mathrm{WF}_{\mathcal{B}}(f)$ are independent of the window function $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ in (6.7).*

We need some preparations for the proof, and start with the following lemma. We omit the proof, since the result can be found in [2].

Lemma 6.6. *Let $f \in \mathscr{E}'(\mathbf{R}^d)$ and $\phi \in \mathscr{S}(\mathbf{R}^d)$. Then for some constant N_0 and every $N \ge 0$, there are constants C_N such that*

$$|V_\phi f(x,\xi)| \le C_N \langle x\rangle^{-N} \langle \xi\rangle^{N_0}.$$

The following result can be found in [15]. Here $\widehat{*}$ is the twisted convolution, given by the formula

$$(F \mathbin{\widehat{*}} G)(x,\xi) = (2\pi)^{-d/2} \iint F(x-y,\xi-\eta) G(y,\eta) e^{-i\langle x-y,\eta\rangle}\, dy d\eta,$$

when $F, G \in \mathscr{S}(\mathbf{R}^{2d})$. The definition of $\widehat{*}$ extends in such way that one may permit one of F and G to belong to $\mathscr{S}'(\mathbf{R}^{2d})$, and in this case it follows that $F \mathbin{\widehat{*}} G$ belongs to $\mathscr{S}' \cap C^\infty$.

Lemma 6.7. *Let $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\phi_j \in \mathscr{S}(\mathbf{R}^d)$ for $j = 1,2,3$. Then*

$$(V_{\phi_1} f) \widehat{*} (V_{\phi_2} \phi_3) = (\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f.$$

Proof of Proposition 6.5. We may assume that $f \in \mathscr{E}'(\mathbf{R}^d)$ and that $\omega(x,\xi) = \omega(\xi)$, since the statements only involve local assertions. Assume that $\phi, \phi_1 \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and let Γ_1 and Γ_2 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The assertion follows if we prove that

$$|f|_{\mathcal{B}(\Gamma_2)} \le C(|f|_{\mathcal{B}_1(\Gamma_1)} + 1) \tag{6.8}$$

for some constant C, where $\mathcal{B}(\Gamma_2) = M^\phi(\omega, \Gamma_2, \mathscr{B})$ and $\mathcal{B}_1(\Gamma_1) = M^{\phi_1}(\omega, \Gamma_1, \mathscr{B})$.

When proving (6.8) we shall mainly follow the proof of (3.5). Let $v \in \mathscr{P}$ be chosen such that ω is v-moderate, and let

$$\Omega_1 = \mathbf{R}^d \times \Gamma_1 \subseteq \mathbf{R}^d \times (\mathbf{R}^d \setminus 0) \qquad \text{and} \qquad \Omega_2 = \complement \Omega_1,$$

with characteristic functions χ_1 and χ_2 respectively. Here the complement is taken with respect to $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$. Also set

$$F_k(x,\xi) = |V_{\phi_1} f(x,\xi)\omega(\xi)\chi_k(x,\xi)| \qquad \text{and} \qquad G = |V_\phi \phi_1(x,\xi) v(\xi)|.$$

By Lemma 6.7, and the fact that ω is v-moderate we get

$$|V_\phi f(x,\xi)\omega(x,\xi)| \le C((F_1 + F_2) * G)(x,\xi),$$

for some constant C, which implies that

$$|f|_{\mathcal{B}(\Gamma_2)} \le C(J_1 + J_2), \tag{6.9}$$

where

$$J_k = \|(F_k * G)(1 \otimes \chi_{\Gamma_2})\|_{\mathscr{B}}$$

and χ_{Γ_2} is the characteristic function of Γ_2. Since G turns rapidly to zero at infinity, (1.4) gives

$$J_1 \le \|F_1 * G\|_{\mathscr B} \le \|G\|_{L^1_{(v)}} \|F_1\|_{\mathscr B} = C|f|_{\mathcal B_1(\Gamma_1)}, \qquad (6.10)$$

where $C = \|G\|_{L^1_{(v)}}$.

Next we consider J_2. Since, for each $N \ge 0$, there are constants C_N such that

$$F_2(x,\xi) = 0, \qquad \text{and} \qquad \langle \xi - \eta\rangle^{-2N} \le C_N \langle \xi\rangle^{-N}\langle \eta\rangle^{-N}$$

when $\xi \in \Gamma_2$ and $\eta \notin \Gamma_1$, Lemma 6.6 and the computations in (3.10) give

$$(F_2 * G)(x,\xi) \le C_N \langle x\rangle^{-N}\langle \xi\rangle^{-N}, \qquad \xi \in \Gamma_2.$$

Hence $(F_2 * G) \in \mathscr B$ in view of Remark 1.3, which implies $J_2 < \infty$. The estimate (6.8) is now a consequence of (6.9) and (6.10). This completes the proof. □

We are now able to prove the following.

Proposition 6.8. *Let $\mathscr B$ be a translation invariant BF-space on $\mathbf R^{2d}$, $\mathscr B_0$ be given by (6.2), $\omega \in \mathscr P(\mathbf R^{2d})$, $\mathcal B = \mathscr F\mathscr B_0(\omega)$ and $\mathcal C = M(\omega,\mathscr B)$. Also let $f \in \mathscr E'(\mathbf R^d)$. Then*

$$\Theta_{\mathcal B}(f) = \Theta_{\mathcal C}(f) \quad \textit{and} \quad \Sigma_{\mathcal B}(f) = \Sigma_{\mathcal C}(f). \qquad (6.11)$$

Proof. Let Γ_j and χ_{Γ_j} for $j = 1,2$ be the same as in the proof of Proposition 6.5. Also let φ and ϕ be chosen such that (1) in Lemma 6.4 is fulfilled. We may assume that $\omega = 1$ in view of Lemma 1.2.

By (6.4) it follows that

$$|V_\phi f(x,\xi)| \le \varphi(x)(|\widehat f| * |\mathscr F\check\phi|)(\xi),$$

where $\check\phi(x) = \phi(-x)$. This gives

$$|f|_{\mathcal C(\Gamma_2)} = \|V_\phi f\,(1\otimes\chi_{\Gamma_2})\|_{\mathscr B} \le C\|\big(\varphi\otimes(|\widehat f| * |\mathscr F\check\phi|)\big)\,(1\otimes\chi_{\Gamma_2})\|_{\mathscr B}$$

$$= C\|(|\widehat f| * |\mathscr F\check\phi|)\chi_{\Gamma_2}\|_{\mathscr B_0} \le C(J_1 + J_2),$$

for some constant C, where J_1 and J_2 are the same as in (3.6) and (3.7) with $\mathscr B_2 = \mathscr B_0$, $\psi = |\mathscr F\check\phi|$ and $F = |\widehat f|$.

A combination of the latter estimate, (3.8) and (3.10) now gives that for each $N \ge 0$, there is a constant C_N such that

$$|f|_{\mathcal C(\Gamma_2)} \le C_N\Big(|f|_{\mathcal B(\Gamma_1)} + \sup_\xi |\widehat f(\xi)\langle\xi\rangle^{-N}|\Big).$$

Hence, by choosing N large enough it follows that $|f|_{\mathcal C(\Gamma_2)}$ is finite when $|f|_{\mathcal B(\Gamma_1)} < \infty$. Consequently,

$$\Theta_{\mathcal B}(f) \subseteq \Theta_{\mathcal C}(f). \qquad (6.12)$$

In order to get a reversed inclusion we choose φ and ϕ such that Lemma 6.4 (2) is fulfilled. Then (6.5) gives

$$
\begin{aligned}
|f|_{\mathcal{B}(\Gamma_1)} &= \|\varphi \otimes (\widehat{f}\chi_{\Gamma_1})\|_{\mathscr{B}} \\
&= \|(\varphi \otimes 1)(V_\phi f\,(1 \otimes \chi_{\Gamma_1}))\|_{\mathscr{B}} \le C_1 \|\varphi\|_{L^\infty} \|V_\phi f (1 \otimes \chi_{\Gamma_1})\|_{\mathscr{B}} \\
&= C_2 |f|_{\mathcal{C}(\Gamma_1)},
\end{aligned}
$$

for some constants $C_1, C_2 > 0$. This proves that (6.12) holds with reversed inclusion. The proof is complete. □

The following result is now an immediate consequence of Proposition 6.8.

Theorem 6.9. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\mathscr{B}_0$ be given by (6.2), $X \subseteq \mathbf{R}^d$ be open, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathcal{B} = \mathscr{F}\mathscr{B}(\omega)$ and $\mathcal{C} = M(\omega, \mathscr{B})$. If $f \in \mathscr{D}'(X)$, then*

$$
\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathcal{C}}(f).
$$

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