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**Review to the paper: A para-differential renormalization technique for nonlinear dispersive equations**

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## UNIVERSITÀ DEGLI STUDI DI TORINO

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Gianluca Garello

Review to the paper: A para-differential renormalization technique for nonlinear dispersive equations.

*AMS MathSciNet, Mathematical Reviews, MR2754070.*

**The definitive version is available at:**

<http://www.ams.org/mathscinet/search/publdoc.html?pg1=RVRI&pg3=authreviews&sl=343862&vpid=2754070>

The initial value problem for the Benjamin-Oto equation with generalized dispersion

$$\begin{cases} \partial_t u + D^\alpha \partial_x u + \partial_x(u^2/2) = 0 & \text{on } \mathbf{R}_x \times \mathbf{R}_t \\ u(0) = \phi \end{cases} \quad (1)$$

is considered, where  $\alpha \in (1, 2)$  and  $D^\alpha$  denotes the operator defined by the Fourier multiplier  $\xi \mapsto |\xi|^\alpha$ . The problem is considered in the function space  $H_r^\sigma = H^\sigma$ ,  $\sigma \in [0, \infty)$ , given by the real functions  $\phi$  with the usual Sobolev norm  $\|\phi\|_{H_r^\sigma} = \|\phi\|_{H^\sigma} := \|(1 + |\xi|^2)^{\sigma/2} \hat{\phi}(\xi)\|_{L_\xi^2}$ ;  $H_r^\infty := \bigcap_{\sigma \in \mathbf{Z}_+} H_r^\sigma$ , with the induced metric. Suitable solutions of (1) satisfy the  $L^2$  conservation law, that is: if  $T_1 < T_2 \in \mathbf{R}$  and  $u \in C((T_1, T_2) : H_r^\infty)$  is a solution of the equation in (1) on  $\mathbf{R} \times (T_1, T_2)$  then  $\|u(t_1)\|_{H_r^0} = \|u(t_2)\|_{H_r^0}$ , for any  $t_1, t_2 \in (T_1, T_2)$ .

The main result concerns the global well-posedness in  $H_r^0$  of the initial-value problem (1), namely it is the following

**Main Theorem**

- (a) Assume  $\phi \in H_r^\infty$ , then there exists a unique global solution  $u = S^\infty(\phi) \in C(\mathbf{R} : H_r^\infty)$  of the initial-value problem (1).
- (b) Assume  $T \in \mathbf{R}_+$ , then the mapping

$$S_T^\infty = \mathbf{1}_{(-T, T)}(t) \cdot S^\infty : H_r^\infty \mapsto C((-T, T) : H_r^\infty)$$

extends uniquely to a continuous mapping

$$S_T^0 : H_r^0 \mapsto C((-T, T) : H_r^0)$$

and  $\|S_T^0(\phi)(t)\|_{H_r^0} = \|\phi\|_{H_r^0}$ , for any  $t \in (-T, T)$ .

One dimensional models as (1) have been extensively studied. The case  $\alpha = 2$  corresponds to the KdV equation, while the case  $\alpha = 1$  corresponds to the Benjamin-Oto equation.

The nonlinearity of (1) is too strong to allow direct perturbative methods (without a low frequency constraint) since the flow map is not locally uniformly continuous in  $H_r^s(\mathbf{R})$ ,  $s > 0$ . Then the proof is technically hard and consists at first in the reduction of the Main Theorem to proving several *a priori* bounds on smooth solutions and differences of smooth solutions of (1), on bounded time intervals. Such a reduction relies on energy-type estimates. Then the authors construct a renormalization which is the key step to further reducing the problem to perturbative analysis. Namely, after subtracting the low frequency component of the solution, which is essentially left unchanged by the evolution structure of the nonlinearity, they further decompose the solution into frequency blocks and multiply each one of them by a suitable bounded factor. This renormalization leads to an infinite system of coupled equations satisfied by the frequency blocks. In the actual

situation each frequency block must be renormalized by a different factor, which leads to substantial technical difficulties in the perturbative analysis. At the end the main normed spaces are defined and it is shown that the Main Theorem can be reduced to proving a number of nonlinear estimates.