Exercises of Mathematics for Economics and Business

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Claudio Mattalia

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Preface

This book is the result of the teaching experience of the last years at the Faculty of Economics of the University of Torino, and it presents the traditional topics of a first course devoted to the mathematics applied to economics and business. After an initial chapter dedicated to inequalities (widely used in all the course) and another chapter devoted to the basic concepts of set theory and of logic, the basic notions concerning real functions of one real variable, limits, differential calculus and integral calculus for such functions are introduced. The final chapters are devoted to linear algebra and to real functions of several real variables.

In each chapter every topic is introduced with a short theoretical recall (keeping in mind, however, that this is essentially a book of exercises, and does not replace the textbook), then a series of examples are illustrated and solved, according to an increasing degree of difficulty. The presentation is kept at a very simple level, with the objective of putting in evidence the reasoning that (beyond the single calculations) is at the basis of the resolution of a given problem. At the end of each chapter, finally, several exercises are collected, and their solution is reported in the final chapter.

I want to thank my colleagues of the Department of Statistics and Applied Mathematics of the Faculty of Economics of the University of Torino for the suggestions while I was writing this book, and the numerous students that, in the last years, have used and appreciated part of the material collected here. It remains clear that the eventual errors still present are my own responsibility. A particular thanks also to the Editor, for the support in the realization of this work.

Torino, September 2011 Claudio Mattalia

Chapter 1

Inequalities

1.1 Definitions

An inequality (with one unknown) is an expression of the kind:

$$A(x) \geq B(x) \qquad \text{or} \qquad A(x) \leq B(x) \qquad x \in \mathbb{R}$$

and, eventually:

$$A(x) > B(x) \qquad \text{or} \qquad A(x) < B(x)$$

Solutions of the inequality are the values of the unknown that satisfy this expression.

Two inequalities are equivalent if they admit the same solutions; with reference to this aspect, two basic principles hold:

1. Adding or subtracting to both members of the inequality the same (constant or variable) quantity, we get an inequality equivalent to the original one:

$$A(x) > B(x) \Rightarrow A(x) + C(x) > B(x) + C(x)$$

2. Multiplying or dividing both members of the inequality for the same (constant) positive quantity, we get an inequality equivalent to the original one; multiplying or dividing them for the same (constant) negative quantity, we get an inequality equivalent to the original one by reversing the direction of the inequality:

$$A(x) > B(x) \Rightarrow \begin{cases} c \cdot A(x) > c \cdot B(x) & \text{if} \quad c > 0 \\ c \cdot A(x) < c \cdot B(x) & \text{if} \quad c < 0 \end{cases}$$

These two principles are used in order to solve an inequality, transforming the initial inequality in a simpler one, equivalent to it.

Example 1 *Solve the inequality:*

$$x - 3 \geq 0$$

Adding the (constant) quantity $+3$ to both members we get (applying the1^{st} principle of equivalence):

$$x - 3 + 3 \geq 0 + 3$$

i.e.:

$$x \geq 3$$

that is the solution.

Example 2 *Solve the inequality:*

$$2x + 5 > x$$

Adding the (variable) quantity $-x$ to both members we get (applying the 1^{st} principle of equivalence):

$$2x + 5 - x > x - x$$

i.e:

$$x + 5 > 0$$

and then adding the (constant) quantity -5 to both members we get (applying again the 1^{st} principle of equivalence):

$$x + 5 - 5 > 0 - 5$$

i.e.:

$$x > -5$$

that is the solution.

Example 3 *Solve the inequality:*

$$3x < 4$$

Multiplying both members for the (constant and positive) quantity $\frac{1}{3}$ we get (applying the 2^{nd} principle of equivalence):

$$\frac{1}{3} \cdot 3x < \frac{1}{3} \cdot 4$$

i.e.:

$$x < \frac{4}{3}$$

that is the solution.

Example 4 *Solve the inequality:*

$$-3x < 4$$

Multiplying both members for the (constant and negative) quantity $-\dfrac{1}{3}$ we get (applying the 2^{nd} principle of equivalence and reversing the inequality):

$$\left(-\frac{1}{3}\right) \cdot (-3x) > \left(-\frac{1}{3}\right) \cdot 4$$

i.e.:

$$x > -\frac{4}{3}$$

that is the solution.

In practice, the two equivalence principles can be applied observing that is possible to move a term from a member to the other one of the inequality changing its sign (1^{st} principle), and that is possible to multiply or divide both members of the inequality for the same quantity, keeping in mind that the direction of the inequality must be preserved if this quantity is positive while it must be reversed if this quantity is negative (2^{nd} principle).

At this point it is possible to introduce the main types of inequalities: rational integer (of 1^{st} and 2^{nd} degree), rational fractional (and containing products of polynomials), with absolute value, irrational, logarithmic and exponential, together with the systems of inequalities.

1.2 Rational integer inequalities of 1^{st} degree

The rational integer inequalities of 1^{st} degree can always be reduced to the canonic form:

$$ax + b \geq 0 \qquad \text{or} \qquad ax + b \leq 0 \qquad \text{with } a > 0$$

where, eventually, the inequalities are strict (and if $a < 0$ it is sufficient to multiply both members of the inequality by -1 and to reverse the direction of the inequality, obtaining the canonic form with $a > 0$). The solution is then easily given by:

$$x \geq -\frac{b}{a} \qquad \text{or} \qquad x \leq -\frac{b}{a}$$

Example 5 *Solve the inequality:*

$$3x - 12 > 0$$

In this case the inequality is already written in canonic form, applying the equivalence principles seen above we get:

$$3x > 12$$

and then:

$$x > 4$$

that is the solution.

Example 6 *Solve the inequality:*

$$-3x - 12 > 0$$

In this case the inequality is not written in canonic form (since $a < 0$), multiplying both members by -1 (and reversing the inequality) we get first of all:

$$3x + 12 < 0$$

that is the inequality written in canonic form (since $a > 0$). We then easily get (applying the equivalence principles):

$$3x < -12$$

and then:

$$x < -4$$

that is the solution.

1.3 Rational integer inequalities of 2^{nd} degree

The rational integer inequalities of 2^{nd} degree can always be reduced to the canonic form:

$$ax^2 + bx + c \geq 0 \qquad \text{or} \qquad ax^2 + bx + c \leq 0 \qquad \text{with } a > 0$$

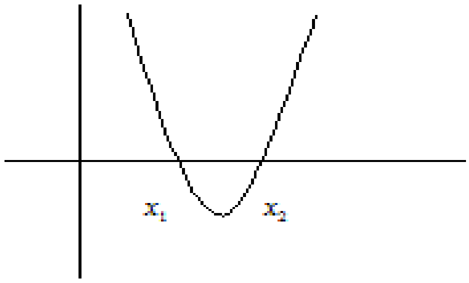
where, eventually, the inequalities are strict (and if $a < 0$ it is sufficient to multiply both members of the inequality by -1 and to reverse the direction of the inequality, obtaining the canonic form with $a > 0$). To solve an inequality of this kind, first of all it is necessary to consider the associated 2^{nd} degree equation:

$$ax^2 + bx + c = 0$$

and to compute its roots x_1 and x_2 (where it is assumed that $x_1 < x_2$ in the case of distinct roots). At this point, the trinomial $ax^2 + bx + c$ is positive for $x < x_1$ and for $x > x_2$, it is negative for $x_1 < x < x_2$ and it is null for $x = x_1$ and for $x = x_2$.

More precisely, keeping in mind that graphically the trinomial $ax^2 + bx + c$ can be represented by a parabola with upward concavity (if $a > 0$), and that a 2^{nd} degree equation can have 2 real distinct roots, 2 real coincident roots or no real roots, the following 3 cases can be distinguished:

1. The discriminant of the 2^{nd} degree equation is positive, that is $\Delta = b^2 - 4ac > 0$. In this case the parabola that represents the trinomial $ax^2 + bx + c$ is as follows:



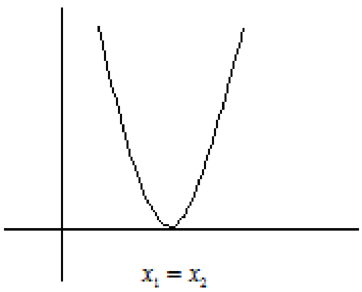
hence the associated equation $ax^2 + bx + c = 0$ has 2 distinct real roots x_1, x_2 (where it is assumed that $x_1 < x_2$) and for the trinomial $ax^2 + bx + c$ we have:

$$ax^2 + bx + c > 0 \qquad \text{for} \qquad x < x_1 \quad \vee \quad x > x_2$$

$$ax^2 + bx + c < 0 \qquad \text{for} \qquad x_1 < x < x_2$$

$$ax^2 + bx + c = 0 \qquad \text{for} \qquad x = x_1 \quad \vee \quad x = x_2$$

2. The discriminant of the 2^{nd} degree equation is null, that is $\Delta = b^2 - 4ac = 0$. In this case the parabola that represents the trinomial $ax^2 + bx + c$ is as follows:

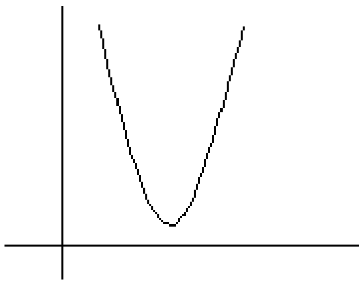


hence the associated equation $ax^2 + bx + c = 0$ has 2 coincident real roots $x_1 = x_2$ and for the trinomial $ax^2 + bx + c$ we have:

$$ax^2 + bx + c > 0 \qquad \text{for} \qquad x \neq x_1, x_2$$

$$ax^2 + bx + c = 0 \qquad \text{for} \qquad x = x_1, x_2$$

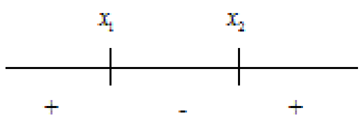
3. The discriminant of the 2^{nd} degree equation is negative, that is $\Delta = b^2 - 4ac < 0$. In this case the parabola that represents the trinomial $ax^2 + bx + c$ is as follows:

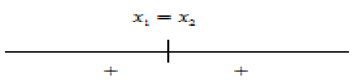
   hence the associated equation $ax^2 + bx + c = 0$ has no real roots and for the trinomial $ax^2 + bx + c$ we have:

$$ax^2 + bx + c > 0 \qquad \forall x \in \mathbb{R}$$

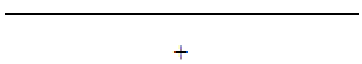
In practice, therefore, if $\Delta > 0$ the sign of the trinomial $ax^2 + bx + c$ is as follows:



while if $\Delta = 0$ the two roots coincide $(x_1 = x_2)$ and the sign of the trinomial is as follows:



and if $\Delta < 0$ there are no (real) roots and the sign of the trinomial is as follows:



Example 7 *Solve the inequality:*

$$x^2 - 3x - 10 > 0$$

In this case the inequality is already written in canonic form, and the associated equation:

$$x^2 - 3x - 10 = 0$$

has two distinct real roots $x_1 = -2$ and $x_2 = 5$. Applying the rule seen above we have that the solution of the inequality is given by:

$$x < -2 \quad \vee \quad x > 5$$

Example 8 *Solve the inequality:*

$$-x^2 + 3x + 10 > 0$$

In this case the inequality is not written in canonic form, hence multipliying both members by -1 (and reversing the inequality) we get:

$$x^2 - 3x - 10 < 0$$

whose associated equation (the same of the previous excercise) has roots $x_1 = -2$ and $x_2 = 5$. Applying the rule seen above we have that the solution of the inequality is given by:

$$-2 < x < 5$$

Example 9 *Solve the inequality:*

$$-3x^2 + 12x < 0$$

Rewriting the inequality in canonic form we get first of all:

$$3x^2 - 12x > 0$$

whose associated equation:

$$3x^2 - 12x = 0$$

has roots $x_1 = 0$ and $x_2 = 4$. The solution of the inequality is then given by:

$$x < 0 \quad \vee \quad x > 4$$

Example 10 *Solve the inequality:*

$$x^2 \geq 4$$

First of all the inequality can be written in canonic form:

$$x^2 - 4 \geq 0$$

then it is possible to observe that the associated equation:

$$x^2 - 4 = 0$$

has roots $x_1 = -2$ and $x_2 = 2$, so that the solution of the inequality is given by:

$$x \leq -2 \quad \vee \quad x \geq 2$$

Example 11 *Solve the inequality:*

$$x^2 - 4x + 4 < 0$$

In this case the inequality is in canonic form, and the associated equation:

$$x^2 - 4x + 4 = 0$$

has two real coincident roots $x_1 = x_2 = 2$, and applying the rule seen above we have that the inequality is never satisfied (the same result can be obtained observing that the first member of the inequality is simply $(x-2)^2$ that, being a square, can never be < 0).

Example 12 *Solve the inequality:*

$$3x^2 + 4 > 0$$

In this case the inequality is in canonic form, and the associated equation:

$$3x^2 + 4 = 0$$

has no real roots, applying the rule seen above we then have that the inequality is satisfied $\forall x \in \mathbb{R}$ (the same result can be obtained observing that the first member of the inequality is the sum of a non-negative term, $3x^2$, and of a positive term, 4, therefore it is strictly positive for any value of x, and the inequality is always satisfied).

1.4 Rational fractional inequalities

The rational fractional inequalities are those in which the unknown appears in the denominator of a fraction, and they can always be reduced to the canonic form:

$$\frac{N(x)}{D(x)} \geq 0 \qquad \text{or} \qquad \frac{N(x)}{D(x)} \leq 0$$

where, eventually, the inequalities are strict (and where the numerator $N(x)$ can also not depend on x, that is $N(x) = N$ constant, while the denominator $D(x)$ must necessarily contain the unknown, otherwise the inequality is not of the rational fractional type).

In this case first of all it is necessary to exclude the values of x that make the denominator of the fraction $D(x)$ equal to 0 (since a fraction with null denominator has no meaning), then it is possible to study separately the sign of $N(x)$ and that of $D(x)$ and, combining them through the “rule of signs”, to determine the sign of the fraction and to solve the inequality.

Example 13 *Solve the inequality:*

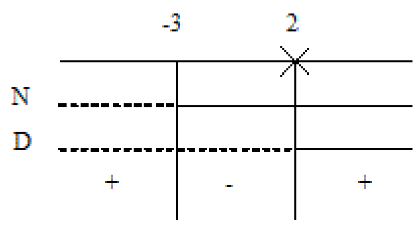
$$\frac{x+3}{2x-4} > 0$$

First of all it must be $2x - 4 \neq 0$, from which $x \neq 2$ (condition of reality of the fraction). Studing separately the sign of the numerator and that of the denominator of the fraction we then have:

$$N(x) > 0 \Rightarrow x + 3 > 0 \Rightarrow x > -3$$

$$D(x) > 0 \Rightarrow 2x - 4 > 0 \Rightarrow x > 2$$

The sign of $N(x)$ and of $D(x)$, together with that of the fraction, can be represented graphically in the following way (where the continuous line denotes the intervals in which the sign is positive and the dashed line the intervals in which the sign is negative, while the cross denotes the value excluded from the existence range):



From the analysis of this graphic we have that the solution of the inequality is given by:

$$x < -3 \quad \vee \quad x > 2$$

Example 14 *Solve the inequality:*

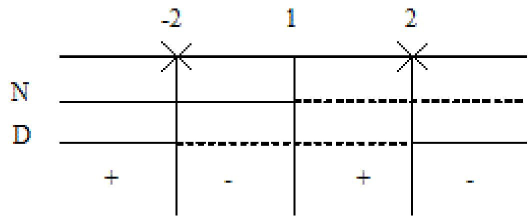
$$\frac{-x+1}{x^2-4} \leq 0$$

First of all it must be $x^2 - 4 \neq 0$, from which $x \neq \mp 2$ (condition of reality of the fraction). Studying the sign of the numerator and of the denominator of the fraction we then have:

$$N(x) \geq 0 \Rightarrow -x+1 \geq 0 \Rightarrow x \leq 1$$

$$D(x) > 0 \Rightarrow x^2 - 4 > 0 \Rightarrow x < -2 \quad \vee \quad x > 2$$

and graphically:



and the solution of the inequality is given by:

$$-2 < x \leq 1 \quad \vee \quad x > 2$$

Since the sign of a product follows the same rules of the sign of a ratio, the same procedure seen to solve rational fractional inequalities can be used also to solve the inequalities containing only products of polynomials. In this case it is possible to study separately the sign of each factor and then, combining them as seen previously, to determine the sign of the product, and to solve the inequality.

Example 15 *Solve the inequality:*

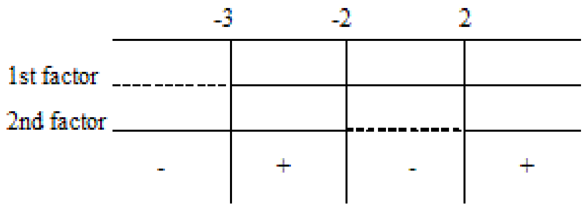
$$(x+3)(x^2-4) > 0$$

Studying separately the sign of the two factors we get:

$$1^{st} \text{ factor} > 0 \Rightarrow x+3 > 0 \Rightarrow x > -3$$

$$2^{nd} \text{ factor} > 0 \Rightarrow x^2 - 4 > 0 \Rightarrow x < -2 \quad \vee \quad x > 2$$

and graphically:



so that the solution of the inequality is given by:

$$-3 < x < -2 \quad \vee \quad x > 2$$

Example 16 *Solve the inequality:*

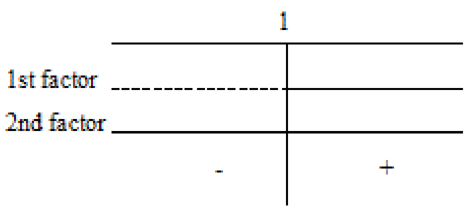
$$(x-1)(x^2-x+3) \leq 0$$

Studing separately the sign of the two factors we get:

$$1^{st} \text{ factor} \geq 0 \Rightarrow x-1 \geq 0 \Rightarrow x \geq 1$$

$$2^{nd} \text{ factor} \geq 0 \Rightarrow x^2-x+3 \geq 0 \Rightarrow \forall x \in \mathbb{R}$$

and graphically:



so that the solution of the inequality is given by:

$$x \leq 1$$

1.5 Syistems of inequalities

A system of inequalities is a set of two or more inequalities that must be satisfied simultaneously. The solution of the system is given by the intersection of the solutions of the single inequalities, and to solve a system of inequalities it is necessary to solve each of the inequalities that compose it and then to consider only the solutions that satisfy at the same time all the inequalities of the system. To this end it is possible to use a graphic representation, in which the solution of each inequality is represented with a continuous line; the system is then satisfied in the intervals in correspondence of which all the lines (as many as the number of inequalities of the system) are continuous.

Example 17 *Solve the system of inequalities:*

$$\begin{cases} \dfrac{x+3}{2x-4} > 0 \\ \\ -x^2 + 3x + 10 > 0 \end{cases}$$

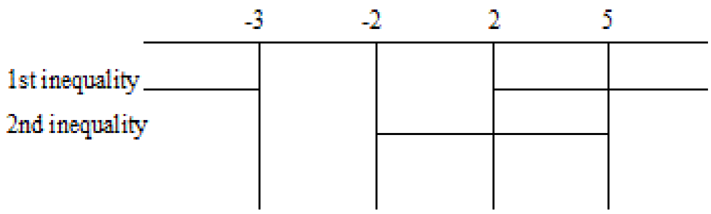
Each of the two inequalities that form the system has already been solved; in particular, the fractional inequality has solution:

$$x < -3 \quad \vee \quad x > 2$$

while the 2^{nd} degree inequality has solution:

$$-2 < x < 5$$

At this point it is possible to represent graphically these sets of solutions, getting:



from which it is possible to deduce that the initial system has solution:

$$2 < x < 5$$

because in correspondence of this interval there are at the same time two continuous lines (as many as the number of inequalities that form the system).

Example 18 *Solve the system of inequalities:*

$$\begin{cases} -3x - 12 > 0 \\ \\ \dfrac{-x+1}{x^2-4} \leq 0 \end{cases}$$

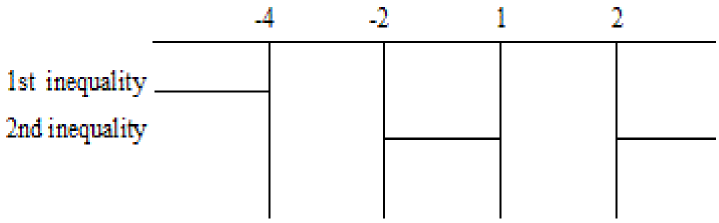
Each of the two inequalities that form the system has already been solved; in particular, the first inequality has solution:

$$x < -4$$

while the second has solution:

$$-2 < x \leq 1 \quad \vee \quad x > 2$$

Representing graphically these sets of solutions we get:



from which it is possible to deduce that the system does not admit solutions (because there are no intervals of the real axis in which there are two continuous lines at the same time), hence it is impossible.

1.6 Inequalities with absolute value

The inequalities with absolute value are those containing the absolute value of one or more expressions in which the unknown appears. Given $x \in \mathbb{R}$, the absolute value of x is defined in the following way:

$$\mid x \mid = \begin{cases} x & \text{if} \quad x \geq 0 \\ \\ -x & \text{if} \quad x < 0 \end{cases}$$

More generally, given an expression $f(x)$ that depends on a variable quantity x, the absolute value of $f(x)$ is defined as:

$$\mid f(x) \mid = \begin{cases} f(x) & \forall x : f(x) \geq 0 \\ \\ -f(x) & \forall x : f(x) < 0 \end{cases}$$

By definition the absolute value of an expression is therefore non-negative, and in particular it is null when the expression contained in the absolute value is null.

To solve an inequality containing one or more absolute values it is necessary "to break it" in two or more (systems of) inequalities corresponding to the intervals of positivity and of negativity of the expressions to which the absolute values are referred, and the solution of the initial inequality is given by the union of the solutions of these single inequalities.

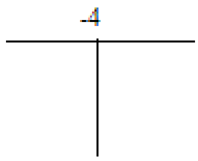
Example 19 *Solve the inequality:*

$$\mid x+4 \mid < 1$$

Applying the definition of absolute value to the expression $\mid x+4 \mid$ we get:

$$\mid x+4 \mid = \begin{cases} x+4 & \text{if} \quad x+4 \geq 0 \quad \Rightarrow \quad x \geq -4 \\ \\ -x-4 & \text{if} \quad x+4 < 0 \quad \Rightarrow \quad x < -4 \end{cases}$$

so that -4 is the "critical" value, that divides the two intervals in correspondence of which the expression that appears in the absolute value changes its sign:



At this point, the initial inequality can be "broken" in the following two systems (one for each of the two intervals in which the expression contained in the absolute value changes its sign):

$$\begin{cases} x < -4 \\ \\ -x-4 < 1 \end{cases} \quad \vee \quad \begin{cases} x \geq -4 \\ \\ x+4 < 1 \end{cases}$$

that become:

$$\begin{cases} x < -4 \\ \\ x > -5 \end{cases} \quad \vee \quad \begin{cases} x \geq -4 \\ \\ x < -3 \end{cases}$$

and then:

$$-5 < x < -4 \quad \vee \quad -4 \leq x < -3$$

and finally:

$$-5 < x < -3$$

that represents the solution of the initial inequality.

Example 20 *Solve the inequality:*

$$| x+3 |<| x-4 |$$

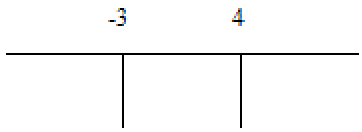
Applying the definition of absolute value to the expressions $| x+3 |$ and $| x-4 |$ we get:

$$| x+3 |= \begin{cases} x+3 & \text{if} \quad x+3 \geq 0 \quad \Rightarrow \quad x \geq -3 \\ -x-3 & \text{if} \quad x+3<0 \quad \Rightarrow \quad x<-3 \end{cases}$$

and then:

$$| x-4 |= \begin{cases} x-4 & \text{if} \quad x-4 \geq 0 \quad \Rightarrow \quad x \geq 4 \\ -x+4 & \text{if} \quad x-4<0 \quad \Rightarrow \quad x<4 \end{cases}$$

so that -3 and 4 are the "critical" values, that divide the intervals in correspondence of which one of the expressions contained in the absolute values changes its sign:



At this point the initial inequality can be "broken" in the following three systems (one for each of the intervals in which one of the expressions contained in the absolute values changes its sign):

$$\begin{cases} x<-3 \\ -x-3<-x+4 \end{cases} \vee \quad \begin{cases} -3 \leq x<4 \\ x+3<-x+4 \end{cases} \vee \quad \begin{cases} x \geq 4 \\ x+3<x-4 \end{cases}$$

that become:

$$\begin{cases} x<-3 \\ -3<4 \end{cases} \vee \quad \begin{cases} -3 \leq x<4 \\ x<\frac{1}{2} \end{cases} \vee \quad \begin{cases} x \geq 4 \\ 3<-4 \end{cases}$$

and then:

$$x<-3 \quad \vee \quad -3 \leq x<\frac{1}{2} \quad \vee \quad \emptyset$$

hence the solution of the initial inequality is:

$$x<\frac{1}{2}$$

Example 21 *Solve the inequality:*

$$\frac{|\ x+5\ |}{2-|\ x\ |} \geq 0$$

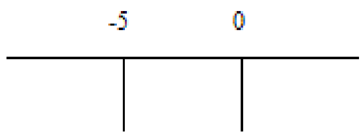
In this case first of all it is necessary to exclude the values of x such that the denominator of the fraction is null, hence we must have $2-|x| \neq 0$, from which $|\ x\ | \neq 2$, that is $x \neq \mp 2$. Applying the definition of absolute value to the expressions $|\ x+5\ |$ and $|\ x\ |$ we then get:

$$|\ x+5\ |= \begin{cases} x+5 & \text{if} \quad x+5 \geq 0 \quad \Rightarrow \quad x \geq -5 \\ \\ -x-5 & \text{if} \quad x+5 < 0 \quad \Rightarrow \quad x < -5 \end{cases}$$

and:

$$|\ x\ |= \begin{cases} x & \text{if} \quad x \geq 0 \\ \\ -x & \text{if} \quad x < 0 \end{cases}$$

so that -5 and 0 are the "critical" values, that divide the intervals in correspondence of which one of the expressions contained in the absolute values changes its sign:



At this point the initial inequality can be "broken" in the following three systems (one for each of the intervals in which one of the expressions contained in the absolute values changes its sign):

$$\begin{cases} x < -5 \\ \\ \dfrac{-x-5}{2-(-x)} \geq 0 \end{cases} \vee \quad \begin{cases} -5 \leq x < 0 \\ \\ \dfrac{x+5}{2-(-x)} \geq 0 \end{cases} \vee \quad \begin{cases} x \geq 0 \\ \\ \dfrac{x+5}{2-x} \geq 0 \end{cases}$$

that become:

$$\begin{cases} x < -5 \\ \\ \dfrac{-x-5}{2+x} \geq 0 \end{cases} \vee \quad \begin{cases} -5 \leq x < 0 \\ \\ \dfrac{x+5}{2+x} \geq 0 \end{cases} \vee \quad \begin{cases} x \geq 0 \\ \\ \dfrac{x+5}{2-x} \geq 0 \end{cases}$$

With simple computations we get:

$$\begin{cases} x < -5 \\ \\ -5 \leq x < -2 \end{cases} \vee \quad \begin{cases} -5 \leq x < 0 \\ \\ x \leq -5 \quad \vee \quad x > -2 \end{cases} \vee \quad \begin{cases} x \geq 0 \\ \\ -5 \leq x < 2 \end{cases}$$

that is:

$$\emptyset \quad \vee \quad (-5 \vee -2 < x < 0) \quad \vee \quad 0 \leq x < 2$$

hence the solution of the initial inequality is:

$$-5 \quad \vee \quad -2 < x < 2$$

Example 22 *Solve the inequality:*

$$| \, x^2 - 9 \, | > -3$$

In this case it is possible to observe immediately that, since the absolute value of an expression is, by definition, non-negative, the first member is always ≥ 0, therefore it is certainly greater than -3 and consequently the inequality is verified $\forall x \in \mathbb{R}$.

1.7 Irrational inequalities

The irrational inequalities are those in which the unknown appears under the sign of a root. In order to solve them first of all it is convenient to isolate the root in one of the two members, then it is necessary to distinguish the case in which the index of the root is odd and the case in which the index is even.

If the root in the inequality has an odd index n, it is possible to obtain an inequality equivalent to the initial one by raising both members to the power n; no other conditions are required, because a root of odd index can have the radical quantity of any sign and can itself assume any sign. If the roots are more than one, eventually with different indexes (always odd), both members of the inequality must be raised to the appropriate power in such a way that the roots are eliminated.

Example 23 *Sove the inequality:*

$$\sqrt[3]{x^2 - 5} \leq -1$$

In this case by raising both members to the cube we get:

$$x^2 - 5 \leq -1$$

and then:

$$x^2 - 4 \leq 0$$

from which:

$$-2 \leq x \leq 2$$

that is the solution of the inequality.

Example 24 *Solve the inequality:*

$$\sqrt[3]{x+1} < \sqrt[9]{x^3+3x^2}$$

In this case by raising both members to the nineth power we get:

$$(x+1)^3 < x^3+3x^2$$

and then:

$$x^3+3x^2+3x+1 < x^3+3x^2$$

from which:

$$x < -\frac{1}{3}$$

that is the solution of the inequality.

If the root in the inequality, on the contrary, has an even index n, it is possible to solve the inequality by using the following procedure:

a We find out the field of existence of the root (by imposing that the radical quantity is non-negative), then we discuss the signs of the two members.

b If the two members are of opposite sign, then we determine immediately the values of the unknown for which the inequality is satisfied.

c If the two members are of the same sign (in particular non-negative, otherwise it is possible to multiply them by -1 changing the direction of the inequality) we raise them to the power n, then we solve the inequality.

d We consider the union of the solutions found at points *b*) and *c*).

It is important to observe that, in the case of root with even index, it would not be correct to raise immediately both members to the power n (without discussing their sign), because in this way there is the risk of introducing uncorrect solutions or of forgetting a part of the solution.

Example 25 *Solve the inequality:*

$$x-3 \leq \sqrt{x}$$

Applying the procedure described above we have:

a) First of all it must be $x \geq 0$ (condition of reality of the root).

b) If $x-3<0$, that is $x<3$, the two members are of opposite sing (the first negative, the second positive or null) and the inequality is always satisfied (since a negative quantity is always less than or equal to a positive or null quantity), provided

$x \geq 0$ (that is the condition of reality) and $x < 3$ (that characterizes the interval considered). This part of the inequality has therefore as a solution:

$$0 \leq x < 3$$

c) If $x-3 \geq 0$, that is $x \geq 3$, the two members are of the same sign (non-negative), therefore they can be raised to the square and the inequality becomes:

$$(x-3)^2 \leq x$$

that is:

$$x^2 - 7x + 9 \leq 0$$

whose solutions are:

$$\frac{7-\sqrt{13}}{2} \leq x \leq \frac{7+\sqrt{13}}{2}$$

provided $x \geq 0$ (that is the condition of reality) and $x \geq 3$ (that characterizes the interval considered). This part of the inequality has therefore as a solution:

$$3 \leq x \leq \frac{7+\sqrt{13}}{2}$$

d) Combining the results found at points b) and c) (i.e. considering their union), finally, it turns out that the solution of the initial inequality is:

$$0 \leq x \leq \frac{7+\sqrt{13}}{2}$$

It is possible to observe that if we raise immediately the two members to the square we get the solution (that takes into account the condition of reality) $\frac{7-\sqrt{13}}{2} \leq x \leq \frac{7+\sqrt{13}}{2}$ that is not correct, since it does not consider a part of the values of the unknown that satisfy the inequality.

Example 26 *Solve the inequality:*

$$x + 3 \geq \sqrt{3x-1}$$

a) First of all it must be $3x - 1 \geq 0$, that is $x \geq \dfrac{1}{3}$ (condition of reality of the root).

b) If $x + 3 < 0$, that is $x < -3$, the two members are of opposite sign (the first negative, the second positive or null) and the inequality is never satisfied (since a negative quantity is never larger than or equal to a positive or null quantity).

c) If $x + 3 \geq 0$, that is $x \geq -3$, the two members are of the same sign (non-negative), therefore they can be raised to the square and the inequality becomes:

$$(x+3)^2 \geq 3x - 1$$

that is:

$$x^2 + 3x + 10 \geq 0$$

which is always satisfied, provided $x \geq \frac{1}{3}$ (which is the condition of reality) and $x \geq -3$ (that characterizes the interval considered). This part of the inequality has therefore as a solution:

$$x \geq \frac{1}{3}$$

d) Combining the results found at points b) and c), finally, it turns out that the solution of the initial inequality is:

$$x \geq \frac{1}{3}$$

Example 27 *Solve the inequality:*

$$\sqrt{x^2 + 2x - 8} < \sqrt{x + 4}$$

First of all we must have, for the conditions of reality of the roots:

$$\begin{cases} x^2 + 2x - 8 \geq 0 \\ \\ x + 4 \geq 0 \end{cases} \Rightarrow \begin{cases} x \leq -4 \quad \vee \quad x \geq 2 \\ \\ x \geq -4 \end{cases} \Rightarrow x = -4 \quad \vee \quad x \geq 2$$

At this point it is possible to observe that both members of the inequality have the same sign (non-negative), since they are two roots with even index, therefore they can be raised to the square getting:

$$x^2 + 2x - 8 < x + 4$$

that is:

$$x^2 + x - 12 < 0$$

whose solution is:

$$-4 < x < 3$$

Combining this result with the condition of reality it turns out that the solution of the initial inequality is:

$$2 \leq x < 3$$

Example 28 *Solve the inequality:*

$$\sqrt[3]{x+x^2} \leq \sqrt{x}$$

a) First of all it must be $x \geq 0$ (condition of reality of the root with even index).

b) If $x + x^2 < 0$, that is $-1 < x < 0$, the two members are of opposite sign (the first negative, the second positive or null) and the inequality is always satisfied (since a negative quantity is always less than or equal to a positive or null quantity) provided $x \geq 0$ (that is the condition of reality) and $-1 < x < 0$ (that characterizes the interval considered). These conditions, however, are not compatible, so that in reality in this case the inequality has no solutions.

c) If $x+x^2 \geq 0$, that is $x \leq -1 \quad \vee \quad x \geq 0$, the two members are of the same sign (non-negative), therefore they can be raised to the sixth power and the inequality becomes:

$$(x+x^2)^2 \leq x^3$$

and then:

$$x^4 + x^3 + x^2 \leq 0$$

and finally:

$$x^2\left(x^2+x+1\right) \leq 0$$

that is verified only for $x = 0$ (compatible with the condition of reality $x \geq 0$ and with the condition $x \leq -1 \quad \vee \quad x \geq 0$ that characterizes the interval considered).

d) Combining the results found at points b) and c), finally, it turns out that the solution of the initial inequality is:

$$x = 0$$

In the case of inequalities with roots with even index it is also possible to transform such inequalities in systems equivalent to them. In particular, considering an inequality of the type:

$$A(x) \geq \sqrt{B(x)}$$

first of all it must be $B(x) \geq 0$ (condition of reality of the root) and also $A(x) \geq 0$ (since the root in the right-hand side is non-negative, and for the inequality to be satisfied also the left-hand side must be non-negative). At this point it is possible to raise to the square both members (that are non-negative) in order to eliminate the root, and therefore the initial inequality is equivalent to the system:

$$\begin{cases} B(x) \geq 0 \\ A(x) \geq 0 \\ [A(x)]^2 \geq B(x) \end{cases}$$

Considering then an inequality of the type:

$$A(x) \leq \sqrt{B(x)}$$

first of all it must be $B(x) \geq 0$ (condition of reality of the root), then if $A(x) \geq 0$ it is possible to raise to the square both members (that are non-negative) in order to eliminate the root, and the initial inequality is verified by the values of x that are solutions of the system:

$$\begin{cases} B(x) \geq 0 \\ A(x) \geq 0 \\ [A(x)]^2 \leq B(x) \end{cases}$$

where the first condition is redundant since it is implied by the third one (in fact if $B(x) \geq [A(x)]^2$ then $B(x) \geq 0$). In this case, furthermore, the initial inequality is verified also when $A(x) < 0$ (because a negative quantity is always less than or equal to a non-negative quantity like $\sqrt{B(x)}$), therefore it is satisfied also by the values of x that are solutions of the system:

$$\begin{cases} B(x) \geq 0 \\ A(x) < 0 \end{cases}$$

and, in conclusion, the initial inequality is equivalent to the union of the two systems:

$$\begin{cases} A(x) \geq 0 \\ [A(x)]^2 \leq B(x) \end{cases} \vee \begin{cases} B(x) \geq 0 \\ A(x) < 0 \end{cases}$$

In these cases, therefore, given the initial inequality first of all it is possible to write the system (or the systems) equivalent to it, then the solution of this system corresponds to the solution of the initial inequality.

Example 29 *Solve the inequality:*

$$x + 3 \geq \sqrt{3x - 1}$$

This inequality (already solved in Example 26) is written in the form $A(x) \geq \sqrt{B(x)}$ and therefore it is equivalent to the system:

$$\begin{cases} 3x - 1 \geq 0 \\ x + 3 \geq 0 \\ (x+3)^2 \geq 3x - 1 \end{cases}$$

from which we get:

$$\begin{cases} 3x-1 \geq 0 \\ x+3 \geq 0 \\ x^2+3x+10 \geq 0 \end{cases}$$

that is:

$$\begin{cases} x \geq \dfrac{1}{3} \\ x \geq -3 \\ \forall x \end{cases}$$

and finally:

$$x \geq \frac{1}{3}$$

that is the solution of the initial inequality, and coincides with the one found previously.

Example 30 *Solve the inequality:*

$$x-3 \leq \sqrt{x}$$

This inequality (already solved in Example 25) is written in the form $A(x) \leq \sqrt{B(x)}$ and therefore it is equivalent to the union of the two systems:

$$\begin{cases} x-3 \geq 0 \\ (x-3)^2 \leq x \end{cases} \quad \vee \quad \begin{cases} x \geq 0 \\ x-3 < 0 \end{cases}$$

from which we get:

$$\begin{cases} x-3 \geq 0 \\ x^2-7x+9 \leq 0 \end{cases} \quad \vee \quad \begin{cases} x \geq 0 \\ x-3 < 0 \end{cases}$$

that is:

$$\begin{cases} x \geq 3 \\ \dfrac{7-\sqrt{13}}{2} \leq x \leq \dfrac{7+\sqrt{13}}{2} \end{cases} \quad \vee \quad \begin{cases} x \geq 0 \\ x < 3 \end{cases}$$

and then:

$$3 \leq x \leq \frac{7+\sqrt{13}}{2} \quad \vee \quad 0 \leq x < 3$$

and finally:

$$0 \leq x \leq \frac{7+\sqrt{13}}{2}$$

that is the solution of the initial inequality, and coincides with the one found previously.

1.8 Logarithmic inequalities

Given two numbers a, $b > 0$ (with $a \neq 1$) the logarithm in basis a of b is defined as the number c at which a must be raised in order to obtain b, that is:

$$\log_a b = c \Leftrightarrow a^c = b$$

By the definition of logarithm we have therefore:

$$\log_a a^b = b \qquad\qquad a^{\log_a b} = b$$

so that any number b can be expressed through the logarithm in any basis $a > 0$ (and different from 1) using one of these two relations (the second can be used only when $b > 0$). The logarithms then satisfy the following properties:

(*i*) $\log_a 1 = 0 \quad \forall a > 0,\ a \neq 1$

(*ii*) if $0 < a < 1$ then $x < x' \Leftrightarrow \log_a x > \log_a x'$ with $x,\ x' > 0$

(*iii*) if $a > 1$ then $x < x' \Leftrightarrow \log_a x < \log_a x'$ with $x,\ x' > 0$

(*iv*) $\log_a x + \log_a y = \log_a (xy)$ with $x,\ y > 0$

(*v*) $\log_a x - \log_a y = \log_a \dfrac{x}{y}$ with $x,\ y > 0$

(*vi*) $\log_a x^p = p \log_a x$ with $x > 0$

(*vii*) $\log_a b = \dfrac{\log_c b}{\log_c a}$ with $a > 0,\ a \neq 1,\ b > 0,\ c > 0,\ c \neq 1$

The logarithmic inequalities are those in which the unknown appears in the argument of a logarithm. To solve them first of all it is necessary to impose that the argument of the logarithm is strictly positive (condition of reality), then it is possible to use the properties listed above in order to obtain the solution.

Example 31 *Solve the inequality:*

$$\log_{\frac{1}{2}} x < -4$$

First of all it must be $x > 0$ (condition of reality of the logarithm), then (simply applying the definition of logarithm) it is possible to write:

$$\log_{\frac{1}{2}} x < \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-4}$$

that is:

$$\log_{\frac{1}{2}} x < \log_{\frac{1}{2}} 16$$

and finally (applying the property (*ii*) seen above – as the basis of the logarithm in this case is lower than 1, so that passing from the inequality between the logarithms to the inequality between the corresponding arguments the direction of the inequality must be reversed –):

$$x > 16$$

that is the solution of the inequality (compatible with the condition of reality $x > 0$).

The same result can be obtained applying the properties of the exponentials (presented in the next Section); in this case first of all it is possible to write (applying to both members of the initial inequality the exponential of basis $\frac{1}{2}$, that requires to reverse the inequality since the basis of the exponential is lower than 1):

$$\left(\frac{1}{2}\right)^{\log_{\frac{1}{2}} x} > \left(\frac{1}{2}\right)^{-4}$$

and then (applying to the left-hand side the definition of logarithm):

$$x > 16$$

that is the solution of the inequality (compatible with the condition of reality $x > 0$).

Example 32 *Solve the inequality:*

$$\log_2 x > 3$$

First of all it must be $x > 0$ (condition of reality of the logarithm), then (applying the definition of logarithm) it is possible to write:

$$\log_2 x > \log_2 (2)^3$$

that is:

$$\log_2 x > \log_2 8$$

and finally (applying the property (*iii*) seen above – as the basis of the logarithm in this case is larger than 1, so that passing from the inequality between the logarithms to the inequality between the corresponding arguments the direction of the inequality is preserved –):

$$x > 8$$

that is the solution of the inequality (compatible with the condition of reality $x > 0$).

The same result can be obtained applying the properties of the exponentials; in this case first of all it is possible to write (applying to both members of the inequality the exponential of basis 2, that requires to preserve the direction of the inequality since the basis of the exponential is larger than 1):

$$2^{\log_2 x} > 2^3$$

and then (applying to the left-hand side the definition of logarithm):

$$x > 8$$

that is the solution of the inequality (compatible with the condition of reality $x > 0$).

Example 33 *Solve the inequality:*

$$\log_{\frac{1}{3}} \frac{x^2}{x+2} < 0$$

First of all we must have, for the conditions of reality of the fraction and of the logarithm:

$$\begin{cases} x + 2 \neq 0 \\ \dfrac{x^2}{x+2} > 0 \end{cases} \Rightarrow -2 < x < 0 \quad \vee \quad x > 0$$

then the initial inequality can be written in the form:

$$\log_{\frac{1}{3}} \frac{x^2}{x+2} < \log_{\frac{1}{3}} 1$$

and passing to the arguments (reversing the direction of the inequality, since the basis of the logarithms is lower than 1):

$$\frac{x^2}{x+2} > 1$$

(the same expression can be obtained if, in the initial inequality, we apply to the two members the exponential of basis $\frac{1}{3}$). This is a rational fractional inequality, solving it as shown above (after reducing it to the canonic form $\frac{x^2}{x+2} - 1 > 0$ and doing the computations) we get:

$$-2 < x < -1 \quad \vee \quad x > 2$$

that is compatible with the condition of reality determined initially ($-2 < x < 0 \quad \vee \quad x > 0$), so that the solution found is also the solution of the initial inequality.

Example 34 *Solve the inequality:*

$$\log \sqrt{x+5} \leq 3$$

First of all we must have, for the conditions of reality of the root and of the logarithm:

$$\begin{cases} x + 5 \geq 0 \\ \sqrt{x+5} > 0 \end{cases} \Rightarrow \begin{cases} x + 5 \geq 0 \\ x + 5 > 0 \end{cases} \Rightarrow x > -5$$

At this point (observing that when the basis is not specified then the logarithm is considered with basis $e = 2.7182...$, therefore larger than 1) the initial inequality can be written as:

$$\log \sqrt{x+5} \leq \log e^3$$

from which:

$$\sqrt{x+5} \leq e^3$$

(the same expression can be obtained if, in the initial inequality, we apply to both members the exponential of basis e). This is an irrational inequality that can be solved as seen previously (in particular, since both members are of the same sign and non-negative they can be raised to the square) getting:

$$x \leq e^6 - 5$$

Combining this solution with the condition of reality found at the beginning ($x > -5$) it is possible to obtain the solution of the initial inequality, that is:

$$-5 < x \leq e^6 - 5$$

1.9 Exponential inequalities

Given a number $a > 0$ (and different from 1) a power of a with real exponent x is denoted by a^x. This is called exponential, and the exponentials satisfy the following properties:

$$(i) \quad a^x > 0 \quad \forall x \text{ with } a > 0$$

$$(ii) \text{ if } 0 < a < 1 \text{ then } x < x' \Leftrightarrow a^x > a^{x'}$$

$$(iii) \text{ if } a > 1 \text{ then } x < x' \Leftrightarrow a^x < a^{x'}$$

The exponential inequalities are those in which the unknown appears in the exponent of a certain expression, and to solve them it is possible to use the properties listed above.

Example 35 *Solve the inequality:*

$$\left(\frac{1}{2}\right)^x < 16$$

First of all the inequality can be written in the form:

$$\left(\frac{1}{2}\right)^x < \left(\frac{1}{2}\right)^{-4}$$

and then (applying the property (*ii*) seen above – as the basis of the exponential in this case is lower than 1, so that passing from the inequality between the exponentials to the inequality between the exponents the direction of the inequality is reversed –):

$$x > -4$$

that is the solution of the inequality.

The same results can be obtained using the logarithms (indeed the exponential inequalities and the logarithmic inequalities are strictly linked, since exponentials and logarithms are inverse functions one with respect to the other). In particular, starting from the initial inequality and applying to both members the logarithm in basis $\frac{1}{2}$ (and reversing the direction of the inequality, since the basis is lower than 1) we get first of all:

$$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^x > \log_{\frac{1}{2}} 16$$

and then (simply applying the definition of logarithm):

$$x > -4$$

that is the solution of the inequality.

Example 36 *Solve the inequality:*

$$3^x > 9$$

First of all the inequality can be written in the form:

$$3^x > 3^2$$

and then (applying the property (*iii*) seen above – as the basis of the exponential in this case is greater than 1, so that passing from the inequality between the exponentials to the inequality between the exponents the direction of the inequality is preserved):

$$x > 2$$

that is the solution of the inequality.

The same result can be obtained applying to both members of the initial inequality the logarithm in basis 3 (preserving the direction of the inequality, since the basis in this case is larger than 1), so that we have:

$$\log_3 3^x > \log_3 9$$

and then (applying the definition of logarithm):

$$x > 2$$

that is the solution of the inequality.

Example 37 *Solve the inequality:*

$$4^{2x+1} + 4^x - 1 < 0$$

First of all this inequality can be written as:

$$4 \cdot 4^{2x} + 4^x - 1 < 0$$

and then, putting $4^x = z$, we have:

$$4z^2 + z - 1 < 0$$

that is an inequality of 2^{nd} degree whose solution is:

$$\frac{-1-\sqrt{17}}{8} < z < \frac{-1+\sqrt{17}}{8}$$

Going back to the initial variable we then have:

$$\frac{-1-\sqrt{17}}{8} < 4^x < \frac{-1+\sqrt{17}}{8}$$

where the first inequality is always satisfied (since 4^x is positive, therefore larger than the negative quantity $\frac{-1-\sqrt{17}}{8}$), while the second inequality (applying to both members the logarithm in basis 4) leads to:

$$x < \log_4 \frac{-1+\sqrt{17}}{8}$$

that is the solution of the initial inequality.

Example 38 *Solve the inequality:*

$$2^{x-1} > 3^{x+1}$$

Applying the logarithm in basis e to both members we get first of all:

$$\log 2^{x-1} > \log 3^{x+1}$$

that is:

$$(x-1)\log 2 > (x+1)\log 3$$

and with some simple computations we get:

$$x(\log 2 - \log 3) > \log 2 + \log 3$$

from which (observing that the quantity $(\log 2 - \log 3)$ is negative, so that when we divide both members for this quantity the direction of the inequality must be reversed):

$$x < \frac{\log 2 + \log 3}{\log 2 - \log 3}$$

that is the solution of the initial inequality.

1.10 Exercises

Solve the following inequalities:

1) $3x + 2 < -1$

2) $3(x+2) - 4(x+3) \leq 1$

3) $x^2 + 4x - 21 > 0$

4) $3x^2 - 15x \leq 0$

5) $x^2 - 6x + 9 > 0$

6) $2x^2 - 15x + 30 < 0$

7) $\dfrac{2x+1}{x-3} \leq 0$

8) $\dfrac{4x-5}{5x} < 4x$

9) $\dfrac{4x-3}{2x} < 3x$

10) $(x-3)^2(x+5) \leq 0$

11) $(x^2+1)(x^2-2x+5) \leq 0$

12) $\begin{cases} 2x-7<0 \\ 3x+18 \geq 0 \end{cases}$

13) $\begin{cases} x^2-1<0 \\ 3x^2-9<0 \end{cases}$

14) $\begin{cases} x^2+4x-21>0 \\ 3x^2-15x \leq 0 \end{cases}$

15) $\begin{cases} x^2-6x+9>0 \\ 2x^2-15x+30<0 \end{cases}$

16) $\left|x^2-1\right| \geq -1$

17) $\mid x^2-2x+5 \mid \leq -3$

18) $\left|x^2-4\right| > x-2$

19) $\left|x^2-4\right| \geq x-2$

20) $\mid x-2 \mid < 2x+2$

21) $\mid x-2 \mid < \mid x \mid$

22) $\sqrt{x^2-4} < x$

23) $\sqrt{x^2-4} < x-4$

24) $\sqrt{x+5} < x-1$

25) $\sqrt{x+5} > x-1$

26) $\sqrt{x+2} \leq x+1$

27) $\sqrt{x^2+7} > -2$

28) $\sqrt{x^2+1} > 1$

29) $\sqrt{x^2-1} \geq 1$

30) $\sqrt{x+1} > \sqrt[3]{x-1}$

31) $\log_3(x+2) < 2$

32) $\log_{\frac{1}{3}}(x+2) < 2$

33) $\log_3(x^2+8) < -2$

34) $\log_4(x^2+7) < -2$

35) $\log_{\frac{1}{4}}(x^2+7) < -2$

36) $\log_{\frac{1}{2}}(x^2+4) < -3$

37) $\log_{\frac{1}{2}}(x^2+4) > -3$

38) $\log_{\frac{1}{2}}(x^2+3) < -2$

39) $\log_{\frac{1}{3}}\sqrt{x-1} \geq -1$

40) $\log_{\frac{1}{2}}\sqrt{x+9} \leq -1$

41) $\log\sqrt{x+3} \leq 1$

42) $\left(\frac{2}{3}\right)^{x^2-4x} \leq 1$

43) $\left(\frac{3}{4}\right)^{x^2-3x} \leq 1$

44) $\left(\frac{1}{3}\right)^{x^2-4x} \leq 1$

45) $\left(\frac{1}{3}\right)^{x^2-4x} \geq 1$

46) $2^{x^2-16} \geq 1$

47) $2^{x^2-3} < 2$

48) $4^x + 2^x - 2 < 0$

49) $4 \cdot 3^x \leq 2 \cdot 4^x$

50) $2^{x+1} > 3^{x-1}$

Chapter 2

Sets and logic

2.1 Sets and their operations

The concept of set is usually assumed as known and used as synonimous of collection, family, class of elements with some characteristic in common. Sets are denoted with capital letters, while their elements are denoted with small letters. A first way to represent a set consists in listing its elements, a second way consists in describing a property that characterizes the elements, a third way consists in using Venn's diagrams, in which the elements of a set are represented as points of the plane.

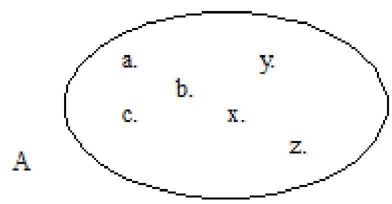
Example 39 *The set A of the letters of the alphabet can be represented listing its elements:*

$$A = \{a, b, c, ..., x, y, z\}$$

or indicating a property that characterizes the elements:

$$A = \{\textit{letters of the alphabet}\}$$

or with a Venn's diagram:



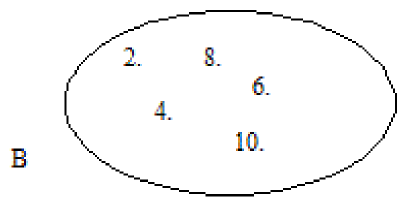
Example 40 *The set B of the first 5 even positive numbers can be represented listing its elements:*

$$B = \{2, 4, 6, 8, 10\}$$

or indicating a property that characterizes the elements:

$$B = \{\textit{first } 5 \textit{ even positive numbers}\}$$

or with a Venn's diagram:



A symbol often used is $\in$ that means "belongs to", while the symbol $\notin$ means "does not belong to", and a particular set is the empty (or null) set, that is without elements, denoted by the symbol $\emptyset$.

Example 41 *Given the set:*

$$X = \{0, 1, 2\}$$

we have that $1 \in X$ (that is 1 is an element of the set), while $4 \notin X$ (that is 4 is not an element of the set).

Given the empty set, then, for any element a we have that $a \notin \emptyset$ (that is a is not an element of the empty set).

Two sets are equal if they have the same elements, and in this case it is not important the order with which they are listed.

Given two sets A and B, we say that A is a subset of B (or that A is included in B) if each element of A is also element of B, and we write $A \subseteq B$. The symbol $\subseteq$ means inclusion, and it does not exclude that the sets A and B coincide, while to exclude this possibility it is possible to use the symbol of strict inclusion $\subset$. In this case we say that A is a proper subset of B (or that A is strictly included in B) if each element of A is also element of B, but there is at least one element of B that is not element of A, and we write $A \subset B$. In particular, the set $\emptyset$ is strictly included in any other set.

Example 42 *Given the sets:*

$$A = \{0, 1, 2\} \qquad B = \{1, 2, 0\}$$

we have $A \subseteq B$ and also $B \subseteq A$, that is A is a subset of B and B is a subset of A, therefore the two sets are equal, that is $A = B$.

Given the sets:

$$A = \{0, 1\} \qquad B = \{0, 1, 2\}$$

then, we have $A \subset B$, that is A is a proper subset of B.

Each set A always contains itself ($A \subseteq A$) and the empty set ($\emptyset \subset A$), that are called improper subsets of A.

Given a set A, the set formed by all its subsets (proper and improper) is called "set of the parts" and is denoted with $\mathcal{P}(A)$. If A has n elements, its sets of the parts is formed by 2^n elements.

Example 43 *Given the set:*

$$A = \{1, 2, 3\}$$

find the corresponding set of the parts.

In this case the set of the parts of A is given by:

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

that has $2^3 = 8$ elements.

It is then possible to define some operations concerning sets. In particular, given two sets A and B, their union is the set, denoted by $A \cup B$, formed by all the elements that belong either to A or to B (or to both), that is:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

while their intersection is the set, denoted by $A \cap B$, formed by all the elements that belong to A and B at the same time, that is:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If the intersection of two sets is empty (that is $A \cap B = \emptyset$) the two sets are said to be disjoint.

Example 44 *Given the sets:*

$$A = \{0, 1, 2\} \qquad B = \{1, 2, 3\}$$

find their union and their intersection.

In this case the union of the two sets is given by:

$$A \cup B = \{0, 1, 2, 3\}$$

while their intersection is given by:

$$A \cap B = \{1, 2\}$$

Given a subset A of U (universe set), the complement of A with respect to U, denoted byA_U^C (or A^C or $\overline{A}$) is the set formed by the elements of U that do not belong to A, that is:

$$A^C = \{x : x \in U \text{ and } x \notin A\}$$

while given two sets A and B their difference, denoted by $A \setminus B$, is the set formed by the elements that belong to A but do not belong to B, that is:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

Example 45 *Given the sets:*

$$A = \{1, 4\} \qquad U = \{1, 2, 3, 4, 5, 6\} \qquad B = \{0, 1, 2\} \qquad C = \{1, 2, 3\}$$

find the complement of A with respect to U and the difference between B and C.

In this case the complement of A with respect to U is:

$$A^C = \{2, 3, 5, 6\}$$

while the difference between B and C is:

$$B \setminus C = \{0\}$$

that is the set formed only by the element 0 (that must not be confused with the empty set $\emptyset$).

Sets operations have a series of properties, in particular:

(i) idempotence property

$$\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}$$

(ii) commutative property:

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

(iii) associative property:

$$\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

(iv) distributive property:

$$\begin{aligned} (A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C) \end{aligned}$$

(v)

$$\begin{aligned} A \cup \emptyset &= A \\ A \cap \emptyset &= \emptyset \end{aligned}$$

(vi) De Morgan's laws:

$$\begin{aligned} (A \cup B)^C &= A^C \cap B^C \\ (A \cap B)^C &= A^C \cup B^C \end{aligned}$$

Given two sets $A = \{a, b\}$ and $B = \{b, a\}$, as seen above they coincide (that is $A = B$), because they have the same elements. In some cases it is necessary to consider ordered couples (ordered pairs), in which it is relevant the order with which the elements are written. Given two sets A and B (not necessarily distinct), we call ordered pair a set (a, b) formed taking an element $a \in A$ and an element $b \in B$ in this order. The set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is called cartesian product of A and B and is denoted by $A \times B$, that is:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

and if $A \neq B$ then $A \times B \neq B \times A$, while if $A = B$ then we write $A \times A = A^2$.

Example 46 *Given the sets:*

$$A = \{0, 1, 2\} \qquad B = \{-1, 1\}$$

find the cartesian products $A \times B$ and $B \times A$.

In this case the cartesian product $A \times B$ is given by:

$$A \times B = \{(0, -1), (0, 1), (1, -1), (1, 1), (2, -1), (2, 1)\}$$

while the cartesian product $B \times A$ is given by:

$$B \times A = \{(-1, 0), (-1, 1), (-1, 2), (1, 0), (1, 1), (1, 2)\}$$

and it turns out to be $A \times B \neq B \times A$.

More generally, given n sets $A_1, A_2, ..., A_n$ we call ordered n-tuple (entuple) a set $(a_1, a_2, ..., a_n)$ formed taking an element $a_1 \in A_1$, an element $a_2 \in A_2$, ..., an element $a_n \in A_n$ in this order. We then call cartesian product of n sets $A_1, A_2, ..., A_n$ the set of the ordered n-tuples of elements belonging respectively to $A_1, A_2, ..., A_n$, that is:

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i \qquad i = 1, 2, ..., n\}$$

2.2 Sets of numbers

Particularly important sets are the numerical sets, more precisely:

1. the set of natural numbers $\mathbb{N}$:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

2. the set of integer relative numbers $\mathbb{Z}$:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$$

3. the set of rational numbers $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \pm \frac{m}{n} \text{ with } m, n \in \mathbb{N}, n \neq 0 \right\}$$

The sets $\mathbb{N}$ and $\mathbb{Z}$ are called “discrete”, because it is not always true that between 2 elements of $\mathbb{N}$ (or of $\mathbb{Z}$) there is another element of $\mathbb{N}$ (or of $\mathbb{Z}$), while considering $\mathbb{Q}$ it is always true that given 2 numbers $q_1, q_2 \in \mathbb{Q}$ there exists a third number q_3 (with $q_1 < q_3 < q_2$) that belongs to $\mathbb{Q}$ (for instance the arithmetic mean between q_1 and q_2). This property (according to which between 2 rational numbers there is always another rational number) can be expressed saying that $\mathbb{Q}$ is “dense”. Nevertheless, $\mathbb{Q}$ is still “discontinuous”, that is between 2 rational numbers there can exist a non rational number. For example, it is possible to prove that the number denoted by $\sqrt{2}$ (that is a number whose square is equal to 2) does not belong to $\mathbb{Q}$. The set $\mathbb{Q}$, therefore, leaves some “holes”, and in order to “fill in” such holes it is necessary to introduce another set of numbers, the set of real numbers $\mathbb{R}$. The difference between real and rational numbers, denoted by $\mathbb{R} \setminus \mathbb{Q}$, then, represents the set of irrational numbers, for instance we have:

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad e \in \mathbb{R} \setminus \mathbb{Q} \qquad \pi \in \mathbb{R} \setminus \mathbb{Q}$$

The set $\mathbb{R}$ “fills in” all the holes on the line, and for this reason it is a continuous set (that is between 2 real numbers there are always all real numbers). Real numbers are therefore in a one-to-one correspondence with the points of the line (each real number corresponds to a point on the line and viceversa), and for this reason the latter is also called “real line”.

2.3 Sets of real numbers and their topology

The numerical sets introduced above are characterized by the following relation:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

where each set is a proper subset of the following one.

Particularly important are then some subsets of $\mathbb{R}$, called intervals. Given two real numbers a, b with $a < b$ we introduce the following sets:

(i) closed and bounded interval with extremes a and b:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

(ii) open and bounded interval with extremes a and b:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

(iii) semi-closed or semi-open (closed to the left and open to the right) and bounded interval with extremes a and b:

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

(*iv*) semi-open or semi-closed (open to the left and closed to the right) and bounded interval with extremes a and b:

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

where the specification "closed" and "open" comes from the fact that the extremes a and b belong or do not belong to the set, while the specification "bounded" comes from the fact that the extremes a and b provide a lower and an upper bound for the elements of the set. All these intervals have a segment of a straight line as geometrical image.

It is then possible to introduce the symbols $+\infty$ (plus infinite) and $-\infty$ (minus infinite), that are such that:

$$-\infty < x < +\infty \qquad \forall x \in \mathbb{R}$$

and then to introduce the set:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$$

In this way it is possible to introduce also unbounded intervals, whose geometrical image is a half-line, and that are sets defined as follows:

(*i*) closed and unbounded to the right interval:

$$[a, +\infty) = \{x \in \mathbb{R} : x \geq a\}$$

(*ii*) open and unbounded to the right interval:

$$(a, +\infty) = \{x \in \mathbb{R} : x > a\}$$

(*iii*) closed and unbounded to the left interval:

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

(*iv*) open and unbounded to the left inteval:

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

The set of real numbers, whose geometrical image is the entire straight line, finally, can be denoted as:

$$\mathbb{R} = (-\infty, +\infty)$$

In the set $\mathbb{R}^*$ it is then possible to introduce a partial arithmetization, in fact the following rules hold:

$$+\infty + x = +\infty \qquad \forall x \in \mathbb{R}$$

$$-\infty + x = -\infty \qquad \forall x \in \mathbb{R}$$

$$+\infty + \infty = +\infty$$

$$-\infty - \infty = -\infty$$

$$+\infty - \infty = ? \qquad \text{not defined}$$

$$x \cdot (\pm\infty) = \pm\infty \qquad \text{if } x > 0$$

$$x \cdot (\pm\infty) = \mp\infty \qquad \text{if } x < 0$$

$$(\pm\infty) \cdot (\pm\infty) = +\infty$$

$$(\pm\infty) \cdot (\mp\infty) = -\infty$$

$$0 \cdot (\pm\infty) = ? \qquad \text{not defined}$$

$$\frac{\pm\infty}{\pm\infty} = ? \qquad \text{not defined}$$

With reference to the sets of real numbers it is then possible to introduce the notions of maximum and minimum. The following definitions hold:

Definition 47 *Given a set $A \subset \mathbb{R}$, a real number M is called maximum of the set A (that is $M = \max A$) if:*

$$(i)\ M \in A$$
$$(ii)\ M \geq a \qquad \forall a \in A$$

while a real number m is called minimum of the set A (that is $m = \min A$) if:

$$(i)\ m \in A$$
$$(ii)\ m \leq a \qquad \forall a \in A$$

As a consequence, the maximum (minimum) of a set is an element that belongs to the set and that is larger (lower) than or equal to all the elements of the set.

Example 48 *Given the sets (intervals) of real numbers:*

$$A = [-1, 3] \qquad\qquad B = (-1, 3)$$

find (if they exist) maximum and minimum.

For the set A maximum and minimum are, respectively:

$$\max A = 3 \qquad\qquad \min A = -1$$

because they satisfy conditions (i) and (ii) of the definitions. For the set B, on the contrary, maximum and minimum do not exist, because the values $x = -1$ and $x = 3$ satisfy condition (ii) of the definitions but they do not satisfy condition (i) (because they do not belong to the set B).

Since maximum and minimum do not always exist, it is possible to introduce the concepts of supremum and infimum (that always exist). First of all, a set $A \subset \mathbb{R}$ is bounded from above if there exists a number h larger than (or equal to) all the elements of A, that is:

$$\exists h \in \mathbb{R} : h \geq a \qquad \forall a \in A$$

while it is bounded from below if there exists a number k lower than (or equal to) all the elements of A, that is:

$$\exists k \in \mathbb{R} : k \leq a \qquad \forall a \in A$$

and it is bounded if it is at the same time bounded from above and from below.

Example 49 *The set:*

$$A = [-1, 3]$$

is bounded from above and from below, hence it is bounded, while the set:

$$B = (-\infty, 3)$$

is bounded from above but not from below, and the set:

$$C = [1, +\infty)$$

is bounded from below but not from above.

We then have the following definition:

Definition 50 *Given a set $A \subset \mathbb{R}$ non-empty and bounded from above, it is called supremum of A the element $S \in \mathbb{R}$ such that:*

$$(i)\ S \geq a \qquad \forall a \in A$$
$$(ii)\ \forall \varepsilon > 0, \exists a \in A : S - \varepsilon < a$$

In practice, condition (i) indicates that S is larger than (or equal to) each element of A, and condition (ii) indicates that it is the smallest element with such property. We then have the following definition:

Definition 51 *Given a set $A \subset \mathbb{R}$ non-empty and bounded from below, it is called infimum of A the element $s \in \mathbb{R}$ such that:*

$$(i)\ s \leq a \qquad \forall a \in A$$
$$(ii)\ \forall \varepsilon > 0, \exists a \in A : s + \varepsilon > a$$

In this case condition (i) indicates that s is lower than (or equal to) each element of A, and condition (ii) indicates that it is the largest element with such property

In these definitions it is not required that $S \in A$ ($s \in A$), and it is precisely this condition that differentiates the concept of supremum (infimum) from that of maximum (minimum). In particular, if $S \in A$ ($s \in A$), then it is the maximum (minimum) of A and also the supremum (infimum) of A, while if $S \notin A$ ($s \notin A$) then it is the supremum (infimum) of A, while the maximum (minimum) does not exist.

Example 52 *Given the sets:*

$$A = [-1, 3] \qquad\qquad B = (-1, 3)$$

find maximum, minimum, supremum and infimum.

In this case for the set A we have:

$$\sup A = \max A = 3 \qquad\qquad \inf A = \min A = -1$$

since $x = 3$ is larger than (or equal to) each element of A and it is the smallest element with such property, moreover it belongs to A (hence it is also the maximum), while $x = -1$ is lower than (or equal to) each element of A and it is the largest element with such property, moreover it belongs to A (hence it is also the minimum). For the set B we then have:

$$\sup B = 3 \qquad\qquad \inf B = -1$$

while maximum and minimum do not exist, since $x = 3$ is larger than each element of B and it is the smallest element with such property, but it does not belong to B, while $x = -1$ is lower than each element of B and it is the largest element with such property, but it does not belong to B.

For a set A that is not bounded from above we can take:

$$\sup A = +\infty$$

and for a set A that is not bounded from below we can take:

$$\inf A = -\infty$$

After the algebraic structure of the set of real numbers it is possible to introduce the metric structure and then the topological one. First of all, given an inequality with absolute value of the kind:

$$|f(x)| \geq k \qquad \text{with } k > 0$$

it corresponds to:

$$f(x) \leq -k \qquad \text{or} \qquad f(x) \geq k$$

while given an inequality of the kind:

$$|f(x)| \leq k \qquad \text{with } k > 0$$

it corresponds to:

$$-k \leq f(x) \leq k$$

We then have the following definition of distance:

Definition 53 *Given $x, y \in \mathbb{R}$, their distance is the absolute value (modulus) of their difference:*

$$d(x,y) = |x - y|$$

As a consequence, we also have that:

$$|x| = |x - 0| = d(x,0)$$

that is the absolute value of a real number is its distance from the origin.

With reference to subsets of real numbers it is then possible to introduce some notions of topology. The starting point is the notion of neighbourhood of a point, that is a particular interval. The following definition holds:

Definition 54 *Given a point $p \in \mathbb{R}$, a (complete) neighbourhood with centre p and radius r (where $r > 0$) is the set of the points whose distance from p is lower than r:*

$$\begin{aligned} U_r(p) &= \{x \in \mathbb{R} : d(x,p) < r\} = \{x \in \mathbb{R} : |x - p| < r\} = \\ &= \{x \in \mathbb{R} : -r < x - p < r\} = \{x \in \mathbb{R} : p - r < x < p + r\} = \\ &= (p - r, p + r) \end{aligned}$$

hence the neighbourhood of a point p with radius r is the open interval $(p - r, p + r)$.

In a similar way it is possible to define right-hand neighbourhoods and left-hand neighbourhoods, that are intervals of the form:

$$U_r^+(p) = [p, p + r) \qquad U_r^-(p) = (p - r, p]$$

and also neighbourhoods of $+\infty$ and of $-\infty$, that are intervals of the form:

$$U(+\infty) = (M, +\infty) \qquad U(-\infty) = (-\infty, -M)$$

Example 55 *The (complete) neighbourhood with centre* 3 *and radius* 1 *is the interval:*

$$U_1(3) = (3-1, 3+1) = (2,4)$$

while the right-hand neighbourhood with centre 3 *and radius* 1 *is the interval:*

$$U_1^+(3) = [3, 3+1) = [3,4)$$

and the left-hand neighbourhood with centre 3 *and radius* 1 *is the interval:*

$$U_1^-(3) = (3-1, 3] = (2,3]$$

Given a set $A \subset \mathbb{R}$ it is then possible to introduce a series of classifications concerning the points that belong (or do not belong) to A. The following definitions hold:

Definition 56 *Given a set $A \subset \mathbb{R}$, a point $p \in \mathbb{R}$ is:*
(i) *interior with respect to A if it belongs to A and there exists at least one neighbourhood of p all included in A;*
(ii) *exterior with respect to A if it is interior with respect to the complement A^C;*
(iii) *a boundary point if each neighbourhood of p contains points of A and points of A^C (in this case the point p can belong or not to A);*
(iv) *an accumulation point for A if each neighbourhood of p contains points of A (different from the point p itself);*
(v) *isolated with respect to A if it belongs to A and there exists at least one neighbourhood of p that does not include points of A (different from p itself).*

It is then possible to introduce some classifications concerning sets, relative to the characteristics of their points. The following definitions hold:

Definition 57 *A set $A \subset \mathbb{R}$ is:*
(i) *open if all its points are interior points;*
(ii) *closed if its complement is open (and also if it contains all its boundary points);*
(iii) *bounded if it is contained in some neighbourhood of the origin.*

It is then possible to observe that sets can be neither open nor closed, for instance:

$$A = (0,1]$$

while the sets $\emptyset$ and $\mathbb{R}$ are considered both open and closed (and their are the only ones with this property).

Example 58 *Given the set:*

$$A = (-3, 1] \cup \{4\}$$

describe its topological characteristics.

In this case we have that the interior points are all those of the interval $(-3, 1)$, while the exterior points are all those of $(-\infty, -3) \cup (1, 4) \cup (4, +\infty)$ and the boundary points are those of the set $\{-3, 1, 4\}$. The accumulation points, then, are all those of the interval $[-3, 1]$ and the only isolated point is $\{4\}$. This set is neither open nor closed and it is bounded.

Example 59 *Given the set:*

$$B = \left\{\frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

describe its topological characteristics.

In this case we have that there are no interior points, while the exterior points are those of $\mathbb{R} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ and the boundary points are those of the set itself, that is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$. In this example, moreover, the only accumulation point is $\{0\}$, while the isolated points are those of the set itself, that is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$. Finally, this set is neither open nor closed and it is bounded.

2.4 Elements of logic

A first concept particularly important in logic is the notion of Proposition, that is a sentence to which is possible to give the value "true" (T) or "false" (F). For instance, the sentence:

"Today it rains"

is a proposition (because it is possible to determine if it is true or false), while the sentence:

"How are you?"

is not a proposition. Propositions are denoted by small letters like p, q, r and can be linked together, obtaining more complex propositions, with the use of logical connectives, that are:

1. the negation, denoted with the symbol $\sim$ (or with a line above the proposition that is negated) and corresponding to "not"; we have for instance:

$p:$ "Today it rains"
$\sim p:$ "Today it does not rain"

2. the conjunction, denoted with the symbol $\wedge$ ("et") and corresponding to "and"; we have for instance:

$$\begin{array}{ll} p: & \text{"I read the newspaper"} \\ q: & \text{"I listen to the radio"} \\ p \wedge q: & \text{"I read the newspaper and I listen to the radio"} \\ & (p \text{ and } q \text{ both hold}) \end{array}$$

3. the disjunction, denoted with the symbol $\vee$ ("vel") and corresponding to "or"; we have for instance:

$$\begin{array}{ll} p: & \text{"I read the newspaper"} \\ q: & \text{"I listen to the radio"} \\ p \vee q: & \text{"I read the newspaper or I listen to the radio"} \\ & (\text{or both, at least one between } p \text{ and } q \text{ holds}) \end{array}$$

4. the implication, denoted with the symbol $\Rightarrow$ and corresponding to "if...then"; we have for instance:

$$\begin{array}{ll} p: & \text{"It's sunny"} \\ q: & \text{"I go to the seaside"} \\ p \Rightarrow q: & \text{"If it's sunny then I go to the seaside"} \end{array}$$

   In this case p is also called sufficient condition for q, and q is also called necessary condition for p.

5. the equivalence or double implication, denoted with the symbol $\Leftrightarrow$ and corresponding to "if and only if"; we have for instance:

$$\begin{array}{ll} p: & \text{"I take the umbrella"} \\ q: & \text{"It rains"} \\ p \Leftrightarrow q: & \text{"I take the umbrella if and only if it rains"} \\ & (\text{that is if I take the umbrella it rains, and if it rains I take the umbrella}) \end{array}$$

   In this case p is also called necessary and sufficient condition for q, and q is also called necessary and sufficient condition for p.

Among logical connectives, as among arithmetic operations, there is an order given by:

$$\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

For instance, the statement:

$$q \vee \sim p \Rightarrow \sim r$$

corresponds to:

$$(q \vee (\sim p)) \Rightarrow (\sim r)$$

Example 60 *Given the propositions:*

$$\begin{array}{ll} p: & \text{“I have time”} \\ q: & \text{“It rains”} \\ r: & \text{“I go to the seaside”} \end{array}$$

the statement:

$$q \vee \sim p \Rightarrow \sim r$$

is equivalent to:

“If it rains or I don’t have time then I don’t go to the seaside”

The value of truth (T or F) of a proposition, obtained from simplest propositions and using the logical connectives, depends on the value of truth of each component proposition according to the following “tables of truth”:

1. for the negation:

p	$\sim p$
T	F
F	T

2. for the conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. for the disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. for the implication:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5. for the equivalence:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Two propositions, then, are "logically equivalent" if they have the same table of truth.

Example 61 *Verify the logical equivalence of the propositions:*

$$\sim (p \Rightarrow q) \qquad \textit{and} \qquad p \wedge (\sim q)$$

In this case the table of truth of $\sim (p \Rightarrow q)$ is given by:

p	q	$p \Rightarrow q$	$\sim (p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

and the table of truth of $p \wedge (\sim q)$ is given by:

p	q	$\sim q$	$p \wedge (\sim q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

that coincides with the previous one, hence the two propositions are logically equivalent.

A sentence in which there is a variable (that represents an element of a given set) is called Predicate and is denoted with $p(x)$, where x represents the variable. The value of truth of a predicate depends on the values assigned to the variables, for instance considering:

$$p(x): \quad \text{"x is a positive real number"}$$

we have:

$$\text{for } x = 5 \qquad p(x) \text{ is true}$$

$$\text{for } x = -1 \qquad p(x) \text{ is false}$$

A predicate hence is not a proposition, but starting from it is possible to get propositions assigning a particular value to the variable.

There exists a correspondence between logical operations on predicates and set operations. In fact, given two predicates $p(x)$ and $q(x)$, where x is an element belonging to a set X, it is possible to consider the subsets of X given by:

$$A = \{x \in X : p(x)\} \qquad B = \{x \in X : q(x)\}$$

so that A is the set of the values of x such that the predicate $p(x)$ is true (we also say that A is the domain of truth of the predicate $p(x)$) and B is the set of the values of

x such that the predicate $q(x)$ is true (we also say that B is the domain of truth of the predicate $q(x)$). We now have:

$$\begin{aligned} A \cup B &= \{x \in X : p(x) \vee q(x)\} \\ A \cap B &= \{x \in X : p(x) \wedge q(x)\} \\ A^c &= \{x \in X : \sim p(x)\} \end{aligned}$$

so that $A \cup B$ is the set of the values of x such that at least one of the two predicates $p(x)$ and $q(x)$ is true, while $A \cap B$ is the set of the values of x such that both predicates $p(x)$ and $q(x)$ are true, and A^c is the set of the values of x such that the negation of the predicate $p(x)$ is true.

Example 62 *Given the predicates:*

$$\begin{aligned} p(x): &\quad \text{“}x \text{ is a real number larger than or equal to } 3\text{”} \\ q(x): &\quad \text{“}x \text{ is a real number lower than or equal to } 5\text{”} \end{aligned}$$

we have:

$$A = \{x \in \mathbb{R} : x \geq 3\} \qquad\qquad B = \{x \in \mathbb{R} : x \leq 5\}$$

and then:

$$\begin{aligned} A \cup B &= \{x \in \mathbb{R} : x \geq 3 \vee x \leq 5\} = \mathbb{R} \\ A \cap B &= \{x \in \mathbb{R} : x \geq 3 \wedge x \leq 5\} = [3, 5] \\ A^c &= \{x \in \mathbb{R} : x < 3\} = (-\infty, 3) \end{aligned}$$

For this reason the disjunction $\vee$ and the conjunction $\wedge$ are also called, respectively, “logical union” and “logical intersection”, since they are the equivalent, from the logical point of view, of the set operations of union and intersection. Logical operations satisfy the properties of set operations, in particular De Morgan's laws, according to which we have:

$$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$$

(that is if we negate that at least one of two propositions is true, this is equivalent to state that both are false) and also:

$$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$$

(that is if we negate that two propositions are both true, this is equivalent to state that at least one of the two is false). This logical equivalences can be easily verified building the tables of truth of the left-hand sides and of the right-hand sides and observing that they are equal.

Starting from predicates, then, it is possible to obtain propositions using the quantifiers, that are:

1. the existential quantifier, denoted by the symbol $\exists$ and that means “there exists”;
2. the universal quantifier, denoted by the symbol $\forall$ and that means “for every”.

For instance, the proposition:

$$\exists x : p(x)$$

means:

"there exists (at least) an x such that $p(x)$ is true"

(while the symbol $\exists!$ means "there exists and is unique"), while the proposition:

$$\forall x : p(x)$$

means:

"for every x, $p(x)$ is true"

There is an important relation between the quantifiers, that is evident passing from a sentence to its negation; we have in fact:

$$\sim (\forall x : p(x)) \Leftrightarrow \exists x : \quad \sim p(x)$$

and also:

$$\sim (\exists x : p(x)) \Leftrightarrow \forall x : \quad \sim p(x)$$

For instance, given the predicate:

$p(x)$: "the student x passes the exam"

the first logical equivalence becomes:

it is not true that all the students pass the exam	$\Leftrightarrow$	there exists (at least) one student that does not pass the exam

while the second one becomes:

it is not true that there exists a student that passes the exam	$\Leftrightarrow$	no student passes the exam (all the students do not pass the exam)

It is therefore evident that the negation of an existential statement is a universal statement and viceversa, that is passing from a sentence to its negation the symbols $\exists$ and $\forall$ exchange each other.

A Tautology, finally, is a proposition that is true for any value of truth of the propositions that form it. Examples of tautologies are the following:

1. $$p \vee (\sim p)$$

2. $$\sim (p \wedge \sim p)$$

3. $$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$$

4.

$$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$$
$$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$$

5.

$$p \wedge (p \Rightarrow q) \Rightarrow q$$

6.

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

7.

$$\sim (p \Rightarrow q) \Leftrightarrow p \wedge (\sim q)$$

In order to verify that the propositions reported above are tautologies it is sufficient to write their tables of truth, that must give the value "true" (T) for any value of truth of the initial propositions. We have for instance, for the first tautology:

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

from which it turns out that the proposition $p \vee (\sim p)$ is always true, hence it is a tautology.

The last 3 tautologies, in particular, are used in the proofs of theorems. With reference to this aspect, given a proposition p (hypothesis) that is assumed true (T) and a proposition q (thesis) of which we have to prove that is true (T), we call "theorem" the proposition $p \Rightarrow q$. In this context, the rules that allow to pass from a proposition that is true to another one logically equivalent are called "rules of deduction", and the set of passages from one proposition to another one logically equivalent is called "proof". Since the proof of a theorem is based on previous theorems, and since it is not possible to proceed backward at infinitum, it is necessary to assume some statements (called "axioms") as conventionally true. An axiom is therefore a proposition that is assumed to be true without any proof.

In the direct proofs of theorems, in particular, we use tautology 5). In this case, if p (hypothesis) is true and we want to show that q (thesis) is true, we proceed showing that $p \Rightarrow q$ is true.

In the proofs by contradiction, then, we use tautology 6). In this case we keep the hypothesis p as true and we negate the thesis q (that is we assume true its negation $\sim q$), and in this way we get a contradiction, that shows the truth of the implication $\sim q \Rightarrow \sim p$ (that is equivalent to $p \Rightarrow q$).

In the proofs by counterexample, finally, we use tautology 7). In this case we consider a proposition of the type:

$$\forall x : p(x) \Rightarrow q(x)$$

and we show that it is false by exhibiting a particular x (the counterexample) for which $p(x)$ is true but $q(x)$ is false, that is we prove that:

$$\exists x : p(x) \wedge (\sim q(x))$$

that guarantees that the initial proposition is false (because there is equivalence between $\sim (p \Rightarrow q)$ and $p \wedge (\sim q)$).

2.5 Exercises

Given the sets A and B, find the sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$:

1) $A = \{0, 1\}$ $\quad B = \{1, 2\}$

2) $A = \{0, 1\}$ $\quad B = \{2, 3\}$

3) $A = \{1, 3, 5\}$ $\quad B = \{2, 3, 4\}$

Represent the following sets of real numbers:

4) $X = A \cup B$ with $A = (-3, 3]$ and $B = (0, 5)$

5) $X = A \cup B$ with $A = (-8, 5]$ and $B = (0, 4)$

6) $X = A \cap B$ with $A = (-\infty, 7)$ and $B = (-3, 7)$

7) $X = A \cap B$ with $A = (-\infty, 3)$ and $B = (6, +\infty)$

8) $X = (A \cup B)^C$ with $A = [1, 4)$ and $B = [3, 8]$

9) $X = (A \cup B)^C$ with $A = (-\infty, 3)$ and $B = (-3, +\infty)$

10) $X = (A \cup B)^C$ with $A = (-\infty, 0)$ and $B = (0, +\infty)$

11) $X = (A \cap B)^C$ with $A = (-7, 7]$ and $B = (0, 5)$

12) $X = (A \cap B)^C$ with $A = (-\infty, 0)$ and $B = (-1, 2]$

13) $X = (A \cap B)^C$ with $A = (-\infty, 2)$ and $B = (2, +\infty)$

Given the sets A and B, find the cartesian products $A \times B$ and $B \times A$:

14) $A = \{0, 1\} \qquad B = \{0, -1\}$

15) $A = \{-1, 1\} \qquad B = \{0, 1\}$

16) $A = \{0, 1\} \qquad B = \{-1, 2, 3\}$

Caracterize from the topological point of view (max and min, sup and inf, interior points, etc.) the following sets of real numbers:

17) $X = A \cup B$ with $A = (-1, 0)$ and $B = [0, 2]$

18) $X = A \cap B$ with $A = (-\infty, -3)$ and $B = (-4, -2)$

19) $X = (A \cup B)^C$ with $A = (-\infty, -5)$ and $B = (3, 4)$

20) $X = (A \cup B)^C$ with $A = [-2, 1)$ and $B = [-1, 2)$

21) $X = (A \cup B)^C$ with $A = [-3, -2)$ and $B = (-1, 0]$

22) $X = (A \cap B)^C$ with $A = [-3, 2)$ and $B = (-2, 3)$

23) $X = (A \cap B)^C$ with $A = (-2, 1]$ and $B = (-1, 2]$

24) $X = (A \cap B)^C$ with $A = (-\infty, 3]$ and $B = (-2, +\infty)$

25) $X = A \cup B$ with $A = (-2, 3]$ and $B = \{4\}$

Find the tables of truth of the following propositions:

26) $\sim p \wedge q$

27) $\sim (\sim p \wedge q)$

28) $\sim (p \Rightarrow \sim q)$

29) $\sim p \Rightarrow \sim q$

30) $p \Leftrightarrow \sim q$

31) $\sim p \Rightarrow q$

32) $\sim q \Rightarrow p$

33) $q \Rightarrow p$

34) $\sim p \vee \sim q$

35) $\sim p \Leftrightarrow \sim q$

36) $p \Leftrightarrow q$

37) $p \Rightarrow \sim q$

38) $\sim p \Leftrightarrow q$

39) $p \wedge \sim q$

40) $\sim q \Rightarrow \sim p$

41) $[(p \vee \sim p) \wedge p] \vee q$

Verify the logical equivalence of the following propositions:

42) $p \Rightarrow q$ and $\sim p \vee q$

43) $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$

44) $\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$

45) $\sim (p \wedge q)$ and $(\sim p) \vee (\sim q)$

Verify that the following propositions are tautologies:

46) $\sim (p \wedge \sim p)$

47) $p \wedge (p \Rightarrow q) \Rightarrow q$

48) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

49) $\sim (p \Rightarrow q) \Leftrightarrow p \wedge \sim q$

50) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

Chapter 3

Functions

3.1 Definitions

Given two (non empty) sets X and Y, a function (or application, or correspondence) of X in Y is a law that associates to each element $x \in X$ one and only one element $y \in Y$. We write in this case:

$$f : X \to Y$$

and also:

$$y = f(x)$$

and we say that y represents the image, through the function f, of x. The set X is called the domain (or definition set, or field of existence), while the set Y is called the codomain, and the subset (proper or improper) of Y constituted by the elements $y \in Y$ for which there exists an element $x \in X$ such that $y = f(x)$ (i.e. the subset of Y formed by the elements that are images of elements of X) is called set of the images (in practice, the codomain is the set in which, a priori, the function can assume values, while the set of the images is the set of the values actually assumed by the function). The variable x, furthermore, is called independent variable, while the variable y is called dependent variable.

The functions considered in this Chapter (and also in Chapters 4, 5 and 6) are defined on subsets of $\mathbb{R}$ and take values in $\mathbb{R}$, i.e. they are real functions of a real variable:

$$f : X \subseteq \mathbb{R} \to \mathbb{R} \qquad \text{and also } y = f(x)$$

while in Chapter 8 we will introduce functions that are defined on subsets of $\mathbb{R}^n$ (that is the set of ordered n-tuples of real numbers) and that take values in $\mathbb{R}$, i.e. real functions of several real variables:

$$f : X \subseteq \mathbb{R}^n \to \mathbb{R} \qquad \text{and also } y = f(x_1, x_2, ..., x_n)$$

It is now possible to analyse the different elements that characterize a function, in order to get the information that allow to study the function itself.

3.2 Domain of a function

The first problem considered in the study of a function is the determination of its domain, that is defined as the largest subset of $\mathbb{R}$ in which are defined the operations reported in the expression $f(x)$ (so that we also talk of natural domain of f). In general, we don't consider the determination of the set of images, that in many cases is not easy to find, and we simply indicate the codomain (that for the functions considered is given by the set $\mathbb{R}$). With reference to this aspect, it is possible to find the following types of problems, that require to impose the corrisponding conditions of reality, in order to determine the domain of the function considered:

1. If the independent variable appears in the denominator of a fraction, we must impose that the denominator is not null.
2. If the independent variable appears under the sign of a root with an even index, we must impose that the quantity under the sign of root is non-negative.
3. If the independent variable appears in the argument of a logarithm, we must impose that the argument is strictly positive.
4. If the independent variable appears both in the basis and in the exponent of a power, i.e we have an expression of the type $f(x)^{g(x)}$, we must impose that the basis of the power is strictly positive (as it turns out rewriting the function in the form $e^{\log f(x)^{g(x)}} = e^{g(x)\log f(x)}$, where $f(x)$ becomes the argument of a logarithm, and therefore must be strictly greater than zero).

Example 63 *Determine the domain of the function:*

$$f(x) = \frac{1}{x+3}$$

In this case it must be $x+3 \neq 0$, that is $x \neq -3$, so that the domain is:

$$D = (-\infty, -3) \cup (-3, +\infty)$$

Example 64 *Determine the domain of the function:*

$$f(x) = \sqrt{x-3} + 5x$$

In this case it must be $x - 3 \geq 0$, that is $x \geq 3$, so that the domain is:

$$D = [3, +\infty)$$

Example 65 *Determine the domain of the function:*

$$f(x) = \log(x^2 - 4)$$

In this case it must be $x^2 - 4 > 0$, that is $x < -2 \quad \vee \quad x > 2$, so that the domain is:

$$D = (-\infty, -2) \cup (2, +\infty)$$

Example 66 *Determine the domain of the function:*

$$f(x) = (3x)^{x+5} + 2$$

In this case it must be $3x > 0$, that is $x > 0$, so that the domain is:

$$D = (0, +\infty)$$

Often some of these situations appear contemporaneously, so that in order to determine the domain of a function it is necessary to consider only those values of the x that satisfy at the same time all the conditions imposed (i.e. it is necessary to solve a system of inequalities).

Example 67 *Determine the domain of the function:*

$$f(x) = \sqrt{\log(x - 3)}$$

In this case we must have at the same time:

$$\begin{cases} x - 3 > 0 \\ \\ \log(x - 3) \geq 0 \end{cases} \Rightarrow \begin{cases} x - 3 > 0 \\ \\ x - 3 \geq 1 \end{cases} \Rightarrow \begin{cases} x > 3 \\ \\ x \geq 4 \end{cases} \Rightarrow x \geq 4$$

so that the domain is:

$$D = [4, +\infty)$$

Example 68 *Determine the domain of the function:*

$$f(x) = \left(\frac{2}{x-3}\right)^{3x}$$

In this case we must have at the same time:

$$\begin{cases} x - 3 \neq 0 \\ \\ \dfrac{2}{x-3} > 0 \end{cases} \Rightarrow \begin{cases} x \neq 3 \\ \\ x > 3 \end{cases} \Rightarrow x > 3$$

so that the domain is:

$$D = (3, +\infty)$$

In some cases, then, it can happen that it is not necessary to use all the natural domain of a function, but only a part of it. If the variable x represents a quantity with economic meaning (e.g. quantity, price), negative values for this variable makes no sense, hence it is necessary to consider, eventually, only a part of the natural domain.

3.3 Intersection with the axes and sign of a function

After the specification of the domain of a function, the following step consists in the determination of the intersections (if they exist) of the function with the cartesian axes and in the study of the sign of the function.

In particular, the intersections with the axes can be obtained solving the system formed by the equation that constitutes the analytical expression of the function and by the equation of the axis considered, while the sign of the function can be found determining first of all the set of values of the x for which the function is positive or null (i.e. solving the inequality $f(x) \geq 0$), after that we have that for the remaining values of the x (belonging to the domain) the function is negative. As a consequence, to find the (eventual) intersections with the horizontal axis it is necessary to solve the system:

$$\begin{cases} y = f(x) \\ \\ y = 0 & \rightarrow \text{equation of the } x\text{-axis} \end{cases}$$

while to find the (eventual) intersection with the vertical axis it is necessary to solve the system:

$$\begin{cases} y = f(x) \\ \\ x = 0 & \rightarrow \text{equation of the } y\text{-axis} \end{cases}$$

observing that the intersection with the vertical axis, if there exists, is unique (by definition of function). To study the sign of the function, then, it is necessary to solve the inequality $f(x) \geq 0$, finding the intervals in which f is positive or null (and, as a consequence, also the intervals in which f is negative).

Example 69 *Determine the intersections with the axes and the sign of the function:*

$$f(x) = x^2 - 5x + 6$$

First of all it is possible to observe that we don't have to impose any restriction to the domain of the function, that therefore coincides with $\mathbb{R}$. At this point the (eventual) intersections with the x-axis can be found by solving the system formed by the equation $y = f(x)$ and by the equation of the x-axis (that is $y = 0$):

$$\begin{cases} y = x^2 - 5x + 6 \\ \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ \\ y = 0 \end{cases} \vee \begin{cases} x = 3 \\ \\ y = 0 \end{cases}$$

so that the function intersects the x-axis in the points $A = (2, 0)$ and $B = (3, 0)$. The (eventual) intersection with the y-axis, then, can be found by solving the system formed by the equation $y = f(x)$ and by the equation of the y-axis (that is $x = 0$):

$$\begin{cases} y = x^2 - 5x + 6 \\ \\ x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ \\ y = 6 \end{cases}$$

so that the function intersects the y-axis in the point $C = (0, 6)$ (by definition of function there is at most one intersection with the y-axis, since given a value of x - in the case considered $x = 0$ - there is at most one value of y correspondent to it, given by $y = f(x)$).

To study the sign of the function on its domain, then, first of all we must solve the inequality:

$$f(x) \geq 0 \Rightarrow x^2 - 5x + 6 \geq 0 \Rightarrow x \leq 2 \quad \vee \quad x \geq 3$$

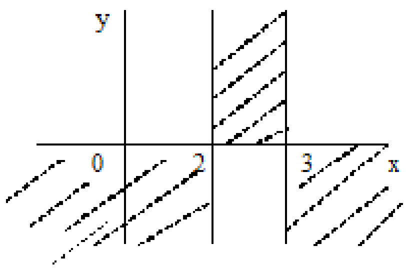
and then we have:

$$f(x) < 0 \quad \text{for} \quad 2 < x < 3$$

$$f(x) = 0 \quad \text{for} \quad x = 2 \quad \vee \quad x = 3$$

$$f(x) > 0 \quad \text{for} \quad x < 2 \quad \vee \quad x > 3$$

The results obtained can also be represented graphically in the following way (the dashed line indicates the parts of the plane where the function cannot be located):



Indeed, the function considered is simply the analitical expression of a parabola with upward concavity and passing through the points A, B and C determined above; this parabola is located in the y-positive half-plane for values of x smaller than 2 and greater than 3, and in the y-negative half-plane for values of x included between 2 and 3, while it intersects the x-axis in correspondence of $x = 2$ and $x = 3$ and the y-axis in correspondence of $y = 6$.

Example 70 *Determine the intersections with the axis and the sign of the function:*

$$f(x) = \frac{\log(2x)}{x^3}$$

First of all it is necessary to determine the domain of the function, that can be obtained imposing the conditions:

$$\begin{cases} 2x > 0 \\ \\ x^3 \neq 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ \\ x \neq 0 \end{cases} \Rightarrow x > 0$$

so that the field of existence of the function is given by the interval $(0, +\infty)$. The (eventual) intersections with the x-axis, then, can be found solving the system:

$$\begin{cases} y = \dfrac{\log(2x)}{x^3} \\ \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ \\ y = 0 \end{cases}$$

so that the function intersects the x-axis in the point $A = \left(\frac{1}{2}, 0\right)$. There are instead no intersections with the y-axis, since for $x = 0$ (that is the equation of the y-axis) the function is not defined (in fact it must be $x > 0$ as seen above).

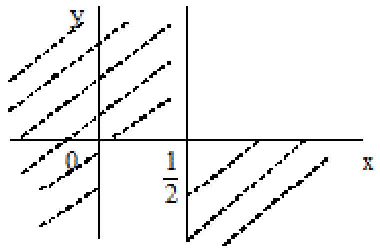
To study the sign of the function on its domain, then, it is necessary first of all to solve the inequality:

$$f(x) \geq 0 \Rightarrow \frac{\log(2x)}{x^3} \geq 0 \Rightarrow x \geq \frac{1}{2}$$

and then we have:

$$\begin{aligned} &f(x) < 0 \quad \text{for} \quad 0 < x < \tfrac{1}{2} \\ &f(x) = 0 \quad \text{for} \quad x = \tfrac{1}{2} \\ &f(x) > 0 \quad \text{for} \quad x > \tfrac{1}{2} \end{aligned}$$

and graphically:



3.4 Even, odd and periodic functions

In the study of a function, after the determination of its domain, of its (eventual) intersections with the axes and of its sign, we are interested in finding eventual symmetries and periodicities. With reference to this aspect, a function is even if it holds:

$$f(-x) = f(x) \qquad \forall x \in D$$

while a function is odd if it holds:

$$f(-x) = -f(x) \qquad \forall x \in D$$

where D is the domain of the function (that must be symmetric with respect to the origin, that is if $x \in D$ also $-x \in D$).

A function then is periodic of period t when t is the smallest positive real number for which it holds:

$$f(x + t) = f(x) \qquad \forall x \in D$$

An even function is characterized by the fact that its graph turns out to be symmetric with respect to the y-axis (i.e. if the point (x, y) belongs to the graph of the function, also the point $(-x, y)$ belongs to the same graph), while an odd function is characterized by the fact that its graph turns out to be symmetric with respect to the origin (i.e. if the point (x, y) belongs to the graph of the function, also the point $(-x, -y)$ belongs to the same graph). A periodic function of period t, on the other hand, is characterized by the fact that its graph repeats itself after each interval of lenght t (i.e. if the point (x, y) belongs to the graph of the function, also the point $(x + t, y)$ belongs to the same graph).

In the case of even or odd functions, therefore, it is sufficient to carry out the study for $x \geq 0$, and then the whole graph of the function can be obtained by reversing the one obtained for non-negative values of the x with respect to the y-axis (in the case of even functions) or with respect to the origin (in the case of odd functions). In the case of periodic functions, then, it is sufficient to carry out the study on an interval, belonging to the domain, of lenght t, and then the whole graph of the function can be obtained repeating several times this one.

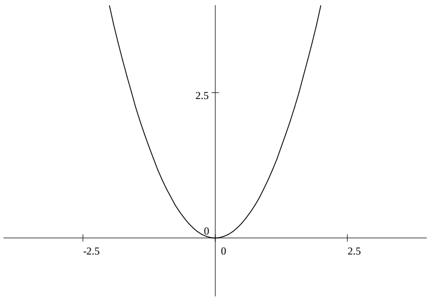
A simple exemple of even function is given by:

$$y = f(x) = x^2$$

for which we have:

$$f(-x) = (-x)^2 = (-x) \cdot (-x) = x^2 = f(x)$$

that is the condition that must be verified in order to have an even function. Its graph is:



and in this case it turns out to be evident the symmetry with respect to the y-axis.

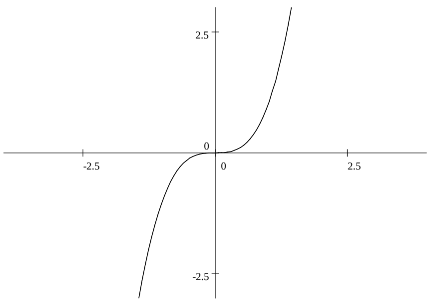
A simple example of odd function is then given by:

$$y = f(x) = x^3$$

for which we have:

$$f(-x) = (-x)^3 = (-x) \cdot (-x) \cdot (-x) = -x^3 = -f(x)$$

that is the condition that must be verified in order to have an odd function. Its graph is:



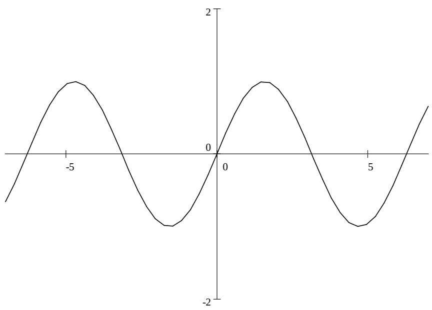
and in this case it turns out to be evident the symmetry with respect to the origin.

Simple examples of periodic functions (of period 2π), then, are the sine and the cosine, for which we have:

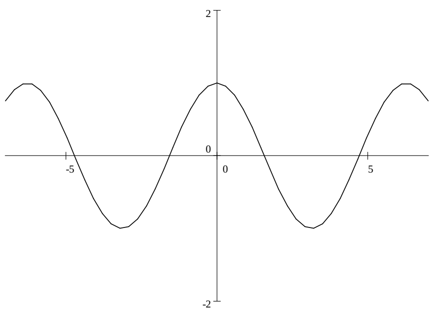
$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x = f(x)$$

$$f(x + 2\pi) = \cos(x + 2\pi) = \cos x = f(x)$$

that is the condition that must be verified in order to have a periodic function. The graph of the sine is:



while the graph of the cosine is:



and it turns out that the graphs of the two funtions repeat themselves after an interval of constant lenght (equal to 2π). In addition, these two functions are also symmetric, in particular the sine is an odd function (in fact $\sin(-x) = -\sin x$) while the cosine is an even function (in fact $\cos(-x) = \cos x$), as it can be observed graphically.

Example 71 *Verify the presence of eventual symmetries or periodicities in the function:*

$$f(x) = 5^x + 5^{-x}$$

In this case we have:

$$f(-x) = 5^{-x} + 5^{-(-x)} = 5^{-x} + 5^x = f(x)$$

so that the function is even.

Example 72 *Verify the presence of eventual symmetries or periodicities in the function:*

$$f(x) = \sqrt{2 - x^3} - \sqrt{2 + x^3}$$

In this case we have:

$$\begin{aligned} f(-x) &= \sqrt{2 - (-x)^3} - \sqrt{2 + (-x)^3} = \sqrt{2 - (-x^3)} - \sqrt{2 + (-x^3)} = \\ &= \sqrt{2 + x^3} - \sqrt{2 - x^3} = -\left(\sqrt{2 - x^3} - \sqrt{2 + x^3}\right) = -f(x) \end{aligned}$$

so that the function is odd.

Example 73 *Verify the presence of eventual symmetries or periodicities in the function:*

$$f(x) = |x| + 3x$$

In this case we have:

$$f(-x) = |-x| + 3(-x) = |x| - 3x$$

and since this expression is not equal neither to $f(x)$ nor to $-f(x)$ the function does not have symmetries (i.e. it is neither even nor odd).

Example 74 *Verify the presence of eventual symmetries or periodicities in the function:*

$$f(x) = \sin(2x)$$

In this case we have first of all:

$$f(-x) = \sin 2(-x) = \sin(-2x) = -\sin(2x) = -f(x)$$

so that the function is odd, furthermore:

$$f(x + \pi) = \sin 2(x + \pi) = \sin(2x + 2\pi) = \sin(2x) = f(x)$$

so that the function is also periodic, of period π.

3.5 Composite functions

Given the real functions of real variable:

$$t = f(x) \qquad \text{with } f : A \subseteq \mathbb{R} \to \mathbb{R}$$

$$y = g(t) \qquad \text{with } g : B \subseteq \mathbb{R} \to \mathbb{R}$$

such that each value assumed by f is in the domain of g (that is $f(A) \subseteq B$), to each $x \in A$ the function f associates a unique element $f(x)$, and since this is an element of B the function g associates to it a unique element $g(f(x))$. At this point the composite function $g \circ f$ is the function (where it is defined):

$$y = h(x) = g(f(x)) \qquad \text{with } h : A \subseteq \mathbb{R} \to \mathbb{R}$$

that at each $x \in A$ associates the element $g(f(x))$.

The condition that must be verified in order to establish if the composite function exists is therefore $f(A) \subseteq B$, and in practice given the two functions f and g it is possible to write immediately the composite function $g \circ f$, then it is necessary to impose the (eventual) conditions of reality required by the analytical expression of the function, and in this way it is possible to determine the domain of the composite function (eventually empty, case in which the composite function does not exist). In a similar way it is possible to find (if it exists) the composite function $f \circ g$. The operation of composition is not commutative, that is also when both $g \circ f$ and $f \circ g$ exist, in general we have $g \circ f \neq f \circ g$.

Example 75 *Given the functions:*

$$f(x) = x^3 + 2 \qquad g(t) = e^t + 5$$

determine the composite functions $g \circ f$ and $f \circ g$.

To determine $g \circ f$ we put:

$$t = f(x) = x^3 + 2$$

and substituting this expression in the function $g(t)$ instead of t we get:

$$g(f(x)) = e^{x^3+2} + 5$$

and, since there are no conditions of reality to impose, this is the composite function $g \circ f$, that turns out to be defined $\forall x \in \mathbb{R}$.

In a similar way, to determine $f \circ g$ we put:

$$x = g(t) = e^t + 5$$

and substituting this expression in the function $f(x)$ instead of x we get:

$$f(g(t)) = \left(e^t + 5\right)^3 + 2$$

and, since also in this case there are no conditions of reality to impose, this is the composite function $f \circ g$, that turns out to be defined $\forall t \in \mathbb{R}$. It is also possible to observe that, although both $g \circ f$ and $f \circ g$ are defined, it is $g \circ f \neq f \circ g$.

Example 76 *Given the functions:*

$$f(x) = \log x \qquad\qquad g(t) = \sqrt{t}$$

determine the composite functions $g \circ f$ and $f \circ g$.

To determine $g \circ f$ we put:

$$t = f(x) = \log x$$

and substituting this expression in the function $g(t)$ instead of t we get:

$$g(f(x)) = \sqrt{\log x}$$

At this point it is necessary to impose the conditions of reality (of the logarithm and of the root):

$$\begin{cases} x > 0 \\ \\ \log x \geq 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ \\ x \geq 1 \end{cases} \Rightarrow x \geq 1$$

so that the composite function is:

$$g \circ f = g(f(x)) = \sqrt{\log x} \qquad \text{for } x \geq 1$$

In a similar way, to determine $f \circ g$ we put:

$$x = g(t) = \sqrt{t}$$

and substituting this expression in the function $f(x)$ instead of x we get:

$$f(g(t)) = \log(\sqrt{t})$$

At this point it is necessary to impose the conditions of reality (of the root and of the logarithm):

$$\begin{cases} t \geq 0 \\ \\ \sqrt{t} > 0 \end{cases} \Rightarrow \begin{cases} t \geq 0 \\ \\ t > 0 \end{cases} \Rightarrow t > 0$$

so that the composite function is:

$$f \circ g = f(g(t)) = \log(\sqrt{t}) \qquad \text{for } t > 0$$

Aso in this case it is possible to observe that, although both $g \circ f$ and $f \circ g$ are defined, it is $g \circ f \neq f \circ g$.

Example 77 *Given the functions:*

$$f(x) = \log x \qquad g(t) = -\sqrt{t}$$

determine the composite functions $g \circ f$ and $f \circ g$.

To determine $g \circ f$ we put:

$$t = f(x) = \log x$$

and substituting this expression in the function $g(t)$ instead of t we get:

$$g(f(x)) = -\sqrt{\log x}$$

At this point it is necessary to impose the conditions of reality:

$$\begin{cases} x > 0 \\ \\ \log x \geq 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ \\ x \geq 1 \end{cases} \Rightarrow x \geq 1$$

so that the composite function is:

$$g \circ f = g(f(x)) = -\sqrt{\log x} \qquad \text{for } x \geq 1$$

In a similar way, to determine $f \circ g$ we put:

$$x = g(t) = -\sqrt{t}$$

and substituting this expression in the function $f(x)$ instead of x we get:

$$f(g(t)) = \log(-\sqrt{t})$$

At this point it is necssary to impose the conditions of reality:

$$\begin{cases} t \geq 0 \\ \\ -\sqrt{t} > 0 \end{cases} \Rightarrow \begin{cases} t \geq 0 \\ \\ \sqrt{t} < 0 \end{cases} \Rightarrow \text{impossible}$$

so that the composite function $f \circ g$ does not exist (since the function g always assumes non-positive values, that therefore do not belong to the domain of the function f, given by strictly positive values). In this case, therefore, the function $g \circ f$ exists, while the function $f \circ g$ does not exist.

Example 78 *Given the functions:*

$$f(x) = \sqrt{x} \qquad\qquad g(t) = -\sqrt{t}$$

determine the composite functions $g \circ f$ and $f \circ g$.

To determine $g \circ f$ we put:

$$t = f(x) = \sqrt{x}$$

and substituting this expression in the function $g(t)$ instead of t we get:

$$g(f(x)) = -\sqrt{\sqrt{x}}$$

At this point it is necessary to impose the conditions of reality:

$$\begin{cases} x \geq 0 \\ \\ \sqrt{x} \geq 0 \end{cases} \Rightarrow x \geq 0$$

so that the composite function is:

$$g \circ f = g(f(x)) = -\sqrt{\sqrt{x}} = -\sqrt[4]{x} \qquad \text{for } x \geq 0$$

In a similar way, to determine $f \circ g$ we put:

$$x = g(t) = -\sqrt{t}$$

and substituting this expression in the function $f(x)$ instead of x we get:

$$f(g(t)) = \sqrt{-\sqrt{t}}$$

At this point it is necessary to impose the conditions of reality:

$$\begin{cases} t \geq 0 \\ \\ -\sqrt{t} \geq 0 \end{cases} \Rightarrow \begin{cases} t \geq 0 \\ \\ \sqrt{t} \leq 0 \end{cases} \Rightarrow t = 0$$

so that the composite function (defined only in a point) is:

$$f \circ g = f(g(t)) = \sqrt{-\sqrt{t}} \qquad \text{for } t = 0$$

that is:

$$f(g(t)) = 0 \qquad \text{in } t = 0$$

3.6 Inverse functions

Given an injective function:

$$y = f(x) \qquad \text{with } f : X \subseteq \mathbb{R} \to \mathbb{R}$$

it is called inverse of $f(x)$ the function:

$$x = f^{-1}(y) \qquad \text{with } f^{-1} : f(X) \to X$$

such that:

$$f(x) = y \qquad \text{that is } f\left(f^{-1}(y)\right) = y$$

It is possible to remind that a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is injective if it holds:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \qquad \forall x_1, x_2 \in X$$

so that distinct elements of the domain have distinct images. The injectivity of f is essential to guarantee that to any element $y \in f(X)$ corresponds a unique element $x \in X$ (so that the correspondence from $f(X)$ into X is a function), and it corresponds to the property according to which every straight line parallel to the x-axis intersects the graph of the function in only one point.

It is also possible to observe that a continuous function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is invertible on X if and only if it is strictly monotonic on X; if, on the contrary, the function is not continuous, then the strict monotonicity is only a sufficient (not necessary) condition for the invertibility, that is we have:

(i) f continuous $\qquad f$ invertible $\Leftrightarrow f$ strictly monotonic

(ii) f not continuous $\qquad f$ strictly monotonic $\Rightarrow f$ invertible

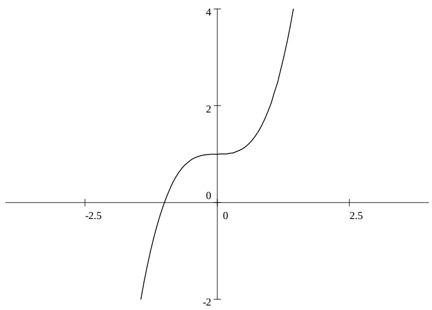
Given a function $y = f(x)$, to establish if it is invertible it is therefore necessary to verify first of all if it is injective. At this point it is possible to obtain the analytical expression of the inverse function expressing the variable x in terms of the variable y, and getting $x = f^{-1}(y)$. It is also necessary to observe that, going from a function to its inverse, the domain and the set of images exchange each other, that is the domain of the direct function becomes the set of images of the inverse function and viceversa. The graphs of a function and of its inverse, finally, turn out to be symmetric with respect to the straight line of equation $y = x$, i.e. if a point (x, y) belongs to the graph of a function, the point (y, x) belongs to the graph of the inverse function.

Example 79 *Given the function:*

$$f(x) = x^3 + 1$$

verify if it is injective and, in this case, determine the inverse function.

The graph of the function is:



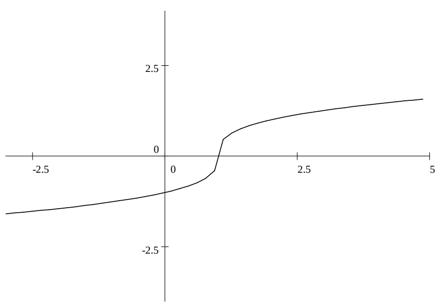
from which it is evident that $f(x)$ is strictly increasing on $\mathbb{R}$, therefore it is injective and invertible on $\mathbb{R}$. The inverse function can be obtained considering:

$$y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = \sqrt[3]{y-1}$$

and it is given by (denoting with x the independent variable):

$$f^{-1}(x) = \sqrt[3]{x-1}$$

The graph of the inverse function is the following:



and for the domain and the set of images of the two functions, finally, we have:

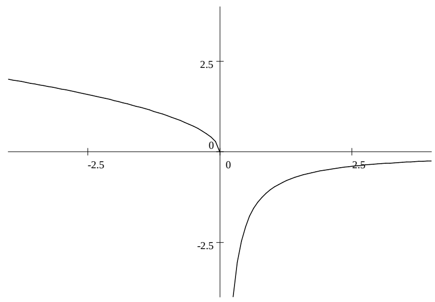
direct function f	$D = \mathbb{R}$	$I = \mathbb{R}$
invers function f^{-1}	$D = \mathbb{R}$	$I = \mathbb{R}$

Example 80 *Given the function:*

$$f(x) = \begin{cases} \sqrt{-x} & \text{if} \quad x \leq 0 \\ -\dfrac{1}{x} & \text{if} \quad x > 0 \end{cases}$$

verify if it is injective and, in this case, determine the inverse function.

The graph of the function is:



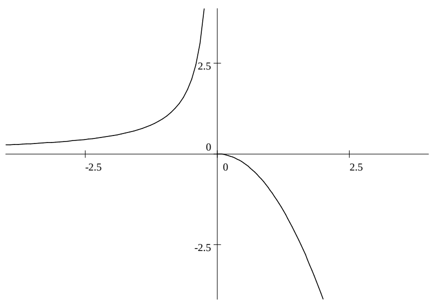
from which it is evident that in this case $f(x)$ is not strictly monotonic on $\mathbb{R}$ (in fact it is strictly monotonic descreasing on the interval $(-\infty, 0)$ and strictly monotonic increasing on the interval $(0, +\infty)$ but it is not monotonic on the whole $\mathbb{R}$), nevertheless it is injective and therefore invertible on all $\mathbb{R}$ (in effect in this case the function is not continuous - in particular it has a discontinuity in the origin - so that, as indicated above, the strict monotonicity is a sufficient, but not necessary, condition for invertibility). The inverse function can be obtained considering:

$$\begin{cases} y = \sqrt{-x} \quad \text{if} \ \ x \leq 0 \Rightarrow x = -y^2 \quad \text{if } y \geq 0 \\ y = -\dfrac{1}{x} \quad \text{if } x > 0 \Rightarrow x = -\dfrac{1}{y} \quad \text{if } y < 0 \end{cases}$$

(where the values taken by the y in each of the two pieces of the inverse function can be determined observing the values assumed by the same variable in the corresponding part of the graph of the direct function) and it is given (denoting with x the independent variable) by:

$$f^{-1}(x) = \begin{cases} -\dfrac{1}{x} & \text{if} \quad x < 0 \\ -x^2 & \text{if} \quad x \geq 0 \end{cases}$$

The graph of the inverse function is the following:



and for the domain and the set of images of the two functions, finally, we have:

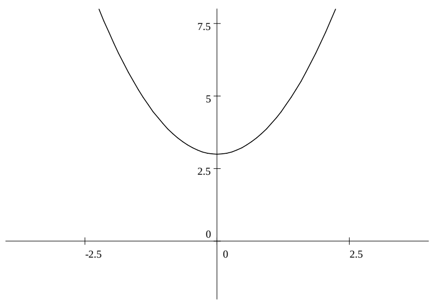
direct function f	$D = \mathbb{R}$	$I = \mathbb{R}$
inverse function f^{-1}	$D = \mathbb{R}$	$I = \mathbb{R}$

Example 81 *Given the function:*

$$f(x) = x^2 + 3$$

verify if it is injective and, in this case, determine the inverse function.

The graph of the function is:



from which it is evident that in this case $f(x)$ is not injective on $\mathbb{R}$ (in fact there are straight lines parallel to the x-axis that intersect the graph of the function in two points), therefore it is not invertible. The function, however, becomes injective (and therefore invertible) considering separately the intervals $(-\infty, 0]$ and $[0, +\infty)$. The corresponding inverse functions (that is the inverse of the restrictions of f to each of the two intervals on which f is injective) can be obtained considering:

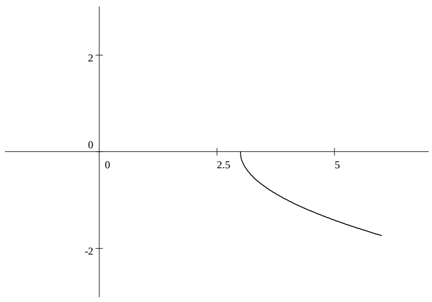
$$\text{on } (-\infty, 0] \qquad y = x^2 + 3 \Rightarrow x^2 = y - 3 \Rightarrow x = -\sqrt{y-3} \quad \text{if } y \geq 3$$

$$\text{on } [0, +\infty) \qquad y = x^2 + 3 \Rightarrow x^2 = y - 3 \Rightarrow x = +\sqrt{y-3} \quad \text{if } y \geq 3$$

In particular, it is possible to observe that on the first of the two intervals the correct expression to be used is $-\sqrt{y-3}$ since the corresponding values of the x are negative (in fact the interval on which the inverse function is computed is $(-\infty, 0]$) and such values can be obtained considering the square root of $(y-3)$ with a minus sign; on the second of the two intervals, on the contrary, the correct expression to be used is $+\sqrt{y-3}$ since the corresponding values of the x are positive (in fact the interval on which the inverse function is computed is $[0, +\infty)$) and such values can be obtained considering the square root of $(y-3)$ with a plus sign. We have therefore that the inverse function of the restriction of f to the interval $(-\infty, 0]$ is given (denoting with x the independent variable) by:

$$f^{-1}(x) = -\sqrt{x-3} \qquad \text{if } x \geq 3$$

its graph is:



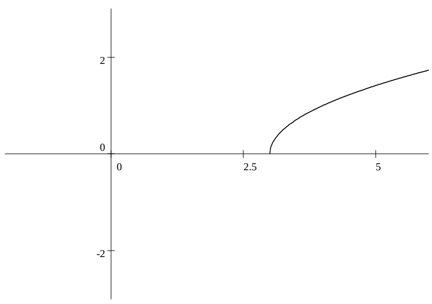
and for the domain and the set of images of the two functions we have:

direct function f	$D = (-\infty, 0]$	$I = [3, +\infty)$
inverse function f^{-1}	$D = [3, +\infty)$	$I = (-\infty, 0]$

The inverse function of the restriction of f to the interval $[0, +\infty)$, then, is given by:

$$f^{-1}(x) = \sqrt{x-3} \qquad \text{if } x \geq 3$$

its graph is:



and for the domain and the set of images of the two functions we have:

direct function f	$D = [0, +\infty)$	$I = [3, +\infty)$
inverse function f^{-1}	$D = [3, +\infty)$	$I = [0, +\infty)$

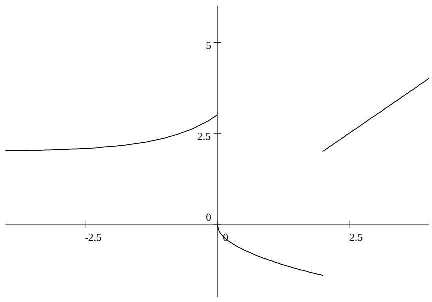
In this case, therefore, the function f is not injective and it is not invertible, while its restrictions to the intervals $(-\infty, 0]$ and $[0, +\infty)$ are invertible, and the corresponding inverse functions are those obtained above.

Example 82 *Given the function:*

$$f(x) = \begin{cases} e^x + 2 & \textit{if} \quad x \leq 0 \\ -\sqrt{x} & \textit{if} \quad 0 < x < 2 \\ x & \textit{if} \quad x \geq 2 \end{cases}$$

verify if it is injective and, in this case, determine the inverse function.

The graph of the function is:



from which it is evident that in this case $f(x)$ is not injective on $\mathbb{R}$, therefore it is not invertible. The function, however, becomes injective (and therefore invertible) considering separately the intervals $(-\infty, 0]$, $(0, 2)$ and $[2, +\infty)$. The corresponding inverse functions (i.e the inverse of the restrictions of f to each of the three intervals on which f is injective) can be obtained considering:

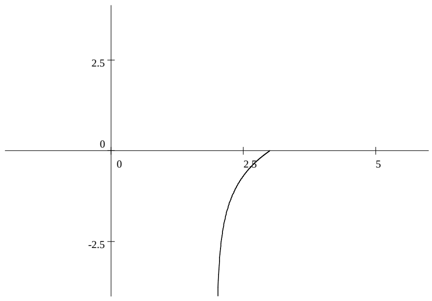
on $(-\infty, 0]$	$y = e^x + 2 \Rightarrow x = \log(y - 2)$	if $2 < y \leq 3$
on $(0, 2)$	$y = -\sqrt{x} \Rightarrow x = y^2$	if $-\sqrt{2} < y < 0$
on $[2, +\infty)$	$y = x \Rightarrow x = y$	if $y \geq 2$

With reference to this aspect it is possible to observe that the definition intervals of each of these inverse functions can be determined observing on the graph the images of the corresponding restrictions of the direct function (that become the domains of the inverse functions, since the domain and the set of images exchange each other going from a function to its inverse).

We have therefore that the inverse function of the restriction of f to the interval $(-\infty, 0]$ is given (denoting with x the independent variable) by:

$$f^{-1}(x) = \log(x - 2) \quad \text{if } 2 < x \leq 3$$

its graph is:



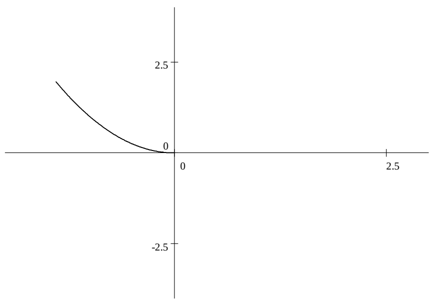
and for the domain and the set of images of the two functions we have:

direct function f	$D = (-\infty, 0]$	$I = (2, 3]$
inverse function f^{-1}	$D = (2, 3]$	$I = (-\infty, 0]$

The inverse function of the restriction of f to the interval $(0, 2)$ then is given by:

$$f^{-1}(x) = x^2 \quad \text{if } -\sqrt{2} < x < 0$$

its graph is:



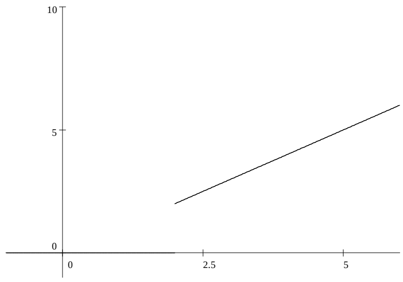
and for the domain and the set of images of the two functions we have:

direct function f	$D = (0, 2)$	$I = (-\sqrt{2}, 0)$
inverse function f^{-1}	$D = (-\sqrt{2}, 0)$	$I = (0, 2)$

The inverse function of the restriction of f to the interval $[2, +\infty)$, finally, is given by:

$$f^{-1}(x) = x \quad \text{if } x \geq 2$$

its graph is:



and for the domain and the set of images of the two functions we have:

direct function f	$D = [2, +\infty)$	$I = [2, +\infty)$
inverse function f^{-1}	$D = [2, +\infty)$	$I = [2, +\infty)$

3.7 Elementary functions and geometric transformations

With the term of elementary functions we indicate the functions (linear, quadratic, power, exponential, logarithmic, trigonometric - together with those that can be obtained from them through the usual algebraic operations and the operation of composition - starting from which it is possible to obtain more complex functions. From the graphs of these functions (and more generally from the graphs of functions that are known), furthermore, it is possible, through simple geometric considerations, to obtain the graphs of other functions, linked to the initial ones by some specific relationships.

In particular, from the graph of $y = f(x)$ it is possible to obtain easily the graphs of:

$$y = -f(x) \qquad\qquad y = f(-x)$$

$$y = f(x) + c \;\text{ with } c \in \mathbb{R} \qquad\qquad y = f(x + c) \;\text{ with } c \in \mathbb{R}$$

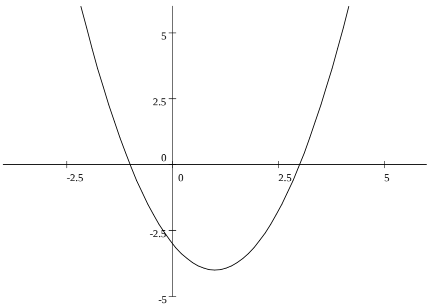
$$y = cf(x) \;\text{ with } c \in \mathbb{R}\setminus\{0\} \qquad\qquad y = f(cx) \;\text{ with } c \in \mathbb{R}\setminus\{0\}$$

$$y = |f(x)| \qquad\qquad y = f(|x|)$$

An example can be illustrated considering the function:

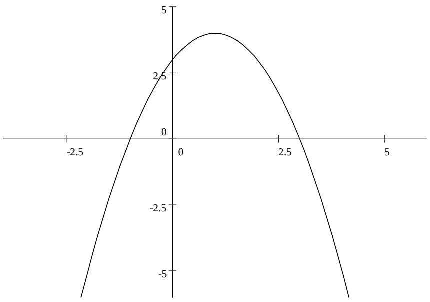
$$y = f(x) = x^2 - 2x - 3$$

that is a parabola with upward concavity that intersects the x-axis in correspondence of the points $A = (-1, 0)$ and $B = (3, 0)$ and the y-axis in correspondence of the point $C = (0, -3)$, and whose graph is the following:

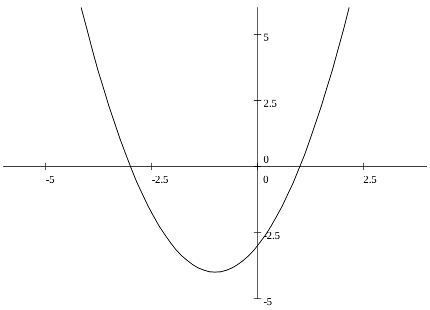


From this graph it is possible to obtain easily the following other graphs:

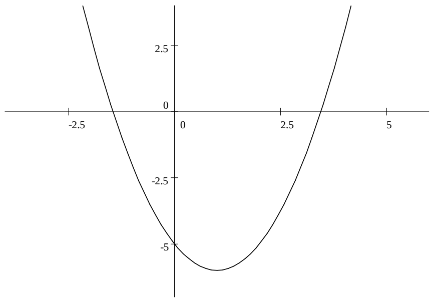
- The graph of $y = -f(x) = -x^2 + 2x + 3$ is obtained from that of $f(x)$ by "reversing" it with respect to the x-axis:



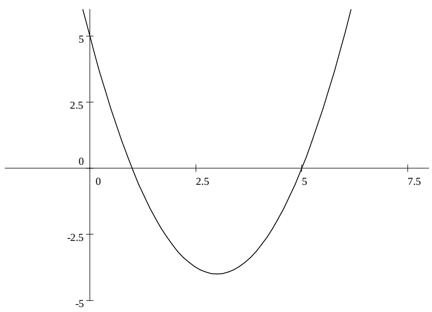
- The graph of $y = f(-x) = x^2 + 2x - 3$ is obtained from that of $f(x)$ by "reversing" it with respect to the y-axis:



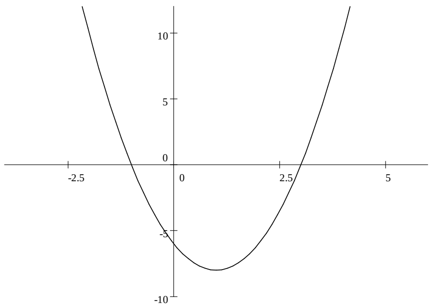
- The graph of $y = f(x) + c$ with $c \in \mathbb{R}$ is obtained from that of $f(x)$ by translating it of the quantity $|c|$ upward (if $c > 0$) or downward (if $c < 0$). For example, in the case of $c = -2$ we get $y = f(x) - 2 = x^2 - 2x - 5$ whose graph is:



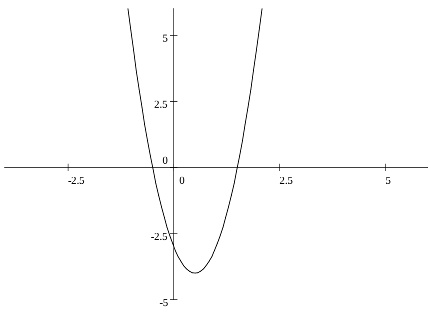
- The graph of $y = f(x + c)$ with $c \in \mathbb{R}$ is obtained from that of $f(x)$ by translating it of the quantity $|c|$ leftward (if $c > 0$) or rightward (if $c < 0$). For example, in the case of $c = -2$ we get $y = f(x - 2) = x^2 - 6x + 5$ whose graph is:



- The graph of $y = cf(x)$ with $c \in \mathbb{R} \setminus \{0\}$ is obtained from that of $f(x)$ by "expanding" it of $|c|$ times in the direction of the y-axis (more precisely, the graph turns out to be enlarged with respect to the initial one if $c > 1$, while it turns out to be compressed with respect to the initial one if $0 < c < 1$ - and if $c < 0$ similar considerations hold, but in addition the graph turns out to be reversed with respect to the x-axis, as in the first transformation considered -). For example, in the case of $c = 2$ we get $y = 2f(x) = 2x^2 - 4x - 6$ whose graph is:

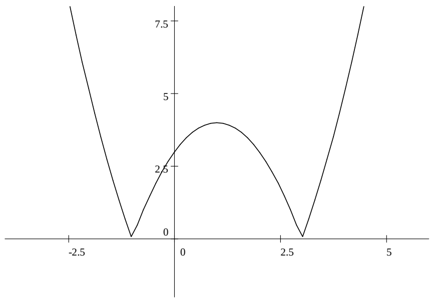


- The graph of $y = f(cx)$ with $c \in \mathbb{R} \setminus \{0\}$ is obtained from that of $f(x)$ by "compressing" it of $|c|$ times in the direction of the x-axis (more precisely, the graph turns out to be compressed with respect to the initial one if $c > 1$, while it turns out to be enlarged with respect to the initial one if $0 < c < 1$ - and if $c < 0$ similar considerations hold, but in addition the graph turns out to be reversed with respect to the y-axis, as in the second transformation considered -). For example, in the case of $c = 2$ we get $y = f(2x) = 4x^2 - 4x - 3$ whose graph is:



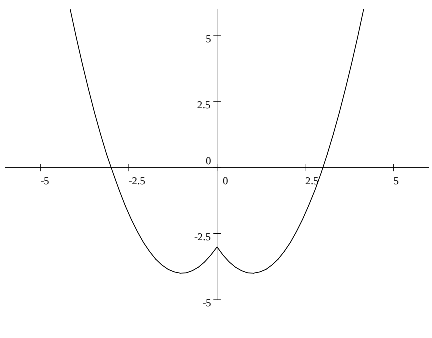
- The graph of $y = |f(x)| = \left|x^2 - 2x - 3\right|$ is obtained from that of $f(x)$ by reversing in the half-plane of the $y > 0$ the part of the graph that is contained in the half-plane of the $y < 0$, since we have:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$



- The graph of $y = f(|x|) = x^2 - 2|x| - 3$ is obtained from that of $f(x)$ by reversing in the half-plane of the $x < 0$ the part of the graph that is contained in the half-plane of the $x > 0$, since we have:

$$f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$



and, whatever the initial function $f(x)$ is, we have that the function $f(|x|)$ is an even function.

3.8 Exercises

Determine the domain of the following functions:

1) $f(x) = \log\sqrt{x^2-3}$

2) $f(x) = e^{\sqrt{x^2-3}}$

3) $f(x) = \log\sqrt{x^2+3}$

4) $f(x) = (3x)^{\sqrt{x^2-4}}$

5) $f(x) = \sqrt{\log(x^2-3)}$

6) $f(x) = \dfrac{\log\sqrt{x^2-3}}{|x^2-4|}$

7) $f(x) = \dfrac{1}{\log\sqrt{x^2+1}}$

8) $f(x) = \left(\dfrac{1}{2}x\right)^{\log(-x)}$

9) $f(x) = \dfrac{\sqrt{x+5}}{\log(x+3)}$

10) $f(x) = (4x)^{\sqrt{2-x}}$

11) $f(x) = \log\left(4-e^{-x}\right)$

12) $f(x) = \dfrac{|x|}{x+|x|}$

13) $f(x) = \sqrt{\log x - 1}$

14) $f(x) = \dfrac{1}{e^{\sqrt{x^2-4}}}$

15) $f(x) = \sqrt{\dfrac{3x}{\log\left(x-3\right)}}$

16) $f(x) = \dfrac{\sqrt{x+5}}{\log\left(x^2-4\right)}$

17) $f(x) = \dfrac{\log\sqrt{x^2-3}}{e^{\sqrt{x^2+3}}}$

18) $f(x) = (\log x)^x$

19) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x}-2}$

20) $f(x) = \dfrac{3\log x^2}{\sqrt{x^2}}$

Determine the intersections with the axes and the sign of the following functions:

21) $f(x) = x(\log x - 3)^2$

22) $f(x) = e^{\frac{2-x}{1-x}}$

23) $f(x) = \dfrac{\mid x^2 - 4 \mid}{e^x}$

24) $f(x) = \dfrac{x^2+2x+5}{x+2}$

25) $f(x) = 3\dfrac{x^2 - 2\,|x| + 1}{|x|+1}$

26) $f(x) = \dfrac{\log x}{x^2}$

27) $f(x) = \log\left(\dfrac{2x-1}{x+2}\right)$

Determine if the following functions have symmetries:

28) $f(x) = -\dfrac{\sqrt{x^2-x^4}}{x^3}$

29) $f(x) = \dfrac{3x}{2x^3+x}$

30) $f(x) = \dfrac{2\,|x| + x^2}{x^3}$

31) $f(x) = \dfrac{\sqrt[3]{x^3-x^5}}{|x|}$

32) $f(x) = -\dfrac{\sqrt{x^2-x^4}}{x^4}$

33) $f(x) = \dfrac{\sqrt[3]{x^2-x}}{x}$

Given the functions f and g, determine the composite functions $g \circ f$ and $f \circ g$:

34) $f(x) = \sqrt[3]{x-1} \qquad g(t) = t^3 + 1$

35) $f(x) = x^3 + 1 \qquad g(t) = \sqrt[3]{t+1}$

36) $f(x) = \log x \qquad g(t) = e^{t+3}$

37) $f(x) = \log x \qquad g(t) = |t-2|$

38) $f(x) = \log(x+1) \qquad g(t) = e^t$

39) $f(x) = \log(x+1) \qquad g(t) = \sqrt{t}$

40) $f(x) = \log x - 2 \qquad g(t) = \sqrt{t}$

41) $f(x) = |x-1| \qquad g(t) = \sqrt{t}$

42) $f(x) = \log x \qquad g(t) = e^{t+1}$

43) $f(x) = e^{x+2} \qquad g(t) = \log t$

Given the following functions, determine the corresponding inverse functions:

44) $f(x) = 2x + 3$

45) $f(x) = x^3 + 3$

46) $f(x) = \sqrt{x+2}$

47) $f(x) = \log|x|$

48) $f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ x^2 + 2 & \text{if } x > 1 \end{cases}$

49) $f(x) = \begin{cases} 2x - 3 & \text{if } x < 1 \\ \log x^3 & \text{if } x \geq 1 \end{cases}$

50) $f(x) = \begin{cases} \sqrt{-x} & \text{if } x \leq -3 \\ x + 3 & \text{if } -3 < x < 3 \\ \log x & \text{if } x \geq 3 \end{cases}$

Chapter 4

Limits and continuity

4.1 Definitions and extended algebra of limits

Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ and given x_0, accumulation point for X, we say that f has limit l as x tends to x_0 and we write:

$$\lim_{x \to x_0} f(x) = l$$

if in correspondence of each neighbourhood of l of radius ε, $U_\varepsilon(l)$, there exists a neighbourhood of x_0 of radius δ, $U_\delta(x_0)$, such that for each x (different from x_0) belonging to the neighbourhood of x_0 (and to the domain of f) the corresponding value of the function belongs to the neighbourhood of l; we have therefore:

$$\forall U_\varepsilon(l) \quad \exists U_\delta(x_0) \quad : \quad x \in U_\delta(x_0) \cap X \ , \quad x \neq x_0 \Rightarrow f(x) \in U_\varepsilon(l)$$

and also, with a different notation:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad : \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

In this definition, both the point x_0 and the value l can be finite or equal to $\pm\infty$. In addition, the one considered is the definition of complete limit, but in an analogous way it is possible to introduce the definitions of right-hand limit and of left-hand limit (considering respectively a right or a left neighbourhood of x_0 and writing $x \to x_0^+$ or $x \to x_0^-$) and of limit from above and from below (considering respectively a right or a left neighbourhood of l and writing l^+ or l^-).

In practice, however, for the calculation of limits the definition is not used, but it is possible to use first of all a number of rules that allow to reduce the calculation of the limit of a function (in whose analytical expression there is a finite number of operations of sum, product, ratio) to the calculation of the limits of its components.

In particular, if f and g are two functions that both admit limit as $x \to x_0$ (where x_0 is an accumulation point for the domains of the two functions), the following equalities hold:

(*i*) $\lim\limits_{x\to x_0} [f(x) + g(x)] = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$

this equality has no meaning if one of the two limits is $+\infty$ and the other one is $-\infty$

(*ii*) $\lim\limits_{x\to x_0} [f(x) \cdot g(x)] = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$

this equality has no meaning if one of the two limits is 0 and the other one is $\pm\infty$

(*iii*) $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x\to x_0} f(x)}$ when $\lim\limits_{x\to x_0} f(x) = m \neq 0$

$\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = \pm\infty$ when $\lim\limits_{x\to x_0} f(x) = 0^{\pm}$

$\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = 0^{\pm}$ when $\lim\limits_{x\to x_0} f(x) = \pm\infty$

(*iv*) $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$

this equality has no meaning if both limits are 0 or $\pm\infty$

According to these equalities we have that the limit of a sum, of a product and of a ratio of functions is equal, respectively, to the sum, to the product and to the ratio of the limits of the single functions.These rules remain true when the limits considered, instead of being numbers, are equal to $\pm\infty$ (with the exception of the cases listed above, in which the equalities have no meaning and originate to the so called "forms of indetermination"), so that it is possible to introduce an "extended algebra" of limits. The following results hold:

a) for the sum:

$$(+\infty) + a = +\infty + a = +\infty \qquad \text{with } a \in \mathbb{R}$$

$$(-\infty) + a = -\infty + a = -\infty \qquad \text{with } a \in \mathbb{R}$$

$$(+\infty) + (+\infty) = +\infty + \infty = +\infty$$

$$(-\infty) + (-\infty) = -\infty - \infty = -\infty$$

b) for the product:

$$(+\infty)\cdot a = \begin{cases} +\infty & \text{if } a>0 \\ -\infty & \text{if } a<0 \end{cases}$$

$$(-\infty)\cdot a = \begin{cases} -\infty & \text{if } a>0 \\ +\infty & \text{if } a<0 \end{cases}$$

$$(\pm\infty)\cdot(\pm\infty) = +\infty$$

$$(\pm\infty)\cdot(\mp\infty) = -\infty$$

c) for the reciprocal:

$$\frac{1}{0^\pm} = \pm\infty$$

$$\frac{1}{\pm\infty} = 0^\pm$$

d) for the ratio:

$$\frac{a}{0^\pm} = a\cdot\frac{1}{0^\pm} = a\cdot(\pm\infty) = \begin{cases} \pm\infty & \text{if } a>0 \\ \mp\infty & \text{if } a<0 \end{cases}$$

$$\frac{a}{\pm\infty} = a\cdot\frac{1}{\pm\infty} = a\cdot 0^\pm = \begin{cases} 0^\pm & \text{if } a>0 \\ 0^\mp & \text{if } a<0 \end{cases}$$

$$\frac{0^\pm}{\pm\infty} = 0^\pm\cdot\frac{1}{\pm\infty} = (0^\pm)\cdot(0^\pm) = 0^+$$

$$\frac{0^\mp}{\pm\infty} = 0^\mp\cdot\frac{1}{\pm\infty} = (0^\mp)\cdot(0^\pm) = 0^-$$

$$\frac{\pm\infty}{0^\pm} = \pm\infty\cdot\frac{1}{0^\pm} = (\pm\infty)\cdot(\pm\infty) = +\infty$$

$$\frac{\mp\infty}{0^\pm} = \mp\infty\cdot\frac{1}{0^\pm} = (\mp\infty)\cdot(\pm\infty) = -\infty$$

It is necessary to observe that, in all these formulae, ∞ does not represent a number but a symbol; an expression like $+\infty+\infty=+\infty$ therefore does not mean that "the sum of $+\infty$ and of $+\infty$ is equal to $+\infty$" (because ∞ is not a number!) but it means "the sum of two functions that tend to $+\infty$ tends to $+\infty$". In a similar way, the expression $\dfrac{1}{0^+}=+\infty$ does not mean "1 divided 0^+ is equal to $+\infty$" (since, algebraically, the division by 0 is not possible!) but it means "the reciprocal of a function that tends to 0^+ tends to $+\infty$". Similar considerations hold for all the other formulae reported above, that represent therefore "abbreviations" of what can be considered an "extended algebra" of limits, and that make sense only considering the notion of limit that is implicit in them.

A fundamental property, that allows to calculate easily the limits in the majority of cases (without having to resort to the definition), is then that of the continuous functions, for which we have (exactly by definition of continuous function):

$$\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right) = f(x_0)$$

so that it becomes possible to calculate the limits simply substituting, in the function $f(x)$, the variable x with the value x_0 to which it tends. Since the elementary functions and the functions obtained from them through the usual algebraic operations and the operation of composition are continuous (on their domain), for all these functions (that represent the majority of the cases that occur) it is possible to calculate in this way the limits (using in addition the rules relative to the limit of a sum, of a product and of a ratio of functions and the rules of the "extended algebra"), without resorting to the definition.

Example 83 *Calculate the following limits:*

1) $\displaystyle\lim_{x\to 3} \frac{x+2}{(4-x)^2}$

2) $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x-2}$

3) $\displaystyle\lim_{x\to 1^+} \frac{x-\sin x+2}{1-x}$

4) $\displaystyle\lim_{x\to 1^-} \frac{x-\sin x+2}{1-x}$

5) $\displaystyle\lim_{x\to +\infty} \frac{x^2+3x-1}{e^{-x}}$

6) $\displaystyle\lim_{x\to 0^+} \frac{x}{\log x}$

All these functions are obtained starting from the elementary functions, therefore they are continuous on their domain and to calculate the limits required it is sufficient to use the rules relative to the limit of a sum, of a product and of a ratio of functions, the property of continuity and, eventually, the rules of the "extended algebra" introduced above. In this way we get:

1) $\lim_{x\to 3} \frac{x+2}{(4-x)^2} = \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}(4-x)^2} = \frac{3+2}{(4-3)^2} = \frac{5}{1} = 5$

2) $\lim_{x\to 0} \frac{e^x-1}{x-2} = \frac{\lim_{x\to 0}(e^x-1)}{\lim_{x\to 0}(x-2)} = \frac{1-1}{0-2} = \frac{0}{-2} = 0$

3) $\lim_{x\to 1^+} \frac{x-\sin x+2}{1-x} = \frac{\lim_{x\to 1^+}(x-\sin x+2)}{\lim_{x\to 1^+}(1-x)} = \frac{1-\sin 1+2}{1-1^+} = \frac{3-\sin 1}{0^-} = -\infty$

4) $\lim_{x\to 1^-} \frac{x-\sin x+2}{1-x} = \frac{\lim_{x\to 1^-}(x-\sin x+2)}{\lim_{x\to 1^-}(1-x)} = \frac{1-\sin 1+2}{1-1^-} = \frac{3-\sin 1}{0^+} = +\infty$

5) $\lim_{x\to +\infty} \frac{x^2+3x-1}{e^{-x}} = \frac{\lim_{x\to +\infty}(x^2+3x-1)}{\lim_{x\to +\infty} e^{-x}} = \frac{+\infty+\infty-1}{0^+} = \frac{+\infty}{0^+} = +\infty$

6) $\lim_{x\to 0^+} \frac{x}{\log x} = \frac{\lim_{x\to 0^+} x}{\lim_{x\to 0^+} \log x} = \frac{0^+}{-\infty} = 0^-$

Another result that, in some cases, is used in the calculation of limits is then the following:

If f is bounded in a neighbourhood of x_0 and if $\lim_{x\to x_0} g(x) = 0$, then:

$$\lim_{x\to x_0} f(x)\cdot g(x) = 0$$

i.e. *the product of a bounded function and of a function that tends to* 0*, also tends to* 0*.*

This property is usually applied in presence of trigonometric functions (in particular sine and cosine, that are bounded functions).

Example 84 *Calculate the following limit:*

$$\lim_{x\to -\infty} e^x \cos x$$

In this case e^x tends to 0 as $x\to -\infty$ while $\cos x$ is bounded in a neighbourhood of $-\infty$ (in fact $|\cos x| \le 1\ \forall x$), therefore we have:

$$\lim_{x\to -\infty} e^x \cos x = 0$$

Example 85 *Calculate the following limit:*

$$\lim_{x \to +\infty} e^x \cos x$$

In this case $\cos x$ is bounded in a neighbourhood of $+\infty$ but e^x tends to $+\infty$ as $x \to +\infty$, so that it is not possible to apply the result reported above (in particular it is possible to show that the limit considered does not exist).

4.2 Forms of indetermination

The rules seen above for the calculation of limits (limit of a sum, of a product, of a ratio) cannot be applied in some cases, that represent the so-called "forms of indetermination" (or "indeterminate forms"). They are characterized by the fact that it is not possible to know a priori the result of the operation of limit that has originated the form of indetermination, since indetermination forms of the same type can give raise to different results.

Considering for instance the functions:

$$f(x) = x^2 + 1 \qquad g(x) = -x^2$$

we have:

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} g(x) = -\infty$$

and therefore the limit as $x \to +\infty$ of $[f(x) + g(x)]$ is in the indeterminate form $+\infty - \infty$. In this case, nevertheless, it is possible to obtain easily:

$$\lim_{x \to +\infty} [f(x) + g(x)] = \lim_{x \to +\infty} \left(x^2 + 1 - x^2\right) = \lim_{x \to +\infty} 1 = 1$$

Considering then the functions:

$$f(x) = x^2 + x \qquad g(x) = -x^2$$

we have:

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} g(x) = -\infty$$

and therefore the limit as $x \to +\infty$ of $[f(x) + g(x)]$ is again in the indeterminate form $+\infty - \infty$, but it is possible to obtain easily:

$$\lim_{x \to +\infty} [f(x) + g(x)] = \lim_{x \to +\infty} \left(x^2 + x - x^2\right) = \lim_{x \to +\infty} x = +\infty$$

Considering then the functions:

$$f(x) = x^2 \qquad g(x) = -x^2 - x$$

we have:

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} g(x) = -\infty$$

and therefore the limit as $x \to +\infty$ of $[f(x)+g(x)]$ is always in the indeterminate form $+\infty - \infty$, but in this case it is possible to obtain easily:

$$\lim_{x\to+\infty}[f(x)+g(x)] = \lim_{x\to+\infty}\left(x^2 - x^2 - x\right) = \lim_{x\to+\infty}(-x) = -\infty$$

From these examples it is therefore evident that the same indeterminate form (in this case $+\infty-\infty$) gives raise to different results, hence it is not possible to determine a priori the result to which it corresponds.

The forms of indetermination, in particular, are:

$$(+\infty)+(-\infty) = +\infty - \infty$$

$$0\cdot(\pm\infty) = 0\cdot\infty$$

$$\frac{0^\pm}{0^\pm} = \frac{0^\pm}{0^\mp} = \frac{0}{0}$$

$$\frac{\pm\infty}{\pm\infty} = \frac{\pm\infty}{\mp\infty} = \frac{\infty}{\infty}$$

Besides these, that are defined "arithmetic", there are also "exponential" indeterminate forms, given by:

$$1^\infty \qquad 0^0 \qquad \infty^0$$

that, in reality, can be easily reduced to the previous ones (in particular to the form $0\cdot\infty$) observing that they are obtained in the calculation of limits of functions of the type $f(x)^{g(x)}$, for which it is always possible to use the following transformation:

$$f(x)^{g(x)} = e^{\log f(x)^{g(x)}} = e^{g(x)\log f(x)}$$

so that the exponential forms of indetermination become:

$$1^\infty \Rightarrow e^{\infty\cdot\log 1} = e^{\infty\cdot 0}$$

$$0^0 \Rightarrow e^{0\cdot\log 0} = e^{0\cdot\infty}$$

$$\infty^0 \Rightarrow e^{0\cdot\log\infty} = e^{0\cdot\infty}$$

and therefore they are all reduced to the arithmetic indeterminate form $0\cdot\infty$.

At this point it is important to describe some techniques that allow to solve the limits in presence of forms of indetermination; in particular, it is possible to distinguish the following methods:

- algebraic manipulations
- infinitesimals and infinities (principle of elimination of negligible terms)
- fundamental limits
- de l'Hospital's rule
- Taylor-Mac Laurin's formula

4.3 Calculation of limits: algebraic manipulations

A first method used to solve the forms of indetermination is represented by algebraic manipulations. In general, they consist in the decomposition of polynomials, rationalizations, use of the properties of the powers, use of the properties of the logarithms, that in some cases allow to solve the form of indetermination that initially appears in the calculation of a limit.

Example 86 *Calculate the limit:*

$$\lim_{x \to 2} \frac{2-x}{x^2-4}$$

The limit is in the form $\frac{0}{0}$, we can then write (decomposing the denominator of the fraction and simplifying):

$$\lim_{x \to 2} \frac{2-x}{x^2-4} = \lim_{x \to 2} \frac{2-x}{(x-2)(x+2)} = \lim_{x \to 2} -\frac{x-2}{(x-2)(x+2)} =$$
$$= \lim_{x \to 2} -\frac{1}{x+2} = -\frac{1}{4}$$

Example 87 *Calculate the limit:*

$$\lim_{x \to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right)$$

The limit is in the form $+\infty - \infty$, we can then write (multiplying and dividing by the same quantity):

$$\lim_{x \to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right) = \lim_{x \to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right) \frac{\sqrt{x^2+1} + \sqrt{x^2+2}}{\sqrt{x^2+1} + \sqrt{x^2+2}} =$$
$$= \lim_{x \to +\infty} \frac{x^2+1-x^2-2}{\sqrt{x^2+1} + \sqrt{x^2+2}} =$$
$$= \lim_{x \to +\infty} \frac{-1}{\sqrt{x^2+1} + \sqrt{x^2+2}} = 0$$

Example 88 *Calculate the limit:*

$$\lim_{x \to +\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+2} \right)$$

The limit is in the form $+\infty - \infty$, we can then write (multiplying and dividing by the same quantity):

$$\lim_{x \to +\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+2} \right) = \lim_{x \to +\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+2} \right) \frac{\sqrt{x^2+x+1} + \sqrt{x^2+2}}{\sqrt{x^2+x+1} + \sqrt{x^2+2}} =$$
$$= \lim_{x \to +\infty} \frac{x^2+x+1-x^2-2}{\sqrt{x^2+x+1} + \sqrt{x^2+2}} =$$
$$= \lim_{x \to +\infty} \frac{x-1}{\sqrt{x^2+x+1} + \sqrt{x^2+2}}$$

At this point we have the indeterminate form $\dfrac{\infty}{\infty}$, it is then possible to factor out both in the numerator and in the denominator of the fraction the power of maximum degree (a technique that is often used when there is an indeterminate form of the type $\dfrac{\infty}{\infty}$ and there are powers of the x both in the numerator and in the denominator), obtaining:

$$\lim_{x\to+\infty}\frac{x-1}{\sqrt{x^2+x+1}+\sqrt{x^2+2}}=\lim_{x\to+\infty}\frac{x\left(1-\dfrac{1}{x}\right)}{\sqrt{x^2\left(1+\dfrac{1}{x}+\dfrac{1}{x^2}\right)}+\sqrt{x^2\left(1+\dfrac{2}{x^2}\right)}}=$$

$$=\lim_{x\to+\infty}\frac{x\left(1-\dfrac{1}{x}\right)}{x\left(\sqrt{1+\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1+\dfrac{2}{x^2}}\right)}=$$

$$=\lim_{x\to+\infty}\frac{1-\dfrac{1}{x}}{\sqrt{1+\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1+\dfrac{2}{x^2}}}=\frac{1}{2}$$

In this case it is possible to observe that the term x^2 factored out the square root should be written in the form $|x|$, but as the limit is calculated for $x\to+\infty$ this means that we are considering values of x that are positive, so that $|x|=x$.

Example 89 *Calculate the limit:*

$$\lim_{x\to+\infty}\left[\log(x^2-2)-\log(x-2)\right]$$

The limit is in the form $+\infty-\infty$, we can then write (using a property of the logarithms):

$$\lim_{x\to+\infty}\left[\log(x^2-2)-\log(x-2)\right]=\lim_{x\to+\infty}\log\frac{x^2-2}{x-2}=\lim_{x\to+\infty}\log\frac{x^2\left(1-\dfrac{2}{x^2}\right)}{x\left(1-\dfrac{2}{x}\right)}=$$

$$=\lim_{x\to+\infty}\log\frac{x\left(1-\dfrac{2}{x^2}\right)}{1-\dfrac{2}{x}}=+\infty$$

4.4 Calculation of limits: infinitesimals and infinities

A second method used to solve the forms of indetermination is represented by the "principle of elimination of negligible terms", that can be applied when the form of indetermination arises in presence of sums of infinitesimals or of inifinities (in particular, in presence of sums of powers). With reference to this aspect, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ with x_0 accumulation point for X, we have first of all:

$$f \text{ is infinitesimal for } x \to x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = 0$$

$$f \text{ is infinity for } x \to x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = \pm\infty$$

that is a function is infinitesimal (for $x \to x_0$) if it tends to 0, while it is infinity (for $x \to x_0$) if it tends to $\pm\infty$. The infinitesimals and the inifinities, then, are characterized by an order (a number), that indicates the speed at which they tend respectively to 0 or to ∞ (a greater order corresponds to a greater speed of convergence to 0 – in the case of infinitesimals – or to ∞ – in the case of infinities –). In particular, considering the power functions $f(x) = kx^\alpha$ with $k \in \mathbb{R}$ and $\alpha > 0$, they are infinitesimals for $x \to 0$ and infinities for $x \to \pm\infty$, and their order is represented by the exponent α.

With reference to the infinitesimals and to the infinities the following rule (known as "principle of elimination of negligible terms") holds (and it can be used in the computation of certain limits, that are in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$):

If f_i $(i = 1, 2, ..., n)$ and g_j $(j = 1, 2, ..., m)$ are infinitesimal or infinity functions as $x \to x_0$, then we have:

$$\lim_{x \to x_0} \frac{f_1(x) + f_2(x) + ... + f_n(x)}{g_1(x) + g_2(x) + ... + g_m(x)} = \lim_{x \to x_0} \frac{f_h(x)}{g_k(x)} \qquad \text{with} \quad \begin{array}{l} 1 \leq h \leq n \\ 1 \leq k \leq m \end{array}$$

where *$f_h(x)$ and $g_k(x)$ represent the infinitesimal of lower order (in the case of infinitesimals) or the infinity of higher order (in the case of infinities) among those that appear in the sum respectively in the numerator and in the denominator of the fraction.*

This result allows therefore to neglect the infinitesimals of higher order (in the calculation of the limit of a ratio among sums of infinitesimals) and the infinities of lower order (in the calculation of the limit of a ratio among sums of infinities).

This criterion is of immediate application in the case of sums of powers (for which higher exponents correspond to infinitesimals - for $x \to 0$ - or infinities - for $x \to \pm\infty$ - of higher order); in this case, in fact, it is sufficient to consider, both in the numerator and in the denominator, the lowest power (in the case of infinitesimals) or the highest power (in the case of infinities), omitting all the others.

Example 90 *Calculate the limit:*

$$\lim_{x\to 0^+} \frac{2x+\sqrt[3]{x}+5x^3}{x^2+\sqrt{x}+2x}$$

The limit is in the form $\frac{0}{0}$, and since both the numerator and the denomianator of the fraction are sums of infinitesimals it is possible to neglect the infinitesimals of higher order (and therefore to consider only the powers of lower degree), obtaining:

$$\lim_{x\to 0^+} \frac{2x+\sqrt[3]{x}+5x^3}{x^2+\sqrt{x}+2x} = \lim_{x\to 0^+} \frac{\sqrt[3]{x}}{\sqrt{x}} = \lim_{x\to 0^+} \frac{(x)^{\frac{1}{3}}}{(x)^{\frac{1}{2}}} = \lim_{x\to 0^+} \frac{1}{(x)^{\frac{1}{6}}} = +\infty$$

Example 91 *Calculate the limit:*

$$\lim_{x\to 0^+} \frac{x^4+3x^3-6\sqrt{x}}{2x^3+x-2\sqrt{x}}$$

The limit is in the form $\frac{0}{0}$, and since both the numerator and the denominator of the fraction are sums of infinitesimals it is possible to neglect the infinitesimals of higher order (and therefore to consider only the powers of lower degree), obtaining:

$$\lim_{x\to 0^+} \frac{x^4+3x^3-6\sqrt{x}}{2x^3+x-2\sqrt{x}} = \lim_{x\to 0^+} \frac{-6\sqrt{x}}{-2\sqrt{x}} = 3$$

Example 92 *Calculate the limit:*

$$\lim_{x\to +\infty} \frac{3\sqrt{x}+2\sqrt[3]{x^2}+x}{3+\sqrt{x^3}+2x}$$

The limit is in the form $\frac{\infty}{\infty}$, and since both the numerator and the denominator of the fraction are sums of inifinities (in particular the value 3 in the denominator can be considered an infinity of order 0) it is possible to neglect the infinities of lower order (and therefore to consider only the powers of higher degree), obtaining :

$$\lim_{x\to +\infty} \frac{3\sqrt{x}+2\sqrt[3]{x^2}+x}{3+\sqrt{x^3}+2x} = \lim_{x\to +\infty} \frac{x}{\sqrt{x^3}} = \lim_{x\to +\infty} \frac{x}{(x)^{\frac{3}{2}}} = \lim_{x\to +\infty} \frac{1}{(x)^{\frac{1}{2}}} = 0^+$$

Example 93 *Calculate the limit:*

$$\lim_{x\to-\infty}\left(3x+2x^2-\sqrt[3]{x}\right)$$

The limit is in the form $-\infty+\infty$, and even if we are not in presence of a fraction it turns out to be a sum of infinities, so that also in this case it is possible to apply the principle of elimination of negligible terms to solve the form of indetermination. In particular, since the expression considered is a sum of infinities, it is possible to neglect the infinities of lower order (and therefore to consider only the power of higher degree, that determines the behaviour of the function for $x\to-\infty$), obtaining:

$$\lim_{x\to-\infty}\left(3x+2x^2-\sqrt[3]{x}\right)=\lim_{x\to-\infty}2x^2=+\infty$$

4.5 Calculation of limits: fundamental limits

A third method used for the resolution of the forms of indetermination is represented by the application of some "fundamental limits", so that it is possible to reduce the calculation of complex limits to that of other limits whose value is known. In particular, the following fundamental limits hold:

(i) $\quad \displaystyle\lim_{x\to0}\frac{\sin x}{x}=1$

(ii) $\quad \displaystyle\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

(iii) $\quad \displaystyle\lim_{x\to0}\frac{a^x-1}{x}=\log a$ $\quad$ in particular $\displaystyle\lim_{x\to0}\frac{e^x-1}{x}=1$

(iv) $\quad \displaystyle\lim_{x\to0}\frac{\log_a(1+x)}{x}=\frac{1}{\log a}$ $\quad$ in particular $\displaystyle\lim_{x\to0}\frac{\log(1+x)}{x}=1$

(v) $\quad \displaystyle\lim_{x\to0}\frac{(1+x)^\alpha-1}{x}=\alpha$ $\quad$ with $\alpha\in\mathbb{R}$

(vi) $\quad \displaystyle\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x=e^a$ $\quad$ in particular $\displaystyle\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=e$

In this way it becomes possible, at least in certain cases, to solve the forms of indetermination that appear initially.

Example 94 *Calculate the limit:*

$$\lim_{x\to 0}\frac{x^2-x^2e^x}{2x^3}$$

The limit is in the form $\frac{0}{0}$, however it is possible to exploit the fundamental limit (*iii*) observing that we have:

$$\lim_{x\to 0}\frac{x^2-x^2e^x}{2x^3}=\lim_{x\to 0}\frac{x^2(1-e^x)}{2x^3}=\lim_{x\to 0}\frac{1-e^x}{2x}=$$
$$=\lim_{x\to 0}-\frac{e^x-1}{2x}=-\frac{1}{2}\lim_{x\to 0}\frac{e^x-1}{x}=-\frac{1}{2}\cdot 1=-\frac{1}{2}$$

Example 95 *Calculate the limit:*

$$\lim_{x\to 0}\frac{2\log(1+x)+x}{x-3\sin x}$$

The limit is in the form $\dfrac{0}{0}$, however it is possible to exploit the fundamental limits (*i*) and (*iv*) observing that we have:

$$\lim_{x\to 0}\frac{2\log(1+x)+x}{x-3\sin x}=\lim_{x\to 0}\frac{\dfrac{2\log(1+x)+x}{x}}{\dfrac{x-3\sin x}{x}}=\lim_{x\to 0}\frac{2\dfrac{\log(1+x)}{x}+1}{1-3\dfrac{\sin x}{x}}=$$
$$=\frac{2\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x}+1}{1-3\lim\limits_{x\to 0}\dfrac{\sin x}{x}}=\frac{2\cdot 1+1}{1-3\cdot 1}=-\frac{3}{2}$$

Example 96 *Calculate the limit:*

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{4x}$$

The limit is in the form $\dfrac{0}{0}$, however it is possible to exploit the fundamental limit (*v*) observing that we have:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{4x}=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1}{4x}=\frac{1}{4}\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1}{x}=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

Example 97 *Calculate the limit:*

$$\lim_{x\to+\infty}\left(\frac{x+1}{x+2}\right)^x$$

The limit has first of all, in the basis, the form $\frac{\infty}{\infty}$, however it is possible to exploit the fundamental limit (vi) observing that we have:

$$\lim_{x\to+\infty}\left(\frac{x+1}{x+2}\right)^x=\lim_{x\to+\infty}\left[\frac{x\left(1+\frac{1}{x}\right)}{x\left(1+\frac{2}{x}\right)}\right]^x=\lim_{x\to+\infty}\frac{\left(1+\frac{1}{x}\right)^x}{\left(1+\frac{2}{x}\right)^x}=$$

$$=\frac{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x}{\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^x}=\frac{e}{e^2}=\frac{1}{e}$$

The fundamental limits can be applied also in more general cases, when instead of x there is a function $f(x)$ that behaves in the same way of the variable x in the expressions examined above. In this case, through opportune changes of variable, it is possible to go back again to the fundamental limits.

Example 98 *Calculate the limit:*

$$\lim_{x\to 0^+}\frac{\sin(2\sqrt{x})}{\sin\sqrt{x}}$$

The limit is in the form $\frac{0}{0}$, furthermore it is not one of the fundamental limits presented above, however it is possible to consider first of all the following transformation (dividing numerator and denominator by $2\sqrt{x}$):

$$\lim_{x\to 0^+}\frac{\sin(2\sqrt{x})}{\sin\sqrt{x}}=\lim_{x\to 0^+}\frac{\frac{\sin(2\sqrt{x})}{2\sqrt{x}}}{\frac{\sin\sqrt{x}}{2\sqrt{x}}}=\frac{\lim_{x\to 0^+}\frac{\sin(2\sqrt{x})}{2\sqrt{x}}}{\lim_{x\to 0^+}\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

At this point the limit that appears in the numerator can be calculated considering the change of variable $2\sqrt{x}=t$, so that we get (observing that if $x\to 0^+$ also $t\to 0^+$):

$$\lim_{x\to 0^+}\frac{\sin(2\sqrt{x})}{2\sqrt{x}}=\lim_{t\to 0^+}\frac{\sin t}{t}=1$$

(computed exploiting the fundamental limit (i)), while the limit that appears in the denominator can be calculated considering the change of variable $\sqrt{x} = t$, so that we get:

$$\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{2\sqrt{x}} = \lim_{t\to 0^+} \frac{\sin t}{2t} = \frac{1}{2} \lim_{t\to 0^+} \frac{\sin t}{t} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

(computed exploiting again the fundamental limit (i)). In conclusion, the limit considered initially is:

$$\lim_{x\to 0^+} \frac{\sin(2\sqrt{x})}{\sin\sqrt{x}} = \frac{\displaystyle\lim_{x\to 0^+} \frac{\sin(2\sqrt{x})}{2\sqrt{x}}}{\displaystyle\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{2\sqrt{x}}} = \frac{1}{\frac{1}{2}} = 2$$

Example 99 *Calculate the limit:*

$$\lim_{x\to+\infty} \frac{\left(1-\dfrac{2}{x}\right)^x}{x\sin\dfrac{1}{x}}$$

First of all the limit can be written in the form:

$$\lim_{x\to+\infty} \frac{\left(1-\dfrac{2}{x}\right)^x}{x\sin\dfrac{1}{x}} = \frac{\displaystyle\lim_{x\to+\infty}\left(1-\frac{2}{x}\right)^x}{\displaystyle\lim_{x\to+\infty}\left(x\sin\frac{1}{x}\right)}$$

and then the limit in the numerator appears in the form 1^∞ while the limit in the denominator appears in the form $0\cdot\infty$. The limit in the numerator can then be written as:

$$\lim_{x\to+\infty}\left(1-\frac{2}{x}\right)^x = \lim_{x\to+\infty}\left(1+\frac{-2}{x}\right)^x = e^{-2}$$

(computed exploiting the fundamental limit (vi)), while the limit in the denominator can be calculated considering the change of variable $\dfrac{1}{x} = t$, so that we get (observing that if $x\to+\infty$ then $t\to 0^+$):

$$\lim_{x\to+\infty}\left(x\sin\frac{1}{x}\right) = \lim_{t\to 0^+}\left(\frac{1}{t}\sin t\right) = \lim_{t\to 0^+}\frac{\sin t}{t} = 1$$

(computed exploting the fundamental limit (i)). In conclusion, the limit considered initially is:

$$\lim_{x\to+\infty} \frac{\left(1-\dfrac{2}{x}\right)^x}{x\sin\dfrac{1}{x}} = \frac{\displaystyle\lim_{x\to+\infty}\left(1-\frac{2}{x}\right)^x}{\displaystyle\lim_{x\to+\infty}\left(x\sin\frac{1}{x}\right)} = \frac{e^{-2}}{1} = \frac{1}{e^2}$$

4.6 Calculation of limits: de l'Hospital's rule

A fourth method used for the resolution of the forms of indetermination is represented by the application of de l'Hospital's rule, that allows to solve indeterminate forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For the application of this rule it is necessary the use of the concept of derivative of a function, that will be introduced in the next Chapter; we refer therefore to such Chapter for the presentation of the rules used for the calculation of the derivative of a function.

With reference to de l'Hospital's rule, the following result holds:

If f and g are two functions defined and with derivative in a neighbourhood of the point x_0, infinitesimals or infinities as $x \to x_0$, and if $g'(x) \neq 0$ in the neighbourhood of x_0 (except at most the point x_0 itself*), then if there exists (finite or infinite) the limit:*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

we have:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

In practice, this rule allows to substitute to the calculation of the limit of the ratio of two functions the calculation of the limit of the ratio of their derivatives; if the latter limit exists, then it is also equal to the initial limit, while if it does not exist it is not possible to conclude that the initial limit does not exist (since that expressed by de l'Hospital's rule is a sufficient - but not necessary - condition for the existence of the limit considered). If also the limit of the ratio of the derivatives of the two functions appears in one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then, it becomes possible (if the hypotheses required for the application of this rule are satisfied) to apply again (eventually several times) de l'Hospital's rule, until the initial form of indetermination is solved.

Example 100 *Calculate the limit:*

$$\lim_{x \to 2} \frac{2-x}{x^2-4}$$

The limit (already solved in Example 85) is in the form $\frac{0}{0}$, however it is possible to apply de l'Hospital's rule obtaining:

$$\lim_{x \to 2} \frac{2-x}{x^2-4} \overset{H}{=} \lim_{x \to 2} \frac{D(2-x)}{D(x^2-4)} = \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}$$

that is the value of the limit (and is equal to the value obtained above using the algebraic manipulations).

Example 101 *Calculate the limit:*

$$\lim_{x\to 0} \frac{x^2 - x^2 e^x}{2x^3}$$

The limit (already solved in Example 93) is in the form $\frac{0}{0}$, however it is possible to apply de l'Hospital's rule obtaining:

$$\lim_{x\to 0} \frac{x^2 - x^2 e^x}{2x^3} \stackrel{H}{=} \lim_{x\to 0} \frac{D(x^2 - x^2 e^x)}{D(2x^3)} = \lim_{x\to 0} \frac{2x - x^2 e^x - 2x e^x}{6x^2}$$

Also this limit is in the form $\frac{0}{0}$, however it is possible to apply again de l'Hospital's rule obtaining:

$$\lim_{x\to 0} \frac{2x - x^2 e^x - 2x e^x}{6x^2} \stackrel{H}{=} \lim_{x\to 0} \frac{D(2x - x^2 e^x - 2x e^x)}{D(6x^2)} = \lim_{x\to 0} \frac{2 - x^2 e^x - 4x e^x - 2e^x}{12x}$$

Also this new limit is in the form $\frac{0}{0}$, and applying for the third time de l'Hospital's rule it is possible to solve the indeterminate form, in fact we get:

$$\lim_{x\to 0} \frac{2 - x^2 e^x - 4x e^x - 2e^x}{12x} \stackrel{H}{=} \lim_{x\to 0} \frac{D(2 - x^2 e^x - 4x e^x - 2e^x)}{D(12x)} =$$
$$\lim_{x\to 0} \frac{-x^2 e^x - 2x e^x - 4x e^x - 4e^x - 2e^x}{12} = -\frac{1}{2}$$

that is the value of the limit (and it is equal to the value obtained above using the fundamental limits).

Example 102 *Calculate the limit:*

$$\lim_{x\to +\infty} \frac{e^x - 1}{3x^3}$$

The limit is in the form $\frac{\infty}{\infty}$, applying de l'Hospital's rule for 3 times we get:

$$\lim_{x\to +\infty} \frac{e^x - 1}{3x^3} \stackrel{H}{=} \lim_{x\to +\infty} \frac{D(e^x - 1)}{D(3x^3)} = \lim_{x\to +\infty} \frac{e^x}{9x^2} \stackrel{H}{=} \lim_{x\to +\infty} \frac{D(e^x)}{D(9x^2)} = \lim_{x\to +\infty} \frac{e^x}{18x} \stackrel{H}{=}$$
$$\stackrel{H}{=} \lim_{x\to +\infty} \frac{D(e^x)}{D(18x)} = \lim_{x\to +\infty} \frac{e^x}{18} = +\infty$$

Example 103 *Calculate the limit:*

$$\lim_{x\to 0^+} x\log x$$

The limit is in the form $0\cdot\infty$, to apply de l'Hospital's rule first of all it is necessary to rewrite it as:

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{\dfrac{1}{x}}$$

that is in the form $\dfrac{\infty}{\infty}$, at this point we have:

$$\lim_{x\to 0^+} \frac{\log x}{\dfrac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{D(\log x)}{D\left(\dfrac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to 0^+} -x = 0^-$$

Example 104 *Calculate the limit:*

$$\lim_{x\to -\infty} xe^x$$

The limit is in the form $0\cdot\infty$, to apply de l'Hospital's rule first of all it is necessary to rewrite it as:

$$\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{e^{-x}}$$

that is in the form $\dfrac{\infty}{\infty}$, at this point we have:

$$\lim_{x\to -\infty} \frac{x}{e^{-x}} \overset{H}{=} \lim_{x\to -\infty} \frac{D(x)}{D(e^{-x})} = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0^-$$

Example 105 *Calculate the limit:*

$$\lim_{x\to 0^+} x^x$$

The limit is in the exponential form 0^0, first of all it is therefore necessary to consider the transformation:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^{\lim\limits_{x\to 0^+} x\log x}$$

The limit in the exponent is in the form $0\cdot\infty$ and it has been computed above (it is equal to 0), therefore we have:

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

De l'Hospital's rule allows also to establish a hierarchy among infinities, comparing the families of exponential, power and logarithmic functions (that, as $x \to +\infty$, are infinity functions). With reference to this aspect, the following result holds:

Each exponential infinity is of higher order than each power infinity, that is ($\forall \alpha > 1$, $\beta > 0$):

$$\lim_{x \to +\infty} \frac{\alpha^x}{x^\beta} = +\infty$$

Each power infinity, then, is of higher order than each logarithmic infinity, that is ($\forall \beta > 0$, $\gamma > 0$):

$$\lim_{x \to +\infty} \frac{x^\beta}{(\log x)^\gamma} = +\infty$$

Example 106 *Calculate the limit:*

$$\lim_{x \to +\infty} \frac{2^x}{x^3}$$

The limit is in the form $\dfrac{\infty}{\infty}$, applying the result stated above we have that 2^x is an infinity of order higher than x^3 and therefore we have:

$$\lim_{x \to +\infty} \frac{2^x}{x^3} = +\infty$$

Example 107 *Calculate the limit:*

$$\lim_{x \to +\infty} \frac{(\log x)^3}{3^x}$$

The limit is in the form $\dfrac{\infty}{\infty}$, applying the result stated above we have that 3^x is an infinity of order higher than $(\log x)^3$ and therefore we have:

$$\lim_{x \to +\infty} \frac{(\log x)^3}{3^x} = \lim_{x \to +\infty} \frac{1}{\frac{3^x}{(\log x)^3}} = \frac{1}{\lim\limits_{x \to +\infty} \frac{3^x}{(\log x)^3}} = 0^+$$

4.7 Calculation of limits: Taylor-Mac Laurin's formula

A fifth method used for the resolution of the forms of indetermination (in particular those of the type $\frac{0}{0}$) is represented by the application of Taylor-Mac Laurin's formula. Such formula will be introduced extensively in the next Chapter (to which we refer for a complete presentation), while here it is possible to illustrate its use in the calculation of some limits, that generate forms of indetermination.

In general, Taylor's formula centered at x_0 up to the order n (with Peano's remainder) allows to approximate a function, in a neighbourhood of the point x_0, through a polynomial of degree n, with an error that turns out to be an infinitesimal of order higher than $(x-x_0)^n$ (i.e. an error that tends to 0 more rapidly than $(x-x_0)^n$); if $x_0 = 0$, then, the formula is called Mac Laurin's formula.

With reference to this aspect, for the calculation of certain limits are of fundamental importance the following Mac Laurin's expansions of some elementary functions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + ... + \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} x^n + o(x^n) \quad \text{with } \alpha \in \mathbb{R}$$

In all these expansions we have the symbol o ("little-o"); with reference to such symbol it is necessary to observe that the expression:

$$f = o(g) \qquad \text{as } x \to x_0$$

(that reads "f is little-o with respect to g as x tends to x_0") is equivalent to:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

and in this case it is also possible to say that f is negligible with respect to g, as $x \to x_0$ (since the idea is that, as $x \to x_0$, f tends to 0 more rapidly than g). The symbol o, furthermore, satisfies the following properties:

$$o(c \cdot g) = o(g) \qquad \forall c \in \mathbb{R},\, c \neq 0$$

$$f \cdot o(g) = o(f \cdot g)$$

$$o(g) \pm o(g) = o(g)$$

that turn out to be particularly useful in the use of Taylor-Mac Laurin's formula for the calculation of certain limits (in particular, the last property indicates that the algebraic sum of two quantities that are negligible with respect to a function g is still a quantity negligible with respect to this function, and it is not possible a simplification of the type $o(g) - o(g) = 0$).

The formulae introduced above can be applied in the calculation of certain limits for the resolution of the forms of indetermination, since they allow to substitute the functions involved with their Taylor-Mac Laurin's expansions. In these calculations, the fundamental problem is represented by the order of the expansion at which it is appropriate to stop. With reference to this aspect, it is necessary to observe that the only rule to follow is the one according to which it is necessary to stop when it becomes possible to eliminate the form of indetermination, since if we stop too early the form of indetermination remains, while if we continue in the expansion introducing terms in excess with respect to those that allow to solve the form of indetermination we make a useless effort (since these terms, being infinitesimals of higher order, will then be omitted in the calculation).

Example 108 *Calculate the limit:*

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2}$$

The limit is in the form $\frac{0}{0}$, applying Mac Laurin's expansion to the function e^x and stopping to the first order we get:

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 + x - 1 - x + o(x)}{x^2} = \lim_{x \to 0} \frac{o(x)}{x^2}$$

and the result of this limit is not known (as $o(x)$ is an infinitesimal of higher order with respect to x as $x \to 0$, so that we would have $\lim_{x \to 0} \frac{o(x)}{x} = 0$, but nothing can be said about its behaviour with respect to x^2). Applying Mac Laurin's expansion up to the second order, on the contrary, we get (neglecting the infinitesimals of higher order, that are incorporated in the term $o(x^2)$):

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 + x - 1 - x - \frac{x^2}{2!} + o(x^2)}{x^2} = \lim_{x \to 0} \frac{-\frac{1}{2}x^2}{x^2} = -\frac{1}{2}$$

and therefore we solve the form of indetermination. The same result can be obtained applying Mac Laurin's expansion up to the third order, so that we have:

$$\lim_{x\to 0}\frac{1+x-e^x}{x^2}=\lim_{x\to 0}\frac{1+x-1-x-\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+o(x^3)}{x^2}=\lim_{x\to 0}\frac{-\dfrac{1}{2}x^2}{x^2}=-\frac{1}{2}$$

from which it turns out that if we continue the expansion besides the order that allows to solve the form of indetermination we make a useless effort (since the terms of higher order, being infinitesimals of higher order, are neglected and therefore they do not contribute to the calculation of the limit).

Example 109 *Calculate the limit:*

$$\lim_{x\to 0}\frac{x^2\left[(1+x)^2-1\right]}{x-\sin x}$$

The limit is in the form $\dfrac{0}{0}$, applying Mac Laurin's expansion to the functions $(1+x)^\alpha$ (with $\alpha=2$) and $\sin x$ we get (stopping the expansion of $(1+x)^2$ to the first order and that of $\sin x$ to the third order and exploiting the properties of the symbol o seen above):

$$\lim_{x\to 0}\frac{x^2\left[(1+x)^2-1\right]}{x-\sin x}=\lim_{x\to 0}\frac{x^2\left[1+2x+o(x)-1\right]}{x-x+\dfrac{x^3}{3!}+o(x^3)}=\lim_{x\to 0}\frac{2x^3+o(x^3)}{\dfrac{1}{6}x^3+o(x^3)}=$$

$$=\lim_{x\to 0}\frac{2x^3}{\dfrac{1}{6}x^3}=12$$

In an analogous way to what has been seen with reference to the fundamental limits, Taylor-Mac Laurin's expansions can be applied also to more general functions than those considered before, obtained through compositions of infinitesimal functions. In this case, using appropriate changes of variable, it becomes possible to go back to the fundamental Taylor-Mac Laurin's expansions.

Example 110 *Calculate the limit:*

$$\lim_{x\to 0}\frac{\sqrt[3]{x}-\sin\sqrt[3]{x}}{x}$$

The limit is in the form $\dfrac{0}{0}$, in this case before applying Mac Laurin's expansion it is convenient to make the change of variable $\sqrt[3]{x}=t$, so that we get (observing that if $x\to 0$ also $t\to 0$):

$$\lim_{x\to 0}\frac{\sqrt[3]{x}-\sin\sqrt[3]{x}}{x}=\lim_{t\to 0}\frac{t-\sin t}{t^3}$$

and then, applying Mac Laurin's expansion to $\sin t$ (up to the third order):

$$\lim_{t\to 0}\frac{t-\sin t}{t^3}=\lim_{t\to 0}\frac{t-t+\dfrac{t^3}{3!}+o(t^3)}{t^3}=\lim_{t\to 0}\frac{\frac{1}{6}t^3}{t^3}=\frac{1}{6}$$

Example 111 *Calculate the limit:*

$$\lim_{x\to 0}\frac{e^{x-\sin x}-1}{x^2\sin x}$$

The limit is in the form $\dfrac{0}{0}$, applying first of all Mac Laurin's expansion to the function $\sin x$ (up to the third order at the numerator and up to the first order at the denominator) we get:

$$\lim_{x\to 0}\frac{e^{x-\sin x}-1}{x^2\sin x}=\lim_{x\to 0}\frac{e^{x-x+\frac{x^3}{3!}+o(x^3)}-1}{x^2(x+o(x))}=\lim_{x\to 0}\frac{e^{\frac{1}{6}x^3}-1}{x^3+o(x^3)}$$

At this point it is possible to make the change of variable $\dfrac{1}{6}x^3=t$ (observing that if $x\to 0$ also $t\to 0$) and then to apply Mac Laurin's expansion to the exponential function, obtaining:

$$\lim_{x\to 0}\frac{e^{\frac{1}{6}x^3}-1}{x^3+o(x^3)}=\lim_{t\to 0}\frac{e^t-1}{6t+o(t)}=\lim_{t\to 0}\frac{1+t+o(t)-1}{6t+o(t)}=\lim_{t\to 0}\frac{t}{6t}=\frac{1}{6}$$

4.8 Asymptotes

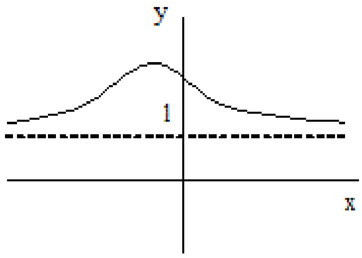
Another topic linked to the calculation of limits is represented by the study of the asymptotes of a function. Indeed, given a function $f: X\subseteq\mathbb{R}\to\mathbb{R}$, the calculation of the limits in correspondence of the extremes of the domain allows to determine the presence of eventual asymptotes. For this reason, proceeding in the study of a function, after the analysis of the domain, of the intersections with the axes, of the sign and of eventual symmetries (as seen in the previous Chapter) it is possible to consider the calculation of the limits at the boundaries of the domain.

With reference to the study of the asymptotes we have the following results:

- If it holds:

$$\lim_{x \to \mp\infty} f(x) = l \quad \text{with } l \in \mathbb{R}$$

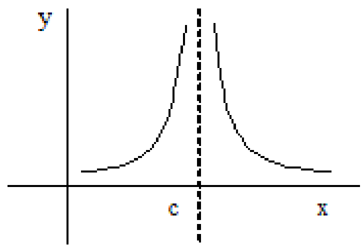
  then $f(x)$ has horizontal asymptote given by the straight line of equation $y = l$. Graphically, we have a situation of this type:



- If it holds:

$$\lim_{x \to c^{\mp}} f(x) = \pm\infty \quad \text{with } c \text{ accumulation point for } X$$

  then $f(x)$ has vertical asymptote given by the straight line of equation $x = c$. Graphically we have a situation of this type:

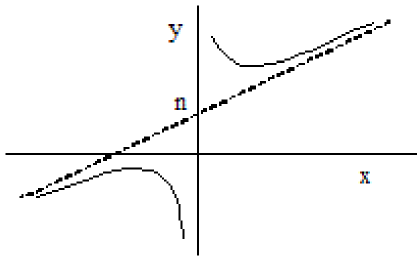


- If it holds:

$$\lim_{x \to \mp\infty} [f(x) - mx - n] = 0$$

  then $f(x)$ has oblique asymptote given by the straight line of equation

$y = mx + n$. Graphically we have a situation of this type:



and this last case is equivalent to the existence of the following limits:

$$\lim_{x \to \mp\infty} f(x) = \pm\infty$$

$$\lim_{x \to \mp\infty} \frac{f(x)}{x} = m \quad \text{(with } m \text{ finite and } \neq 0\text{)}$$

$$\lim_{x \to \mp\infty} [f(x) - mx] = n \quad \text{(with } n \text{ finite)}$$

If the limits listed above hold only as $x \to -\infty$ or as $x \to c^-$ we have a left (horizontal, vertical, oblique) asymptote, if they hold only as $x \to +\infty$ or as $x \to c^+$ we have a right (horizontal, vertical, oblique) asymptote. It is also possible to observe that the horizontal asymptote and the oblique asymptote exclude each other (therefore they cannot be present at the same time) , while the vertical asymptote is compatible with both the horizontal one and the oblique one.

Example 112 *Find the eventual asymptotes of the function:*

$$f(x) = \frac{3}{x-2}$$

First of all we have that the domain of the function is given by:

$$D = (-\infty, 2) \cup (2, +\infty)$$

so that the limits to calculate are those as $x \to \mp\infty$ and as $x \to 2$ (in fact the limits of a function must be calculated in correspondence of the boundaries of its domain, so that the study of the latter is important also to understand which are the limits to be computed). We then have:

$$\lim_{x \to \mp\infty} \frac{3}{x-2} = 0$$

$$\lim_{x \to 2^\mp} \frac{3}{x-2} = \mp\infty$$

from which we deduce that $y = 0$ is an horizontal asymptote (as $x \to -\infty$ and also as $x \to +\infty$), while $x = 2$ is a vertical asymptote.

Example 113 *Find the eventual asymptotes of the function:*

$$f(x) = \frac{x^3 + 5x^2 + 1}{x^2 - 9}$$

First of all we have that the domain of the function is given by:

$$D = (-\infty, -3) \cup (-3, 3) \cup (3, +\infty)$$

so that the limits to calculate are those as $x \to \mp\infty$ and as $x \to \mp 3$. With reference to this aspect we have:

$$\lim_{x \to -\infty} \frac{x^3 + 5x^2 + 1}{x^2 - 9} = \lim_{x \to -\infty} \frac{x^3}{x^2} = \lim_{x \to -\infty} x = -\infty$$

$$\lim_{x \to -3^{\mp}} \frac{x^3 + 5x^2 + 1}{x^2 - 9} = \pm\infty$$

$$\lim_{x \to 3^{\mp}} \frac{x^3 + 5x^2 + 1}{x^2 - 9} = \mp\infty$$

$$\lim_{x \to +\infty} \frac{x^3 + 5x^2 + 1}{x^2 - 9} = \lim_{x \to +\infty} \frac{x^3}{x^2} = \lim_{x \to +\infty} x = +\infty$$

We can therefore conclude that the straight lines $x = -3$ and $x = 3$ are two vertical asymptotes, while there are no horizontal asymptotes, and to verify the presence of oblique asymptotes we consider:

$$\lim_{x \to \mp\infty} \frac{f(x)}{x} = \lim_{x \to \mp\infty} \frac{x^3 + 5x^2 + 1}{x^3 - 9x} = \lim_{x \to \mp\infty} \frac{x^3}{x^3} = 1 = m$$

and then:

$$\begin{aligned} \lim_{x \to \mp\infty} [f(x) - mx] &= \lim_{x \to \mp\infty} \left[\frac{x^3 + 5x^2 + 1}{x^2 - 9} - x \right] = \\ &= \lim_{x \to \mp\infty} \frac{5x^2 + 9x + 1}{x^2 - 9} = \lim_{x \to \mp\infty} \frac{5x^2}{x^2} = 5 = n \end{aligned}$$

so that the straight line $y = x + 5$ is an oblique asymptote as $x \to \mp\infty$.

4.9 Continuous functions

A last use of the limits is that linked to the notion of continuity of a function. With reference to this aspect, a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in X$, accumulation point for X, if it holds:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

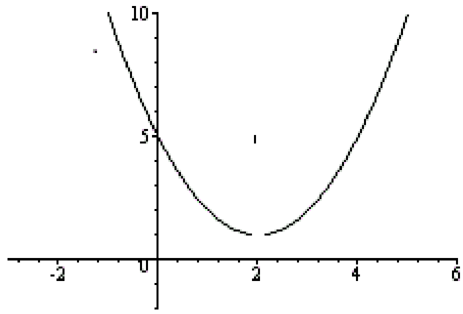
that is the left-hand limit (as $x \to x_0$) of $f(x)$ is finite and equal to the right-hand limit, and both are equal to the value assumed by the function in the point x_0.

If f is not continuous it has a discontinuity in x_0, that can be of 3 types:

- eliminable discontinuity, if it holds:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) \neq f(x_0)$$

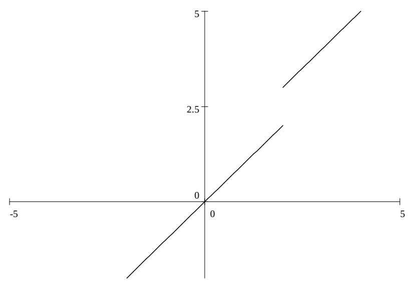
  that is the left-hand limit and the right-hand limit (as $x \to x_0$) of $f(x)$ are finite and equal, but they are different from the value assumed by the function in x_0. Graphically we have a situation of this type:



- discontinuity of the first kind, if it holds:

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) \qquad \text{and both are finite}$$

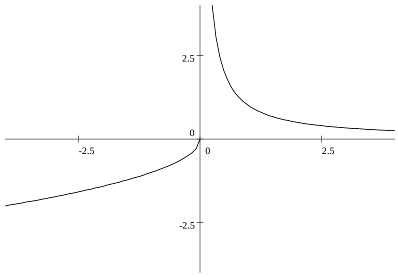
  that is the left-hand limit and the right-hand limit (as $x \to x_0$) of $f(x)$ exist finite but they are different from each other. In this case we talk also of "jump", and graphically we have a situation of this type:



- discontinuity of the second kind, if it holds:

  al least one of the two limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ does not exist or it is $\pm\infty$

  Graphically in this case we have a situation of this type:



If only one of the two limits $\lim_{x \to x_0^-} f(x)$ or $\lim_{x \to x_0^+} f(x)$ is equal to the value of the function in x_0, $f(x_0)$, then, we say that f is continuous respectively from the left or from the right at x_0.

In practice, with reference to the elementary functions (and to the functions obtained from them through the usual algebraic operations, the operation of composition and the calculation of the inverse) we know that they are continuous on their domain. The problem of existence of eventual discontinuities can arise in the case of piecewise functions, with reference to the points in correspondence of which the analytical expression of the function changes. In these points it is therefore necessary to study explicitely the continuity, applying the definition reported above.

Example 114 *Discuss the continuity of the following function:*

$$f(x) = \begin{cases} 2x(x+5) & \textit{if } x \leq 0 \\ \\ \log(1+\sqrt{3x}) & \textit{if } x > 0 \end{cases}$$

First of all we have that $f(x)$ is continuous $\forall x \neq 0$ (since it is defined through elementary functions, that are continuous on their domain), to verify the continuity also at $x = 0$ it is necessary to consider:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} [2x(x+5)] = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \log(1+\sqrt{3x}) = 0$$

$$f(0) = 0$$

and since we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

then we can conclude that the function is continuous also at $x = 0$.

It is possible to observe that, in the computation of the left-hand limit and of the right-hand limit relative to the point x_0 (in this case $x_0 = 0$), it is necessary to pay attention to which is the correct expression of the function that must be used. In the example considered, for the calculation of the left-hand limit it is necessary to use $2x(x+5)$ because this is the expression of $f(x)$ for $x < 0$, while for the calculation of the right-hand limit it is necessary to use $\log(1+\sqrt{3x})$ because this is the expression of $f(x)$ for $x > 0$.

Example 115 *Discuss the continuity of the following function:*

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x \leq 0 \\ \dfrac{2x}{4-x} + 3 & \text{if } 0 < x < 4 \\ e^x + 1 & \text{if } x \geq 4 \end{cases}$$

First of all we have that $f(x)$ is continuous for $x \neq 0$ and $x \neq 4$. To verify the continuity at $x = 0$ it is necessary to consider:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 + 3\right) = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{2x}{4-x} + 3\right) = 3$$

$$f(0) = 3$$

and since we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

then we can conclude that the function is continuous also at $x = 0$. To verify the continuity at $x = 4$, then, it is necessary to consider:

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \left(\frac{2x}{4-x} + 3\right) = +\infty$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (e^x + 1) = e^4 + 1$$

$$f(4) = e^4 + 1$$

from which we deduce that $f(x)$ has a discontinuity of the second kind at $x = 4$ (where it is continuous only from the right since we have $\lim_{x \to 4^+} f(x) = f(4)$).

Example 116 *Discuss the continuity of the following function:*

$$f(x) = \begin{cases} 3\alpha + 2x & \text{if } x < 1 \\ \\ 3x^2 + 5 & \text{if } x \geq 1 \end{cases} \qquad \text{with } \alpha \in \mathbb{R}$$

First of all we have that $f(x)$ is continuous $\forall x \neq 1$, to verify the continuity also at $x = 1$ it is necessary to consider:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3\alpha + 2x) = 3\alpha + 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(3x^2 + 5\right) = 8$$

$$f(1) = 8$$

and for f to be continuous also at $x = 1$ we must have:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

that is:

$$3\alpha + 2 = 8 \Rightarrow \alpha = 2$$

so that we can conclude that the function is continuous also at $x = 1$ if $\alpha = 2$.

Example 117 *Discuss the continuity of the following function:*

$$f(x) = \begin{cases} \alpha e^{x-1} & \text{if } x < 1 \\ \\ \alpha \log x + \beta & \text{if } x \geq 1 \end{cases} \qquad \text{with } \alpha, \beta \in \mathbb{R}$$

First of all we have that $f(x)$ is continuous $\forall x \neq 1$, to verify the continuity also at $x = 1$ it is necessary to consider:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \alpha e^{x-1} = \alpha$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (\alpha \log x + \beta) = \beta$$

$$f(1) = \beta$$

and for f to be continuous also at $x = 1$ we must have:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

that is:

$$\alpha = \beta$$

so that we can conclude that the function is continuous also at $x = 1$ if $\alpha = \beta$.

4.10 Exercises

Calculate the following limits:

1) $\lim\limits_{x\to 1} \dfrac{1-x}{1-x^2}$

2) $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

3) $\lim\limits_{x\to +\infty} \left(\sqrt{x+1}-\sqrt{x-1}\right)$

4) $\lim\limits_{x\to +\infty} \left(\sqrt{x^2+3}-\sqrt{x^2-3}\right)$

5) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$

6) $\lim\limits_{x\to +\infty} \dfrac{\sqrt{x^2+2}+\sqrt{x^2+3}}{\sqrt[3]{x^3+2x}}$

7) $\lim\limits_{x\to +\infty} \dfrac{x^3+x^2+5x}{x^3+2x+3}$

8) $\lim\limits_{x\to +\infty} \dfrac{x^5-x^4+2}{x^3-x^2+5}$

9) $\lim\limits_{x\to +\infty} \dfrac{-2x^2-5x}{x^3+6x^2}$

10) $\lim\limits_{x\to +\infty} \dfrac{\sqrt{x^2-3x}}{\sqrt{4x^2+5x}}$

11) $\lim\limits_{x\to +\infty} \dfrac{\sqrt{4x^2+x+3}}{\sqrt{x^2+2}}$

12) $\lim\limits_{x\to +\infty} \dfrac{2^x+\log x+1}{3^x+x^2+3}$

13) $\lim\limits_{x\to +\infty} \left(\dfrac{x+2}{x+1}\right)^x$

14) $\lim\limits_{x\to +\infty} \left(\dfrac{x+2}{x+3}\right)^x$

15) $\lim\limits_{x\to 0^+} \dfrac{\sin(3\sqrt{x})}{\sin\sqrt{x}}$

16) $\lim_{x\to 0} \dfrac{(e^x - 1)\, x}{\sin x}$

17) $\lim_{x\to 1} \dfrac{(x+1)\log x}{x-1}$

18) $\lim_{x\to 0^+} x\left(1+\sqrt{\dfrac{1}{x}}\right)$

19) $\lim_{x\to +\infty} \dfrac{2x\log x}{x+\log x}$

20) $\lim_{x\to 0} \dfrac{e^x - 1 + x}{x}$

21) $\lim_{x\to 1} \dfrac{x-1+\log x}{x-1-2\sin(x-1)}$

22) $\lim_{x\to 0} \dfrac{3(1-\cos x)}{\sin x - \log(1+x)}$

23) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$

24) $\lim_{x\to 0} \dfrac{\sin^2 x - x^2}{x^2}$

25) $\lim_{x\to +\infty} \dfrac{\sin x}{x}$

Determine the eventual asymptotes (horizontal, vertical, oblique) of the following functions:

26) $f(x) = \dfrac{2x+3}{x-1}$

27) $f(x) = \dfrac{x+5}{x-1}$

28) $f(x) = \dfrac{x+3}{x-3}$

29) $f(x) = \dfrac{e^{2x}}{x-1}$

30) $f(x) = \dfrac{e^{3x}}{x-2}$

31) $f(x) = \dfrac{x^2 + 5x}{x + 3}$

32) $f(x) = \sqrt{x^2 - 2} - x$

33) $f(x) = xe^x$

34) $f(x) = xe^{-\frac{2}{x}}$

35) $f(x) = e^{\frac{3-x}{1-x}}$

36) $f(x) = e^{\frac{2-x}{1-x}}$

37) $f(x) = \dfrac{x^2 + 2x + 5}{x + 3}$

38) $f(x) = \dfrac{x^2 + 3x + 4}{x + 1}$

39) $f(x) = \dfrac{x^3 + 2x^2 + 1}{x^2 - 1}$

40) $f(x) = \log \dfrac{x^2}{(x+2)^2}$

Discuss the continuity of the following functions on their domain:

41) $f(x) = \begin{cases} \dfrac{e}{2}\left(3 - x^2\right) & \text{if } x \leq 1 \\ e^{\frac{1}{x}} + \alpha & \text{if } x > 1 \end{cases}$ with $\alpha \in \mathbb{R}$

42) $f(x) = \begin{cases} e^{x-1} + \alpha & \text{if } x \leq 1 \\ \log x + 1 + \beta & \text{if } x > 1 \end{cases}$ with $\alpha, \beta \in \mathbb{R}$

43) $f(x) = \begin{cases} \alpha x + \beta & \text{if } x \leq 1 \\ \alpha x^2 + 1 & \text{if } x > 1 \end{cases}$ with $\alpha, \beta \in \mathbb{R}$

44) $f(x) = \begin{cases} \alpha x + 5 & \text{if } x < 0 \\ x^2 + 5 & \text{if } x \geq 0 \end{cases}$ with $\alpha \in \mathbb{R}$

45) $f(x) = \begin{cases} e^{x-1} + \alpha(x-1) & \text{if } x < 1 \\ (x-1)^2 - 3(x-1) & \text{if } x \geq 1 \end{cases}$ with $\alpha \in \mathbb{R}$

46) $f(x) = \begin{cases} \left(x^2+1\right)^2 + \alpha & \text{if } x < -1 \\ e^{x^2-1} & \text{if } x \geq -1 \end{cases}$ with $\alpha \in \mathbb{R}$

47) $f(x) = \begin{cases} \dfrac{e^{x-1}-1}{x-1} & \text{if } x \neq 1 \\ x^{\alpha} & \text{if } x = 1 \end{cases}$ with $\alpha \in \mathbb{R}$

48) $f(x) = \begin{cases} \alpha x \sin \frac{1}{x} - \alpha & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$ with $\alpha \in \mathbb{R}$

49) $f(x) = \begin{cases} e^x + \alpha x & \text{if } x < 0 \\ x^2 - 2x & \text{if } x \geq 0 \end{cases}$ with $\alpha \in \mathbb{R}$

50) $f(x) = \begin{cases} (x+\alpha)^2 - 1 & \text{if } x < 0 \\ \log(1+x) & \text{if } x \geq 0 \end{cases}$ with $\alpha \in \mathbb{R}$

Chapter 5

Differential calculus

5.1 Definitions and rules of differentiation

A concept of great importance for the development of the differential calculus is that of derivative. With reference to this aspect, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ and a point x_0 interior with respect to X, the derivative of f at x_0 (denoted with $f'(x_0)$) is defined as the limit of the difference quotient of f obtained starting from the point x_0, provided this limit exists finite:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

With a different notation, the same derivative can be expressed as:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

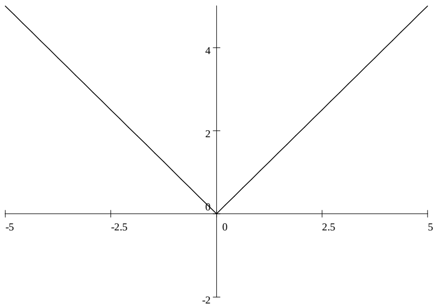
In a similar way it is possible to define the left derivative and the right derivative of f at x_0 (denoted with $f'_-(x_0)$ and $f'_+(x_0)$) as the left-hand limit and the right-hand limit, respectively, of the difference quotient of f obtained starting from the point x_0, provided this limit exists finite. The (complete) derivative of f at x_0 is then the common value of the left derivative and of the right derivative, that is we have $f'(x_0) = f'_-(x_0) = f'_+(x_0)$.

If, on the contrary, in the point x_0 there exist the left derivative and the right derivative, but they are not equal (or they are not finite), we say that in x_0 the function f has:

- a corner if:

$$f'_-(x_0) \neq f'_+(x_0)$$

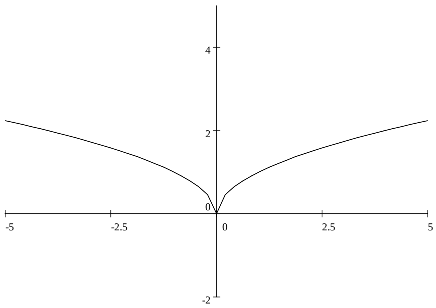
  and graphically we have a situation of this type:



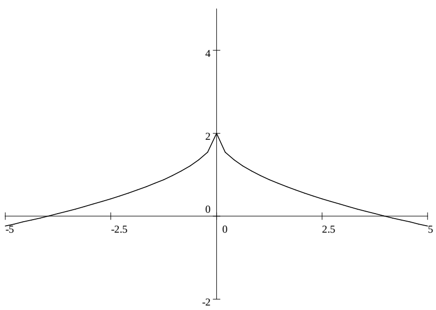
- a peak if:

  $f'_-(x_0)$ and $f'_+(x_0)$ are infinite with different sign (one is $+\infty$ and the other is $-\infty$)

  and graphically we have a situation of this type:



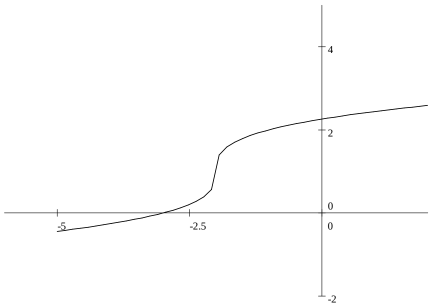
  or of this type:



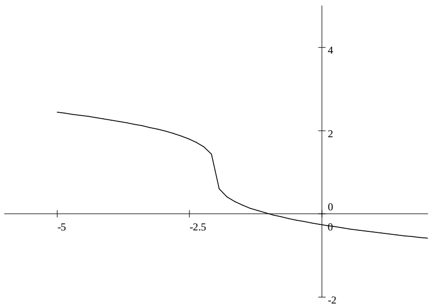
- a flex with vertical tangent if:

$$f'_-(x_0) \text{ and } f'_+(x_0) \text{ are infinite with the same sign (both } +\infty \text{ or } -\infty)$$

  and graphically we have a situation of this type:



  or of this type:



If f is a function defined in a certain interval and it has a derivative at each point x interior with respect to this interval, then, it is possible to associate at each x the derivative of f at that point, $f'(x)$, obtaining a function f' that is called first derivative of f.

In conclusion, to compute the derivative of a function resorting to the definition it is necessary to proceed in two steps: first we calculate the difference quotient and then we compute the limit of this ratio as $h \to 0$. In practice, however, for the calculation of the derivatives we don't resort to the definition (similarly to what happens for the calculation of limits) but we use a series of rules that allow to reduce the calculation of the derivative of a generic function to that of the derivatives of the elementary functions (that are known).

The starting point for the calculation of the derivatives is therefore represented by the derivatives of the elementary functions, that are summarized in the following table:

Primitive function $f(x)$	Derivative function $f'(x)$
c	0
x^α	$\alpha x^{\alpha-1}$
a^x	$a^x \log a \quad \text{with } a > 0$
e^x	e^x
$\log_a \lvert x \rvert$	$\dfrac{1}{x \log a} \quad \text{with } x \neq 0, a > 0$
$\log \lvert x \rvert$	$\dfrac{1}{x} \quad \text{with } x \neq 0$
$\log \lvert f(x) \rvert$	$\dfrac{f'(x)}{f(x)} \quad \text{with } f(x) \neq 0$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
tgx	$\dfrac{1}{\cos^2 x} = 1 + tg^2 x$

It is then possible to introduce a series of rules of derivation. First of all, if f and g are two functions that have a derivative at a generic point x, then also their sum, their product and their quotient (the latter if $g(x) \neq 0$) have a derivative at x, and the following rules hold (where D denotes the derivative of a function):

$$(i) \quad D\,[f(x) + g(x)] = D\,[f(x)] + D\,[g(x)]$$

$$(ii) \quad D\,[f(x) \cdot g(x)] = D\,[f(x)] \cdot g(x) + f(x) \cdot D\,[g(x)]$$

in particular if c is a constant we have $D\,[c \cdot f(x)] = c \cdot D\,[f(x)]$

$$(iii) \quad D\left[\frac{f(x)}{g(x)}\right] = \frac{D\,[f(x)] \cdot g(x) - f(x) \cdot D\,[g(x)]}{[g(x)]^2} \quad \text{with } g(x) \neq 0$$

Rules (i) and (ii), then, extend to the case of any number $n > 2$ of functions. Together with these rules, that concern the derivative of a sum, of a product and of a ratio of functions, we also have the following rules, that concern the derivative of the composite function and the derivative of the inverse function:

(iv) Given the functions $y = f(t)$ and $t = g(x)$ such that it is possible to consider the composite function $y = f \circ g = f(g(x))$, if g has a derivative at x and f has a derivative at $t = g(x)$, then $f \circ g$ has a derivative at x and we have:

$$D[f(g(x))] = D[f(t)]_{t=g(x)} \cdot D[g(x)]$$

(v) Given a function f continuous and strictly monotonic, if f has a derivative at x_0 with $D[f(x)]_{x=x_0} \neq 0$, then the inverse function f^{-1} has a derivative at $y_0 = f(x_0)$ and we have:

$$D\left[f^{-1}(y)\right]_{y=y_0} = \frac{1}{D[f(x)]_{x=x_0}}$$

With reference to this latter rule, it is possible to observe that it allows to calculate the derivative in a point of the inverse of a certain function without having to calculate the inverse itself (that in certain cases cannot be computed explicitely). If the inverse can be easily calculated, however, it can be more convenient, for the calculation of the derivative, first of all to obtain the inverse function and then to derive it, evaluating such derivative in the point of interest.

Example 118 *Calculate the derivative of the function:*

$$f(x) = 2x \log x + x^3$$

Applying the rules (i) and (ii) seen above we have:

$$D\left[2x \log x + x^3\right] = D(2x \log x) + D(x^3) = D(2x) \cdot \log x + 2x \cdot D(\log x) + D(x^3) =$$
$$= 2 \log x + 2x \frac{1}{x} + 3x^2 = 2 \log x + 2 + 3x^2$$

and therefore:

$$f'(x) = 3x^2 + 2 \log x + 2$$

Example 119 *Calculate the derivative of the function:*

$$f(x) = \frac{e^x - 3}{e^x + 3}$$

Applying the rule (iii) seen above we have:

$$D\left[\frac{e^x - 3}{e^x + 3}\right] = \frac{D(e^x - 3) \cdot (e^x + 3) - (e^x - 3) \cdot D(e^x + 3)}{(e^x + 3)^2} =$$
$$= \frac{e^x(e^x + 3) - (e^x - 3)e^x}{(e^x + 3)^2} = \frac{e^x(e^x + 3 - e^x + 3)}{(e^x + 3)^2} = \frac{6e^x}{(e^x + 3)^2}$$

and therefore:

$$f'(x) = \frac{6e^x}{(e^x + 3)^2}$$

Example 120 *Calculate the derivative of the function:*

$$z(x) = \sqrt{\log x}$$

The function is obtained from the composition $f \circ g$ where f and g are given by:

$$f(t) = \sqrt{t} \qquad\qquad t = g(x) = \log x$$

and applying the rule (iv) seen above we have:

$$D\left[f(g(x))\right] = D\left[f(t)\right]_{t=g(x)} \cdot D\left[g(x)\right] = \left(\frac{1}{2\sqrt{t}}\right)_{t=\log x} \cdot \frac{1}{x} =$$
$$= \frac{1}{2\sqrt{\log x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\log x}}$$

and therefore:

$$z'(x) = \frac{1}{2x\sqrt{\log x}}$$

In practice, then, it is possible to avoid the intermediate passages observing that they are equivalent to derive the "exterior" function evaluating the derivative in correspondence of the "interior" function, and to multiply this result for the derivative of the "interior" function. In the example considered we have therefore directly:

$$z'(x) = \frac{1}{2\sqrt{\log x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\log x}}$$

where the term $\dfrac{1}{2\sqrt{\log x}}$ represents the derivative of the "exterior" function (that is $\sqrt{()}$) evaluated in correspondence of the "interior" function (that is $\log x$), while the term $\dfrac{1}{x}$ represents the derivative of the "interior" function (that is of $\log x$).

Example 121 *Calculate the derivative of the function:*

$$z(x) = e^{\sqrt{\sin x}}$$

The function is obtained from the composition $f \circ g \circ h$ where f, g and h are given by:

$$f(t) = e^t \qquad\qquad t = g(u) = \sqrt{u} \qquad\qquad u = h(x) = \sin x$$

so that applying the rule (iv) seen above (that extends in the case of more than two functions) we have:

$$D\left[f(g(h(x)))\right] = D\left[f(t)\right]_{t=g(h(x))} \cdot D\left[g(u)\right]_{u=h(x)} \cdot D\left[h(x)\right] =$$
$$= \left(e^t\right)_{t=\sqrt{\sin x}} \cdot \left(\frac{1}{2\sqrt{u}}\right)_{u=\sin x} \cdot \cos x =$$
$$= e^{\sqrt{\sin x}} \cdot \left(\frac{1}{2\sqrt{\sin x}}\right) \cdot \cos x = \frac{e^{\sqrt{\sin x}}\cos x}{2\sqrt{\sin x}}$$

and therefore:

$$z'(x) = \frac{e^{\sqrt{\sin x}} \cos x}{2\sqrt{\sin x}}$$

Also in this case it is possible to avoid the intermediate passages observing that the rule described is equivalent to calculate the derivative of each function that appears in the composition (starting from the more exterior), evaluating it in correspondence of the interior function (or of the interior functions, if there are several compositions), and to multiply these different derivatives. In the example considered we have therefore directly:

$$z'(x) = e^{\sqrt{\sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x = \frac{e^{\sqrt{\sin x}} \cos x}{2\sqrt{\sin x}}$$

where the term $e^{\sqrt{\sin x}}$ represents the derivative of the most exterior function (that is of $e^{()}$) evaluated in correspondence of the interior functions (that is of $\sqrt{\sin x}$), while the term $\dfrac{1}{2\sqrt{\sin x}}$ represents the derivative of the intermediate function (that is of $\sqrt{()}$) evaluated in correspondence of the most interior function (that is of $\sin x$), and the term $\cos x$ represents the derivative of the most interior function (that is of $\sin x$).

Example 122 *Calculate the derivative of the function:*

$$z(x) = x^x$$

In this case first of all it is necessary to rewrite the function using the transformation:

$$x^x = e^{\log x^x} = e^{x \log x}$$

and then the derivative can be calculated applying the rule (iv) seen above, observing that the function is obtained from the composition $f \circ g$ where f and g are given by:

$$f(t) = e^t \qquad t = g(x) = x \log x$$

so that we have:

$$\begin{aligned} D\left[f(g(x))\right] &= D\left[f(t)\right]_{t=g(x)} \cdot D\left[g(x)\right] = \left(e^t\right)_{t=x\log x} \cdot (1 + \log x) = \\ &= e^{x \log x}(1 + \log x) = x^x(1 + \log x) \end{aligned}$$

and therefore:

$$z'(x) = x^x(1 + \log x)$$

Example 123 *Given the function:*

$$f(x) = \sqrt{e^{x-3}}$$

calculate the derivative of the inverse function in the point $y_0 = f(x_0)$ with $x_0 = 2$ and in the point $y_0 = \sqrt{e}$.

Applying the rule (v) seen above we know that:

$$D\left[f^{-1}(y)\right]_{y=y_0} = \frac{1}{D\left[f(x)\right]_{x=x_0}}$$

In the case considered we have first of all that the derivative of the function $f(x)$ is given by:

$$f'(x) = \frac{1}{2\sqrt{e^{x-3}}} \cdot e^{x-3} = \frac{\sqrt{e^{x-3}}}{2}$$

At this point, to calculate the derivative of the inverse function in correspondence of $y_0 = f(x_0)$ with $x_0 = 2$ first of all it is necessary to observe that $y_0 = f(2) = \sqrt{e^{-1}} = \frac{1}{\sqrt{e}}$, then applying the formula seen above we get:

$$D\left[f^{-1}(y)\right]_{y=\frac{1}{\sqrt{e}}} = \frac{1}{D\left[f(x)\right]_{x=2}} \Rightarrow D\left[f^{-1}\left(\frac{1}{\sqrt{e}}\right)\right] = \frac{1}{\frac{\sqrt{e^{2-3}}}{2}} = 2\sqrt{e}$$

To calculate the derivative of the inverse function in correspondence of $y_0 = \sqrt{e}$, then, it is necessary to observe that $y_0 = \sqrt{e}$ corresponds to $x_0 = 4$ (in fact $f(4) = \sqrt{e}$), and applying again the formula seen above we get:

$$D\left[f^{-1}(y)\right]_{y=\sqrt{e}} = \frac{1}{D\left[f(x)\right]_{x=4}} \Rightarrow D\left[f^{-1}\left(\sqrt{e}\right)\right] = \frac{1}{\frac{\sqrt{e^{4-3}}}{2}} = \frac{2}{\sqrt{e}}$$

In the case considered the derivative of the inverse function could also be calculated obtaining first of all the inverse function itself and then deriving it. With reference to this aspect we have:

$$\begin{aligned} &f(x) = \sqrt{e^{x-3}} \Rightarrow y = \sqrt{e^{x-3}} \Rightarrow y^2 = e^{x-3} \Rightarrow \log y^2 = x - 3 \Rightarrow \\ &\quad \Rightarrow x = f^{-1}(y) = 3 + 2\log y \end{aligned}$$

and the derivative of this function is:

$$D\left[f^{-1}(y)\right] = \frac{2}{y}$$

In particular, then, this derivative evaluated in the point $y_0 = f(2) = \frac{1}{\sqrt{e}}$ is given by:

$$D\left[f^{-1}\left(\frac{1}{\sqrt{e}}\right)\right] = \frac{2}{\frac{1}{\sqrt{e}}} = 2\sqrt{e}$$

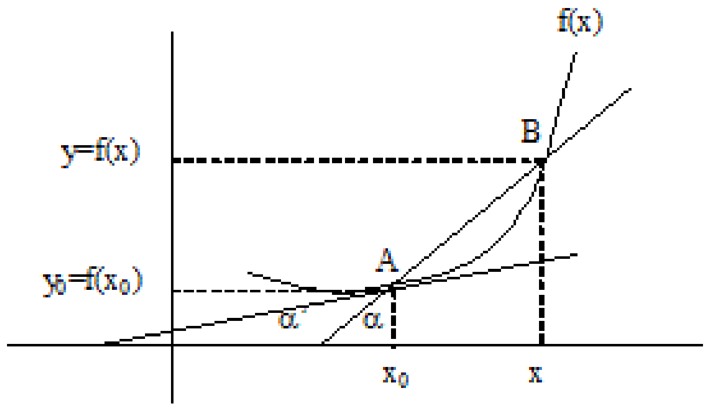
while the same derivative evaluated in the point $y_0 = \sqrt{e}$ is given by:

$$D\left[f^{-1}\left(\sqrt{e}\right)\right] = \frac{2}{\sqrt{e}}$$

and these are exactly the results obtained above exploiting the formula of the derivative of the inverse function.

5.2 Geometric interpretation of the derivative

Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ to provide a geometric interpretation of the derivative it is possible to consider the graph of the function, given for example by:



The equation of the straight line passing through the points $A = (x_0, y_0)$ and $B = (x, y)$ is:

$$y - y_0 = m(x - x_0)$$

where m is the angular coefficient of such straight line, that can therefore be expressed as:

$$m = \frac{y - y_0}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} = tg\alpha$$

that is the angular coefficient of the straight line turns out to be the trigonometric tangent of the corner α that the same straight line forms with the positive direction of the x-axis. When $x \to x_0$ the straight line passing through the points A and B assumes a limit position, constituted by the straight line tangent to the function in correspondence of the point $A = (x_0, y_0)$, and the angular coefficient of this tangent straight line is given by the limit of the angular coefficient of the straight line passing through the points A and B, that is:

$$m' = tg\alpha' = \lim_{x \to x_0} m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

We have therefore that the derivative of the function f in correspondence of the point x_0 represents the angular coefficient of the straight line tangent to the function in that point (that is it represents the trigonometric tangent of the corner α' that the tangent straight line forms with the positive direction of the x-axis) and indicates the slope of the function in x_0. The equation of the straight line tangent to the graph of f in the point $(x_0, f(x_0))$, finally, is:

$$y - f(x_0) = m'(x - x_0)$$

that is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

and also:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$, then, is differentiable at a point x_0 interior with respect to X if we can write:

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + R \qquad \text{for } x \to x_0$$

i.e. the increment of the function in correspondence of a small variation of the independent variable can be expressed as the sum of a linear term and of a negligible term. In particular, in this formula $f(x) - f(x_0)$ represents the increment of the dependent variable as a consequence of a variation of the independent variable, increment measured on the graph of the function f, while $f'(x_0)(x - x_0)$ represents the analogous increment measured on the straight line tangent to the function in x_0 and R is a negligible quantity (in particular we have $R = o(x - x_0)$ for $x \to x_0$). This formula can also be written as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R \qquad \text{for } x \to x_0$$

from which it turns out that, close to the point x_0, the function $f(x)$ can be expressed as the sum of the straight line tangent to the same function in the point x_0 and of a negligible quantity, i.e. the function $f(x)$ can be approximated, close to the point x_0, by the straight line tangent to the function itself in x_0, with an error that turns out to be negligible.

The term $f'(x_0)(x - x_0)$ is called differential of the function f in the point x_0, and in the case of infinitesimal increment of the independent variable x it is denoted by:

$$dy = f'(x_0)dx$$

This quantity represents the increment of the dependent variable (the y) when the independent variable (the x) varies of a small quantity, increment measured on the straight line tangent to the function in x_0 and, as seen above, for values of x close to x_0 such increment constitutes a good approximation of the variation of the y observed on the function f.

In the case of real functions of a real variable (that is $f : X \subseteq \mathbb{R} \to \mathbb{R}$) the notion of differentiability is equivalent to that of derivability (that is a function is differentiable in a point if and only if it has a derivative in that point), while in the case of real functions of several real variables (that is $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$) this equivalence is not anymore valid (in particular, the notion of differentiability implies that of derivability but the viceversa does not necessarily hold).

Example 124 *Determine the equation of the straight line tangent to the graph of the function:*

$$f(x) = x^3 e^{x-1}$$

in correspondence to the point $x_0 = 1$ and the differential of the function in correspondence to the same point.

The equation of the straight line tangent to the graph of f in correspondence to the point $(1, f(1))$ is given by:

$$y = f(1) + f'(1)(x-1)$$

so that we must consider:

$$f(x) = x^3 e^{x-1} \Rightarrow f(1) = 1$$

$$f'(x) = x^3 e^{x-1} + 3x^2 e^{x-1} = x^2 e^{x-1}(x+3) \Rightarrow f'(1) = 4$$

and then we get:

$$y = 1 + 4(x-1) \Rightarrow y = 4x - 3$$

that is the equation of the straight line.

The differential of the function then is:

$$dy = f'(x)dx$$

that is:

$$dy = \left[x^2 e^{x-1}(x+3)\right] dx$$

and in the point $x_0 = 1$ such differential is:

$$dy = 4dx$$

Example 125 *Determine the equation of the straight line tangent to the graph of the function:*

$$f(x) = \frac{x^3 - 3x + 4}{x - 4}$$

in correspondence to the point $x_0 = 2$ and the differential of the function in correspondence to the same point.

The equation of the straight line tangent to the graph of f in correspondence to the point $(2, f(2))$ is given by::

$$y = f(2) + f'(2)(x-2)$$

so that we must consider:

$$f(x) = \frac{x^3 - 3x + 4}{x - 4} \Rightarrow f(2) = -3$$

$$f'(x) = \frac{(x-4)(3x^2-3) - x^3 + 3x - 4}{(x-4)^2} = \frac{2x^3 - 12x^2 + 8}{(x-4)^2} \Rightarrow f'(2) = -6$$

and we get:

$$y = -3 - 6(x-2) \Rightarrow y = 9 - 6x$$

that is the equation of the straight line.

The differential of the function then is:

$$dy = f'(x)dx$$

that is:

$$dy = \left[\frac{2x^3 - 12x^2 + 8}{(x-4)^2}\right] dx$$

and in the point $x_0 = 2$ such differential is:

$$dy = -6dx$$

5.3 Derivability and continuity

The continuity and the derivability of a function represent important conditions of regularity that allow to obtain pieces of information on the behaviour of the same function. With reference to the link between these two notions, we have that if a function has a derivative at a point then it is also continuous at that point. The viceversa instead doesn't necessarily hold (that is, a function can be continuous at a point without having a derivative at that point), therefore we can deduce that the derivability represents a more restrictive condition than the continuity. As consequence of the first relationship, then, we have that if a function is not continuous at a point, then it has not a derivative at that point.

As shown in the previous Chapter with reference to the continuity, in general the elementary functions (and those obtained from them through the usual algebraic operations, the operation of composition and the calculation of the inverse) are continous and have a derivative on all their domain (except eventually in single points). The problem arises in the case of piecewise defined functions, relatively to the points in correspondence of which the analytical expression of the function changes, so that in these points it is necessary to study explicitely the continuity and the derivability of the functions considered. With reference to this aspect, a sufficient condition that can be used to verify the derivability of a function at a point (without having to resort to the definition) is the following:

If f is defined and continuous in a neighbourhood of x_0 (included x_0) and it has a derivative at each point $x \neq x_0$, and if there exists finite the limit of $f'(x)$ as $x \to x_0$, then f has a derivative also at x_0 and we have:

$$f'(x_0) = \lim_{x \to x_0^-} f'(x) = \lim_{x \to x_0^+} f'(x)$$

This result expresses the continuity, at the point x_0, of the first derivative, as the double equality reported above is exactly the definition of continuity at the point x_0 applied to the function $f'(x)$. As a consequence, the functions for which the derivability at x_0 can be checked using this result not only have a derivative but also have a continuous derivative at x_0 (we use in this case the notation $f \in \mathcal{C}^1(X)$ to denote the class of functions that have a continuous derivative at X). If the limits $\lim_{x \to x_0^\mp} f'(x)$ do not exist, however, it is not possible to conclude that $f'(x_0)$ does not exist, since that considered is a sufficient (but not necessary) condition for the derivability.

Example 126 *Discuss the continuity and the derivability of the following function:*

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

With reference to the continuity we have first of all that $f(x)$ is continuous $\forall x \neq 0$ (since it is defined through elementary functions that are continuous on their domain). To verify the continuity also at $x = 0$ then we must consider:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{-\frac{1}{x}} = 0$$

$$f(0) = 0$$

and since we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

then $f(x)$ is continuous also at $x = 0$.

With reference to the derivability, on the other hand, we have first of all that, for $x \neq 0$, the function has a derivative with:

$$f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{1}{x^2} e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

(it can be observed that, in the case of a piecewise defined function, passing from the expression of the function $f(x)$ to that of its derivative $f'(x)$ a point is "lost", that is the point in correspondence of which the analytical expression of the function changes - in this case it is the origin -). At this point, to verify if $f(x)$ has a derivative also at the origin (keeping in mind that in that point it is continuous, since if it were not it could not have a derivative) it is possible to apply the sufficient condition reported above, considering:

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 0 = 0$$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{1}{x^2}e^{-\frac{1}{x}} = 0$$

Since these two limits are equal, then, it is possible to conclude that $f(x)$ has a derivative also at $x = 0$, and furthermore:

$$f'(0) = \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$$

Example 127 *Discuss the continuity and the derivability of the following function:*

$$f(x) = \begin{cases} \alpha e^x & \textit{if} \quad x \leq 0 \\ \\ x^2 + 3x + \beta & \textit{if} \quad x > 0 \end{cases} \qquad \textit{with } \alpha, \beta \in \mathbb{R}$$

With reference to the continuity we have first of all that $f(x)$ is continuous $\forall x \neq 0$ (since it is defined through elementary functions that are continuous on their domain). To verify the continuity also at $x = 0$ we must then consider:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \alpha e^x = \alpha$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^2 + 3x + \beta\right) = \beta$$

$$f(0) = \alpha$$

and in order to have $f(x)$ continuous also at $x = 0$ we must have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

that is:

$$\alpha = \beta$$

With reference to the derivability, we have first of all that, for $x \neq 0$, the function has a derivative with:

$$f'(x) = \begin{cases} \alpha e^x & \text{if} \quad x < 0 \\ \\ 2x + 3 & \text{if} \quad x > 0 \end{cases}$$

For $f(x)$ to have a derivative also at $x = 0$ first of all the function must be continuous in $x = 0$, therefore it must be $\alpha = \beta$, furthermore since we have:

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \alpha e^x = \alpha$$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (2x + 3) = 3$$

for the sufficient condition of derivability it must be $\alpha = 3$. In conclusion, $f(x)$ has a derivative also at $x = 0$ if $\alpha = \beta = 3$ and in this case we have:

$$f'(0) = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x) = 3$$

5.4 Taylor-Mac Laurin's formula

As seen above, if a function f is differentiable at a point x_0, in a neighbourhood of this point the increment $f(x) - f(x_0)$ is well approximated by the differential $f'(x_0)(x - x_0)$, and as a consequence the function can be approximated by the first degree polynomial $f(x_0) + f'(x_0)(x - x_0)$, that is by the straight line tangent to the function in the point x_0. Sometimes, however, this linear approximation (also called "first-order approximation") is too broad and does not give sufficient information, so that it is possible to try to approximate the graph of the function f, close to the point x_0, with a curve that follows its behaviour better than a straight line. This possibility is offered by the so-called Taylor's formula, centered at x_0 and up to the order n, that allows to approximate locally (in a neighbourhood of the point x_0) a function, with an opportune number of derivatives, through a polynomial of degree n, making in this way a negligible error. With reference to this aspect, the following result holds:

If f is a function with $n - 1$ derivatives at a neighbourhood U of x_0 and with n derivatives at x_0, then $\forall x \in U$ the following Taylor's expansion (with Peano's remainder) holds:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n$$

as $x \to x_0$.

In the particular case in which $x_0 = 0$ the formula is called Mac Laurin's expansion (with Peano's remainder).

Taylor's formula expresses therefore a function as the sum of a polynomial of degree n (called Taylor's polynomial) and of a term (the remainder) that turns out to be negligible, and the approximation that it allows can be improved increasing the degree n of the approximating polynomial.

Example 128 *Determine Taylor's expansion centered at $x_0 = 3$ and up to the third order, and then Mac Laurin's expansion up to the third order, of the function:*

$$f(x) = e^x + \cos x$$

Taylor's expansion centered in $x_0 = 3$ and up to the third order is:

$$f(x) = f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+o\left((x-3)^3\right) \qquad \text{for } x \to 3$$

and as we have:

$$\begin{aligned}
f(x) &= e^x + \cos x &\Rightarrow\quad f(3) &= e^3 + \cos 3 \\
f'(x) &= e^x - \sin x &\Rightarrow\quad f'(3) &= e^3 - \sin 3 \\
f''(x) &= e^x - \cos x &\Rightarrow\quad f''(3) &= e^3 - \cos 3 \\
f'''(x) &= e^x + \sin x &\Rightarrow\quad f'''(3) &= e^3 + \sin 3
\end{aligned}$$

it becomes:

$$\begin{aligned}
f(x) = e^3 + \cos 3 + (e^3 - \sin 3)(x-3) + \frac{1}{2}(e^3 - \cos 3)(x-3)^2 + \\
+ \frac{1}{6}(e^3 + \sin 3)(x-3)^3 + o\left((x-3)^3\right) \qquad \text{for } x \to 3
\end{aligned}$$

where, eventually, it is possible to explicit the computations concerning $(x-3)^2$ and $(x-3)^3$.

Mac Laurin's expansion up to the third order, then, is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o\left(x^3\right) \qquad \text{for } x \to 0$$

and as we have:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = 1$$

it becomes:

$$f(x) = 2 + x + \frac{1}{6}x^3 + o\left(x^3\right) \qquad \text{for } x \to 0$$

Example 129 *Determine Mac Laurin's expansion up to the third order of the function:*

$$f(x) = e^{\sin x}$$

The expansion is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o\left(x^3\right) \qquad \text{for } x \to 0$$

and as we have:

$$\begin{aligned}
&f(x) = e^{\sin x} && \Rightarrow \quad f(0) = 1 \\
&f'(x) = e^{\sin x}\cos x && \Rightarrow \quad f'(0) = 1 \\
&f''(x) = e^{\sin x}(\cos^2 x - \sin x) && \Rightarrow \quad f''(0) = 1 \\
&f'''(x) = e^{\sin x}\cos x(\cos^2 x - 1 - 3\sin x) && \Rightarrow \quad f'''(0) = 0
\end{aligned}$$

it becomes:

$$f(x) = 1 + x + \frac{1}{2}x^2 + o\left(x^3\right) \qquad \text{for } x \to 0$$

The same result can be obtained resorting to Mac Laurin's expansions of the elementary functions introduced in the previous Chapter. In particular, given the function $f(x) = e^{\sin x}$ it is possible to define $\sin x = t$, and then the function can be expanded according to Taylor's formula in the following way:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + o\left(t^3\right) \qquad \text{for } t \to 0$$

and substituting then to t the expression $\sin x$ we have:

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + o\left(\sin^3 x\right) \qquad \text{for } x \to 0$$

At this point it is possible to expand the function $\sin x$ obtaining (observing that in the computations all the terms of degree higher than three are not written explicitely since they are encompassed in the symbol $o\left(x^3\right)$):

$$\begin{aligned}
e^{\sin x} &= 1 + \left(x - \frac{x^3}{3!} + o\left(x^3\right)\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + o\left(x^3\right)\right)^2 + \\
&= + \frac{1}{6}\left(x - \frac{x^3}{3!} + o\left(x^3\right)\right)^3 + o\left(x^3\right) = \\
&= 1 + x - \frac{1}{6}x^3 + o\left(x^3\right) + \frac{1}{2}\left(x^2 + o\left(x^3\right)\right) + \frac{1}{6}\left(x^3 + o\left(x^3\right)\right) + o\left(x^3\right) = \\
&= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o\left(x^3\right) = \\
&= 1 + x + \frac{1}{2}x^2 + o\left(x^3\right) \qquad \text{for } x \to 0
\end{aligned}$$

Alternatively, given the function $f(x) = e^{\sin x}$ it is possible to expand initially $\sin x$ obtaining:

$$e^{\sin x} = e^{x-\frac{x^3}{3!}+o\left(x^3\right)} \qquad \text{for } x \to 0$$

then it is possible to define $x - \dfrac{x^3}{3!} + o\left(x^3\right) = t$ and to expand the exponential function obtaining:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + o\left(t^3\right) \qquad \text{for } t \to 0$$

and substituting again to t the expression $x - \dfrac{x^3}{3!} + o\left(x^3\right)$ we have also in this case:

$$\begin{aligned}
e^{\sin x} &= e^{x-\frac{x^3}{3!}+o\left(x^3\right)} = 1 + \left(x - \frac{x^3}{3!} + o\left(x^3\right)\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + o\left(x^3\right)\right)^2 = \\
&= +\frac{1}{6}\left(x - \frac{x^3}{3!} + o\left(x^3\right)\right)^3 + o\left(x^3\right) = \\
&= 1 + x - \frac{1}{6}x^3 + o\left(x^3\right) + \frac{1}{2}\left(x^2 + o\left(x^3\right)\right) + \frac{1}{6}\left(x^3 + o\left(x^3\right)\right) + o\left(x^3\right) = \\
&= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o\left(x^3\right) = \\
&= 1 + x + \frac{1}{2}x^2 + o\left(x^3\right) \qquad \text{for } x \to 0
\end{aligned}$$

As seen in the previous Chapter, finally, Mac Laurin's expansions of the elementary functions can be used in the calculation of certain limits, as they allow to solve the indeterminate forms that appear in the resolution of these limits.

5.5 Derivatives and behaviour of a function

The analysis of the derivatives of a function gives a series of information useful to establish the behaviour of the same function, and eventually to obtain a graphical representation. In order to apply the results described below the functions considered must have a derivative, but (as observed before) the elementary functions and those obtained from them through algebraic operations and compositions have a derivative (except, eventually, in single points), so that the criteria illustrated are applicable in the great majority of cases (while in the eventual points of non derivability it is necessary, for the control of certain properties, to use the corresponding definition).

The results that can be obtained from the study of the derivatives of a function concern in particular:

- monotonicity (and invertibility) of the function
- extremum points (maxima and minima) of the function
- concavity and convexity (and inflection points) of the function

5.5.1 Monotonicity

In order to obtain information relative to the monotonicity of a function it is possible to use the first derivative of the function itself. With reference to this aspect, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ with derivative in X, and given an interval $I = (a, b)$ included in X, the following results hold:

(a) Necessary and sufficient conditions

$$f \text{ increasing on } I \Leftrightarrow f'(x) \geq 0 \quad \forall x \in I$$

$$f \text{ decreasing on } I \Leftrightarrow f'(x) \leq 0 \quad \forall x \in I$$

(b) Sufficient conditions

$$f'(x) > 0 \quad \forall x \in I \Rightarrow f \text{ strictly increasing on } I$$

$$f'(x) < 0 \quad \forall x \in I \Rightarrow f \text{ strictly decreasing on } I$$

It is necessary to observe that the use of these conditions in order to verify the monotonicity of a function on its domain requires that it is an interval; if this is not true, the results can be applied on each of the intervals that constitute the domain of the function, but they cannot be applied globally.

The conditions linked to the monotonicity of a function can be used also to obtain information concerning its invertibility. As observed in Chapter 3, the strict monotonicity of a function over an interval is a sufficient condition for its invertibility over that interval. In the case of functions with a derivative we have therefore that a sufficient condition for a function f to be invertible over an interval I is that it has over this interval first derivative of constant sign (different from 0), that is we have:

$$f'(x) > 0 \text{ (or } f'(x) < 0) \;\; \forall x \in I \Rightarrow f \text{ invertible over } I$$

5.5.2 Maxima and minima

The study of the sign of the first derivative of a function allows also to obtain information with respect to the presence of eventual extremum points (maxima and minima) of the same function. With reference to this aspect, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ with a derivative over X, with x_0 interior point with respect to X, the following results hold:

(a) Necessary conditions

$$x_0 \text{ point of relative maximum} \Rightarrow f'(x_0) = 0$$

$$x_0 \text{ point of relative minimum} \Rightarrow f'(x_0) = 0$$

Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ continuous at $x_0 \in X$ and with a derivative at a neighbourhood of x_0, $U(x_0)$, except eventually the point x_0, then, the following results hold (where x_0 is a stationary point, i.e. such that $f'(x_0) = 0$):

(b) Sufficient conditions

$$\left.\begin{array}{ll} f'(x) \geq 0 & \forall x \in U_-(x_0) \\ & \\ f'(x) \leq 0 & \forall x \in U_+(x_0) \end{array}\right\} \Rightarrow x_0 \text{ point of relative maximum}$$

$$\left.\begin{array}{ll} f'(x) \leq 0 & \forall x \in U_-(x_0) \\ & \\ f'(x) \geq 0 & \forall x \in U_+(x_0) \end{array}\right\} \Rightarrow x_0 \text{ point of relative minimum}$$

It must be observed that the necessary conditions illustrated in (a) to verify the presence of maxima and minima require that such points be interior to the domain. The sufficient conditions illustrated in (b), instead, allow to find such extremum points also when the necessary conditions cannot be applied (i.e in the case of points not interior to the domain, or in the case of isolated points or of points of non derivability).

5.5.3 Concavity and convexity

In order to obtain information relative to the concativity and convexity of a function it is necessary to introduce the second derivative of the function itself. This is simply the derivative of the first derivative (that is the limit of the difference quotient built using the function f' instead of the function f, provided this limit exists finite), and it is calculated using the same rules seen before, applied to the function f' instead of the function f.

At this point, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ with second derivative over X, and given an interval $I = (a, b)$ included in X, the following results hold:

(a) Necessary and sufficient conditions

$$f \text{ convex on } I \Leftrightarrow f''(x) \geq 0 \quad \forall x \in I$$

$$f \text{ concave on } I \Leftrightarrow f''(x) \leq 0 \quad \forall x \in I$$

(b) Sufficient conditions

$$f''(x) > 0 \quad \forall x \in I \Rightarrow f \text{ strictly convex on } I$$

$$f''(x) < 0 \quad \forall x \in I \Rightarrow f \text{ strictly concave on } I$$

The study of the second derivative allows also to find the presence of eventual inflection points. With reference to this aspect, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ with second derivative over X, with x_0 interior point with respect to X, the following results hold:

(a) Necessary conditions

$$x_0 \text{ inflection point} \Rightarrow f''(x_0) = 0$$

Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ continuous at $x_0 \in X$ and with second derivative in a neighbourhood of x_0, $U(x_0)$, except eventually the point x_0, the following results then hold (where x_0 is a point such that $f''(x_0) = 0$):

(b) Sufficient conditions

$$\left.\begin{array}{ll} f''(x) \geq 0 & \forall x \in U_-(x_0) \\ f''(x) \leq 0 & \forall x \in U_+(x_0) \end{array}\right\} \Rightarrow x_0 \text{ descreasing inflection point}$$

$$\left.\begin{array}{ll} f''(x) \leq 0 & \forall x \in U_-(x_0) \\ f''(x) \geq 0 & \forall x \in U_+(x_0) \end{array}\right\} \Rightarrow x_0 \text{ increasing inflection point}$$

The inflection points can also be classified according to the value of the first derivative of the function calculated in correspondence of such points; if x_0 is an inflection point we have in fact:

inflection point with horizontal tangent if $f'(x_0) = 0$

inflection point with vertical tangent if $\lim_{x \to x_0} f'(x) = \pm\infty$

oblique inflection point in the other cases

In conclusion, to obtain information on the behaviour of a function using its derivatives, it is possible to compute these derivatives (first derivative and second derivative) and then to study their sign. Applying the criteria illustrated above, finally, it becomes possible to obtain indications on the monotonicity and the presence of eventual extremum points (through the first derivative) and on the concavity/convexity and the presence of eventual inflection points (through the second derivative) of the function considered.

Example 130 *Discuss the monotonicity and the invertibility of the function:*

$$f(x) = \log \frac{3-x}{x}$$

The function is defined for:

$$\begin{cases} \dfrac{3-x}{x} > 0 \\ \\ x \neq 0 \end{cases} \Rightarrow 0 < x < 3$$

and its first derivative is:

$$f'(x) = \frac{\dfrac{-x-3+x}{x^2}}{\dfrac{3-x}{x}} = -\frac{3}{x^2} \cdot \frac{x}{3-x} = -\frac{3}{x(3-x)}$$

For $0 < x < 3$ we have $f'(x) < 0$, therefore $f(x)$ is strictly decreasing on its domain, and hence it is invertible.

Example 131 *Discuss the monotonicity and the invertibility of the function:*

$$f(x) = x^3 - 3x^2 + 2x - 4$$

The function is defined for each value of $x \in \mathbb{R}$ and its first derivative is:

$$f'(x) = 3x^2 - 6x + 2$$

We then have:

$$\begin{aligned} f'(x) < 0 \quad &\text{for} \quad \frac{3-\sqrt{3}}{3} < x < \frac{3+\sqrt{3}}{3} \\ f'(x) = 0 \quad &\text{for} \quad x = \frac{3-\sqrt{3}}{3} \quad \vee \quad x = \frac{3+\sqrt{3}}{3} \\ f'(x) > 0 \quad &\text{for} \quad x < \frac{3-\sqrt{3}}{3} \quad \vee \quad x > \frac{3+\sqrt{3}}{3} \end{aligned}$$

so that $f(x)$ is strictly decreasing on the interval $\left(\frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3}\right)$ and strictly increasing on each of the intervals $\left(-\infty, \frac{3-\sqrt{3}}{3}\right)$ and $\left(\frac{3+\sqrt{3}}{3}, +\infty\right)$. As a consequence, $f(x)$ is not invertible on $\mathbb{R}$, while the restrictions to each of the three intervals listed above are invertible.

Example 132 *Determine eventual maxima and minima of the function:*

$$f(x) = 3x(\log x - 2)^2$$

The function is defined on the interval $(0, +\infty)$ and its first derivative is:

$$f'(x) = 6x(\log x - 2)\frac{1}{x} + 3(\log x - 2)^2 = 3(\log x - 2)(2 + \log x - 2) =$$
$$= 3\log x(\log x - 2)$$

At this point, studying the sign of the two factors ans combining the results we get:

$$f'(x) < 0 \quad \text{for} \quad 1 < x < e^2$$

$$f'(x) = 0 \quad \text{for} \quad x = 1 \quad \vee \quad x = e^2$$

$$f'(x) > 0 \quad \text{for} \quad 0 < x < 1 \quad \vee \quad x > e^2$$

from which we deduce that $f(x)$ is strictly decreasing on the interval $(1, e^2)$ and strictly increasing on the intervals $(0, 1)$ and $(e^2, +\infty)$, as a consequence $x = 1$ is a point of relative maximum and $x = e^2$ is a point of relative minimum (and also of absolute minimum as $f(x) \geq 0$ on its domain and $f(e^2) = 0$).

Example 133 *Determine eventual maxima and minima of the function:*

$$f(x) = \sqrt{x} + 3 - 2x$$

over the interval $[0, 10]$.

The function is defined over the interval $[0, +\infty)$, furthermore since f is continuous over the closed and bounded interval $[0, 10]$ by Weierstrass Theorem it admits absolute maximum and minimum over this interval. Its first derivative is:

$$f'(x) = \frac{1}{2\sqrt{x}} - 2 = \frac{1 - 4\sqrt{x}}{2\sqrt{x}}$$

for which we have:

$$f'(x) < 0 \quad \text{for} \quad x > \frac{1}{16}$$

$$f'(x) = 0 \quad \text{for} \quad x = \frac{1}{16}$$

$$f'(x) > 0 \quad \text{for} \quad 0 < x < \frac{1}{16}$$

from which we deduce that $f(x)$ is strictly decreasing on the interval $\left(\frac{1}{16}, +\infty\right)$ and strictly increasing on the interval $\left(0, \frac{1}{16}\right)$ and $x = \frac{1}{16}$ is a point of relative maximum.

To find absolute maximum and minimum on the interval $[0, 10]$ it is then necessary to consider also the extremes of this interval, calculating the value of the function in correspondence of these extremes (and of the point of relative maximum found before). As we have:

$$f(0) = 3 \qquad f\left(\frac{1}{16}\right) = \frac{25}{8} \qquad f(10) = \sqrt{10} - 17$$

we conclude that $x = \dfrac{1}{16}$ represents the point of absolute maximum and $x = 10$ the point of absolute minimum of $f(x)$ on the interval $[0, 10]$. Over this interval, as it turns out from the previous analysis, it is not sufficient the use of the first derivative to find maxima and minima, since not all of them are interior to the interval itself; it becomes therefore necessary a separate study of the points that constitute the extremes of such interval.

Example 134 *Discuss convexity and concavity of the function:*

$$f(x) = 3x^2 - 2x + 4$$

The function is defined for every value of $x \in \mathbb{R}$ and its first derivative is:

$$f'(x) = 6x - 2$$

while its second derivative is:

$$f''(x) = 6$$

As $f''(x) > 0$ $\forall x \in \mathbb{R}$ we can conclude that $f(x)$ is strictly convex over all its domain.

Example 135 *Discuss convexity and concavity of the function:*

$$f(x) = 3x(\log x - 2)^2$$

The function (the same of Example 131) is defined on the interval $(0, +\infty)$ and its first derivative is:

$$f'(x) = 3\log x(\log x - 2)$$

while its second derivative is:

$$f''(x) = \frac{1}{x}3\log x + \frac{3}{x}(\log x - 2) = \frac{6}{x}(\log x - 1)$$

Over the domain, the sign of $f''(x)$ depends only on the factor $(\log x - 1)$ (as $\dfrac{6}{x}$ is always positive) and we have:

$$f''(x) < 0 \quad \text{for} \quad 0 < x < e$$

$$f''(x) = 0 \quad \text{for} \quad x = e$$

$$f''(x) > 0 \quad \text{for} \quad x > e$$

from which we deduce that $f(x)$ is strictly concave over the interval $(0, e)$ and strictly convex over the interval $(e, +\infty)$ and $x = e$ is an inflection point (in particular it is an increasing oblique inflection point).

5.6 Study of a function

The results concerning the functions obtained previously (relatively to the domain, the sign, the intersections with the axes - presented in Chapter 3 -, the limits - presented in Chapter 4 -, the relationships between derivatives and monotonicity, extremum points and convexity - presented in this Chapter -) can be used jointly to obtain the study of a function, that consists in the analysis of all the elements that characterize a function, starting from its analytical expression, in order to obtain its graphical representation.

In particular, in the study of a function it is possible to proceed considering the following elements:

1. domain
2. sign of the function, intersections with the axes, symmetries
3. behaviour at the frontier (limits) and asymptotes
4. first derivative, monotonicity, local extrema
5. second derivative, concavity, inflection points
6. graph of the function

If the study of the sign of the function or that of the second derivative turns out to be particularly complex, then, it is possible to omit it, deducing the behaviour of the function from the other elements.

Example 136 *Study the function:*

$$f(x) = x^2 e^{3-x}$$

In this case we have:

- Domain of the function

In this case there are no restrictions to impose, therefore the domain is:

$$D = \mathbb{R}$$

- Sign of the function, intersections with the axes, symmetries

We have $f(x) > 0 \ \ \forall x \neq 0$ and $f(0) = 0$, therefore the function intersects the axes in correspondence of the origin (that is a global minimum), furthermore it does not present symmetries.

- Behaviour at the frontier and asymptotes

We have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^2e^{3-x} = +\infty$$

and then:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^2e^{3-x} = \lim_{x\to+\infty} \frac{x^2}{e^{x-3}} = 0^+$$

so that $y = 0$ is an horizontal asymptote for $x \to +\infty$, and furthermore since:

$$\lim_{x\to\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{x^2e^{3-x}}{x} = \lim_{x\to-\infty} xe^{3-x} = -\infty$$

the function does not present oblique asymptotes.

- First derivative, monotonicity, local extrema

The function $f(x)$ has a derivative $\forall x \in D$ and the first derivative is:

$$f'(x) = -x^2e^{3-x} + 2xe^{3-x} = xe^{3-x}(2-x)$$

The sign of this derivative depends on that of $x(2-x)$ and we have:

$$f'(x) < 0 \quad \text{for} \quad x < 0 \quad \vee \quad x > 2$$

$$f'(x) = 0 \quad \text{for} \quad x = 0 \quad \vee \quad x = 2$$

$$f'(x) > 0 \quad \text{for} \quad 0 < x < 2$$

so that $f(x)$ is strictly decreasing on the intervals $(-\infty, 0)$ and $(2, +\infty)$ and strictly increasing on the interval $(0, 2)$, furthermore $x = 0$ is a local (and also global) point of minimum and $x = 2$ is a local point of maximum.

- Second derivative, concavity, inflection points

The function $f(x)$ has second derivative $\forall x \in D$ and this derivative is:

$$\begin{aligned} f''(x) &= -xe^{3-x} + (2-x)(-xe^{3-x} + e^{3-x}) = -xe^{3-x} + e^{3-x}(2-x)(1-x) = \\ &= e^{3-x}(x^2 - 4x + 2) \end{aligned}$$

The sign of this derivative depends only on the second factor and we have:

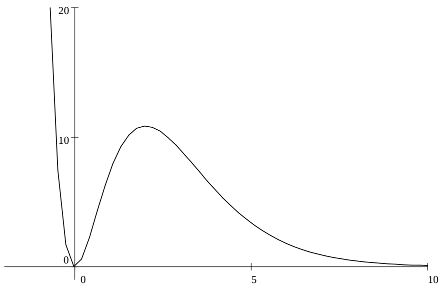
$$f''(x) < 0 \quad \text{for} \quad 2-\sqrt{2} < x < 2+\sqrt{2}$$

$$f''(x) = 0 \quad \text{for} \quad x = 2-\sqrt{2} \quad \vee \quad x = 2+\sqrt{2}$$

$$f''(x) > 0 \quad \text{for} \quad x < 2-\sqrt{2} \quad \vee \quad x > 2+\sqrt{2}$$

so that $f(x)$ is strictly concave on the interval $(2-\sqrt{2}, 2+\sqrt{2})$ and strictly convex on the intervals $(-\infty, 2-\sqrt{2})$ and $\left(2+\sqrt{2}, +\infty\right)$, furthermore $x = 2-\sqrt{2}$ and $x = 2+\sqrt{2}$ are inflection points.

- Graph of the function



Example 137 *Study the function:*

$$f(x) = \frac{x^2 - 2|x| + 1}{|x| + 1}$$

In this case we have:

- Domain of the function

In this case it must be $|x| + 1 \neq 0$, that is always true, therefore the domain is:

$$D = \mathbb{R}$$

- Sign of the function, intersections with the axes, symmetries

First of all it is possible to observe that:

$$f(-x) = \frac{(-x)^2 - 2|-x| + 1}{|-x| + 1} = \frac{x^2 - 2|x| + 1}{|x| + 1} = f(x)$$

so that $f(x)$ is even. It is therefore sufficient to study it for $x \geq 0$ (as its graph will be symmetric with respect to the y-axis) and we have:

$$f(x) = \frac{x^2 - 2x + 1}{x + 1} = \frac{(x-1)^2}{x+1} \qquad \text{with } x \geq 0$$

from which it turns out that $f(x) \geq 0\ \forall x \geq 0$, furthermore $f(0) = 1$ and $f(1) = 0$, so that the function intersects the x-axis in the point $(1, 0)$ and the y-axis in the point $(0, 1)$. The point $x = 1$, finally, is a global minimum point.

- Behaviour at the frontier and asymptotes

We have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2-2x+1}{x+1} = \lim_{x\to+\infty} \frac{x^2\left(1-\dfrac{2}{x}+\dfrac{1}{x^2}\right)}{x\left(1+\dfrac{1}{x}\right)} = +\infty$$

and then:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2-2x+1}{x^2+x} = \lim_{x\to+\infty} \frac{x^2\left(1-\dfrac{2}{x}+\dfrac{1}{x^2}\right)}{x^2\left(1+\dfrac{1}{x}\right)} = 1$$

and finally:

$$\lim_{x\to+\infty} (f(x)-x) = \lim_{x\to+\infty} \left(\frac{x^2-2x+1}{x+1} - x\right) = \lim_{x\to+\infty} \frac{x^2-2x+1-x^2-x}{x+1} =$$
$$= \lim_{x\to+\infty} \frac{-3x+1}{x+1} = \lim_{x\to+\infty} \frac{x\left(-3+\dfrac{1}{x}\right)}{x\left(1+\dfrac{1}{x}\right)} = -3$$

so that the function $f(x)$ admits, for $x \to +\infty$, oblique asymptote of equation $y = x - 3$.

- First derivative, monotonicity, local extrema

The function $f(x)$ has a derivative $\forall x \neq 0$ and the first derivative (for $x > 0$) is:

$$f'(x) = \frac{(x+1)2(x-1)-(x-1)^2}{(x+1)^2} = \frac{(x-1)\left[2(x+1)-(x-1)\right]}{(x+1)^2} =$$
$$= \frac{(x-1)(2x+2-x+1)}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}$$

The sign of $f'(x)$ depends (for $x > 0$) only on $(x-1)$, we have therefore:

$$f'(x) < 0 \quad \text{for} \quad 0 < x < 1$$

$$f'(x) = 0 \quad \text{for} \quad x = 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1$$

so that $f(x)$ is strictly decreasing on the interval $(0, 1)$ and strictly increasing on the interval $(1, +\infty)$, furthermore $x = 1$ is a local (and also global) minimum point.

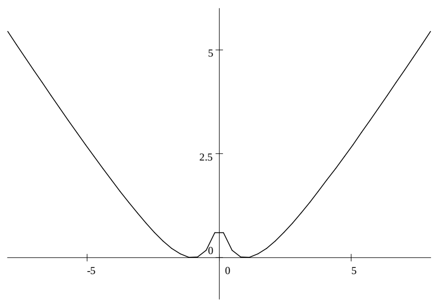
- Second derivative, concavity, inflection points

The function $f(x)$ has second derivative $\forall x \neq 0$ and this derivative (for $x > 0$) is:

$$f^{''}(x) = \frac{(x+1)^2(2x+2) - (x^2+2x-3)2(x+1)}{(x+1)^4} =$$
$$= \frac{2(x+1)^3 - 2(x+1)(x^2+2x-3)}{(x+1)^4} = \frac{2(x+1)\left[(x+1)^2 - x^2 - 2x + 3\right]}{(x+1)^4} =$$
$$= \frac{2(x+1)(x^2+2x+1-x^2-2x+3)}{(x+1)^4} = \frac{8}{(x+1)^3}$$

and for $x > 0$ we have $f''(x) > 0$, so that $f(x)$ is strictly convex on the interval $(0, +\infty)$.

- Graph of the function



5.7 Exercises

Calculate the derivatives of the following functions:

1) $f(x) = \log\left(\dfrac{x^2}{2-x}\right)$

2) $f(x) = \dfrac{1}{e^{\sqrt{x}}}$

3) $f(x) = \dfrac{\log(3x)}{1 + \log(3x)}$

4) $f(x) = e^{\sqrt{\log x}}$

5) $f(x) = \sqrt{\log(x^2+1)}$

6) $f(x) = e^{\sqrt{\cos x}}$

7) $f(x) = \sqrt[3]{x}e^{-2x}$

8) $f(x) = \dfrac{x^2 - 3x}{6x - 4}$

9) $f(x) = \dfrac{1 + \log x}{\log x}$

10) $f(x) = xe^{\sqrt{\sin x}}$

11) $f(x) = e^{\frac{4-x}{1-x}}$

12) $f(x) = \log \sqrt{x^2 + 1}$

13) $f(x) = xe^{\sin x}$

14) $f(x) = x\sqrt{1 + x}$

15) $f(x) = (xe^x)^x$

Given the function $y = f(x)$, calculate the derivative of the inverse function $x = f^{-1}(y)$ in correspondence of the point $y_0 = f(x_0)$:

16) $y = f(x) = e^{x^2-2}$ with $x_0 = 1$

17) $y = f(x) = \log(x^2 + 2)$ with $x_0 = -1$

18) $y = f(x) = \sqrt{e^x}$ with $x_0 = 2$

19) $y = f(x) = \sqrt{e^{x^2-2}}$ with $x_0 = 2$

20) $y = f(x) = e^{x^2+2}$ with $x_0 = 1$

Given the function $f(x)$, determine the equation of the straight line tangent to $f(x)$ in correspondence of the point x_0:

21) $f(x) = 2\sqrt{x} + 5x$ with $x_0 = 1$

22) $f(x) = x^2 - x + 3$ with $x_0 = 1$

23) $f(x) = \log x + 5x$ with $x_0 = 1$

24) $f(x) = \dfrac{x + 2}{x}$ with $x_0 = 2$

25) $f(x) = \log x + \sqrt{x}$ with $x_0 = 1$

26) $f(x) = \sqrt{x} + x^2$ with $x_0 = 1$

27) $f(x) = e^x + \sin x$ with $x_0 = 0$

28) $f(x) = x^3 e^{x-1}$ with $x_0 = 1$

Discuss the continuity and the derivability of the following functions on their domain:

29) $f(x) = \begin{cases} 4x & \text{if } x < 1 \\ \alpha x^2 + 2 & \text{if } x \geq 1 \end{cases}$ with $\alpha \in \mathbb{R}$

30) $f(x) = \begin{cases} \alpha e^{x-1} & \text{if } x < 1 \\ \alpha \log x + \beta & \text{if } x \geq 1 \end{cases}$ with $\alpha, \beta \in \mathbb{R}$

31) $f(x) = \begin{cases} \alpha x + \beta & \text{if } x \leq 0 \\ \log(1+x) + \dfrac{3}{1+x} & \text{if } x > 0 \end{cases}$ with $\alpha, \beta \in \mathbb{R}$

Determine Taylor-Mac Laurin's expansion, centred at x_0 and up to the order n, of the following functions:

32) $f(x) = e^{\cos x}$ $x_0 = 0$ $n = 3$

33) $f(x) = e^x - \sin x$ $x_0 = 0$ $n = 3$

34) $f(x) = e^x \log x$ $x_0 = 1$ $n = 2$

35) $f(x) = e^x - x^2$ $x_0 = 0$ $n = 3$

36) $f(x) = \sqrt{1+2x}$ $x_0 = 0$ $n = 3$

Discuss the monotonicity and determine eventual maxima and minima of the following functions:

37) $f(x) = x^3 - 2x^2 + x + 5$

38) $f(x) = \sqrt{x} + x$ on the interval $[1, 2]$

39) $f(x) = \sqrt{x} + 2 + \frac{1}{2}x$ on the interval $[0,1]$

40) $f(x) = \log x - \sqrt{x}$ on the interval $[2,4]$

41) $f(x) = \sqrt{x} + 1 - x$ on the interval $[0,1]$

42) $f(x) = \log \sqrt{x}$ on the interval $(1,2]$

43) $f(x) = 2\log x + x^2$

Discuss the concavity and convexity of the following functions:

44) $f(x) = x^4 + 2x^3 + 6$

45) $f(x) = \dfrac{x^2 + 2x + 5}{x + 3}$

46) $f(x) = e^{x^2 - 2}$

47) $f(x) = xe^x$

48) $f(x) = e^{4 - x^2}$

Study the following functions:

49) $f(x) = e^{-x^2 + \log x + 2}$

50) $f(x) = \dfrac{e^{|x| - 3}}{x}$

Chapter 6

Integral calculus

6.1 Primitives and indefinite integral

In the differential calculus, introduced in Chapter 5, given a function f it is possible to associate to it a new function, called derivative function, that is denoted with f'. In the integral calculus the first step consists in proceeding in the opposite direction, so that given a function f it is necessary to determine a function F whose derivative is the initial function. In this way it is possible to introduce the notion of primitive (or antiderivative) of a function and, then, that of indefinite integral.

Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ and given an interval $I = (a, b)$ included in X, a function $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is a primitive (or antiderivative) of f if it holds:

$$F'(x) = f(x) \qquad \forall x \in I$$

If a function has a primitive, moreover, it has infinite ones, that differ for an arbitrary constant, that is if F is a primitive of f also $F + c$ (where c is a constant) is a primitive of f, in fact we have:

$$(F(x) + c)' = F'(x) + 0 = f(x) \qquad \forall x \in I$$

that indicates that also $F + c$ is a primitive of f. On the contrary, it is unique the primitive of f that takes a given value y_0 in correspondence of a given $x_0 \in I$, that is the primitive of f passing through the point $P = (x_0, y_0)$.

Considering in fact the generic primitive $F(x)+c$ of the function $f(x)$ and imposing that it passes through the point P, that is imposing the condition:

$$F(x_0) + c = y_0$$

we get:

$$c = y_0 - F(x_0)$$

and substituting this value of the constant in $F(x) + c$ we have that the primitive is:

$$F(x) + c = F(x) + y_0 - F(x_0)$$

The set of all the primitives of a function f has particular importance and takes the name of indefinite integral of f, that is denoted with the symbol:

$$\int f(x)dx$$

so that we have:

$$\int f(x)dx = F(x) + c \qquad \text{with } c \in \mathbb{R}$$

The function f is called integrand function, while x constitutes the variable of integration.

The problem of the indefinite integration of a function f consists in the determination of its indefinite integral, therefore in finding one of its primitives. In order to get this result it is possible to use some rules that allow to reduce the calculation of the primitive of a generic function to that of the primitives of the elementary functions (that are known), in an analogous manner to what we have seen with reference to the calculation of derivatives. In fact, a first result is constituted by the so-called immediate integrals, that are obtained simply interpreting in a reverse way the table of derivatives of the elementary functions, introduced in the previous Chapter.

We have therefore the following results:

Derivation formula	Integration formula
$Dx = 1$	$\int 1dx = x + c$
$D\dfrac{x^{\alpha+1}}{\alpha+1} = x^{\alpha}$ with $\alpha \neq -1$	$\int x^{\alpha}dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c$
$D\dfrac{a^x}{\log a} = a^x$ with $0 < a \neq 1$	$\int a^x dx = \dfrac{a^x}{\log a} + c$
$De^x = e^x$	$\int e^x dx = e^x + c$
$D\log\lvert x\rvert = \dfrac{1}{x}$ with $x \neq 0$	$\int \dfrac{1}{x}dx = \log\lvert x\rvert + c$
$D\log\lvert f(x)\rvert = \dfrac{f'(x)}{f(x)}$ with $f(x) \neq 0$	$\int \dfrac{f'(x)}{f(x)}dx = \log\lvert f(x)\rvert + c$
$D\sin x = \cos x$	$\int \cos x dx = \sin x + c$
$D\cos x = -\sin x$	$\int \sin x dx = -\cos x + c$
$Dtgx = \dfrac{1}{\cos^2 x}$	$\int \dfrac{1}{\cos^2 x}dx = tgx + c$

Starting from these immediate integrals, then, to calculate the integral of a generic function it is possible to use two properties of the indefinite integral and two methods of integration, in order to reduce the integrals to be computed to one or more immediate integrals (that are known).

The two properties used are the following:

(i) *Homogeneity.* If $f(x)$ is a continuous function and k is a constant, then:

$$\int kf(x)dx = k\int f(x)dx$$

(ii) *Additivity with respect to the integrand function.* If $f(x)$ and $g(x)$ are continuous functions, then:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

(and the same holds in the case of a sum of more than two functions).

These two properties indicate that the multiplicative constants can be “taken out” from the sign of integral and that the integral of a sum of functions is equal to the sum of the integrals of the single functions. Considering jointly the properties (i) and (ii) we have that the indefinite integral satisfies the property of linearity, that can be summarized in the expression:

$$\int \left(\sum_{i=1}^{n} k_i f_i(x) \right) dx = \sum_{i=1}^{n} k_i \int f_i(x) dx$$

according to which the integral of a linear combination of functions is equal to the linear combination of the integrals of the single functions, and that expresses the so-called procedure of integration by decomposition.

The two methods of integration used are then the following:

(i) *Integration by substitution.* If $f(x)$ is a continuous function and $x = \varphi(t)$ is a function with continuous derivative, such that there exists the composite function $f(\varphi(t))$, then:

$$\int f(x)dx = \int f(\varphi(t)) \cdot \varphi'(t)dt$$

In practice, this method is used considering, in the initial integral, the substitution $x = \varphi(t)$ and observing that, differentiating both members, we have $dx = \varphi'(t)dt$ that must also be substituted in the initial integral (where, therefore, the symbol dx is interpreted as the symbol of differential).

(ii) *Integration by parts.* If $f(x)$ and $g(x)$ are continuous functions with continuous derivative, then:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

In practice, this method is used to calculate the indefinite integral of a function that can be expressed as the product of two functions $f(x)$ and $g'(x)$, one of which is the derivative of a known function. In this case, in the initial integral, $f(x)$ takes the name of finite factor while $g'(x)dx$ takes the name of differential factor.

In conclusion, the aim of these methods of integration is to reduce the calculation of a given integral to that of a simpler integral (through an appropriate substitution, or through an appropriate choice of the finite factor and of the differential factor), that therefore can be solved.

With reference to the conditions that guarantee the integrability of a function, finally, it is possible to show that each continuous function over an interval has primitives on that interval. However, not always can these primitives be expressed through elementary functions; when this is possible the function considered is said to be elementarily integrable.

Example 138 *Determine the generic primitive of the function:*

$$f(x) = x^2 + 3x + 2$$

In this case, using the procedure of integration by decomposition, we get first of all:

$$\begin{aligned}\int \left(x^2 + 3x + 2\right) dx &= \int x^2 dx + \int 3x dx + \int 2dx = \\ &= \int x^2 dx + 3\int x dx + 2\int dx\end{aligned}$$

At this point each of the integrals in which the original one has been decomposed can be easily calculated (as they are immediate integrals), we have therefore:

$$\begin{aligned}\int x^2 dx + 3\int x dx + 2\int dx &= \frac{x^3}{3} + 3\frac{x^2}{2} + 2x + c = \\ &= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + c \qquad \text{with } c \in \mathbb{R}\end{aligned}$$

that represents the generic primitive of the function $f(x)$. It is also possible to observe that in the calculation of an integral it is possible to check the result, simply deriving the function obtained from the calculations; since this function represents a primitive of the initial function, by definition its derivative must be exactly the function considered at the beginning, that is the function that appears under the sign of integral. In the example considered we have:

$$D\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + c\right) = x^2 + 3x + 2$$

that is exactly the initial function, so that the function obtained is effectively its generic primitive (that is its indefinite integral).

Example 139 *Determine the primitive of the function:*

$$f(x) = x^2 + 3x + 2$$

passing through the point $P = (2, 3)$.

In this case first of all it is necessary to determine the generic primitive of $f(x)$, that is (as it has been calculated in the previous exercise):

$$\int \left(x^2 + 3x + 2\right) dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + c$$

At this point we impose that this function passes through the point $P = (2, 3)$, that is:

$$\frac{1}{3}(2)^3 + \frac{3}{2}(2)^2 + 2 \cdot 2 + c = 3$$

from which we get:

$$\frac{38}{3} + c = 3 \Rightarrow c = -\frac{29}{3}$$

and substituting in the generic primitive the value of c just found we get:

$$F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x - \frac{29}{3}$$

that represents the primitive of the function $f(x)$ passing through the point $P = (2, 3)$.

Example 140 *Determine the generic primitive of the function:*

$$f(x) = \frac{3x^3 - \sqrt{x} - 2}{x}$$

and then find the primitive of the same function passing through the point $P = (1, 0)$.

In this case, using the procedure of integration by decomposition, we get first of all:

$$\begin{aligned}\int \frac{3x^3 - \sqrt{x} - 2}{x} dx &= \int \frac{3x^3}{x} dx - \int \frac{\sqrt{x}}{x} dx - \int \frac{2}{x} dx = \\ &= 3\int x^2 dx - \int x^{-\frac{1}{2}} dx - 2\int \frac{1}{x} dx\end{aligned}$$

At this point each of the integrals obtained can be easily computed (as they are immediate integrals), we have therefore:

$$\begin{aligned}3\int x^2 dx - \int x^{-\frac{1}{2}} dx - 2\int \frac{1}{x} dx &= 3\frac{x^3}{3} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2\log|x| + c = \\ &= x^3 - 2\sqrt{x} - 2\log|x| + c\end{aligned}$$

that represents the generic primitive of the function $f(x)$. Imposing then that this primitive passes through the point $P = (1, 0)$ we get:

$$(1)^3 - 2\sqrt{1} - 2\log|1| + c = 0$$

from which:

$$1 - 2 + c = 0 \Rightarrow c = 1$$

and substituting in the generic primitive the value of c just found we have:

$$F(x) = x^3 - 2\sqrt{x} - 2\log|x| + 1$$

that represents the primitive of the function $f(x)$ passing through the point $P = (1, 0)$.

Example 141 *Calculate the indefinite integral:*

$$\int \sin(2x+3)dx$$

In this case it is possible to use the method of integration by substitution, to this end we consider first of all:

$$2x+3=t$$

from which we also have:

$$x=\frac{t-3}{2}$$

and differentiating both members we get:

$$dx=\frac{1}{2}dt$$

At this point it is possible to substitute the expressions obtained for $2x+3$ and for dx in the initial integral, that becomes:

$$\int \sin(2x+3)dx=\int \sin t\cdot\frac{1}{2}dt$$

This integral can be easily solved, we have in fact:

$$\int \sin t\cdot\frac{1}{2}dt=\frac{1}{2}\int \sin t dt=-\frac{1}{2}\cos t+c$$

and finally, substituting to t its original expression, we get:

$$-\frac{1}{2}\cos(2x+3)+c$$

that represents the solution.

Example 142 *Calculate the indefinite integral:*

$$\int \log x dx$$

In this case it is possible to use the method of integration by parts, to this end first of all the integral can be written as:

$$\int 1\cdot\log x dx$$

then it is necessary to choose, between the two factors appearing under the sign of integral, the finite factor and the differential factor (that is, in practice, the functions $f(x)$ and $g'(x)$ that appear, together with $f'(x)$ and $g(x)$, in the formula of integration by parts). In this case the choice to make is:

$$f(x)=\log x \qquad g'(x)=1$$

so that the functions to use in the formula of integration by parts are:

$$f(x) = \log x \qquad f'(x) = \frac{1}{x}$$

$$g(x) = x \qquad g'(x) = 1$$

and applying this formula of integration we get:

$$\int 1 \cdot \log x dx = x \log x - \int \frac{1}{x} x dx = x \log x - \int dx =$$
$$= x \log x - x + c = x(\log x - 1) + c$$

that represents the solution.

It is possible to observe that, choosing $f(x) = 1$ and $g'(x) = \log x$, it wouldn't have been possible to solve the integral as it wouldn't have been possible to find the function $g(x)$ that appears in the formula of integration by parts (in this case, being $g'(x) = \log x$, the function $g(x)$ is the primitive of $\log x$, that is exactly the function we are trying to calculate). In practice, the rule to follow is the one that consists in choosing as the function $g'(x)$ the more "complicated" but that, at the same time, is immediately integrable between the two functions that appear in the initial integral (in the case under examination the more "complicated" of the two functions that appear in $\int 1 \cdot \log x dx$ is $\log x$, that however is not immediately integrable, so that it is necessary to choose $g'(x) = 1$).

Example 143 *Calculate the indefinite integral:*

$$\int x e^x dx$$

In this case it is possible to use the method of integration by parts, considering first of all:

$$f(x) = x \qquad g'(x) = e^x$$

then we have:

$$f(x) = x \qquad f'(x) = 1$$

$$g(x) = e^x \qquad g'(x) = e^x$$

and then:

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c =$$
$$= e^x (x - 1) + c$$

that represents the solution.

In this case it is possible to observe that, considering the choice $f(x) = e^x$ and $g'(x) = x$, it would have been necessary to calculate (applying the formula of integration by parts) an integral more complex than the initial one (precisely $\int x^2 e^x dx$) so that it wouldn't have been possible to solve the problem. It remains therefore valid the rule mentioned above, according to which it is opportune to choose as the function $g'(x)$ the more "complicated" but that at the same time is immediately integrable between the two functions that appear in the initial integral (in this case the more "complicated" of the two functions that appear in $\int xe^x dx$ is e^x, that is immediately integrable, so that we consider $g'(x) = e^x$).

Example 144 *Calculate the indefinite integral:*

$$\int x^2 e^x dx$$

In this case it is possible to use the method of integration by parts, considering first of all:

$$f(x) = x^2 \qquad f'(x) = 2x$$

$$g(x) = e^x \qquad g'(x) = e^x$$

then we have:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

At this point the new integral $\int xe^x dx$ can itself be solved by parts (it is the integral calculated in the previous exercise), so that in the end we get:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2\left[e^x(x-1) + c\right] = x^2 e^x - 2xe^x + 2e^x + c =$$
$$= e^x(x^2 - 2x + 2) + c$$

that represents the solution (in this case it is possible to observe that the product $-2c$ that appears developing the computations continues to be denoted with c since this is an arbitrary constant, that therefore can assume every value, exactly as $-2c$).

Example 145 *Calculate the indefinite integral:*

$$\int e^x \sin x dx$$

In this case it is possible to use the method of integration by parts, considering first of all:

$$f(x) = \sin x \qquad f'(x) = \cos x$$

$$g(x) = e^x \qquad g'(x) = e^x$$

then we have:

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

At this point the new integral $\int e^x \cos x dx$ can itself be solved by parts considering:

$$f(x) = \cos x \qquad f'(x) = -\sin x$$
$$g(x) = e^x \qquad g'(x) = e^x$$

so that we get:

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x dx\right] =$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

and then, observing that in the left and in the right-hand side there is the same integral:

$$2\int e^x \sin x dx = e^x \sin x - e^x \cos x + c$$

and finally:

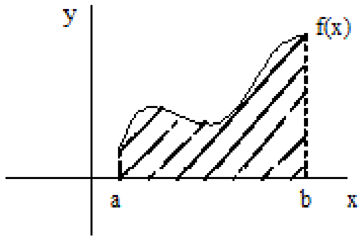
$$\int e^x \sin x dx = \frac{1}{2}e^x (\sin x - \cos x) + c$$

that represents the solution (it is possible to observe that in the right-hand side the constant c appears when the symbol of integral disappears, since up to this point it is "embedded" in the symbol of integral itself).

In this case the integral could have been solved, always by parts, also choosing at the beginning $f(x) = e^x$ and $g'(x) = \sin x$ and, in the second integration by parts, $f(x) = e^x$ and $g'(x) = \cos x$ (i.e. it is necessary to choose in both integrations by parts as $f(x)$ the exponential function and as $g'(x)$ the trigonometric function, or viceversa, because otherwise we get the identity $0 = 0$ that does not allw to calculate the initial integral).

6.2 Definite integral

A different concept with respect to that of indefinite integral introduced in the previous Section is that of definite integral. Given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ and given an interval $I = [a, b]$ included in X, the definite integral of f between a and b is the number that represents the area (with sign) of the surface included between the graph of f and the x-axis relatively to the interval $[a, b]$, that is:



This integral is denoted with the symbol:

$$\int_a^b f(x)dx$$

where f is called integrand function, x is the variable of integration, while a and b constitute respectively the lower and the upper extreme of integration.

In the case of a function $f(x) \geq 0\ \forall x \in [a, b]$ (that is of a function whose graph is never below the x-axis) the value of the integral is itself ≥ 0, while in the case of a function $f(x) \leq 0\ \forall x \in [a, b]$ (that is of a function whose graph is never above the x-axis) the value of the integral is itself ≤ 0. In the case of a function $f(x)$ that changes sign in the interval $[a, b]$, on the contrary, the value of the integral represents the difference between the (positive) area of the surface included between the part of the graph that lies above the x-axis and the x-axis itself and the (negative) area of the surface included between the part of the graph that lies below the x-axis and the x-axis itself, that is in this case the value of the integral represents the result of the "compensation" among positive and negative areas.

With reference to the conditions that guarantee the existence of the definite integral of a function, it is possible to prove that each continous function on a closed interval $[a, b]$, or bounded on the interval $[a, b]$ (with eventually a finite number of points of discontinuity), or monotonic on the interval $[a, b]$ (with eventually a countable infinity of points of discontinuity) turns out to be integrable over such interval.

The definite integral enjoys a series of properties; first of all we put by definition:

$$\int_a^a f(x)dx = 0 \qquad\qquad \int_a^b f(x)dx = -\int_b^a f(x)dx \quad \text{with } b < a$$

and then the following properties hold (where f and g are continuous functions, therefore integrable, over the interval $[a, b]$):

(i) *Homogeneity.* If k is a constant, then:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

(ii) *Additivity with respect to the integrand function.* We have:

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(iii) *Linearity.* From properties (i) and (ii) we get:

$$\int_a^b \left(\sum_{i=1}^n k_i f_i(x)\right) dx = \sum_{i=1}^n k_i \int_a^b f_i(x)dx$$

(iv) *Additivity with respect to the interval of integration.* We have:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad \forall c \in (a,b)$$

(v) *Positivity.* If $f(x) \geq 0\ \forall x \in [a,b]$, then:

$$\int_a^b f(x)dx \geq 0$$

(vi) *Monotonicity.* If $f(x) \leq g(x)\ \forall x \in [a,b]$, then:

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(vii) We have:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$$

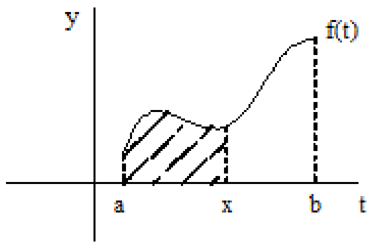
The fundamental Theorem of integral calculus (or Torricelli-Barrow theorem) establishes then the link between differential calculus and integral calculus (and also between definite integral and indefinite integral), and it gives the rule that allows, in practice, to calculate a definite integral. According to this theorem, given a function $f(x)$ continuous over $[a,b]$, the corresponding integral function, that is defined as:

$$F(x) = \int_a^x f(t)dt \qquad \text{with } x \in [a,b]$$

is continuous and has a derivative $\forall x \in [a,b]$ and we have:

$$F'(x) = f(x)$$

The integral function is a function that indicates the value of the area (with sign) of the surface included between the graph of f and the x-axis in the interval $[a, x]$, with x that varies in $[a, b]$ (therefore when x changes, the value of the area changes, and for this reason we get a function), that is we have:



and by the fundamental theorem of integral calculus the integral function of f turns out to be a primitive of f, more precisely the one that in $x = a$ is equal to 0 (as $F(a) = \int_a^a f(t)dt = 0$). From this theorem we then have, as a Corollary, that the calculation of a definite integral can be done using the following formula:

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

where G is a generic primitive of f. This means that the definite integral of a (continuous) function over an interval is equal to the difference between the values that a generic primitive of the same function assumes in the upper extreme and in the lower extreme of integration.

Example 146 *Calculate the definite integral of* $f(x) = x$ *first on the interval* $[0, 1]$ *and then on the interval* $[-1, 1]$.

In this case we have (observing that a primitive of x is $\frac{x^2}{2}$):

$$\int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

that represents the value of the integral on the interval $[0, 1]$, and then:

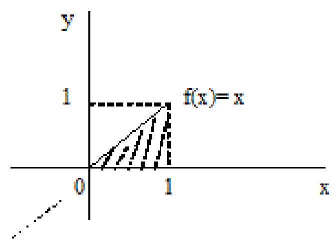
$$\int_{-1}^1 xdx = \left[\frac{x^2}{2}\right]_{-1}^1 = \frac{1^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

the represents the value of the integral on the interval $[-1, 1]$.

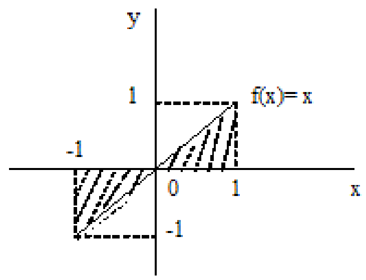
It is possible to observe that, in the choice of the primitive to use for the calculation, it is possible to omit the arbitrary constant c (that is, in practice, to choose

$c = 0$) since this constant, in any case, cancels out considering the difference between the values that the primitive assumes in correspondence of the upper and of the lower extreme of integration.

This example can be used also to underline the geometric meaning of the definite integral; in particulat, the first integral constitutes the (positive) area of the triangle (with basis and height equal to 1) represented in the picture:



This area is equal to $\frac{1 \cdot 1}{2} = \frac{1}{2}$ as in fact it turns out from the calculation of the definite integral. The second integral considered, on the other hand, constitutes, from the geometrical point of view, the algebraic sum of the areas of the two triangles represented in the picture:



Each of these two areas is equal to $\frac{1}{2}$, but since one is positive and the other is negative their algebraic sum is equal to 0, as indeed turns out from the calculation of the definite integral (that therefore in this case expresses the compensation between positive areas and negative areas). In practice, to use the definite integral for the calculation of an area (without sign) it is necessary to consider the areas of the parts of the plane that are below the x-axis taking their absolute value. For example, in the last example, to calculate the area of the part of the plane dashed in the picture it is necessary to proceed in the following way:

$$\int_{-1}^{1} x dx = \left| \int_{-1}^{0} x dx \right| + \int_{0}^{1} x dx = \left| -\frac{1}{2} \right| + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

that represents exactly the sum of the (positive) areas of the two triangles.

Example 147 *Calculate the definite integral:*

$$\int_0^2 \left(x^2+3x+2\right) dx$$

In this case we have:

$$\int_0^2 \left(x^2+3x+2\right) dx = \left[\frac{x^3}{3}+3\frac{x^2}{2}+2x\right]_0^2 = \left[\frac{1}{3}x^3+\frac{3}{2}x^2+2x\right]_0^2 =$$
$$= \left(\frac{8}{3}+6+4\right)-(0+0+0) = \frac{8}{3}+10 = \frac{38}{3}$$

that represents the value of the definite integral.

Example 148 *Calculate the definite integral:*

$$\int_1^3 \frac{3x^3-\sqrt{x}-2}{x} dx$$

In this case we have:

$$\int_1^3 \frac{3x^3-\sqrt{x}-2}{x} dx = \int_1^3 \left(\frac{3x^3}{x}-\frac{\sqrt{x}}{x}-\frac{2}{x}\right) dx = \int_1^3 \left(3x^2-x^{-\frac{1}{2}}-\frac{2}{x}\right) dx =$$
$$= \left[3\frac{x^3}{3}-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}-2\log|x|\right]_1^3 = \left[x^3-2\sqrt{x}-2\log|x|\right]_1^3 =$$
$$= \left(27-2\sqrt{3}-2\log 3\right)-(1-2-0) = 28-2\sqrt{3}-2\log 3$$

that represents the value of the definite integral.

Example 149 *Calculate the definite integral:*

$$\int_0^1 \frac{2x}{x^2+1} dx$$

In this case the integral can be calculated considering first of all the substitution:

$$x^2+1=t$$

from which we also have:

$$x^2=t-1$$

and differentiating both sides:

$$2xdx=dt$$

At this point the integral becomes:

$$\int_0^1 \frac{2x}{x^2+1}dx = \int_1^2 \frac{1}{t}dt$$

where, passing from the integral in the variable x to the integral in the new variable t, also the extremes of integration must be changed. More precisely, the new extremes of integration are obtained substituting those relative to the integral in x (that is 0 and 1) in the expression that defines the variable t (that is $t = x^2 + 1$) obtaining in such a way the extremes relative to the integral in t (that is 1 and 2). At this point the new integral can be easily calculated obtaining:

$$\int_1^2 \frac{1}{t}dt = [\log|t|]_1^2 = \log 2 - \log 1 = \log 2$$

that represents the value of the initial definite integral.

The same result can be obtained calculating first of all the indefinite integral corresponding to the initial definite one, that is (considering the same substitution used above):

$$\int \frac{2x}{x^2+1}dx = \int \frac{1}{t}dt = \log|t| + c$$

then substituting to t its original expression:

$$\int \frac{2x}{x^2+1}dx = \log(x^2+1) + c$$

and finally passing to the definite integral (keeping the original extremes of integration since the primitive found is a function of the original variable x), obtaining:

$$\int_0^1 \frac{2x}{x^2+1}dx = \left[\log(x^2+1)\right]_0^1 = \log 2 - \log 1 = \log 2$$

that is the same result found above.

In this case, finally, the same result can be obtained also using the immediate integral:

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

since in the example considered the numerator of the fraction that appears under the sign of integral is exactly the derivative of the denominator, we have therefore:

$$\int_0^1 \frac{2x}{x^2+1}dx = \left[\log\left|x^2+1\right|\right]_0^1 = \log 2 - \log 1 = \log 2$$

that is the same result found above.

Example 150 *Calculate the definite integral:*

$$\int_{-3}^{1} e^{\sqrt{x+3}}dx$$

In this case it is possible to consider first of all the substitution:

$$\sqrt{x+3} = t$$

from which we get:

$$x = t^2 - 3$$

and also:

$$dx = 2tdt$$

At this point the initial integral becomes (observing that the extremes of integration must be changed):

$$\int_{-3}^{1} e^{\sqrt{x+3}}dx = \int_{0}^{2} e^t 2tdt = 2\int_{0}^{2} te^t dt$$

This integral can be calculated by parts considering:

$$f(t) = t \qquad f'(t) = 1$$

$$g(t) = e^t \qquad g'(t) = e^t$$

and then we have

$$2\int_{0}^{2} te^t dt = 2\left[\left[te^t\right]_0^2 - \int_{0}^{2} e^t dt\right] = 2\left[\left[te^t\right]_0^2 - \left[e^t\right]_0^2\right] =$$
$$= 2\left[\left(2e^2 - 0\right) - \left(e^2 - 1\right)\right] = 2\left(e^2 + 1\right)$$

that represents the value of the definite integral considered at the beginning.

Example 151 *Given the function $f : \mathbb{R} \to \mathbb{R}$ defined by:*

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 2 \\ \\ x^2 + 1 & \text{if } 2 < x \leq 3 \end{cases}$$

calculate the definite integral $\int_0^3 f(x)dx$.

In this case, since the integrand function is defined piecewise, it is necessary to apply the property of additivity with respect to the interval of integration, "breaking" the integral in the following way:

$$\int_{0}^{3} f(x)dx = \int_{0}^{2} f(x)dx + \int_{2}^{3} f(x)dx$$

and then, using on each interval the expression of $f(x)$ corresponding to such interval:

$$\begin{aligned}\int_0^3 f(x)dx &= \int_0^2 2xdx + \int_2^3 \left(x^2+1\right)dx = \left[2\frac{x^2}{2}\right]_0^2 + \left[\frac{x^3}{3}+x\right]_2^3 = \\ &= \left[x^2\right]_0^2 + \left[\frac{1}{3}x^3+x\right]_2^3 = (4-0) + \left(\frac{27}{3}+3-\frac{8}{3}-2\right) = \\ &= 4+9+3-\frac{8}{3}-2 = \frac{34}{3}\end{aligned}$$

that represents the value of the initial definite integral.

6.3 Exercises

Calculate the following indefinite integrals:

1) $\int \left(x+2x^2+3x^3\right)dx$

2) $\int \left(2x+3\sqrt{x}-\frac{4}{x}\right)dx$

3) $\int \frac{x+5}{\sqrt{x}}dx$

4) $\int e^{\cos x}\sin x dx$

5) $\int \sin\sqrt{x}dx$

6) $\int x\log x dx$

7) $\int \frac{\log x}{x}dx$

8) $\int \frac{\log^2 x}{x}dx$

9) $\int x^2e^{2x}dx$

10) $\int \sqrt{1-x}dx$

11) $\displaystyle\int \left(1+e^{2x}\right) dx$

12) $\displaystyle\int (x+1)e^{-x} dx$

13) $\displaystyle\int e^{\sqrt{x}} dx$

14) $\displaystyle\int \left(\sqrt{x}+x^2\right) dx$

15) $\displaystyle\int (x+\sin x)\, dx$

16) $\displaystyle\int 2x^2 e^{2x} dx$

17) $\displaystyle\int \sqrt[3]{x} dx$

18) $\displaystyle\int \sqrt{1+x} dx$

19) $\displaystyle\int \frac{3x^2}{x^3+1} dx$

20) $\displaystyle\int e^{1-x} dx$

Determine the primitive of the function $f(x)$ passing through the point $P=(x_0,y_0)$:

21) $f(x)=3x^2+2x+5$ with $P=(-1,1)$

22) $f(x)=\cos x+x+3x^2$ with $P=(0,1)$

23) $f(x)=\sqrt{x}+\dfrac{1}{x}$ with $P=(1,1)$

24) $f(x)=(2x+3)^2$ with $P=(0,-1)$

25) $f(x)=\dfrac{e^{\sqrt{x}}+2}{\sqrt{x}}$ with $P=(1,1)$

26) $f(x)=\dfrac{2x-1}{x^2-x-1}$ with $P=(1,-1)$

27) $f(x) = e^{\sin x} \cos x \quad \text{with } P = (0, 2)$

28) $f(x) = e^{2x-3} \quad \text{with } P = (1, 0)$

29) $f(x) = e^{\sqrt{x}} \quad \text{with } P = (0, 1)$

30) $f(x) = \sqrt{x} - x^2 \quad \text{with } P = (1, 0)$

Calculate the following definite integrals:

31) $\displaystyle\int_1^2 e^{x-1} dx$

32) $\displaystyle\int_{\frac{1}{2}}^1 \sqrt{2x-1} dx$

33) $\displaystyle\int_{-1}^0 \sqrt{1+x} dx$

34) $\displaystyle\int_0^1 \cos \sqrt{x} dx$

35) $\displaystyle\int_0^3 \frac{2x}{x^2+1} dx$

36) $\displaystyle\int_0^1 \frac{x}{x^2+1} dx$

37) $\displaystyle\int_0^2 \frac{3x^2}{x^3+1} dx$

38) $\displaystyle\int_0^1 (x + \sqrt{x})\, dx$

39) $\displaystyle\int_0^1 \left(\sqrt{x} + x^2\right) dx$

40) $\displaystyle\int_0^5 e^x \sin x dx$

41) $\displaystyle\int_0^1 (x + \cos x)\, dx$

42) $\displaystyle\int_0^2 \sqrt[3]{x}dx$

43) $\displaystyle\int_1^2 xe^x dx$

44) $\displaystyle\int_1^2 \frac{x+3}{2}dx$

45) $\displaystyle\int_1^2 x\log x dx$

46) $\displaystyle\int_0^1 e^{1-x}dx$

47) $\displaystyle\int_1^2 \log\left(3x\right)dx$

48) $\displaystyle\int_3^4 \frac{x-1}{x^2-2x}dx$

49) *Given the function:*

$$f(x) = \begin{cases} 3x^2 & \textit{if } -1 \leq x \leq 0 \\ \\ x+2 & \textit{if } 0 < x \leq 1 \end{cases}$$

calculate the definite integral $\displaystyle\int_{-1}^1 f(x)dx$.

50) *Given the function:*

$$f(x) = \begin{cases} 4x^3 & \textit{if } 0 \leq x \leq 1 \\ \\ \dfrac{1}{x}+1 & \textit{if } 1 < x \leq 2 \end{cases}$$

calculate the definite integral $\displaystyle\int_0^2 f(x)dx$.

Chapter 7

Linear algebra

7.1 Vectors: definitions and properties

Linear algebra is a set of rules of calculus that are applied to objects called vectors and matrices, that are substantially numerical tables used in many applications. The starting point of the analysis is therefore represented by the definition of the concepts of vector and then of matrix and by the study of the operations that it is possible to introduce among them.

A vector is an ordered n-tuple of real numbers, that is denoted with:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \qquad x_i \in \mathbb{R} \qquad i = 1, 2, ..., n$$

where $x_1, x_2, ..., x_n$ are the components of the vector. The same vector (that is called column vector) can be represented in the following way:

$$\mathbf{x}^T = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}$$

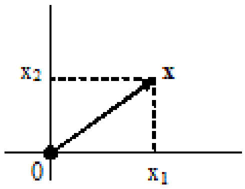
that is the transpose vector of $\mathbf{x}$ (and it is also called row vector).

The set of all vectors with n real components is denoted by $\mathbb{R}^n$, that is given by the cartesian product of $\mathbb{R}$ considered n times, that is we have:

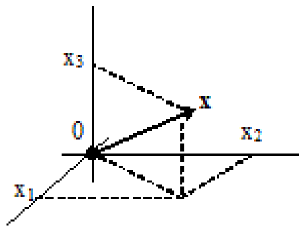
$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{R},\ i = 1, 2, ..., n\}$$

The real numbers can be seen as a particular type of vectors, with a unique component, and as it is well known they can be represented on a straight line, that constitutes therefore the geometric image of the set $\mathbb{R}$:

In a similar way, the vectors with two components (that is the pairs of real numbers) can be represented on a plane, that constitutes therefore the geometric image of $\mathbb{R}^2$:



and the vectors with three components (that is the terns of real numbers) can be represented in the tridimensional space, that constitutes therefore the geometric image of $\mathbb{R}^3$:



while for $n \geq 4$ it is not possible a geometric representation.

Among the vectors of $\mathbb{R}^n$ particularly important are the so-called fundamental vectors (or "versors"), denoted with $\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^n$. The generic fundamental vector $\mathbf{e}^i$ has all the components equal to 0 except the i-th, that is equal to 1, that is we have:

$$\mathbf{e}^i = \begin{pmatrix} 0 \\ 0 \\ ... \\ 1 \\ ... \\ 0 \end{pmatrix}$$

For example, in $\mathbb{R}^3$ the fundamental vectors are:

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Two vectors with the same number of components are equal when the components of equal position coincide, that is:

$$\text{Given} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \quad \text{then} \quad \mathbf{x} = \mathbf{y} \Leftrightarrow x_i = y_i \quad \forall i = 1, 2, ..., n$$

For example, the vectors:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

are clearly equal, while the vectors:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

are not equal (since in the vectors it is important the order of the components).

Among vectors it is then possible to introduce a (partial) order, so that we have:

$$\text{Given} \quad x, \mathbf{y} \in \mathbb{R}^n \quad \text{then} \quad \mathbf{x} > \mathbf{y} \Leftrightarrow x_i > y_i \quad \forall i = 1, 2, ..., n$$

(in an analogous way it is possible to introduce the relations of $<$, $\geqq$, $\leqq$) . The order is partial because (differently from what happens with numbers) given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ it is not necessarily true that $\mathbf{x} > \mathbf{y}$ (eventually $\mathbf{x} \geqq \mathbf{y}$) or $\mathbf{x} < \mathbf{y}$ (eventually $\mathbf{x} \leqq \mathbf{y}$) or $\mathbf{x} = \mathbf{y}$. Considering for example the vectors:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

we have $\mathbf{x} < \mathbf{y}$, while considering the vectors:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we have $\mathbf{x} > \mathbf{y}$, but if we consider the vectors:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

in this case $\mathbf{x}$ and $\mathbf{y}$ cannot be compared.

It is then possible to introduce the notion of positive vector in the following way:

$$\text{Given} \quad \mathbf{x} \in \mathbb{R}^n \quad \text{then} \quad \mathbf{x} > \mathbf{0} \Leftrightarrow x_i > 0 \quad \forall i = 1, 2, ..., n$$

where $\mathbf{0}$ is the null vector (that is the vector with all the components equal to 0). In a similar way it is possible to define negative vectors ($\mathbf{x} < \mathbf{0}$) and also non-negative vectors ($\mathbf{x} \geqq \mathbf{0}$) and non-positive vectors ($\mathbf{x} \leqq \mathbf{0}$).

7.2 Operations with vectors

At this point it is possible to introduce the operations among vectors, that are 3. Using these operations, then, it is possible to introduce other two notions of particular importance, that of norm of a vector (and of distance among vectors) and that of linear combination of vectors (with the concepts of linear dependence and independence).

7.2.1 Sum of vectors

The first operation that can be introduced is the sum of vectors. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix}$$

the sum is the vector $\mathbf{x} + \mathbf{y} \in \mathbb{R}^n$ given by:

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ ... \\ x_n + y_n \end{pmatrix}$$

that is the vector obtained summing each component of the first vector with the corresponding component of the second vector.

7.2.2 Multiplication vector-scalar

The second operation is the multiplication of a vector by a scalar (that is a real number). Given $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ with:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

the product of the vector $\mathbf{x}$ by the scalar c is the vector $c\mathbf{x} \in \mathbb{R}^n$ given by:

$$c\mathbf{x} = \begin{pmatrix} cx_1 \\ cx_2 \\ ... \\ cx_n \end{pmatrix}$$

that is the vector obtained multiplying each component of the inial vector by the scalar.

7.2.3 Scalar product of vectors

The third operation is the scalar product (or inner product) among vectors. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix}$$

the scalar product (or inner product) is the number given by:

$$(\mathbf{x} \mid \mathbf{y}) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

that is the number obtained multiplying each component of the first vector by the corresponding component of the second vector and summing the various products so obtained. Differently from what happens in the case of product among numbers, for the inner product the "cancellation law" does not hold (the scalar product of two vectors, one of which is the null vector, is equal to zero, but the scalar product can be equal to zero even if the two factors are both different from the null vector). Two vectors whose inner product is null, then, are said to be orthogonal (and we write $\mathbf{x} \perp \mathbf{y}$).

Example 152 *Given the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ *calculate their sum.*

In this case we have:

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix}$$

that represents the sum of the two vectors.

Example 153 *Given the vector* $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ *and the scalar* $c = -5$ *calculate the product* $c\mathbf{x}$*.*

In this case we have:

$$c\mathbf{x} = (-5) \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ -5 \\ 15 \end{pmatrix}$$

that represents the product $-5\mathbf{x}$.

Example 154 *Given the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ *calculate their scalar product.*

In this case we have:

$$(\mathbf{x} \mid \mathbf{y}) = 1 \cdot (-3) + 2 \cdot 4 + 0 \cdot 2 = 5$$

that represents the scalar product of the two vectors.

Example 155 *Determine for which value of* $\alpha \in \mathbb{R}$ *the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 2 \\ \alpha \\ 5 \end{pmatrix}$ *are orthogonal.*

In this case we have:

$$(\mathbf{x} \mid \mathbf{y}) = 1 \cdot 2 + (-3) \cdot \alpha + 2 \cdot 5 = 12 - 3\alpha$$

and then:

$$\mathbf{x} \perp \mathbf{y} \Leftrightarrow 12 - 3\alpha = 0 \Leftrightarrow \alpha = 4$$

so that for $\alpha = 4$ the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. This is an example from which it turns out clearly that for the scalar product the "cancellation law" does not hold, in fact (if $\alpha = 4$) the inner product between $\mathbf{x}$ and $\mathbf{y}$ is null even if the two vectors are both different from the null vector.

7.2.4 Norm of a vector, distance among vectors, linear combination of vectors, linear dependence and independence

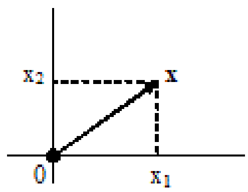
The operation of inner product (or scalar product) among vectors allows to introduce the notion of norm of a vector. Given $\mathbf{x} \in \mathbb{R}^n$ with:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

the norm of $\mathbf{x}$ is the number given by:

$$\| \mathbf{x} \| = \sqrt{(\mathbf{x} \mid \mathbf{x})} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

that is by the square root of the scalar product of $\mathbf{x}$ with itself. Geometrically the norm of a vector represents the length of the segment that corresponds to the same vector, as it is evident in the case $n = 2$:



in which, applying Pythagorean's Theorem, we have that the length of the hypotenuse of the rectangular triangle that has vertices $0, x_1$ and $\mathbf{x}$ is given by $\sqrt{x_1^2 + x_2^2}$, that is exactly the norm of the vector considered.

In the case $n = 1$, then, we have:

$$\| \mathbf{x} \| = \sqrt{(\mathbf{x} \mid \mathbf{x})} = \sqrt{x_1^2} = |x_1|$$

i.e. the norm coincides with the absolute value of the number considered.

Using the notion of norm it is possible to introduce that of distance among vectors of $\mathbb{R}^n$. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the distance between $\mathbf{x}$ and $\mathbf{y}$ is defined as the norm of the difference $\mathbf{x} - \mathbf{y}$, i.e. the lenght of the segment with extremes $\mathbf{x}$ and $\mathbf{y}$:

$$d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

The operations of sum of vectors and of product of a vector by a scalar, on the other hand, allow to introduce the notion of linear combination of vectors. Given k vectors $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k \in \mathbb{R}^n$ and k scalars (that is numbers) $c_1, c_2, ..., c_k \in \mathbb{R}$, the vector given by:

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + ... + c_k\mathbf{x}^k = \sum_{i=1}^{k} c_i\mathbf{x}^i$$

is called linear combination of the vectors $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k$ with weights (or coefficients) $c_1, c_2, ..., c_k$.

In particular, it is possible to prove that each vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ can be expressed as linear combination of the fundamental vectors (or versors) $\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^n$ with coefficients equal to the components x_1, $x_2, ...,$ x_n of the vector, we have in fact:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ ... \\ 0 \end{pmatrix} + ... + x_n \begin{pmatrix} 0 \\ 0 \\ ... \\ 1 \end{pmatrix} =$$

$$= x_1\mathbf{e}^1 + x_2\mathbf{e}^2 + ... + x_n\mathbf{e}^n = \sum_{i=1}^{n} x_i\mathbf{e}^i$$

The notion of linear combination allows then to introduce that of linear dependence (and independence) among vectors. With reference to this aspect, we say that the vectors $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k \in \mathbb{R}^n$ are linearly dependent if (at least) one of them can be expressed as linear combination of the others, that is if there exist scalars $c_1, c_2, ..., c_{k-1} \in \mathbb{R}$ such that it is possible to write:

$$\mathbf{x}^k = \sum_{i=1}^{k-1} c_i \mathbf{x}^i$$

(it is always possible to think that the last vector is the one that can be expressed as linear combination of the others, eventually changing the order). When, on the contrary, this representation is not possible, we say that the vectors considered are linearly independent.

An equivalent definition (that is used in practice to verify the linear dependence or independence) is that according to which the vectors $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k \in \mathbb{R}^n$ are linearly dependent if and only if there exists a linear combination of these vectors with coefficients not all equal to zero that corresponds to the null vector, that is it is possible to write:

$$\sum_{i=1}^{k} c_i \mathbf{x}^i = \mathbf{0} \qquad \text{with at least one} \quad c_i \neq 0$$

while the vectors are linearly independent if and only the unique linear combination of these vectors equal to the null vector is the one with coefficients all equal to zero, that is it is possible to write:

$$\sum_{i=1}^{k} c_i \mathbf{x}^i = \mathbf{0} \qquad \text{only when } c_i = 0 \quad \forall i$$

(it is possible to observe that this solution always exists, as it is evident that a linear combination of vectors with coefficients all equal to zero is always equal to the null vector).

Example 156 *Given the vector* $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$ *calculate its norm and its distance from the vector* $\mathbf{y} = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$.

In this case we have first of all:

$$\| \mathbf{x} \| = \sqrt{(\mathbf{x} \mid \mathbf{x})} = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{3^2 + 0^2 + (-4)^2} = \sqrt{9 + 0 + 16} = \sqrt{25} = 5$$

that represents the norm of the vector $\mathbf{x}$, and then:

$$d(\mathbf{x},\mathbf{y}) = \| \mathbf{x} - \mathbf{y} \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} =$$
$$= \sqrt{(3-2)^2 + (0-(-3))^2 + (-4-(-5))^2} = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

that represents the distance between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Example 157 *Given the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ *and the scalars* $\alpha = 3$ *and* $\beta = -2$ *determine the linear combination* $\alpha\mathbf{x} + \beta\mathbf{y}$.

We have in this case:

$$\alpha\mathbf{x} + \beta\mathbf{y} = 3 \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + (-2) \cdot \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

that represents the linear combination $3\mathbf{x} - \mathbf{2y}$.

Example 158 *Given the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ *determine if they are linearly dependent or independent.*

We have in this case:

$$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0} \qquad \text{with } c_1, c_2 \in \mathbb{R}$$

that is we consider the generic linear combination of the 2 vectors and we equal it to the null vector. With some calculations we then get:

$$c_1 \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} c_1 \\ 3c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} c_1 \\ 3c_1 + 2c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that we must have:

$$\begin{cases} c_1 = 0 \\ 3c_1 + 2c_2 = 0 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

and since the unique linear combination of the 2 vectors that gives the null vector is the one with null coefficients, the vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

Example 159 *Given the vectors* $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ *and* $\mathbf{z} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ *determine if they are linearly dependent or independent.*

We have in this case:

$$c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0} \qquad \text{with } c_1, c_2, c_3 \in \mathbb{R}$$

and developing the calculations we get:

$$\begin{aligned} c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \\ \begin{pmatrix} c_1 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ c_2 \\ -c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -c_3 \\ 3c_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \\ \begin{pmatrix} c_1 + 2c_2 \\ c_2 - c_3 \\ c_1 - c_2 + 3c_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

so that we must have:

$$\begin{cases} c_1 + 2c_2 = 0 \\ c_2 - c_3 = 0 \\ c_1 - c_2 + 3c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -2c_3 \\ c_2 = c_3 \\ c_2 = c_3 \end{cases} \qquad \text{with } c_3 \in \mathbb{R}$$

All the linear combinations of the 3 vectors with coefficients $(-2c_3, c_3, c_3)$, where $c_3 \in \mathbb{R}$, are equal to the null vector, and since there are also linear combinations with coefficients not all equal to zero (it is sufficiento to take $c_3 \neq 0$) the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent.

7.3 Matrices: definitions and properties

A matrix is a table of real numbers, that is denoted by:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \qquad a_{ij} \in \mathbb{R} \qquad \begin{array}{l} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{array}$$

and it is a matrix with m rows and n columns, that can also be represented in a compact form in the following way:

$$A = (a_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}$$

A matrix of dimension $m \times n$, then, can be obtained writing m row vectors of $\mathbb{R}^n$ one above the other, so that we have:

$$A = \begin{matrix} \left(a_{11} \quad a_{12} \quad ... \quad a_{1n} \right) \\ \left(a_{21} \quad a_{22} \quad ... \quad a_{2n} \right) \\ ... \\ \left(a_{m1} \quad a_{m2} \quad ... \quad a_{mn} \right) \end{matrix} = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ ... \\ \mathbf{a}^m \end{pmatrix} \qquad \begin{matrix} \mathbf{a}^i \in \mathbb{R}^n \\ \\ i = 1, 2, ..., m \end{matrix}$$

or it can be obtained writing n column vectors of $\mathbb{R}^m$ one side by side to the other, so that we have:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ ... \\ a_{m2} \end{pmatrix} ... \begin{pmatrix} a_{1n} \\ a_{2n} \\ ... \\ a_{mn} \end{pmatrix} = \left(\mathbf{b}^1 \quad \mathbf{b}^2 \quad ... \quad \mathbf{b}^n \right) \qquad \begin{matrix} \mathbf{b}^j \in \mathbb{R}^m \\ \\ j = 1, 2, ..., n \end{matrix}$$

The set of matrices with real components and with m rows and n columns is denoted by $\mathbb{R}^{m,n}$, and the vectors can be seen as particular matrices, with only one column (in the case of column vectors, that therefore can be denoted in general by $\mathbb{R}^{m,1}$) or only one row (in the case of row vectors, that therefore can be denoted in general by $\mathbb{R}^{1,n}$).

Given a matrix $A \in \mathbb{R}^{m,n}$ with:

$$A = (a_{ij})_{\substack{i=1,2,...,m \\ j=1,2,...,n}}$$

it is called transpose of A the matrix $A^T \in \mathbb{R}^{n,m}$ given by:

$$A^T = (a_{ji})_{\substack{j=1,2,...,n \\ i=1,2,...,m}}$$

that is the matrix that we get from A exchanging between them the rows and the columns. For example, given the matrix 2×3:

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix}$$

its transpose is the matrix 3×2:

$$A^T = \begin{pmatrix} 2 & 0 \\ 1 & 4 \\ -3 & 2 \end{pmatrix}$$

We also have $\left(A^T\right)^T = A$, that is the transpose of the transpose of a given matrix is the initial matrix.

A matrix $A \in \mathbb{R}^{n,n}$ (that is with the same number of rows and columns) is called square matrix of order n, and the elements with the same indeces of row and column $(a_{11}, a_{22}, ..., a_{nn})$ constitute the main diagonal.

A square matrix that coincides with its transpose, then, is said symmetric, and in this matrix the elements symmetric with respect to the main diagonal are equal, we have therefore:

$$A \text{ symmetric} \Rightarrow A = A^T \Rightarrow a_{ij} = a_{ji} \quad \forall i \neq j$$

The matrix:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

for example is symmetric.

A matrix $A \in \mathbb{R}^{n,n}$ with all the elements below the main diagonal that are null is called upper triangular matrix:

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

while a matrix $A \in \mathbb{R}^{n,n}$ with all the elements above the main diagonal that are null is called lower triangular matrix:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$

A matrix $A \in \mathbb{R}^{n,n}$ with any elements along the main diagonal and zero elements elsewhere is called diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & ... & \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

and finally the diagonal matrix that has on the main diagonal elements all equal to 1 is called unit matrix or identity matrix (of order n) and is denoted by I_n:

$$I_n = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & & ... & \\ 0 & 0 & ... & 1 \end{pmatrix}$$

7.4 Operations with matrices

At this point it is possible to introduce the operations among matrices, that are 3 and are completely analogous to those introduced for the vectors (actually, in the particular case in which the matrices have a unique row or a unique column - so that they are in reality vectors - the operations become exactly those introduced previously with reference to the vectors).

7.4.1 Sum of matrices

The first operation that can be introduced is the sum of matrices. Given $A, B \in \mathbb{R}^{m,n}$ with:

$$A = (a_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} \qquad B = (b_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}$$

the sum is the matrix $A + B \in \mathbb{R}^{m,n}$ given by:

$$A + B = (a_{ij} + b_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}$$

that is the matrix obtained summing each component of the first matrix with the corresponding component of the second matrix.

7.4.2 Multiplication matrix-scalar

The second operation is the multiplication of a matrix by a scalar (that is a real number). Given $c \in \mathbb{R}$ and $A \in \mathbb{R}^{m,n}$ with:

$$A = (a_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}$$

the product of the matrix A by the scalar c is the matrix $cA \in \mathbb{R}^{m,n}$ given by:

$$cA = (ca_{ij})_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}$$

that is the matrix obtained multiplying each component of the initial matrix by the scalar.

7.4.3 Product of matrices

The third operation is the product of matrices. Given $A \in \mathbb{R}^{m,n}$ and $B \in \mathbb{R}^{n,p}$ with:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \ldots \\ \mathbf{a}^m \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1p} \\ b_{21} & b_{22} & \ldots & b_{2p} \\ \ldots & & & \\ b_{n1} & b_{n2} & \ldots & b_{np} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^1 & \mathbf{b}^2 & \ldots & \mathbf{b}^p \end{pmatrix}$$

the product is the matrix $C = AB \in \mathbb{R}^{m,p}$ given by:

$$C = AB = \begin{bmatrix} \mathbf{a}^1\mathbf{b}^1 & \mathbf{a}^1\mathbf{b}^2 & ... & \mathbf{a}^1\mathbf{b}^p \\ \mathbf{a}^2\mathbf{b}^1 & \mathbf{a}^2\mathbf{b}^2 & ... & \mathbf{a}^2\mathbf{b}^p \\ ... & & & \\ \mathbf{a}^m\mathbf{b}^1 & \mathbf{a}^m\mathbf{b}^2 & ... & \mathbf{a}^m\mathbf{b}^p \end{bmatrix}$$

Using a different notation we have that, given $A \in \mathbb{R}^{m,n}$ and $B \in \mathbb{R}^{n,p}$ with:

$$A = (a_{ij})_{\substack{i=1,2,...,m \\ j=1,2,...,n}} \qquad B = (b_{jr})_{\substack{j=1,2,...,n \\ r=1,2,...,p}}$$

the product is the matrix $C = AB \in \mathbb{R}^{m,p}$ given by:

$$C = (c_{ir})_{\substack{i=1,2,...,m \\ r=1,2,...,p}}$$

where:

$$c_{ir} = a_{i1}b_{1r} + a_{i2}b_{2r} + ... + a_{in}b_{nr} = \sum_{s=1}^{n} a_{is}b_{sr}$$

that is the generic element in position (i, r) of the product matrix is equal to the scalar product of the i-th row of the first matrix with the r-th column of the second matrix.

In order for the product of two matrices to be defined, therefore, it is necessary that the number of the columns of the first matrix is equal to the number of the rows of the second matrix; in this case the product is a matrix that has the same number of rows of the first matrix and the same number of columns of the second matrix. In particular, then, the product between a matrix $1 \times n$ (a row vector with n components) and a matrix $n \times 1$ (a column vector with n components) is a number, and it is the scalar product introduced previously.

The product among matrices does not enjoy the commutative property (changing the order of the factors the product may not be defined, and in any case even if it is defined in general we have $AB \neq BA$), furthermore (as already seen for the scalar product) the "cancellation law" does not hold (the product of two matrices, one of which is the null matrix - that is with all the elements equal to zero - is the null matrix, however a product of matrices can be null with all the factors different from the null matrix).

Example 160 *Given the matrices* $A = \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & -4 \end{pmatrix}$ *and* $B = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ *calculate their sum.*

We have in this case:

$$A + B = \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & -4 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 4 & -2 \\ 1 & 2 & -5 \end{pmatrix}$$

that represents the sum of the two matrices.

Example 161 *Given the matrix* $A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$ *and the scalar* $c = 3$ *calculate the product* cA.

We have in this case:

$$cA = 3 \cdot \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 9 & 0 \end{pmatrix}$$

that represents the product $3A$.

Example 162 *Given the matrices* $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix}$ *calculate the product* $C = AB$.

In this case, since $A \in \mathbb{R}^{2,3}$ and $B \in \mathbb{R}^{3,3}$, the product AB exists (because the number of columns of the first matrix is equal to the number of rows of the second one) - while the product BA does not exist - and we have $C = AB \in \mathbb{R}^{2,3}$ with:

$$\begin{aligned} C &= \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix} = \\ &= \begin{pmatrix} 2 \cdot 1 + 1 \cdot 0 + (-1) \cdot 0 & 2 \cdot (-2) + 1 \cdot 3 + (-1) \cdot 2 & 2 \cdot 0 + 1 \cdot 1 + (-1) \cdot (-1) \\ 0 \cdot 1 + 1 \cdot 0 + 3 \cdot 0 & 0 \cdot (-2) + 1 \cdot 3 + 3 \cdot 2 & 0 \cdot 0 + 1 \cdot 1 + 3 \cdot (-1) \end{pmatrix} = \\ &= \begin{pmatrix} 2 & -3 & 2 \\ 0 & 9 & -2 \end{pmatrix} \end{aligned}$$

that represents the product AB.

Example 163 *Given the matrices* $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$ *calculate the product* AB *and the product* BA.

In this case, since $A \in \mathbb{R}^{2,2}$ and $B \in \mathbb{R}^{2,2}$, there exist both the product AB and the product BA and we have $AB \in \mathbb{R}^{2,2}$ and $BA \in \mathbb{R}^{2,2}$ with:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 3 & 1 \cdot 1 + 1 \cdot 3 \\ (-1) \cdot 1 + (-1) \cdot 3 & (-1) \cdot 1 + (-1) \cdot 3 \end{pmatrix} = \\ &= \begin{pmatrix} 4 & 4 \\ -4 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot (-1) & 1 \cdot 1 + 1 \cdot (-1) \\ 3 \cdot 1 + 3 \cdot (-1) & 3 \cdot 1 + 3 \cdot (-1) \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

from which it is evident that the commutative property of the product between matrices does not hold (in fact $AB \neq BA$) and the "cancellation law" of the product does not hold (in fact BA is equal to the null matrix even if neither the matrix A nor the matrix B is the null matrix).

7.5 Exercises

Compare (if it is possible) the vectors $\mathbf{x}$ *and* $\mathbf{y}$*:*

1) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

2) $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3) $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

4) $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

5) $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

6) $\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

7) $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$

8) $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$

Given the vectors $\mathbf{x}$ *and* $\mathbf{y}$ *calculate their sum, their scalar product and the product* $2\mathbf{x}$*:*

9) $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

10) $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

11) $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

12) $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$

13) $\mathbf{x} = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

14) $\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

15) $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

16) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

Calculate for which value of the parameter $\alpha \in \mathbb{R}$ *the vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthogonal. Putting* $\alpha = 1$*, then, calculate the norm of the vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *and the distance between them:*

17) $\mathbf{x} = \begin{pmatrix} 2 \\ \alpha \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

18) $\mathbf{x} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

19) $\mathbf{x} = \begin{pmatrix} 3 \\ \alpha \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$

20) $\mathbf{x} = \begin{pmatrix} \alpha \\ -2 \\ 3 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ \alpha \\ 0 \end{pmatrix}$

21) $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ \alpha \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} \alpha \\ 0 \\ 1 \end{pmatrix}$

22) $\mathbf{x} = \begin{pmatrix} \alpha \\ 2 \\ 3\alpha \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}$

23) $\mathbf{x} = \begin{pmatrix} \alpha \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

24) $\mathbf{x} = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} \alpha \\ 1 \\ 0 \end{pmatrix}$

Determine if the vectors given are linearly dependent or linearly independent:

25) $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

26) $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix}$

27) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

28) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$

29) $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

30) $\mathbf{x} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Determine for which values of $\alpha \in \mathbb{R}$ the vectors given are linearly dependent and for which values they are linearly independent:

31) $\mathbf{x} = \begin{pmatrix} 1 \\ \alpha \\ 3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$

32) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} \alpha \\ -1 \\ -3 \end{pmatrix}$

33) $\mathbf{x} = \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ \alpha \end{pmatrix} \quad \mathbf{z} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

34) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \quad \mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix}$

Given the matrices A and B, calculate their sum and the product $-2A$:

35) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$

36) $A = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$

37) $A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$

38) $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 \\ 1 & 0 \end{pmatrix}$

39) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

40) $A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$

41) $A = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

42) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Given the matrices A and B, calculate the product AB and the product BA:

43) $A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

44) $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

45) $A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

46) $A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

47) $A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \end{pmatrix}$

48) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

49) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix}$

50) $A = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$

Chapter 8

Functions of several variables

8.1 Definitions and domain

In the previous Chapters we have examined the real functions of one real variable, that is functions of the type:

$$f : X \subseteq \mathbb{R} \to \mathbb{R} \qquad \text{and also } y = f(x)$$

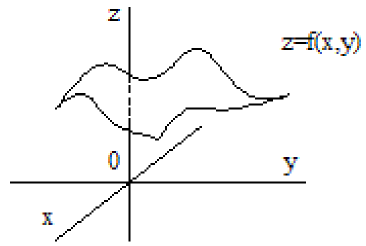
which are defined on subsets of $\mathbb{R}$ and assume values on $\mathbb{R}$. At this point it is possible to introduce some concepts concerning real functions of several real variables, that is functions of the type:

$$f : X \subseteq \mathbb{R}^n \to \mathbb{R} \qquad \text{and also } y = f(x_1, x_2, ..., x_n)$$

which are defined on subsets of $\mathbb{R}^n$ and assume values on $\mathbb{R}$. In particular, it is possible to consider real functions of 2 real variables, that is functions of the type:

$$f : X \subseteq \mathbb{R}^2 \to \mathbb{R} \qquad \text{and also } y = f(x_1, x_2) \text{ or } z = f(x, y)$$

As for real functions of one real variable, also for real functions of several real variables the starting point is represented by the determination of the domain (or field of existence). With reference to this aspect, it is possible to apply the same rules seen for functions of one variable (the denominator of a fraction cannot be null, the expression under the sign of a root with even index must be non-negative, the argument of a logarithm must be strictly positive, tha basis of a power with variable basis and exponent must be strictly positive), observing that in this case there are n independent variables. Considering the functions of 2 variables, furthermore, it is possible to give the graphic representation of the domain (that is a subset of $\mathbb{R}^2$, eventually coincident with $\mathbb{R}^2$). For these functions it is also possible a graphic representation of the functions themselves (in the tridimensional space, that is in $\mathbb{R}^3$, even if in general it is not easy to determine the graph of a function of this type), while this is not possible when we consider functions of n variables with $n > 2$.



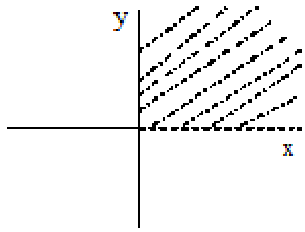
Example 164 *Determine the domain of the function:*

$$f(x,y) = \frac{\sqrt{2x}}{\sqrt{3y}}$$

In this case each expression under the sign of root must be non-negative, therefore we must have $2x \geq 0$ (that is $x \geq 0$) and $3y \geq 0$ (that is $y \geq 0$), moreover the denominator of the fraction must be different from 0 (that happens when $3y \neq 0$, i.e. when $y \neq 0$), so that the domain of the function is given by:

$$D = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0 \ \wedge \ y > 0\right\}$$

Graphically this set can be represented in the following way:



and it is formed by all the points that belong to the first quadrant, with the exclusion of those that are on the positive x half-axis.

Example 165 *Determine the domain of the function:*

$$f(x,y) = \sqrt{\frac{5x}{2y}}$$

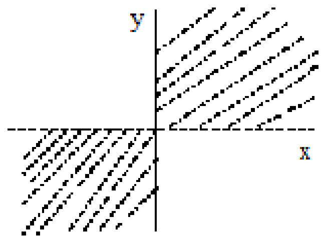
In this case the expression under the sign of root must be non-negative, therefore we must have $\dfrac{5x}{2y} \geq 0$ (that is $\dfrac{x}{y} \geq 0$), that happens when we have:

$$(x \geq 0 \ \wedge \ y > 0) \vee (x \leq 0 \ \wedge \ y < 0)$$

so that the domain of the function is given by:

$$D = \left\{(x, y) \in \mathbb{R}^2 : \frac{x}{y} \geq 0\right\} = \big\{(x, y) \in \mathbb{R}^2 : (x \geq 0 \ \wedge \ y > 0) \vee (x \leq 0 \ \wedge \ y < 0)\big\}$$

Graphically this set can be represented in the following way:



and it is formed by all the points that belong to the first and to the third quadrant of the cartesian plane, with the exclusion of those that are on the x-axis.

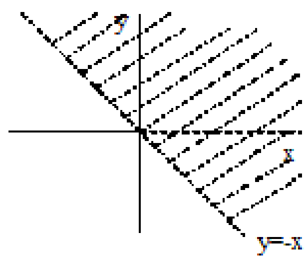
Example 166 *Determine the domain of the function:*

$$f(x, y) = \log(x + y)$$

In this case the argument of the logarithm must be strictly positive, therefore we must have $x + y > 0$, that happens when $y > -x$, so that the domain of the function is given by:

$$D = \big\{(x, y) \in \mathbb{R}^2 : x + y > 0\big\} = \big\{(x, y) \in \mathbb{R}^2 : y > -x\big\}$$

Graphically this set can be represented in the following way:



and it is formed by all the points that are in the half-plane above the straight line of equation $y = -x$, with the exclusion of the points that belong to the same straight line.

Example 167 *Determine the domain of the function:*

$$f(x,y) = \log(x^2 - y^2)$$

In this case the argument of the logarithm must be strictly positive, therefore we must have $x^2 - y^2 > 0$, and therefore:

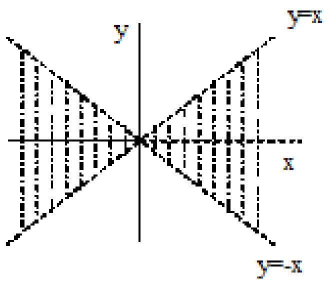
$$x^2 - y^2 > 0 \Rightarrow (x-y)(x+y) > 0 \Rightarrow \begin{cases} x - y > 0 \\ x + y > 0 \end{cases} \vee \begin{cases} x - y < 0 \\ x + y < 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y < x \\ y > -x \end{cases} \vee \begin{cases} y > x \\ y < -x \end{cases}$$

so that the domain of the function is given by:

$$D = \left\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 0\right\} = \left\{(x,y) \in \mathbb{R}^2 : (-x < y < x) \quad \vee \quad (x < y < -x)\right\}$$

Graphically this set can be represented in the following way:



and it is formed by all the points that are included between the straight lines of equation $y = x$ and $y = -x$, with the exclusion of the points that belong to the same straight lines.

8.2 Partial derivatives, differential, tangent plane

Also for functions of several variables it is possible to introduce the notions of derivative and differential, and since in this case there are several independent variables there are several derivatives of the same order, that are called partial derivatives.

Given a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ and given a point (x_0, y_0) interior with respect to X, the partial derivative of f with respect to the variable x in (x_0, y_0) (denoted with $\dfrac{\partial f}{\partial x}(x_0, y_0)$ or with $f_x(x_0, y_0)$ or with $D_x f(x_0, y_0)$) is defined as the limit of the difference quotient of f built starting from the point (x_0, y_0) increasing only the variable x, provided this limit exists finite:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = D_x f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

In a similar way it is possible to define the partial derivative of f with respect to the variable y in (x_0, y_0) (denoted with $\dfrac{\partial f}{\partial y}(x_0, y_0)$ or with $f_y(x_0, y_0)$ or with $D_y f(x_0, y_0)$) as the limit of the difference quotient of f built starting from the point (x_0, y_0) increasing only the variable y, provided this limit exists finite:

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = D_y f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

If f is a function defined in a certain set and has first partial derivatives in each point (x, y) interior with respect to that set, then, at each pair (x, y) it is possible to associate the first partial derivatives of f in that point, $f_x(x, y)$ and $f_y(x, y)$, obtaining in this way two functions that are called first partial derivatives of f.

It is then possible to define the second partial derivatives, obtained with a similar procedure starting from first partial derivatives. The second partial derivatives turn out to be 4 and they are denoted with the symbols:

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) \qquad \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \qquad \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \qquad \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

or:

$$f_{xx}(x_0, y_0) \qquad f_{xy}(x_0, y_0) \qquad f_{yx}(x_0, y_0) \qquad f_{yy}(x_0, y_0)$$

or also:

$$D_{xx} f(x_0, y_0) \qquad D_{xy} f(x_0, y_0) \qquad D_{yx} f(x_0, y_0) \qquad D_{yy} f(x_0, y_0)$$

where f_{xx} and f_{yy} are called pure second derivatives (in particular f_{xx} is obtained starting from f_x and deriving it again with respect to x, while f_{yy} is obtained starting from f_y and deriving it again with respect to y), while f_{xy} and f_{yx} are called mixed second derivatives (in particular f_{xy} is obtained starting from f_x and deriving it with respect to y, while f_{yx} is obtained starting from f_y and deriving it with respect to x, furthermore if such derivatives are continuous we have $f_{xy} = f_{yx}$, that is the order of derivation is irrelevant).

Also in this case if f is defined in a certain set and has second partial derivatives in each point (x, y) interior with respect to that set it is possible to introduce the functions called second partial derivatives of f.

The first partial derivatives can then be collected in a (row) vector that is called gradient of the function f:

$$\bigtriangledown f(x, y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix}$$

while the second partial derivatives can be collected in a (square and symmetric) matrix that is called hessian matrix of the function f:

$$\bigtriangledown^2 f(x, y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

In practice, for the calculation of the partial derivatives of a function of several variables we don't use the definition (as already seen for the derivatives of real functions of one real variable) but we use the rules valid for the calculation of the derivatives of functions of one variable, observing that, when we derive with respect to one variable, the other variables must be considered as constants.

Example 168 *Calculate the gradient and the hessian matrix of the function:*

$$f(x,y) = 4x^2y$$

and evaluate them in the point $P = (1,0)$.

In this case the first partial derivatives of the function are (observing that when we derive with respect to x the variable y must be considered as a constant and when we derive with respect to y the variable x must be considered as a constant):

$$\frac{\partial f}{\partial x} = 4y \cdot 2x = 8xy \qquad \frac{\partial f}{\partial y} = 4x^2 \cdot 1 = 4x^2$$

and the second partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 8y \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 8x \qquad \frac{\partial^2 f}{\partial y^2} = 0$$

The gradient of the function is then given by:

$$\bigtriangledown f(x,y) = \begin{pmatrix} 8xy & 4x^2 \end{pmatrix}$$

while the hessian matrix is given by:

$$\bigtriangledown^2 f(x,y) = \begin{pmatrix} 8y & 8x \\ 8x & 0 \end{pmatrix}$$

In particular, in the point $P = (1,0)$ the gradient is:

$$\bigtriangledown f(1,0) = (0 \quad 4)$$

while the hessian matrix is:

$$\bigtriangledown^2 f(1,0) = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}$$

Example 169 *Calculate the gradient and the hessian matrix of the function:*

$$f(x, y) = 3x^2(x - 2y)$$

and evaluate them in the point $P = (1, 2)$.

In this case we have first of all:

$$f(x, y) = 3x^3 - 6x^2 y$$

then the first partial derivatives of the function are:

$$\frac{\partial f}{\partial x} = 9x^2 - 12xy \qquad \frac{\partial f}{\partial y} = -6x^2$$

and the second partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 18x - 12y \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -12x \qquad \frac{\partial^2 f}{\partial y^2} = 0$$

The gradient of the function is then given by:

$$\bigtriangledown f(x, y) = \begin{pmatrix} 9x^2 - 12xy & -6x^2 \end{pmatrix}$$

while the hessian matrix is given by:

$$\bigtriangledown^2 f(x, y) = \begin{pmatrix} 18x - 12y & -12x \\ -12x & 0 \end{pmatrix}$$

In particular, in the point $P = (1, 2)$ the gradient is:

$$\bigtriangledown f(1, 2) = (-15 \quad -6)$$

while the hessian matrix is:

$$\bigtriangledown^2 f(1, 2) = \begin{pmatrix} -6 & -12 \\ -12 & 0 \end{pmatrix}$$

Example 170 *Calculate the gradient and the hessian matrix of the function:*

$$f(x,y) = 2x\log y + 3xy$$

In this case the first partial derivatives of the function are:

$$\frac{\partial f}{\partial x} = 2\log y + 3y \qquad \frac{\partial f}{\partial y} = \frac{2x}{y} + 3x$$

and the second partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{y} + 3 \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{2x}{y^2}$$

The gradient of the function is then given by:

$$\bigtriangledown f(x,y) = \begin{pmatrix} 2\log y + 3y & \frac{2x}{y} + 3x \end{pmatrix}$$

while the hessian matrix is given by:

$$\bigtriangledown^2 f(x,y) = \begin{pmatrix} 0 & \frac{2}{y} + 3 \\ \frac{2}{y} + 3 & -\frac{2x}{y^2} \end{pmatrix}$$

Example 171 *Calculate the gradient and the hessian matrix of the function:*

$$f(x,y) = x^y$$

In this case first of all the function must be rewritten in the following way:

$$f(x,y) = x^y = e^{y\log x}$$

then the first partial derivatives of the function are:

$$\frac{\partial f}{\partial x} = e^{y\log x} \cdot \frac{y}{x} \qquad \frac{\partial f}{\partial y} = e^{y\log x} \cdot \log x$$

and the second partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = e^{y\log x} \cdot \left(-\frac{y}{x^2}\right) + \frac{y}{x} \cdot e^{y\log x} \cdot \frac{y}{x} = \frac{y}{x^2} \cdot e^{y\log x} \cdot (y-1)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = e^{y\log x} \cdot \frac{1}{x} + \frac{y}{x} \cdot e^{y\log x} \cdot \log x = \frac{1}{x} \cdot e^{y\log x} \cdot (1 + y\log x)$$

$$\frac{\partial^2 f}{\partial y^2} = \log x \cdot e^{y\log x} \cdot \log x = e^{y\log x} \cdot \log^2 x$$

The gradient of the function is then given by:

$$\bigtriangledown f(x,y) = \left(x^y \cdot \frac{y}{x} \qquad x^y \cdot \log x\right)$$

while the hessian matrix is given by:

$$\bigtriangledown^2 f(x,y) = \begin{pmatrix} \dfrac{y}{x^2} \cdot x^y \cdot (y-1) & \dfrac{1}{x} \cdot x^y \cdot (1 + y \log x) \\ \\ \dfrac{1}{x} \cdot x^y \cdot (1 + y \log x) & x^y \cdot \log^2 x \end{pmatrix}$$

As we have seen in Chapter 5, given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$, that is $y = f(x)$, we say that this function is differentiable at a point x_0 interior with respect to X if it is possible to write:

$$f(x) - f(x_0) = f'(x_0)\,(x - x_0) + R \qquad \text{as } x \to x_0$$

In this case the expression $f'(x_0)\,(x - x_0)$ is called differential of the function in the point x_0, denoted by:

$$dy = f'(x_0)dx$$

and, close to the point x_0, a good approximation of the function f is given by the straight line tangent to the function itself in correspondence of x_0, whose equation is:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

In a similar way, given a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, that is $z = f(x,y)$, we say that this function is differentiable at a point (x_0, y_0) interior with respect to X if it is possible to write:

$$f(x,y) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + R \qquad \text{as } (x,y) \to (x_0,y_0)$$

In this case the expression $\frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$ is called total differential of the function in the point (x_0, y_0), denoted by:

$$dz = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy$$

while each of the two terms $\dfrac{\partial f}{\partial x}(x_0,y_0)dx$ and $\dfrac{\partial f}{\partial y}(x_0,y_0)dy$ constitutes a partial differential (the first with respect to the variable x and the second with respect to the variable y). Close to the point (x_0, y_0), then, a good approximation of the function f is given by the plane tangent to the function itself in correspondence of (x_0, y_0), whose equation is:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

In the case of a function of two (or more) variables, then, the notion of derivability and that of differentiability are not equivalent (contrary to what happens for functions of one variable). In particular, the notion of differentiability implies that of derivability but the viceversa is not necessarily true, i.e. if a function is differentiable at a point then it has partial derivatives in that point, while if it has partial derivatives it is not guaranteed that it is differentiable (but if these derivatives are continuous then the differentiability is guaranteed).

Example 172 *Given the function:*

$$f(x,y) = 5 - 3x^2 - 2y^2$$

write the expression of its total differential, evaluate it in correspondence to the point $P = (1,2)$ *and determine the equation of the plane tangent to the function in correspondence to this point.*

In this case we have first of all:

$$\frac{\partial f}{\partial x} = -6x \qquad \frac{\partial f}{\partial y} = -4y$$

and the total differential of f is given by:

$$dz = -6xdx - 4ydy$$

while in correspondence to the point $P = (1,2)$ such differential is:

$$dz = -6dx - 8dy$$

The equation of the plane tangent to the function in correspondence to the point P, then, is given by:

$$z = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

and since we have:

$$f(1,2) = -6 \qquad \frac{\partial f}{\partial x}(1,2) = -6 \qquad \frac{\partial f}{\partial y}(1,2) = -8$$

we get:

$$\begin{aligned} z &= -6 - 6(x-1) - 8(y-2) = -6 - 6x + 6 - 8y + 16 = \\ &= 16 - 6x - 8y \end{aligned}$$

so that the equation of the plane tangent in the point $P = (1,2)$ is $z = 16 - 6x - 8y$.

Example 173 *Given the function:*

$$f(x, y) = 3e^x + 2y^2$$

write the expression of its total differential, evaluate it in correspondence to the point $P = (0, 1)$ and determine the equation of the plane tangent to the function in correspondence to this point.

In this case we have first of all:

$$\frac{\partial f}{\partial x} = 3e^x \qquad \frac{\partial f}{\partial y} = 4y$$

and the total differential of f is given by:

$$dz = 3e^x dx + 4ydy$$

while in correspondence to the point $P = (0, 1)$ such differential is:

$$dz = 3dx + 4dy$$

The equation of the plane tangent to the function in correspondence to the point P, then, is given by:

$$z = f(0, 1) + \frac{\partial f}{\partial x}(0, 1)(x - 0) + \frac{\partial f}{\partial y}(0, 1)(y - 1)$$

and since we have:

$$f(0, 1) = 5 \qquad \frac{\partial f}{\partial x}(0, 1) = 3 \qquad \frac{\partial f}{\partial y}(0, 1) = 4$$

we get:

$$\begin{aligned} z &= 5 + 3(x - 0) + 4(y - 1) = 5 + 3x + 4y - 4 = \\ &= 1 + 3x + 4y \end{aligned}$$

so that the equation of the plane tangent in the point $P = (0, 1)$ is $z = 1 + 3x + 4y$.

8.3 Unconstrained maxima and minima

With reference to the functions of several variables a problem of great interest is represented by the analysis of eventual extrema (maxima and minima). In particular it is possible to examine, with reference to a function of 2 variables, the analysis of unconstrained (i.e. without any constraint) maxima and minima. With reference to this aspect, given a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ with first derivatives on X, with (x_0, y_0) interior point with respect to X, the following results hold:

(a) Necessary conditions

$$(x_0, y_0) \text{ point of relative maximum } \Rightarrow \bigtriangledown f(x_0, y_0) = \mathbf{0}$$

$$(x_0, y_0) \text{ point of relative minimum} \Rightarrow \bigtriangledown f(x_0, y_0) = \mathbf{0}$$

Given a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ with second derivatives on X, with second partial derivatives continuous in a point (x_0, y_0) interior with respect to X, the following results then hold (where (x_0, y_0) is a stationary point, i.e. such that $\bigtriangledown f(x_0, y_0) = \mathbf{0}$):

(b) Sufficient conditions

$$\left.\begin{array}{l} \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0 \\ \\ \det \bigtriangledown^2 f(x_0, y_0) > 0 \end{array}\right\} \Rightarrow (x_0, y_0) \text{ point of relative maximum}$$

$$\left.\begin{array}{l} \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0 \\ \\ \det \bigtriangledown^2 f(x_0, y_0) > 0 \end{array}\right\} \Rightarrow (x_0, y_0) \text{ point of relative minimum}$$

$$\det \bigtriangledown^2 f(x_0, y_0) < 0 \ \Rightarrow (x_0, y_0) \text{ saddle point}$$

where $\det \bigtriangledown^2 f(x, y)$ is the determinant of the hessian matrix of the function f:

$$\bigtriangledown^2 f(x, y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

that is defined by the expression:

$$\det \bigtriangledown^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial^2 f}{\partial x \partial y}$$

constituted by the difference between the product of the two terms that are on the main diagonal and the product of the two terms that are on the secondary diagonal.

In practice, to find the presence of eventual unconstrained maxima or minima of a function of 2 variables first of all we look for the points that make the gradient of the same function equal to zero (and that are called stationary points), then we compute, in correspondence to these points, the second derivative $\dfrac{\partial^2 f}{\partial x^2}$ and the determinant of the hessian matrix. If the sequence of the signs of these two quantities is $-$, $+$ we can conclude that we have a relative maximum, while if it is $+$, $+$ we have a relative minimum, and if the determinant is negative (independently by the sign of $\dfrac{\partial^2 f}{\partial x^2}$) we have a saddle point (that is a point which is a maximum with respect to one of the two variables and a minimum with respect to the other). In the case in which the determinant of the hessian matrix is equal to 0 (independently by the sign of $\frac{\partial^2 f}{\partial x^2}$), finally, it is not possible to conclude anything with certainty.

Example 174 *Determine the eventual maxima and minima of the function:*

$$f(x, y) = 3x^2 + 4y^2$$

In this case we have first of all that the first partial derivatives of the function are given by:

$$\frac{\partial f}{\partial x} = 6x \qquad \frac{\partial f}{\partial y} = 8y$$

and using the necessary conditions we get:

$$\nabla f(x, y) = \mathbf{0} \Rightarrow \begin{cases} 6x = 0 \\ 8y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

so that $A = (0, 0)$ represents the unique stationary point for the function considered.

At this point to establish the nature of the stationary point it is necessary to use the sufficient conditions, with reference to this aspect we have first of all that the second partial derivatives of the function are given by:

$$\frac{\partial^2 f}{\partial x^2} = 6 \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 8$$

so that the hessian matrix is:

$$\nabla^2 f(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}$$

This is also the hessian matrix in the point $A = (0, 0)$, and since we have:

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = 6 \qquad \det \nabla^2 f(0, 0) = 48$$

the point $A = (0, 0)$ is a point of relative minimum for the function (while there are no relative maxima).

Example 175 *Determine the eventual maxima and minima of the function:*

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 - 4x + y$$

In this case the first partial derivatives of the function are given by:

$$\frac{\partial f}{\partial x} = x^2 - 4 \qquad \frac{\partial f}{\partial y} = y + 1$$

and using the necessary conditions we get:

$$\bigtriangledown f(x,y) = \mathbf{0} \Rightarrow \begin{cases} x^2 - 4 = 0 \\ \\ y + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ \\ y = -1 \end{cases} \vee \begin{cases} x = 2 \\ \\ y = -1 \end{cases}$$

so that $A = (-2, -1)$ and $B = (2, -1)$ are the stationary points for the function considered.

We then have that the second partial derivatives of the function are given by:

$$\frac{\partial^2 f}{\partial x^2} = 2x \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 1$$

so that the hessian matrix is:

$$\bigtriangledown^2 f(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

In the point $A = (-2, -1)$ this matrix is:

$$\bigtriangledown^2 f(-2,-1) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}(-2,-1) = -4 \qquad \det \bigtriangledown^2 f(-2,-1) = -4$$

the point $A = (-2, -1)$ is a saddle point.

In the point $B = (2, -1)$, then, the hessian matrix is:

$$\bigtriangledown^2 f(2,-1) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}(2,-1) = 4 \qquad \det \bigtriangledown^2 f(2,-1) = 4$$

the point $B = (2, -1)$ is a point of relative minimum. In conclusion, the function has a relative minimum and a saddle point, while it doesn't have relative maxima.

Example 176 *Determine the eventual maxima and minima of the function:*

$$f(x,y) = -x^3 - y^2 - xy$$

In this case the first partial derivatives of the function are given by:

$$\frac{\partial f}{\partial x} = -3x^2 - y \qquad \frac{\partial f}{\partial y} = -2y - x$$

and using the necessary conditions we get:

$$\nabla f(x,y) = \mathbf{0} \Rightarrow \begin{cases} -3x^2 - y = 0 \\ -2y - x = 0 \end{cases} \Rightarrow \begin{cases} -3x^2 - y = 0 \\ y = -\frac{1}{2}x \end{cases} \Rightarrow \begin{cases} -3x^2 + \frac{1}{2}x = 0 \\ y = -\frac{1}{2}x \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x(-3x + \frac{1}{2}) = 0 \\ y = -\frac{1}{2}x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = \dfrac{1}{6} \\ y = -\dfrac{1}{12} \end{cases}$$

so that $A = (0,0)$ and $B = \left(\frac{1}{6}, -\frac{1}{12}\right)$ are the statioanry points for the function considered.

We then have that the second partial derivatives of the function are given by:

$$\frac{\partial^2 f}{\partial x^2} = -6x \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -1 \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

so that the hessian matrix is:

$$\nabla^2 f(x,y) = \begin{pmatrix} -6x & -1 \\ -1 & -2 \end{pmatrix}$$

In the point $A = (0,0)$ this matrix is:

$$\nabla^2 f(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0 \qquad \det \nabla^2 f(0,0) = -1$$

the point $A = (0,0)$ is a saddle point.

In the point $B = \left(\frac{1}{6}, -\frac{1}{12}\right)$, then, the hessian matrix is:

$$\nabla^2 f\left(\frac{1}{6}, -\frac{1}{12}\right) = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{6},-\frac{1}{12}\right)=-1 \qquad\qquad \det \nabla^2 f\left(\frac{1}{6},-\frac{1}{12}\right)=1$$

the point $B=\left(\frac{1}{6},-\frac{1}{12}\right)$ is a point of relative maximum. In conclusion, the function has a relative maximum and a saddle point, while it has no relative minima.

Example 177 *Determine the eventual maxima and minima of the function:*

$$f(x,y)=\frac{1}{3}x^3-4x-y^3+3y$$

In this case the first partial derivatives of the function are given by:

$$\frac{\partial f}{\partial x}=x^2-4 \qquad\qquad \frac{\partial f}{\partial y}=-3y^2+3$$

and using the necessary conditions we get:

$$\nabla f(x,y)=\mathbf{0}\Rightarrow\left\{\begin{array}{l} x^2-4=0 \\ \\ -3y^2+3=0 \end{array}\right. \Rightarrow\left\{\begin{array}{l} x^2=4 \\ \\ y^2=1 \end{array}\right. \Rightarrow$$
$$\Rightarrow\left\{\begin{array}{l} x=-2 \\ \\ y=-1 \end{array}\right. \vee\left\{\begin{array}{l} x=-2 \\ \\ y=1 \end{array}\right. \vee\left\{\begin{array}{l} x=2 \\ \\ y=-1 \end{array}\right. \vee\left\{\begin{array}{l} x=2 \\ \\ y=1 \end{array}\right.$$

so that $A=(-2,-1)$, $B=(-2,1)$, $C=(2,-1)$ and $D=(2,1)$ are the stationary points for the function considered.

We have then that the second partial derivatives of the function are given by:

$$\frac{\partial^2 f}{\partial x^2}=2x \qquad\qquad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}=0 \qquad\qquad \frac{\partial^2 f}{\partial y^2}=-6y$$

so that the hessian matrix is:

$$\nabla^2 f(x,y)=\left(\begin{array}{cc} 2x & 0 \\ 0 & -6y \end{array}\right)$$

In the point $A=(-2,-1)$ this matrix is:

$$\nabla^2 f(-2,-1)=\left(\begin{array}{cc} -4 & 0 \\ 0 & 6 \end{array}\right)$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}(-2,-1)=-4 \qquad\qquad \det \nabla^2 f(-2,-1)=-24$$

the point $A = (-2, -1)$ is a saddle point.
In the point $B = (-2, 1)$, then, the hessian matrix is:

$$\nabla^2 f\,(-2,1) = \begin{pmatrix} -4 & 0 \\ 0 & -6 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}\,(-2,1) = -4 \qquad\qquad \det \nabla^2 f(-2,1) = 24$$

the point $B = (-2, 1)$ is a point of relative maximum.
In the point $C = (2, -1)$, then, the hessian matrix is:

$$\nabla^2 f\,(2,-1) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}\,(2,-1) = 4 \qquad\qquad \det \nabla^2 f(2,-1) = 24$$

the point $C = (2, -1)$ is a point of relative minimum.
In the point $D = (2, 1)$, finally, the hessian matrix is:

$$\nabla^2 f\,(2,1) = \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}$$

and since we have:

$$\frac{\partial^2 f}{\partial x^2}\,(2,1) = 4 \qquad\qquad \det \nabla^2 f(2,1) = -24$$

the point $D = (2, 1)$ is a saddle point. In conclusion, the function has a relative minimum, a relative maximum and two saddle points.

8.4 Exercises

Determine the domain of the following functions:

1) $f(x, y) = \sqrt{xy}$

2) $f(x, y) = \sqrt{x}\sqrt{y}$

3) $f(x, y) = \log(xy)$

4) $f(x, y) = \log x \cdot \log y$

5) $f(x, y) = \dfrac{xy}{\log(xy)}$

6) $f(x,y) = \sqrt[3]{xy} + \sqrt{x^2+y^2}$

7) $f(x,y) = e^{\sqrt{x^2+y^2}}$

8) $f(x,y) = \log\sqrt{\dfrac{x^2}{y}}$

9) $f(x,y) = \log\sqrt{xy^2}$

10) $f(x,y) = \log(x^2+y^2)$

11) $f(x,y) = e^{\log\left(x^2+y^2\right)}$

12) $f(x,y) = \log\sqrt{x^2y}$

13) $f(x,y) = \log\left(x^2+y^2+1\right)$

14) $f(x,y) = \dfrac{\log(xy)}{xy}$

15) $f(x,y) = \dfrac{x}{\sqrt{x+y}}$

Determine the gradient and the hessian matrix of the following functions in the point $P = (x_0, y_0)$:

16) $f(x,y) = \log(x+y)$ with $P = (1,0)$

17) $f(x,y) = e^{x+y} + xy + y^2$ with $P = (1,-1)$

18) $f(x,y) = \dfrac{6}{x} + xy$ with $P = (1,0)$

19) $f(x,y) = e^{x^2+y}$ with $P = (1,-1)$

20) $f(x,y) = e^x + e^y$ with $P = (1,1)$

21) $f(x,y) = e^{x^2+y^2}$ with $P = (0,0)$

22) $f(x,y) = \sqrt{x+y}$ with $P = (2,2)$

23) $f(x,y) = \log\left(e^x + y^2\right)$ with $P = (0,0)$

24) $f(x,y) = y^{\log x}$ with $P = (1,1)$

25) $f(x,y) = 3x^2y - 5xy^2$ with $P = (1,0)$

Determine the total differential of the following functions in the point $P = (x_0, y_0)$:

26) $f(x, y) = e^x + 2y$ with $P = (0, 1)$

27) $f(x, y) = \log(xy) + e^{x+y}$ with $P = (2, -2)$

28) $f(x, y) = e^x + e^y$ with $P = (0, 1)$

29) $f(x, y) = \sqrt{x^2 + y^2}$ with $P = (3, 4)$

30) $f(x, y) = e^{x^2+y}$ with $P = (1, -1)$

31) $f(x, y) = \dfrac{x}{\sqrt{x + y}}$ with $P = (2, 2)$

32) $f(x, y) = \log \sqrt{xy^2}$ with $P = (3, 2)$

Determine the equation of the plane tangent to the following functions in the point $P = (x_0, y_0)$:

33) $f(x, y) = e^x + y^2$ with $P = (0, -1)$

34) $f(x, y) = \sqrt{x + y}$ with $P = (1, 3)$

35) $f(x, y) = e^x + y^3$ with $P = (0, 1)$

36) $f(x, y) = e^x + e^y$ with $P = (0, 0)$

37) $f(x, y) = e^{x+y^2}$ with $P = (-1, 1)$

38) $f(x, y) = \log(xy)$ with $P = (1, 1)$

39) $f(x, y) = \sqrt{x^2 + y^2}$ with $P = (3, 4)$

40) $f(x, y) = \dfrac{1}{x + y}$ with $P = (1, 0)$

Determine eventual maxima and minima of the following functions:

41) $f(x,y) = x^3 - 12x - y^2$

42) $f(x,y) = \dfrac{1}{2}x^2 + y^3 - xy$

43) $f(x,y) = x^4 - 2x^2 + y^2 - y$

44) $f(x,y) = x^3 + 3x^2 + 4y^2 + 1$

45) $f(x,y) = \left(x^2 - 1\right)(y - 1)$

46) $f(x,y) = xe^x + y^2$

47) $f(x,y) = \dfrac{3}{x} + \dfrac{1}{y} + xy$

48) $f(x,y) = x^3 - 6x - y^2$

49) $f(x,y) = \log(1 + x + y) - 5x - y^2$

50) $f(x,y) = -x^2 + xy - y^2$

Chapter 9

Solutions of the exercises

9.1 Exercises Chapter 1

1) $x < -1$

2) $x \geq -7$

3) $x < -7 \quad \vee \quad x > 3$

4) $0 \leq x \leq 5$

5) $x \neq 3$

6) no value of $x \in \mathbb{R}$

7) $-\dfrac{1}{2} \leq x < 3$

8) $x > 0$

9) $x > 0$

10) $x \leq -5 \quad \vee \quad x = 3$

11) no value of $x \in \mathbb{R}$

12) $-6 \leq x < \dfrac{7}{2}$

13) $-1 < x < 1$

14) $3 < x \leq 5$

15) no value of $x \in \mathbb{R}$

16) each value of $x \in \mathbb{R}$

17) no value of $x \in \mathbb{R}$

18) $x \neq 2$

19) each value of $x \in \mathbb{R}$

20) $x > 0$

21) $x > 1$

22) $x \geq 2$

23) no value of $x \in \mathbb{R}$

24) $x > 4$

25) $-5 \leq x < 4$

26) $x \geq \dfrac{-1+\sqrt{5}}{2}$

27) each value of $x \in \mathbb{R}$

28) $x \neq 0$

29) $x \leq -\sqrt{2} \quad \vee \quad x \geq \sqrt{2}$

30) $x \geq -1$

31) $-2 < x < 7$

32) $x > -\dfrac{17}{9}$

33) no value of $x \in \mathbb{R}$

34) no value of $x \in \mathbb{R}$

35) $x < -3 \quad \vee \quad x > 3$

36) $x < -2 \quad \vee \quad x > 2$

37) $-2 < x < 2$

38) $x < -1 \quad \vee \quad x > 1$

39) $1 < x \leq 10$

40) $x \geq -5$

41) $-3 < x \leq e^2 - 3$

42) $x \leq 0 \quad \vee \quad x \geq 4$

43) $x \leq 0 \quad \vee \quad x \geq 3$

44) $x \leq 0 \quad \vee \quad x \geq 4$

45) $0 \leq x \leq 4$

46) $x \leq -4 \quad \vee \quad x \geq 4$

47) $-2 < x < 2$

48) $x < 0$

49) $x \geq \dfrac{\log 4 - \log 2}{\log 4 - \log 3}$

50) $x < -\dfrac{\log 2 + \log 3}{\log 2 - \log 3}$

9.2 Exercises Chapter 2

1) $A \cup B = \{0, 1, 2\}$ $\quad A \cap B = \{1\}$ $\quad A \setminus B = \{0\}$ $\quad B \setminus A = \{2\}$

2) $A \cup B = \{0, 1, 2, 3\}$ $\quad A \cap B = \emptyset$ $\quad A \setminus B = \{0, 1\}$ $\quad B \setminus A = \{2, 3\}$

3) $A \cup B = \{1, 2, 3, 4, 5\}$ $\quad A \cap B = \{3\}$ $\quad A \setminus B = \{1, 5\}$ $\quad B \setminus A = \{2, 4\}$

4) $(-3, 5)$

5) $(-8, 5]$

6) $(-3, 7)$

7) $\emptyset$ (empty set)

8) $(-\infty, 1) \cup (8, +\infty)$

9) $\emptyset$ (empty set)

10) $\{0\}$

11) $(-\infty, 0] \cup [5, +\infty)$

12) $(-\infty, -1] \cup [0, +\infty)$

13) $\mathbb{R}$

14) $A \times B = \{(0, 0), (0, -1), (1, 0), (1, -1)\}$ and $B \times A = \{(0, 0), (0, 1), (-1, 0), (-1, 1)\}$

15) $A \times B = \{(-1, 0), (-1, 1), (1, 0), (1, 1)\}$ and $B \times A = \{(0, -1), (0, 1), (1, -1), (1, 1)\}$

16) $A \times B = \{(0, -1), (0, 2), (0, 3), (1, -1), (1, 2), (1, 3)\}$ and $B \times A = \{(-1, 0), (-1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$

17) $X = (-1, 2]$, $\max = 2$, $\min \nexists$, $\sup = 2$, $\inf = -1$, the interior points are those of $(-1, 2)$, the exterior points are those of $(-\infty, -1) \cup (2, +\infty)$, the boundary points

are those of $\{-1, 2\}$, the accumulation points are those of $[-1, 2]$, the set is neither open nor closed and it is bounded.

18) $X = (-4, -3)$, $\max \nexists$, $\min \nexists$, $\sup = -3$, $\inf = -4$, the interior points are those of $(-4, -3)$, the exterior points are those of $(-\infty, -4) \cup (-3, +\infty)$, the boundary points are those of $\{-4, -3\}$, the accumulation points are those of $[-4, -3]$, the set is open and it is bounded.

19) $X = [-5, 3] \cup [4, +\infty)$, $\max \nexists$, $\min = -5$, $\sup = +\infty$, $\inf = -5$, the interior points are those of $(-5, 3) \cup (4, +\infty)$, the exterior points are those of $(-\infty, -5) \cup (3, 4)$, the boundary points are those of $\{-5, 3, 4\}$, the accumulation points are those of $[-5, 3] \cup [4, +\infty)$, the set is closed and it is unbounded (from above).

20) $X = (-\infty, -2) \cup [2, +\infty)$, $\max \nexists$, $\min \nexists$, $\sup = +\infty$, $\inf = -\infty$, the interior points are those of $(-\infty, -2) \cup (2, +\infty)$, the exterior points are those of $(-2, 2)$, the boundary points are those of $\{-2, 2\}$, the accumulation points are those of $(-\infty, -2] \cup [2, +\infty)$, the set is neither open nor closed and it is unbounded.

21) $X = (-\infty, -3) \cup [-2, -1] \cup (0, +\infty)$, $\max \nexists$, $\min \nexists$, $\sup = +\infty$, $\inf = -\infty$, the interior points are those of $(-\infty, -3) \cup (-2, -1) \cup (0, +\infty)$, the exterior points are those of $(-3, -2) \cup (-1, 0)$, the boundary points are those of $\{-3, -2, -1, 0\}$, the accumulation points are those of $(-\infty, -3] \cup [-2, -1] \cup [0, +\infty)$, the set is neither open nor closed and it is unbounded.

22) $X = (-\infty, -2] \cup [2, +\infty)$, $\max \nexists$, $\min \nexists$, $\sup = +\infty$, $\inf = -\infty$, the interior points are those of $(-\infty, -2) \cup (2, +\infty)$, the exterior points are those of $(-2, 2)$, the boundary points are those of $\{-2, 2\}$, the accumulation points are those of $(-\infty, -2] \cup [2, +\infty)$, the set is closed and it is unbounded.

23) $X = (-\infty, -1] \cup (1, +\infty)$, $\max \nexists$, $\min \nexists$, $\sup = +\infty$, $\inf = -\infty$, the interior points are those of $(-\infty, -1) \cup (1, +\infty)$, the exterior points are those of $(-1, 1)$, the boundary points are those of $\{-1, 1\}$, the accumulation points are those of $(-\infty, -1] \cup [1, +\infty)$, the set is neither open nor closed and it is unbounded.

24) $X = (-\infty, -2] \cup (3, +\infty)$, $\max \nexists$, $\min \nexists$, $\sup = +\infty$, $\inf = -\infty$, the interior points are those of $(-\infty, -2) \cup (3, +\infty)$, the exterior points are those of $(-2, 3)$, the boundary points are those of $\{-2, 3\}$, the accumulation points are those of $(-\infty, -2] \cup [3, +\infty)$, the set is neither open nor closed and it is unbounded.

25) $X = (-2, 3] \cup \{4\}$, $\max = 4$, $\min \nexists$, $\sup = 4$, $\inf = -2$, the interior points are those of $(-2, 3)$, the exterior points are those of $(-\infty, -2) \cup (3, 4) \cup (4, +\infty)$, the boundary points are those of $\{-2, 3, 4\}$, the accumulation points are those of $[-2, 3]$, the isolated point is $\{4\}$, the set is neither open nor closed and it is bounded.

26) The table of truth of $\sim p \wedge q$ is:

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

27) The table of truth of $\sim (\sim p \wedge q)$ is:

p	q	$\sim p$	$\sim p \wedge q$	$\sim (\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

28) The table of truth of $\sim (p \Rightarrow \sim q)$ is:

p	q	$\sim q$	$p \Rightarrow \sim q$	$\sim (p \Rightarrow \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

29) The table of truth of $\sim p \Rightarrow \sim q$ is:

p	q	$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

30) The table of truth of $p \Leftrightarrow \sim q$ is:

p	q	$\sim q$	$p \Leftrightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

31) The table of truth of $\sim p \Rightarrow q$ is:

p	q	$\sim p$	$\sim p \Rightarrow q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

32) The table of truth of $\sim q \Rightarrow p$ is:

p	q	$\sim q$	$\sim q \Rightarrow p$
T	T	F	T
T	F	T	T
F	T	F	T
F	F	T	F

33) The table of truth of $q \Rightarrow p$ is:

p	q	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

34) The table of truth of $\sim p \vee \sim q$ is:

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

35) The table of truth of $\sim p \Leftrightarrow \sim q$ is:

p	q	$\sim p$	$\sim q$	$\sim p \Leftrightarrow \sim q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

36) The table of truth of $p \Leftrightarrow q$ is:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

37) The table of truth of $p \Rightarrow \sim q$ is:

p	q	$\sim q$	$p \Rightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

38) The table of truth of $\sim p \Leftrightarrow q$ is:

p	q	$\sim p$	$\sim p \Leftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

39) The table of truth of $p\wedge \sim q$ is:

p	q	$\sim q$	$p\wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

40) The table of truth of $\sim q \Rightarrow \sim p$ is:

p	q	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

41) The table of truth of $[(p\vee \sim p) \wedge p] \vee q$ is:

p	q	$\sim p$	$p\vee \sim p$	$(p\vee \sim p) \wedge p$	$[(p\vee \sim p) \wedge p] \vee q$
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	T
F	F	T	T	F	F

42) The tables of truth of $p \Rightarrow q$ and of $\sim p \vee q$ are:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

43) The tables of truth of $p \Leftrightarrow q$ and of $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

44) The tables of truth of $\sim(p \vee q)$ and of $(\sim p) \wedge (\sim q)$ are:

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

45) The tables of truth of $\sim(p \wedge q)$ and of $(\sim p) \vee (\sim q)$ are:

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

46) The table of truth of $\sim(p \wedge \sim p)$ is:

p	$\sim p$	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	T
F	T	F	T

47) The table of truth of $p \wedge (p \Rightarrow q) \Rightarrow q$ is:

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$p \wedge (p \Rightarrow q) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

48) The table of truth of $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is:

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

49) The table of truth of $\sim(p \Rightarrow q) \Leftrightarrow p \wedge \sim q$ is:

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$\sim q$	$p \wedge \sim q$	$\sim(p \Rightarrow q) \Leftrightarrow p \wedge \sim q$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

50) The table of truth of $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ is:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

9.3 Exercises Chapter 3

1) $D = \left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, +\infty\right)$

2) $D = \left(-\infty, -\sqrt{3}\right] \cup \left[\sqrt{3}, +\infty\right)$

3) $D = \mathbb{R}$

4) $D = [2, +\infty)$

5) $D = (-\infty, -2] \cup [2, +\infty)$

6) $D = (-\infty, -2) \cup \left(-2, -\sqrt{3}\right) \cup \left(\sqrt{3}, 2\right) \cup (2, +\infty)$

7) $D = \mathbb{R} \setminus \{0\}$

8) $D = \emptyset$ (empty set)

9) $D = (-3, -2) \cup (-2, +\infty)$

10) $D = (0, 2]$

11) $D = (-\log 4, +\infty)$

12) $D = (0, +\infty)$

13) $D = [e, +\infty)$

14) $D = (-\infty, -2] \cup [2, +\infty)$

15) $D = (4, +\infty)$

16) $D = \left[-5, -\sqrt{5}\right) \cup \left(-\sqrt{5}, -2\right) \cup \left(2, \sqrt{5}\right) \cup \left(\sqrt{5}, +\infty\right)$

17) $D = \left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, +\infty\right)$

18) $D = (1, +\infty)$

19) $D = [0, 4) \cup (4, +\infty)$

20) $D = \mathbb{R} \setminus \{0\}$

21) f intersects the x-axis in the point $(e^3, 0)$ while it doesn't intersect the y-axis, furthermore $f(x) \geq 0 \ \forall x \in D = (0, +\infty)$.

22) f doesn't intersect the x-axis while it intersects the y-axis in the point $(0, e^2)$, furthermore $f(x) > 0 \ \forall x \in D = \mathbb{R} \setminus \{1\}$.

23) f intersects the x-axis in the points $(-2, 0)$ and $(2, 0)$ and the y-axis in the point $(0, 4)$, furthermore $f(x) \geq 0 \ \forall x \in D = \mathbb{R}$.

24) f doesn't intersect the x-axis while it intersects the y-axis in the point $\left(0, \dfrac{5}{2}\right)$, furthermore $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$ (observing that the domain is given by $D = \mathbb{R} \setminus \{-2\}$).

25) f intersects the x-axis in the points $(-1, 0)$ and $(1, 0)$ and the y-axis in the point $(0, 3)$, furthermore $f(x) \geq 0 \ \forall x \in D = \mathbb{R}$.

26) f intersects the x-axis in the point $(1, 0)$ while it doesn't intersect the y-axis, furthermore $f(x) < 0$ for $0 < x < 1$ and $f(x) > 0$ for $x > 1$ (observing that the domain is given by $D = (0, +\infty)$).

27) f intersects the x-axis in the point $(3, 0)$ while it doesn't intersect the y-axis, furthermore $f(x) > 0$ for $x < -2$ and for $x > 3$ and $f(x) < 0$ for $\frac{1}{2} < x < 3$ (observing that the domain is given by $D = (-\infty, -2) \cup \left(\frac{1}{2}, +\infty\right)$).

28) f is odd.

29) f is even.

30) f is odd.

31) f is odd.

32) f is even.

33) f is neither odd nor even.

34) $g \circ f = g(f(x)) = x$ while $f \circ g = f(g(t)) = t$.

35) $g \circ f = g(f(x)) = \sqrt[3]{x^3+2}$ while $f \circ g = f(g(t)) = t+2$.

36) $g \circ f = g(f(x)) = e^{\log x+3} = xe^3$ for $x > 0$ while $f \circ g = f(g(t)) = \log e^{t+3} = t+3$.

37) $g \circ f = g(f(x)) = |\log x - 2|$ for $x > 0$ while $f \circ g = f(g(t)) = \log|t-2|$ $\forall t \neq 2$.

38) $g \circ f = g(f(x)) = e^{\log(x+1)} = x+1$ for $x > -1$ while $f \circ g = f(g(t)) = \log(e^t+1)$.

39) $g \circ f = g(f(x)) = \sqrt{\log(x+1)}$ for $x \geq 0$ while $f \circ g = f(g(t)) = \log\left(\sqrt{t}+1\right)$ for $t \geq 0$.

40) $g \circ f = g(f(x)) = \sqrt{\log x - 2}$ for $x \geq e^2$ while $f \circ g = f(g(t)) = \log\sqrt{t} - 2$ for $t > 0$.

41) $g \circ f = g(f(x)) = \sqrt{|x-1|}$ while $f \circ g = f(g(t)) = \left|\sqrt{t}-1\right|$ for $t \geq 0$.

42) $g \circ f = g(f(x)) = xe$ for $x > 0$ while $f \circ g = f(g(t)) = t+1$.

43) $g \circ f = g(f(x)) = x+2$ while $f \circ g = f(g(t)) = te^2$ for $t > 0$.

44) $f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$

45) $f^{-1}(x) = \sqrt[3]{x-3}$

46) $f^{-1}(x) = x^2 - 2 \quad \text{for } x \geq 0$

47) The inverse of the restriction of f to the interval $(-\infty, 0)$ is $f^{-1}(x) = -e^x$. The inverse of the restriction of f to the interval $(0, +\infty)$ is $f^{-1}(x) = e^x$.

48) $$f^{-1}(x) = \begin{cases} \log_2 x & \text{if} \quad 0 < x \leq 2 \\ \sqrt{x-2} & \text{if} \quad x > 3 \end{cases}$$

49) $$f^{-1}(x) = \begin{cases} \dfrac{1}{2}x + \dfrac{3}{2} & \text{if} \quad x < -1 \\ \sqrt[3]{e^x} & \text{if} \quad x \geq 0 \end{cases}$$

50) The inverse of the restriction of f to the interval $(-\infty, -3]$ is $f^{-1}(x) = -x^2$ if $x \geq \sqrt{3}$. The inverse of the restriction of f to the interval $(-3, 3)$ is $f^{-1}(x) = x - 3$ if $0 < x < 6$. The inverse of the restriction of f to the interval $[3, +\infty)$ is $f^{-1}(x) = e^x$ if $x \geq \log 3$.

9.4 Exercises Chapter 4

1) $\frac{1}{2}$

2) 5

3) 0^+

4) 0

5) $\frac{1}{2\sqrt{2}}$

6) 2

7) 1

8) $+\infty$

9) 0^-

10) $\frac{1}{2}$

11) 2

12) 0

13) e

14) $\frac{1}{e}$

15) 3

16) 0

17) 2

18) 0^+

19) $+\infty$

20) 2

21) -2

22) 3

23) 0

24) 0

25) 0

26) $f(x)$ has horizontal asymptote $y = 2$ and vertical asymptote $x = 1$.

27) $f(x)$ has horizontal asymptote $y = 1$ and vertical asymptote $x = 1$.

28) $f(x)$ has horizontal asymptote $y = 1$ and vertical asymptote $x = 3$.

29) $f(x)$ has horizontal asymptote (as $x \to -\infty$) $y = 0$ and vertical asymptote $x = 1$.

30) $f(x)$ has (left) horizontal asymptote $y = 0$ and vertical asymptote $x = 2$.

31) $f(x)$ has vertical asymptote $x = -3$ and oblique asymptote $y = x + 2$.

32) $f(x)$ has horizontal asymptote (as $x \to +\infty$) $y = 0$ and oblique asymptote (as $x \to -\infty$) $y = -2x$.

33) $f(x)$ has (left) horizontal asymptote $y = 0$.

34) $f(x)$ has (left) vertical asymptote $x = 0$ and oblique asymptote $y = x - 2$.

35) $f(x)$ has horizontal asymptote $y = e$ and (left) vertical asymptote $x = 1$.

36) $f(x)$ has horizontal asymptote $y = e$ and (left) vertical asymptote $x = 1$.

37) $f(x)$ has vertical asymptote $x = -3$ and oblique asymptote $y = x - 1$.

38) $f(x)$ has vertical asymptote $x = -1$ and oblique asymptote $y = x + 2$.

39) $f(x)$ has vertical asymptotes $x = -1$ and $x = 1$ and oblique asymptote $y = x + 2$.

40) $f(x)$ has horizontal asymptote $y = 0$ and vertical asymptotes $x = -2$ and $x = 0$.

41) $f(x)$ is continuous $\forall x \neq 1$ and it is continuous also at $x = 1$ for $\alpha = 0$.

42) $f(x)$ is continuous $\forall x \neq 1$ and it is continuous also at $x = 1$ for $\alpha = \beta$.

43) $f(x)$ is continuous $\forall x \neq 1$ and it is continuous also at $x = 1$ for any value of α and for $\beta = 1$.

44) $f(x)$ is continuous on $\mathbb{R}$ for any value of α.

45) $f(x)$ is continuous $\forall x \neq 1$ but it is continuous at $x = 1$ for no value of α.

46) $f(x)$ is continuous $\forall x \neq -1$ and it is continuous also at $x = -1$ for $\alpha = -3$.

47) $f(x)$ is continuous on $\mathbb{R}$ for any value of α.

48) $f(x)$ is continuous $\forall x \neq 0$ and it is continuous also at $x = 0$ for $\alpha = -3$.

49) $f(x)$ is continuous $\forall x \neq 0$ and it is continuous also at $x = 0$ for no value of α.

50) $f(x)$ is continuous $\forall x \neq 0$ and it is continuous also at $x = 0$ for $\alpha = \mp 1$.

9.5 Exercises Chapter 5

1) $f'(x) = \dfrac{4-x}{x(2-x)}$

2) $f'(x) = -\dfrac{1}{2\sqrt{x}e^{\sqrt{x}}}$

3) $f'(x) = \dfrac{1}{x\left[1+\log(3x)\right]^2}$

4) $f'(x) = \dfrac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}}$

5) $f'(x) = \dfrac{x}{(x^2+1)\sqrt{\log(x^2+1)}}$

6) $f'(x) = -\dfrac{e^{\sqrt{\cos x}}\sin x}{2\sqrt{\cos x}}$

7) $f'(x) = e^{-2x}\dfrac{1-6x}{3\sqrt[3]{x^2}}$

8) $f'(x) = \dfrac{6x^2-8x+12}{(6x-4)^2}$

9) $f'(x) = -\dfrac{1}{x(\log x)^2}$

10) $f'(x) = e^{\sqrt{\sin x}} \left(\frac{x \cos x}{2\sqrt{\sin x}} + 1 \right)$

11) $f'(x) = \frac{3e^{\frac{4-x}{1-x}}}{(1-x)^2}$

12) $f'(x) = \frac{x}{x^2+1}$

13) $f'(x) = e^{\sin x} (1 + x \cos x)$

14) $f'(x) = \frac{3x+2}{2\sqrt{1+x}}$

15) $f'(x) = (xe^x)^x [x + 1 + \log(xe^x)]$

16) $D\left[f^{-1}\left(\frac{1}{e}\right)\right] = \frac{1}{2}e$

17) $D\left[f^{-1}(\log 3)\right] = -\frac{3}{2}$

18) $D\left[f^{-1}(e)\right] = \frac{2}{e}$

19) $D\left[f^{-1}(e)\right] = \frac{1}{2e}$

20) $D\left[f^{-1}\left(e^3\right)\right] = \frac{1}{2e^3}$

21) $y = 6x + 1$

22) $y = x + 2$

23) $y = 6x - 1$

24) $y = -\frac{1}{2}x + 3$

25) $y = \frac{3}{2}x - \frac{1}{2}$

26) $y = \frac{5}{2}x - \frac{1}{2}$

27) $y = 2x + 1$

28) $y = 4x - 3$

29) f is continuous and it has a derivative $\forall x \neq 1$, furthermore it is continuous and it has a derivative also at $x = 1$ if $\alpha = 2$.

30) f is continuous and it has a derivative $\forall x \neq 1$, furthermore it is continuous and it has a derivative also at $x = 1$ if $\alpha = \beta$.

31) f is continuous and it has a derivative $\forall x \neq 0$, furthermore it is continuous also at $x = 0$ if $\beta = 3$ and it has a derivative also at $x = 0$ if $\alpha = -2$ and $\beta = 3$.

32) $f(x) = e - \frac{e}{2}x^2 + o(x^3)$

33) $f(x) = 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)$

34) $f(x) = -\frac{e}{2} + \frac{e}{2}x^2 + o((x-1)^2)$

35) $f(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$

36) $f(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)$

37) f is strictly increasing on the intervals $\left(-\infty, \frac{1}{3}\right)$ and $(1, +\infty)$ and strictly decreasing on the interval $\left(\frac{1}{3}, 1\right)$, furthermore it has a maximum in $x = \frac{1}{3}$ and a minimum in $x = 1$.

38) f is strictly increasing on the interval $[1, 2]$, furthermore it has a minimum in $x = 1$ and a maximum in $x = 2$.

39) f is strictly increasing on the interval $[0, 1]$, furthermore it has a minimum in $x = 0$ and a maximum in $x = 1$.

40) f is strictly increasing on the interval $[2, 4]$, furthermore it has a minimum in $x = 2$ and a maximum in $x = 4$.

41) f is strictly increasing on the interval $\left[0, \frac{1}{4}\right]$ and strictly decreasing on the interval $\left[\frac{1}{4}, 1\right]$, furthermore it has a maximum in $x = \frac{1}{4}$ and a minimum in $x = 0$ and in $x = 1$.

42) f is strictly increasing on the interval $(1, 2]$, furthermore it has a maximum in $x = 2$ while it has no minimum.

43) f is strictly increasing on its domain $(0, +\infty)$, furthermore it has neither minimum nor maximum.

44) f is strictly convex on the intervals $(-\infty, -1)$ and $(0, +\infty)$ and strictly concave on the interval $(-1, 0)$, furthermore it has an inflection point in $x = -1$ and in $x = 0$.

45) f is strictly concave on the interval $(-\infty, -3)$ and strictly convex on the interval $(-3, +\infty)$, furthermore in this case the point $x = -3$ is not an inflection point (since here the function is not defined).

46) f is strictly convex on all $\mathbb{R}$.

47) f is strictly concave on the interval $(-\infty, -2)$ and strictly convex on the interval $(-2, +\infty)$, furthermore it has an inflection point in $x = -2$.

48) f is strictly convex on the intervals $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, +\infty\right)$ and strictly concave on the interval $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, furthermore it has an inflection point in $x = -\frac{1}{\sqrt{2}}$ and in $x = \frac{1}{\sqrt{2}}$.

49) We have in this case:

$$f(x) = e^{-x^2 + \log x + 2} = xe^{-x^2+2}$$

• Domain of the function

It must be $x > 0$, therefore the domain is:

$$D = (0, +\infty)$$

• Sign of the function, intersections with the axes, symmetries

We have $f(x) > 0 \quad \forall x \in D$, furthermore there are no intersections with the axes and the function does not have symmetries.

• Behaviour at the frontier and asymptotes

We have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} xe^{-x^2+2} = 0^+$$

and then:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} xe^{-x^2+2} = \lim_{x \to +\infty} \frac{x}{e^{x^2-2}} = 0^+$$

so that $y = 0$ is an horizontal asymptote.

- First derivative, monotonicity, local extrema

The function $f(x)$ has a derivative $\forall x \in D$ and the first derivative is:

$$f'(x) = -2x^2 e^{-x^2+2} + e^{-x^2+2} = \left(-2x^2 + 1\right) e^{-x^2+2}$$

The sign of $f'(x)$ depends only on that of $\left(-2x^2+1\right)$, we have therefore:

$$f'(x) < 0 \qquad \text{for} \qquad x > \frac{1}{\sqrt{2}}$$

$$f'(x) = 0 \qquad \text{for} \qquad x = \frac{1}{\sqrt{2}}$$

$$f'(x) > 0 \qquad \text{for} \qquad 0 < x < \frac{1}{\sqrt{2}}$$

so that $f(x)$ is strictly decreasing on the interval $\left(\frac{1}{\sqrt{2}}, +\infty\right)$ and strictly increasing on the interval $\left(0, \frac{1}{\sqrt{2}}\right)$ and the point $x = \frac{1}{\sqrt{2}}$ is a global maximum point.

- Second derivative, concavity, inflection points

The function $f(x)$ has second derivative $\forall x \in D$ and this derivative is:

$$\begin{aligned} f''(x) &= -2x(-2x^2+1)e^{-x^2+2} - 4xe^{-x^2+2} = \\ &= 2x(2x^2-3)e^{-x^2+2} \end{aligned}$$

The sign of $f''(x)$ depends only on $\left(2x^2-3\right)$, we have therefore:

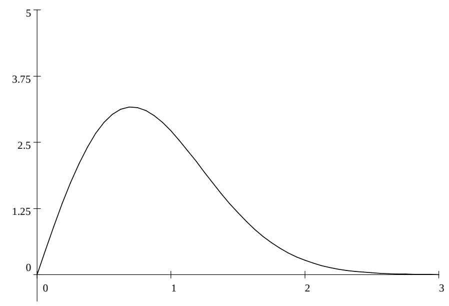
$$f''(x) < 0 \qquad \text{for} \qquad 0 < x < \sqrt{\frac{3}{2}}$$

$$f''(x) = 0 \qquad \text{for} \qquad x = \sqrt{\frac{3}{2}}$$

$$f''(x) > 0 \qquad \text{for} \qquad x > \sqrt{\frac{3}{2}}$$

so that $f(x)$ is strictly concave on the interval $\left(0, \sqrt{\frac{3}{2}}\right)$ and strictly convex on the interval $\left(\sqrt{\frac{3}{2}}, +\infty\right)$ and the point $x = \sqrt{\frac{3}{2}}$ is an inflection point.

- Graph of the function



50) We have in this case:

$$f(x) = \frac{e^{|x|-3}}{x}$$

- Domain of the function

It must be $x \neq 0$, therefore the domain is:

$$D = \mathbb{R} \setminus \{0\}$$

- Sign of the function, intersections with the axes, symmetries

First of all it is possible to observe that we have:

$$f(-x) = \frac{e^{|-x|-3}}{-x} = -\frac{e^{|x|-3}}{x} = -f(x)$$

so that $f(x)$ is odd. It is therefore sufficient to study it for $x > 0$ (since its graph will be symmetric with respect to the origin), where $f(x) = \frac{e^{x-3}}{x}$. We have $f(x) > 0$ $\forall x > 0$ and there are no intersections with the axes.

- Behaviour at the frontier and asymptotes

We have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{x-3}}{x} = +\infty$$

so that $x = 0$ is a vertical asymptote, furthermore:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{x-3}}{x} = +\infty$$

and then:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{e^{x-3}}{x^2} = +\infty$$

so that there are no oblique asymptotes.

- First derivative, monotonicity, local extrema

The function $f(x)$ has a derivative $\forall x \in D$ and the first derivative (for $x > 0$) is:

$$f'(x) = \frac{xe^{x-3} - e^{x-3}}{x^2} = \frac{e^{x-3}(x-1)}{x^2}$$

The sign of $f'(x)$ depends only on $(x-1)$, we have therefore (on the interval $(0, +\infty)$):

$$f'(x) < 0 \quad \text{for} \quad 0 < x < 1$$

$$f'(x) = 0 \quad \text{for} \quad x = 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1$$

so that $f(x)$ is strictly decreasing on the interval $(0, 1)$ and strictly increasing on the interval $(1, +\infty)$, furthermore $x = 1$ is a local minimum point (and therefore $x = -1$ is a local maximum point).

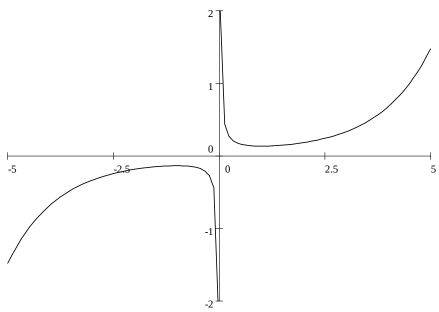
- Second derivative, concavity, inflection points

The function $f(x)$ has second derivative $\forall x \in D$ and this derivative (for $x > 0$) is:

$$f''(x) = \frac{x^2\left[e^{x-3} + e^{x-3}(x-1)\right] - 2xe^{x-3}(x-1)}{x^4} = \frac{xe^{x-3}(x^2-2x+2)}{x^4} = $$
$$= \frac{e^{x-3}(x^2-2x+2)}{x^3}$$

for which we have (on the interval $(0, +\infty)$) $f''(x) > 0$ for $x > 0$, so that $f(x)$ is strictly convex on the interval $(0, +\infty)$.

- Graph of the function



9.6 Exercises Chapter 6

1) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + c \quad \text{with } c \in \mathbb{R}$

2) $x^2 + 2x\sqrt{x} - 4\log|x| + c \quad \text{with } c \in \mathbb{R}$

3) $\frac{2}{3}x\sqrt{x} + 10\sqrt{x} + c \quad \text{with } c \in \mathbb{R}$

4) $-e^{\cos x} + c \quad \text{with } c \in \mathbb{R}$

5) $2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + c \quad \text{with } c \in \mathbb{R}$

6) $\frac{1}{4}x^2\left(2\log x - 1\right) + c \quad \text{with } c \in \mathbb{R}$

7) $\frac{1}{2}\log^2 x + c \quad \text{with } c \in \mathbb{R}$

8) $\frac{1}{3}\log^3 x + c \quad \text{with } c \in \mathbb{R}$

9) $\frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + c \quad \text{with } c \in \mathbb{R}$

10) $-\frac{2}{3}\left(1-x\right)\sqrt{1-x} + c \quad \text{with } c \in \mathbb{R}$

11) $x + \frac{1}{2}e^{2x} + c \quad \text{with } c \in \mathbb{R}$

12) $e^{-x}(-x-2) + c \quad \text{with } c \in \mathbb{R}$

13) $2e^{\sqrt{x}}(\sqrt{x} - 1) + c \quad \text{with } c \in \mathbb{R}$

14) $\frac{2}{3}x\sqrt{x} + \frac{1}{3}x^3 + c \quad \text{with } c \in \mathbb{R}$

15) $\frac{1}{2}x^2 - \cos x + c \quad \text{with } c \in \mathbb{R}$

16) $e^{2x}\left(x^2 - x + \frac{1}{2}\right) + c \quad \text{with } c \in \mathbb{R}$

17) $\frac{3}{4}x\sqrt[3]{x} + c \quad \text{with } c \in \mathbb{R}$

18) $\frac{2}{3}\left(1+x\right)\sqrt{1+x} + c \quad \text{with } c \in \mathbb{R}$

19) $\log\left|x^3+1\right|+c \quad \text{with } c \in \mathbb{R}$

20) $-e^{1-x}+c \quad \text{with } c \in \mathbb{R}$

21) $F(x)=x^3+x^2+5x+6$

22) $F(x)=\sin x+\frac{1}{2}x^2+x^3+1$

23) $F(x)=\frac{2}{3}x\sqrt{x}+\log|x|+\frac{1}{3}$

24) $F(x)=\frac{4}{3}x^3+6x^2+9x-1$

25) $F(x)=2e^{\sqrt{x}}+4\sqrt{x}-3-2e$

26) $F(x)=\log\left|x^2-x-1\right|-1$

27) $F(x)=e^{\sin x}+1$

28) $F(x)=\frac{1}{2}e^{2x-3}-\frac{1}{2}e^{-1}$

29) $F(x)=2e^{\sqrt{x}}\left(\sqrt{x}-1\right)+3$

30) $F(x)=\frac{2}{3}x\sqrt{x}-\frac{1}{3}x^3-\frac{1}{3}$

31) $e-1$

32) $\frac{1}{3}$

33) $\frac{2}{3}$

34) $2\left(\sin 1+\cos 1-1\right)$

35) $\log 10$

36) $\frac{1}{2}\log 2$

37) $\log 9$

38) $\frac{7}{6}$

39) 1

40) $\dfrac{e^5(\sin 5 - \cos 5) + 1}{2}$

41) $\dfrac{1}{2} + \sin 1$

42) $\dfrac{3}{2}\sqrt[3]{2}$

43) e^2

44) $\dfrac{9}{4}$

45) $2\log 2 - \dfrac{3}{4}$

46) $e - 1$

47) $\log 12 - 1$

48) $\dfrac{1}{2}\log\dfrac{8}{3}$

49) $\dfrac{7}{2}$

50) $2 + \log 2$

9.7 Exercises Chapter 7

1) $\mathbf{x} \leqq \mathbf{y}$

2) $\mathbf{x} \geqq \mathbf{y}$

3) $\mathbf{x}$ and $\mathbf{y}$ cannot be compared.

4) $\mathbf{x} > \mathbf{y}$

5) $\mathbf{x} < \mathbf{y}$

6) $\mathbf{x}$ and $\mathbf{y}$ cannot be compared.

7) $\mathbf{x} > \mathbf{y}$

8) $\mathbf{x} < \mathbf{y}$

9) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = -2 \quad 2\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

10) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = 0 \quad 2\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

11) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = -2 \quad 2\mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

12) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = 0 \quad 2\mathbf{x} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$

13) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = -1 \quad 2\mathbf{x} = \begin{pmatrix} -2 \\ 8 \\ -4 \end{pmatrix}$

14) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = -4 \quad 2\mathbf{x} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$

15) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = 1 \quad 2\mathbf{x} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}$

16) $\mathbf{x}+\mathbf{y} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ $\quad (\mathbf{x} \mid \mathbf{y}) = -10 \quad 2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$

17) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal for $\alpha = -\frac{6}{5}$; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{5}$, $\| \mathbf{y} \| = \sqrt{34}$ and $d(\mathbf{x}, \mathbf{y}) = \sqrt{17}$.

18) $\mathbf{x}$ and $\mathbf{y}$ are never orthogonal; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{2}$, $\| \mathbf{y} \| = 3$ and $d(\mathbf{x}, \mathbf{y}) = \sqrt{5}$.

19) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal for $\alpha = 0$; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{10}$, $\| \mathbf{y} \| = \sqrt{2}$ and $d(\mathbf{x}, \mathbf{y}) = 2$.

20) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal $\forall \alpha \in \mathbb{R}$; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{14}$, $\| \mathbf{y} \| = \sqrt{5}$ and $d(\mathbf{x}, \mathbf{y}) = \sqrt{19}$.

21) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal $\forall \alpha \in \mathbb{R}$; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{2}$, $\| \mathbf{y} \| = \sqrt{2}$ and $d(\mathbf{x}, \mathbf{y}) = 2$.

22) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal for $\alpha = 0$ and for $\alpha = -\frac{1}{3}$; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{14}$, $\| \mathbf{y} \| = \sqrt{2}$ and $d(\mathbf{x}, \mathbf{y}) = \sqrt{8}$.

23) $\mathbf{x}$ and $\mathbf{y}$ are never orthogonal; for $\alpha = 1$ we have $\| \mathbf{x} \| = \sqrt{6}$, $\| \mathbf{y} \| = \sqrt{2}$ and $d(\mathbf{x}, \mathbf{y}) = \sqrt{2}$.

24) $\mathbf{x}$ and $\mathbf{y}$ are orthogonal for $\alpha = 0$; for $\alpha = 1$ we have $\| \mathbf{x} \| = 1$, $\| \mathbf{y} \| = \sqrt{2}$ and $d(\mathbf{x}, \mathbf{y}) = 1$.

25) $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

26) $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

27) $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

28) $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent.

29) $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent.

30) $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent.

31) $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent for $\alpha = 1$ and linearly independent for $\alpha \neq 1$.

32) $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent for $\alpha = 0$ and linearly independent for $\alpha \neq 0$.

33) $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent for $\alpha = 2$ and linearly independent for $\alpha \neq 2$.

34) $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent for $\alpha = 0$ and linearly independent for $\alpha \neq 0$.

35) $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad -2A = \begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}$

36) $A + B = \begin{pmatrix} -1 & 1 \\ 2 & 5 \end{pmatrix} \qquad -2A = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}$

37) $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad -2A = \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$

38) $A + B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \qquad -2A = \begin{pmatrix} -6 & 4 \\ -2 & 0 \end{pmatrix}$

39) $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \qquad -2A = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$

40) $A+B=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad -2A=\begin{pmatrix} -2 & -2 \\ 2 & -4 \end{pmatrix}$

41) $A+B=\begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix}$ $\quad -2A=\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

42) $A+B=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad -2A=\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

43) $AB=\begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix}$ $\quad BA=\begin{pmatrix} 3 & 8 \\ 3 & 8 \end{pmatrix}$

44) $AB=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad BA=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

45) $AB=\begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix}$ $\quad BA=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

46) $AB=\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad BA=\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$

47) $AB=\begin{pmatrix} 8 & 6 & 31 \\ 11 & 3 & 40 \end{pmatrix}$ $\quad BA$ does not exist

48) $AB=\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 2 \end{pmatrix}$ $\quad BA$ does not exist

49) AB does not exist $\quad BA=\begin{pmatrix} 2 & 2 \\ -2 & -4 \\ 3 & 4 \end{pmatrix}$

50) $AB=\begin{pmatrix} 0 & 8 \\ 1 & 4 \end{pmatrix}$ $\quad BA=\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$

9.8 Exercises Chapter 8

1) $D=\left\{(x,y)\in\mathbb{R}^2 : (x\geq 0\ \wedge\ y\geq 0)\ \vee\ (x\leq 0\ \wedge\ y\leq 0)\right\}$

2) $D=\left\{(x,y)\in\mathbb{R}^2 : x\geq 0\ \wedge\ y\geq 0\right\}$

3) $D=\left\{(x,y)\in\mathbb{R}^2 : (x>0\ \wedge\ y>0)\ \vee\ (x<0\ \wedge\ y<0)\right\}$

4) $D=\left\{(x,y)\in\mathbb{R}^2 : x>0\ \wedge\ y>0\right\}$

5) $D = \left\{(x,y) \in \mathbb{R}^2 : ((x > 0 \ \wedge \ y > 0) \ \vee \ (x < 0 \ \wedge \ y < 0)) \wedge y \neq \dfrac{1}{x}\right\}$

6) $D = \mathbb{R}^2$

7) $D = \mathbb{R}^2$

8) $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \ \wedge \ y > 0\}$

9) $D = \{(x,y) \in \mathbb{R}^2 : x > 0 \ \wedge \ y \neq 0\}$

10) $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \ \vee \ y \neq 0\}$

11) $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \ \vee \ y \neq 0\}$

12) $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \ \wedge \ y > 0\}$

13) $D = \mathbb{R}^2$

14) $D = \{(x,y) \in \mathbb{R}^2 : (x > 0 \ \wedge \ y > 0) \ \vee \ (x < 0 \ \wedge \ y < 0)\}$

15) $D = \{(x,y) \in \mathbb{R}^2 : y > -x\}$

16) $\nabla f(1,0) = (1 \quad 1) \qquad \nabla^2 f(1,0) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

17) $\nabla f(1,-1) = (0 \quad 0) \qquad \nabla^2 f(1,-1) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

18) $\nabla f(1,0) = (-6 \quad 1) \qquad \nabla^2 f(1,0) = \begin{pmatrix} 12 & 1 \\ 1 & 0 \end{pmatrix}$

19) $\nabla f(1,-1) = (2 \quad 1) \qquad \nabla^2 f(1,-1) = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}$

20) $\nabla f(1,1) = (e \quad e) \qquad \nabla^2 f(1,1) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$

21) $\nabla f(0,0) = (0 \quad 0) \qquad \nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

22) $\nabla f(2,2) = \left(\dfrac{1}{4} \quad \dfrac{1}{4}\right) \qquad \nabla^2 f(2,2) = \begin{pmatrix} -\dfrac{1}{32} & -\dfrac{1}{32} \\ -\dfrac{1}{32} & -\dfrac{1}{32} \end{pmatrix}$

23) $\nabla f(0,0) = (1 \quad 0) \qquad \nabla^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

24) $\nabla f(1,1) = (0 \quad 0) \qquad \nabla^2 f(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

25) $\nabla f(1,0) = (0 \quad 3) \qquad \nabla^2 f(1,0) = \begin{pmatrix} 0 & 6 \\ 6 & -10 \end{pmatrix}$

26) $df(0,1) = dx + 2dy$

27) $df(2,-2) = \frac{3}{2}dx + \frac{1}{2}dy$

28) $df(0,1) = dx + edy$

29) $df(3,4) = \frac{3}{5}dx + \frac{4}{5}dy$

30) $df\,(1,-1) = 2dx + dy$

31) $df(2,2) = \frac{3}{8}dx - \frac{1}{8}dy$

32) $df\,(3,2) = \frac{1}{6}dx + \frac{1}{2}dy$

33) $z = x - 2y$

34) $z = \frac{1}{4}x + \frac{1}{4}y + 1$

35) $z = x + 3y - 1$

36) $z = x + y + 2$

37) $z = x + 2y$

38) $z = x + y - 2$

39) $z = \frac{3}{5}x + \frac{4}{5}y$

40) $z = -x - y + 2$

41) The function has a relative maximum in $A = (-2, 0)$, while there are no relative minima.

42) The function has a relative minimum in $A = \left(\frac{1}{3}, \frac{1}{3}\right)$ and a saddle in $B = (0, 0)$, while there are no relative maxima.

43) The function has a relative minimum in $A = \left(-1, \frac{1}{2}\right)$ and another relative minimum in $B = \left(1, \frac{1}{2}\right)$, while there are no relative maxima.

44) The function has a relative minimum in $A = (0, 0)$ and a saddle in $B = (-2, 0)$, while there are no relative maxima.

45) The function has a saddle in $A = (-1, 1)$ and another saddle in $B = (1, 1)$, while there are neither relative maxima nor relative minima.

46) The function has a relative minimum in $A = (-1, 0)$, while there are no relative maxima.

47) The function has a relative minimum in $A = \left(\sqrt[3]{9}, \frac{1}{\sqrt[3]{3}}\right)$, while there are no relative maxima.

48) The function has a relative maximum in $A = \left(-\sqrt{2}, 0\right)$ and a saddle in $B = \left(\sqrt{2}, 0\right)$, while there are no relative minima.

49) The function has a relative maximum in $A = \left(-\frac{33}{10}, \frac{5}{2}\right)$, while there are no relative minima.

50) The function has a relative maximum in $A = (0, 0)$, while there are no relative minima.