SIAM J. Financial Math.
Vol. 2, pp. 383–403 © 2011 Society for Industrial and Applied Mathematics

Pricing Discretely Monitored Asian Options by Maturity Randomization*

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Abstract. We present a new methodology based on maturity randomization to price discretely monitored arithmetic Asian options when the underlying asset evolves according to a generic Lévy process. Our randomization technique considers the option expiry to be a random variable distributed according to a geometric distribution of a parameter independent of the underlying process. This allows one to transform the pricing backward procedure into a set of independent integral equations. Numerical procedures for a fast and accurate solution of the pricing problem are provided.

Key words. Asian option, discrete monitoring, fast Fourier transform, integral equation, Lévy process, option pricing, quadrature formula

AMS subject classifications. 60G51, 65D30, 65R20, 65T50

DOI. 10.1137/09076115X

1. Introduction.

This paper introduces a new numerical approach for pricing discretely monitored arithmetic Asian options under Lévy processes. Asian options have become very popular instruments for hedging transactions whose costs are related to the average price of the underlying asset, being much cheaper than the corresponding options on the underlying asset and less subject to price manipulations near settlement.

An extensive literature deals with the pricing problem under continuous monitoring, providing analytical solutions, such as in Geman and Yor [21]. A review can be found in Boyle and Potapchik [7]. In the discrete monitoring case, where the arithmetic mean is updated only at prefixed points in time, the pricing of Asian options is not an easy task, and even in the Black–Scholes setting no analytical solution is available. Several approaches have been proposed to tackle this problem: Monte Carlo simulations [34], partial integro-differential equation approaches [36], lattices [15], and approximations of the distribution of the average [27]. However, the recent literature on option pricing considers extensions of the Gaussian framework in order to overcome the limits of the Black–Scholes setting, such as the volatility smile in the implied volatility curve. In this sense a recent approach is represented by the Lévy framework, a compromise between flexibility in modeling the smile and analytical tractability (see [14] and [35]).

Recent contributions to Asian option pricing in the exponential Lévy setting are Albrecher [2], Albrecher and Predota [3], Benhamou [6], Černý and Kyriakou [11], Fusai and Meucci

*Received by the editors June 4, 2009; accepted for publication (in revised form) February 6, 2011; published electronically June 7, 2011.
http://www.siam.org/journals/sifin/2/76115.html

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[20], and Iseger and Oldenkamp [23]. Albrecher [2] and Albrecher and Predota [3] explore approximations of the arithmetic option price based on the moments of the average, but in general it is not clear which approximate distribution to choose, and the approximation error is difficult to evaluate.

The fast Fourier transform (FFT) approach was introduced by Carverhill and Clewlow [10] in the Black–Scholes framework. Their forward density convolution algorithm requires a large discretization grid, resulting in slow convergence. Benhamou [6] extends the algorithm of Clewlow and Carverhill to some non-lognormal density and improves the numerical efficiency by recentering the density at each monitoring date, thus reducing the size of the grid. In the Lévy setting, Fusai and Meucci [20] solve the valuation problem by recursive Gaussian quadrature and derive a formula for the moments to check accuracy. Černý and Kyriakou [11] introduce a fast and accurate algorithm, using a backward price convolution, and provide an analytical upper bound for the pricing error due to truncations. Finally, Iseger and Oldenkamp [23] propose an algorithm based on the Laplace inversion technique.

This paper extends the randomization technique introduced in Fusai, Abrahams, and Sgarra [17] to Lévy processes and the pricing of Asian options. The implementation of other path-dependent options such as barrier and lookback can be found in [18] and [19].

In the randomization technique we consider the option expiry to be a random variable distributed according to a geometric distribution of the parameter q, independent of the underlying process. This allows us to transform the usual backward procedure into a set of independent integral equations parametrized by q. These integral equations in general do not allow an analytical solution; thus a contribution of this paper is to devise an accurate numerical solution based on a quadrature formula. Thereafter, the option price can be obtained as a weighted sum of the solutions of $N+1$ integral equations, with N the number of monitoring dates. Therefore, this procedure appears to have a computational cost that is linear in the number of monitoring dates, just as other numerical methods presented in the literature. However, exploiting an accelerating technique for alternating sums, we can make the computational cost nearly independent of the number of monitoring dates. Finally, given that the integral equations can be solved independently, we can exploit distributed computing.

Extensive numerical experiments are conducted to compare our results with Monte Carlo simulations and other pricing methods presented in the literature.

The paper is organized as follows: Section 2 models the underlying process. Section 3 summarizes the recursive algorithm, introduced in [10], for the valuation of arithmetic Asian options. Sections 4 and 5 introduce the numerical procedure, and section 6 presents the numerical results. Section 7 presents the conclusions.

2. The exponential Lévy model.

The risk-neutral process for the stock price $(S_t)_{t\geq 0}$ is assumed to be described by

$$S(t) = S(0)\exp\left((r-d+g)\,t + L(t)\right),$$

where r is the continuously compounded interest rate, d is the dividend yield, L_t is a Lévy process, and g is the so-called compensator chosen to ensure that the discounted price process is a martingale. The Lévy process is fully identified by its characteristic exponent $\psi(\omega) = \log\mathbb{E}\left(e^{i\omega L_1}\right)$, where $i=\sqrt{-1}$. Following [35], under the mean-correcting martingale measure,

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we set $g = -\psi(-i)$. For a discussion on the choice of martingale measure in the setting of exponential Lévy models, see [12] and [22].

We are interested in pricing arithmetic Asian options under a discrete monitoring rule; that is, prices contributing to the arithmetic average are observed at equally spaced monitoring dates $t_0 = 0$, $t_1 = \Delta, \dots,$ $t_n = n\Delta$, $\dots$, $t_N = N\Delta = T$. The log-return on a time interval of length Δ is defined by

$$Z_n \equiv \log \frac{S_n}{S_{n-1}} = (r - d + g)\Delta + L_n - L_{n-1}, \tag{2.1}$$

where $S_n = S(n\Delta)$ and the Lévy increments $L_n - L_{n-1} = L(n\Delta) - L((n-1)\Delta)$ are independent and identically distributed. It follows that Z_n has a characteristic function that does not depend on the monitoring time index n,

$$\phi_Z(\omega) = e^{(\psi(\omega)+i\omega(r-d+g))\Delta},$$

and its density f_Z can be obtained by numerical inversion of the characteristic function using the FFT or the fractional FFT, as explained in [13].

3. Recursion for arithmetic Asian options.

The starting point of our numerical approach is based on a recursive formulation for the Asian option price. In [16], floating and fixed strike Asian options, within the framework of exponential Lévy models, are proved to be related by a symmetry result; moreover, a put-call parity result holds (see section 3.3). In section 3.1 we present a general price recursion for Asian options, while in section 3.2 we introduce a density recursion.

3.1. Price recursion.

Under the unified framework of [36], the payoff of an arithmetic Asian option depends on the following path-dependent random variable:

$$I_N \equiv \sum_{n=0}^{N} \lambda_n S_n, \tag{3.1}$$

where λ_n are deterministic. For example, the payoff of an Asian call option is given by

$$V(S_N, I_N; N) \equiv (I_N - cS_N)^+,$$

where $(\cdot)^+$ is the positive part function.

By suitable choices of $\{\lambda_n\}_{n=0}^N$ and c, we can describe a wide class of Asian options. In particular, standard cases are

$$\lambda_0 = \frac{\gamma}{N+\gamma} - \frac{K}{S_0}; \quad \lambda_n = \frac{1}{N+\gamma}, \quad n = 1, \dots, N; \quad c = 0 \tag{3.2}$$

for fixed strike call options and

$$\lambda_0 = -\frac{\alpha\gamma}{N+\gamma}; \quad \lambda_n = -\frac{\alpha}{N+\gamma}, \quad n = 1, \dots, N; \quad c = -1 \tag{3.3}$$

for floating strike calls, where $\gamma = 1$ if S_0 is included in the average and 0 otherwise, and α is a coefficient of partiality for the floating strike case.

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For pricing purposes, we can combine (2.1) and (3.1) and observe that

$$I_{n+1} = I_n + \lambda_{n+1} S_n e^{Z_{n+1}}, \quad n = 0, \dots, N-1. \tag{3.4}$$

Therefore, using (3.4) and the standard backward pricing procedure, we obtain the following recursion for the option price:

$$V(S_N, I_N; N) = (I_N - cS_N)^+,$$
$$V(S_n, I_n; n) = e^{-r\Delta} \int_{-\infty}^{+\infty} f_Z(s)\, V(S_n e^s, I_n + \lambda_{n+1} S_n e^s; n+1)\, ds$$

for $n = N-1, \dots, 0$.

Since the return distribution is independent of the current stock level and the payoff function is a homogeneous function of the spot price, then the price function itself is a homogeneous function of degree one (see [24] for further details). Thus, we can write

$$V(S_n, I_n; n) = S_n\, V\left(1, \frac{I_n}{S_n}; n\right).$$

If we set $x = I_n/S_n$, we can define

$$v_n(x) \equiv V(1, x; n),$$

where we have omitted the dependence of x on n. The function v satisfies the recursion

$$v_N(x) = (x - c)^+,$$
$$v_n(x) = e^{-r\Delta} \int_{-\infty}^{+\infty} f_Z(s)\, e^s v_{n+1}\big(xe^{-s} + \lambda_{n+1}\big)\, ds, \quad n = N-1 \dots, 0.$$

Notice that if $x = 0$, then $v_n(0) = e^{-d\Delta} v_{n+1}(\lambda_{n+1})$. Otherwise, by a change of variables in the integration ($y = xe^{-s} + \lambda_{n+1}$, i.e., $\log\left(x/\left(y - \lambda_{n+1}\right)\right) = s$), we obtain

$$v_n(x) = \begin{cases} e^{-r\Delta} \int_{\lambda_{n+1}}^{+\infty} f_Z\Big(\log\Big(\frac{x}{y-\lambda_{n+1}}\Big)\Big) \frac{x}{(y-\lambda_{n+1})^2}\, v_{n+1}(y)\, dy & \text{if } x > 0, \\ -e^{-r\Delta} \int_{-\infty}^{\lambda_{n+1}} f_Z\Big(\log\Big(\frac{x}{y-\lambda_{n+1}}\Big)\Big) \frac{x}{(y-\lambda_{n+1})^2}\, v_{n+1}(y)\, dy & \text{if } x < 0 \end{cases} \tag{3.5}$$

for $n = N-1, \dots, 0$. The desired option price will be $S_0 v_0(\lambda_0)$. Notice that if $x \geq 0$, $c \leq 0$, and $\lambda_n > 0$ for $n = 1, \dots, N$, then

$$v_n(x) = e^{-r(N-n)\Delta} \left[x + \sum_{i=0}^{N-n-1} \lambda_{N-i} e^{(N-n-i)(r-d)\Delta} - ce^{(N-n)(r-d)\Delta}\right] \tag{3.6}$$

for $n = N, \dots, 0$. See Appendix A.

Summing up, for a fixed strike call option, being $\lambda_n = \lambda := 1/(N+\gamma) > 0$, for $n = 1, \dots, N$, the recursion will be

$$v_N(x) = (x)^+,$$
$$v_n(x) = \begin{cases} e^{-r(N-n)\Delta} \left[x + \lambda e^{(r-d)\Delta} \frac{1-e^{(N-n)(r-d)\Delta}}{1-e^{(r-d)\Delta}}\right] & \text{if } x \geq 0, \\ -e^{-r\Delta} \int_{-\infty}^{\lambda} f_Z\Big(\log\Big(\frac{x}{y-\lambda}\Big)\Big) \frac{x}{(y-\lambda)^2}\, v_{n+1}(y)\, dy & \text{if } x < 0 \end{cases} \tag{3.7}$$

for $n = N-1, \dots, 0$.

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For a floating strike call option, being $\lambda_n = \lambda := -\alpha/(N+\gamma) < 0$, for $n = 1, \dots, N$, we always have $x < 0$; thus the recursion can be written as

$$
\begin{aligned}
&v_N(x) = (x+1)^+ \,, \\
&v_n(x) = -e^{-r\Delta} \int_{-\infty}^{\lambda} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right) \frac{x}{(y-\lambda)^2} v_{n+1}(y)\, dy
\end{aligned}
\tag{3.8}
$$

for $n = N-1, \dots, 0$

3.2. Density recursion. We can also perform a recursion on the density, as in [20]. In fact, if $\lambda_n > 0$ for $n = 1, \dots, N$, and $c = 0$, we can factorize expression (3.1) as follows:[1]

$$
I_N = S_0 \left[\lambda_0 + e^{Z_1}\left(\lambda_1 + e^{Z_2}\left(\cdots\left(\lambda_{N-1} + \lambda_N e^{Z_N}\right)\right)\right)\right].
$$

Letting $l_N \equiv \lambda_N e^{Z_N}$ and defining recursively the quantities

$$
l_n \equiv e^{Z_n}\left(\lambda_n + l_{n+1}\right), \quad n = N-1, \dots, 1,
$$

we obtain

$$
I_N = S_0 \left(\lambda_0 + l_1\right).
$$

The density of l_1 or, equivalently, the density of $B_1 \equiv \log(l_1)$ turns out to be the key variable in pricing Asian options. Since Z_n and l_{n+1} are independent, the density of $B_n \equiv \log(l_n) = Z_n + \log(\lambda_n + l_{n+1})$ is the convolution of the density f_Z of Z_n and that of $\log\left(\lambda_n + e^{B_{n+1}}\right)$. In particular, B_n is well defined, being $\lambda_n > 0$, for $n = 1, \dots, N$. With a change of variables, we obtain that the density of B_n, which we denote with g_n, satisfies the recursion

$$
g_n(x) = \int_{-\infty}^{+\infty} f_Z\left(x - \log\left(\lambda_n + e^y\right)\right) g_{n+1}(y)\, dy, \quad n = N-1, \dots, 1,
\tag{3.9}
$$

where the initial condition g_N can be obtained by numerically inverting the characteristic function $\phi_{B_N}(\omega) \equiv \phi_Z(\omega)\, e^{i\omega \log(\lambda_N)}$.

Once the density g_1 has been computed, call option prices can be obtained by the following additional numerical integration:

$$
e^{-rN\Delta} \int_{-\infty}^{+\infty} S_0 \left(\lambda_0 + e^x\right)^+ g_1(x)\, dx.
\tag{3.10}
$$

3.3. Put-call parity and symmetry for Asian options. In the previous section we showed a recursion to price Asian call option (C) with arithmetic average and discrete monitoring dates. The corresponding put price (P) can be computed by the following put-call parity condition: since

$$
(I_N - cS_N)^+ - (cS_N - I_N)^+ = I_N - cS_N,
$$

it holds that

$$
\begin{aligned}
C(S_0, I_0; 0) - P(S_0, I_0; 0) &= e^{-rN\Delta}\mathbb{E}_0(I_N - cS_N) \\
&= S_0 \left(\sum_{n=0}^{N} e^{-r\Delta(N-n)}\lambda_n - c\right).
\end{aligned}
$$

[1] This factorization first appeared in Carverhill and Clewlow [10] and is due to an insight by Stewart Hodges.

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Given the price of a fixed strike call option, we can easily compute the price of the corresponding put option using the put-call parity above. There exists also a symmetry relationship between fixed and floating options. This symmetry is possible by exploiting a change of numéraire and a time reversal of the Lévy process that does not affect the underlying Lévy structure but only the Lévy triplet. This is discussed in [16]. The equivalent result is not valid for in-progress Asian options.

3.4. Remarks.

Remark 1. If the density recursion is considered, the option Greeks for an Asian fixed call option can be easily computed from (3.10):

$$\text{Delta} = e^{-rN\Delta}\int_{d_1}^{+\infty}\left(\frac{\gamma}{N+\gamma}+e^x\right)g_1(x)dx, \quad \text{Gamma} = e^{-rN\Delta}\frac{K^2}{S_0^3 e^{d_1}}g_1(d_1),$$

with $d_1 = \log(K/S_0 - \gamma/(N+\gamma))$ if $K/S_0 - \gamma/(N+\gamma) > 0$, and

$$\text{Delta} = e^{-rN\Delta}\int_{-\infty}^{+\infty}\left(\frac{\gamma}{N+\gamma}+e^x\right)g_1(x)dx, \quad \text{Gamma} = 0$$

otherwise. We stress that if the price recursion is considered, these Greeks can be computed only by finite difference.

Finally, an analytical formula for the moments of the arithmetic average is available:

$$\mathbb{E}_0\left\{I_N^k\right\} \equiv (S_0)^k \sum_{j=0}^{k}\binom{k}{j}\lambda_0^{k-j}\mathbb{E}_0\left\{l_1^j\right\}, \tag{3.11}$$

where

$$\mathbb{E}_0\left\{l_n^j\right\} = \mathbb{E}_0\left\{\left(e^{Z_n}\left(\lambda_n + l_{n+1}\right)\right)^j\right\} = \phi_Z\left(-ij\right)\sum_{q=0}^{j}\binom{j}{q}\lambda_n^{j-q}\mathbb{E}_0\left\{\left(l_{n+1}\right)^q\right\}$$

for $n = N-1,\dots,1$, and the recursion starts with $\mathbb{E}_0\left\{l_N^j\right\} = \lambda_N^j\phi_Z\left(-ik\right)$.

Remark 2. In order to avoid nonsmoothness of the initial condition $v_N(x)$, we observe that

$$\begin{aligned} v_{N-1}\left(x\right) &= e^{-r\Delta}\int_{-\infty}^{+\infty} f_Z\left(s\right)e^s v_N\left(xe^{-s}+\lambda_N\right)ds \\ &= e^{-r\Delta}\int_{-\infty}^{+\infty} f_Z(s)\left((\lambda_N - c)e^s + x\right)^+ ds. \end{aligned}$$

In the fixed strike call option case, since $\lambda_N > 0$ and $c = 0$, we obtain

$$v_{N-1}\left(x\right) = \begin{cases} \lambda_N\, C^{pv}\left(1, -\frac{x}{\lambda_N}, \Delta\right) & \text{if } x < 0, \\ e^{-r\Delta}\left[x + \lambda_N e^{(r-d)\Delta}\right] & \text{if } x \geq 0, \end{cases} \tag{3.12}$$

where C^{pv} is the price of the plain vanilla call option with spot price 1, strike $-\frac{x}{\lambda_N}$, and maturity Δ. Otherwise, for floating strike call options, if $\lambda_N + 1 > 0$ and $x < 0$, we obtain

$$v_{N-1}\left(x\right) = \left(1+\lambda_N\right)C^{pv}\left(1, -\frac{x}{1+\lambda_N}, \Delta\right). \tag{3.13}$$

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The plain vanilla option prices can be computed by exploiting the Carr–Madan formula [9]. Thus recursions (3.7) and (3.8) can start at $N-2$ with initial condition (3.12) and (3.13), respectively.

Remark 3. Černý and Kyriakou [11] introduce a recursive pricing algorithm for the fixed strike Asian call option, suggesting the recursion [11, Theorem 3.1]

$$(3.14)\qquad \begin{aligned} q_{N-1}(x) &= \int_{-\infty}^{+\infty} f_Z(s)\left(e^{x+s}+\lambda_0\right)^+ ds,\\ q_n(x) &= \int_{-\infty}^{+\infty} f_Z(s)\, q_{n+1}\big(\log\left(e^{x+s}+\lambda_{N-n-1}\right)\big)\, ds \end{aligned}$$

for $n = N-2, \ldots, 0$ and the fixed strike call option price turns out to be

$$e^{-rN\Delta} S_0\, q_0\left(\log\left(\lambda_N\right)\right).$$

By a change of variables in the above integral, we obtain a recursion similar to (3.8):

$$q_n(x) = \int_{\log(\lambda_{N-n-1})}^{+\infty} f_Z\left(\log\left(e^y - \lambda_{N-n-1}\right) - x\right) \frac{e^y}{e^y - \lambda_{N-n-1}} q_{n+1}(y)\, dy$$

for $n = N-2, \ldots, 0$. The maturity randomization procedure that we will introduce in the following section can be applied to this recursion as well, with comparable numerical results in terms of accuracy and computational cost. Černý and Kyriakou introduce their recursion considering the reverse filtration, because of the Markovian properties of the obtained processes (see [11, Proposition 2.1]). In fact, the essential difference between the two approaches is that Černý and Kyriakou, starting from a generalized version of the Caverhill–Clewlow–Hodges factorization, introduce a new process which is Markov in the reverse filtration. Our price recursion is based on a change of measure that makes the process I_n/S_n Markov in the natural filtration.

4. The maturity randomization algorithm. In this section, we show how the above recursions can be solved by exploiting the so-called z-transform when $\lambda_n = \lambda$ for $n = 1, \ldots, N$; i.e., they do not depend on n. In practice this is the standard situation previously considered in the literature and in the market.

We start summarizing recursive equations (3.5) and (3.9) for an Asian call price into

$$(4.1)\qquad \begin{aligned} h(x,k) &= \int_\Omega K(x,y)\, h(y,k-1)\, dy, \quad k = 1, \ldots, M,\\ h(x,0) &= \phi(x). \end{aligned}$$

For ease of exposition, we will consider only fixed and floating Asian options. The price recursions (3.7) and (3.8) are equivalent to (4.1), if $x<0$, by setting

$$\begin{aligned} &h(x,k) = v_{N-k}(x);\; K(x,y) = -e^{-r\Delta} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right)\frac{x}{(y-\lambda)^2};\\ &\phi(x) = (x-c)^+;\; \Omega = (-\infty, \lambda);\; M = N. \end{aligned}$$

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Notice that if recursion (3.7) is considered, the value of $h(x,k)$, $k = 1, \ldots, M$, is known analytically for $x \geq 0$. See Remark 4 for further details.

In a similar way, for recursion (3.9), we set

$$h(x,k) = g_{N-k}(x); \; K(x,y) = f_Z(x - \log(\lambda + e^y)); \\ \phi(x) = f_{B_N}(x); \; \Omega = (-\infty, +\infty); \; M = N-1.$$

In these expressions the density $f_{B_N}(x)$ is obtained by numerically inverting the characteristic function of B_N using, for example, the FFT algorithm.[2]

The z-transform is a standard technique for solving difference equations, and it also has a nice probabilistic interpretation. The approach is very similar to the Laplace transform technique initially adopted by Geman and Yor [21] to solve the pricing problem of Asian options with continuous monitoring. There the maturity was randomized according to an exponential distribution. Instead, here, since we are considering a discretely monitored option, we deal with the z-transform that is equivalent to randomizing the option expiry according to a geometric distribution.

The randomization technique consists in making the expiry date T be random according to a geometric distribution of the parameter q and then defining

$$H(x,q) := (1-q) \sum_{k=0}^{+\infty} q^k h(x,k).$$

If we multiply both sides of (4.1) by $(1-q)\,q^k$ and then sum over k, $k \geq 1$, interchanging the order of integration and summation and finally adding $(1-q)\phi(x)$ to both sides, we get that the function $H(x,q)$ satisfies the integral equation

$$H(x,q) = q \int_\Omega K(x,y)\, H(y,q)\, dy + (1-q)\phi(x). \tag{4.2}$$

Therefore a recursive integral equation for $h(x,n)$ has been transformed into an integral equation for $H(x,q)$.

Remark 4. For fixed Asian options, if we consider the price recursion, the integral equation becomes

$$H(x,q) = q \int_{-\infty}^{0} K(x,y)\, H(y,q)\, dy + (1-q)\widetilde{\phi}(x),$$

with

$$\widetilde{\phi}(x) = \phi(x) + \frac{q}{1-q} \int_0^\lambda K(x,y)\, H(y,q)\, dy, \tag{4.3}$$

where, if $y \geq 0$,

$$H(y,q) = (1-q)\left(\frac{y}{1-qe^{-r\Delta}} + \frac{e^{(r-d)\Delta}}{(N+\gamma)(1-e^{(r-d)\Delta})} \left(\frac{1}{1-qe^{-r\Delta}} - \frac{1}{1-qe^{-d\Delta}} \right) \right).$$

[2] In our numerical experiments, we implemented the fractional FFT algorithm with 2^{17} points.

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The unknown function $h(x,n)$ can be obtained by derandomizing function $H(x,q)$ exploiting the complex inversion integral

$$h(x,n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi} \frac{H\left(x,\rho e^{iu}\right)}{1-\rho e^{iu}} e^{-inu} du \tag{4.4}$$

for $n = 0, 1, \ldots$. In particular, we approximate numerically (4.4) using (see [1])

$$\widetilde{h}(x,n) = \frac{\frac{H(x,\rho)}{1-\rho} + (-1)^n \frac{H(x,-\rho)}{1+\rho} + 2\sum_{j=1}^{n-1}(-1)^j \operatorname{Re}\left(\frac{H\left(x,\rho e^{ij\pi/n}\right)}{1-\rho e^{ij\pi/n}}\right)}{2n\rho^n}, \tag{4.5}$$

where $\operatorname{Re}(\cdot)$ denotes the real part function and ρ is set equal to $10^{-4/M}$ (see section 5.3).

In conclusion, our procedure consists of the following steps:

1. solve (4.2) when q is equal to $q_j = \rho e^{ij\pi/n}$, $j = 0, \ldots, n$;
2. approximate the desired quantity $h(x,M)$ by $\widetilde{h}(x,M)$ as in (4.5).

5. Implementation.

This section aims to describe how the randomization algorithm can be efficiently implemented. In particular, section 5.1 deals with the numerical solution of the integral equation (4.2), while section 5.2 presents a fast way to compute $\widetilde{h}(x,N)$ in (4.5). Finally, section 5.3 deals with the approximation error.

5.1. The Reichel approach for integral equations.

In order to solve the integral equation (4.2), we have to

1. truncate the domain Ω to a finite one $\Omega_T = (a,b)$, as discussed in section 5.3.2,
2. discretize the integral equation (4.2) by applying an appropriate quadrature formula; see [4] and [32].

If the chosen quadrature rule provides nodes x_i and weights w_i, $i = 1, \ldots, m$, (4.2) is approximated by

$$H(x_i,q) = q\sum_{j=1}^{m} w_j K(x_i,x_j) H(x_j,q) + (1-q)\phi(x_i), \quad i = 1, \ldots, m,$$

and thus we obtain the linear system

$$(\mathbf{I}_m - q\mathbf{K}_m\mathbf{D}_m)\mathbf{H}_m = \mathbf{\Phi}_m, \tag{5.1}$$

where the following hold:

- $\mathbf{I}_m$ is the identity matrix of size m;
- $\mathbf{K}_m$ is the square matrix with elements $K_{ij} = K(x_i,x_j)$, $i,j = 1,\ldots,m$;
- $\mathbf{D}_m$ is the diagonal matrix containing the weights w_i, $i = 1,\ldots,m$;
- $\mathbf{H}_m$ is the unknown solution vector, $H_i = H(x_i,q)$, $i = 1,\ldots,m$;
- $\mathbf{\Phi}_m$ is the right-hand side vector, $\Phi_i = (1-q)\phi(x_i)$, $i = 1,\ldots,m$.

Gaussian elimination (GE) or iterative methods are standard procedures for the numerical solution of (5.1). However, these algorithms are computationally demanding. For example, GE has a cost proportional to $2m^3/3$, whereas a standard iterative method such as the generalized

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minimal residual (GMRes) method has a cost proportional to $m^2 iter$, where $iter$ is the number of iterations necessary for the method to converge. Therefore, in order to speed up the solution of the above linear system, we consider an algorithm due to Reichel [33], who proposes a fast solution method for the one-dimensional Fredholm integral equation (4.2) on $\Omega_T = (a, b)$ for any $q \in \mathbb{C}$. The idea consists in discretizing the integral equation using a Nystrom quadrature rule based on the m nodes

$$x_j = \frac{1}{2}\left((b+a) + (b-a)\cos\left(\frac{\pi\,(j-1)}{m-1}\right)\right), \quad j = 1, \dots, m,$$

and the corresponding weights

$$w_j = \frac{\pi}{m-1}\sin\left(\frac{\pi\,(j-1)}{m-1}\right), \quad j = 1, \dots, m.$$

Therefore, we obtain a linear system of the form (5.1), but as proved in [33], the matrix $\mathbf{K}_m$ can be well approximated by a matrix $\mathbf{K}'_m$ of rank much smaller than m, returning the new linear system

$$\left(\mathbf{I}_m - q\mathbf{K}'_m\mathbf{D}_m\right)\mathbf{V}'_m = \mathbf{\Phi}_m. \tag{5.2}$$

Thus $\mathbf{V}'_m$ is an approximation to $\mathbf{V}_m$.

In addition, Reichel proposes an iterative algorithm for the solution of (5.2), exploiting a suitable preconditioner [33, section 4]. This algorithm is based on the solution of an $l+1 \times l+1$ linear system rather than $m \times m$, provided that the conditions

$$c_1 m^{\delta} \le l \le c_2 m^{2/3}, \quad m > m_1, \tag{5.3}$$

are satisfied for some constants m_1, $0 < c_1 < c_2 < +\infty$, and $0 < \delta < \frac{2}{3}$. In our numerical experiments we set $l = m^{2/3}$, and thus conditions (5.3) hold. Referring to the original Reichel paper [33] for implementation details, we stress that the number of floating point operations (flops) used for this method is $O(m^2)$. This means evident advantages with respect to the GMRes method. This is shown in the numerical experiments presented in section 6.2.

Remark 5. Notice that the same domain is used for all integral equations, independently of the value of q. Thus, the construction of matrices and vectors necessary for the Reichel iterative algorithm can be performed only once, always with $O(m^2)$ flops.

5.2. Euler acceleration. Euler summation is a convergence-acceleration technique well suited for evaluating alternating series, such as the one appearing in (4.5), which can be written as

$$\widetilde{h}\,(x, M) = \frac{1}{\rho^M}\sum_{j=0}^{M}(-1)^j a_j\, H\left(x, \rho e^{ij\pi/M}\right).$$

The idea of the Euler acceleration technique is to approximate the above sum with

$$\widetilde{h}\,(x, M) \approx \widetilde{h}_{n_e, m_e}(x, M) := \frac{1}{2^{m_e}\rho^M}\sum_{j=0}^{m_e}\binom{m_e}{j} b_{n_e+j}(x, M),$$

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where

$$b_k(x,M) = \sum_{j=0}^{k} (-1)^j a_j H\left(x, \rho e^{ij\pi/M}\right),$$

with n_e and m_e suitably chosen, such that $n_e + m_e < M$. For further details see [29] and [31].

Since the Euler summation works only if $n_e + m_e < M$, we can apply this technique only when more than $n_e + m_e$ monitoring dates are considered. Notice that in this case we need the values of $H(x, \rho e^{ij\pi/N})$ for $j = 0, \dots, n_e + m_e$; that is, we need to solve $n_e + m_e + 1$ integral equations (4.2) instead of $M + 1$ as in (4.5). Given that the choices of n_e and m_e can be done independently of M, this accelerating technique makes our method very competitive when pricing with a large number of monitoring dates is required.

5.3. Approximation error. The approximation error will arise mainly from three sources: the z-transform inversion error and its Euler acceleration; the domain truncation; and the numerical solution of the integral equations. We now discuss the different contributions.

5.3.1. Z-transform error. The numerical approximation error of (4.4) by (4.5) depends on the free parameter ρ, according to the following discussion. Abate and Whitt [1] provide an error bound to their discretization formula:

$$\left| h(x,M) - \widetilde{h}(x,M) \right| \leq \frac{C\rho^{2M}}{1-\rho^{2M}}.$$

For practical purposes, this error bound, when ρ^{2M} is small, is approximately equal to ρ^{2M}. Hence, to have an accuracy of $10^{-2\gamma}$, say, we require $\rho = 10^{-\gamma/M}$. Note that in practice it is important to adjust ρ in this way for each value of M, so that ρ^M stays small but bounded away from zero as $M \to \infty$. Otherwise small computational errors in the numerical evaluation of the denominator in (4.5) are magnified and will lead to gross errors in the evaluation of $h(x,M)$. In all our numerical experiments we set $\gamma = 4$ and $\rho = 10^{-\gamma/M}$ (which gives accuracy 10^{-8}).

If the Euler acceleration technique is considered, we also have to take into account the error $|\widetilde{h}(x,M) - \widetilde{h}_{n_e,m_e}(x,M)|$. A bound of this error is provided in [29, Theorem 1]. In our numerical experiments, setting $n_e = 12$ and $m_e = 10$ appears to guarantee sufficient accuracy at a low computational cost, as discussed in section 6.

5.3.2. Truncation error. The error due to the truncation of the domain Ω in (4.2) to a finite set Ω_T can be made arbitrary small. We recall the following:

- $\Omega = (-\infty, \lambda)$ for the floating strike price recursion (3.8);
- $\Omega = (-\infty, 0)$ for the fixed strike price recursion (3.7) (see Remark 4);
- $\Omega = (-\infty, +\infty)$ for the fixed strike density recursion (3.9).

Numerical experiments show that the choice of Ω_T cannot be done in a naive way. In fact, to keep the truncation error small, we need a large Ω_T, but this implies the use of a large number of quadrature nodes and thus a high computational cost. On the other hand, if Ω_T is too small, the accuracy is not great enough.

In our numerical experiments, for the fixed strike case based on density recursion, the domain of the integral equation is truncated by exploiting the moment bound (see [30] for further

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details). This bound is based on the arithmetic average's moments, computed according to (3.11), and allows us to obtain a tight bound on the tail probabilities. Numerical experiments have shown that a 10^{-8} tolerance is enough.

When price recursions are considered, the moment bound is no longer useful. In this case numerical experiments show that the best choice is to consider bounds that decrease in absolute value with respect to the number of monitoring dates. More precisely, we set heuristically $\Omega_T = (-\beta, \lambda)$ ($\Omega_T = (-\beta, 0)$) for the floating (fixed) strike price recursion, with $\beta = \frac{3}{2} + \frac{30}{N}$. This choice seems to provide a robust bound across all the considered distributions. In fact, our numerical experiments show that this domain truncation does not affect the accuracy up to the fourth decimal digit of the price estimates.

5.3.3. Numerical solution error. To compute the numerical approximation of the option price, we approximate $\widetilde{h}(x, M)$ in (4.5) (or its Euler acceleration) replacing the exact solution of the jth integral equation, $H(x, q_j)$, with its numerical approximation, $H_m(x, q_j)$, computed with m quadrature nodes, as discussed in section 5.1. Let us consider (4.5). The error in a quadrature node x is controlled by

$$E(x) = \frac{1}{M\rho^M} \left| \sum_{j=0}^{M} \frac{(-1)^j}{1 - \rho e^{ij\pi/M}} \left(H(x, q_j) - H_m(x, q_j) \right) \right|.$$

A similar discussion holds if the Euler acceleration technique is considered.

The error terms $|H(x, q_j) - H_m(x, q_j)|$, $j = 1, \ldots, M$, can be made arbitrary small by increasing the quadrature nodes m, since

$$||H(\cdot, q) - H_m(\cdot, q)||_{L^\infty(\Omega_T)} \leq C(q) m^{-\delta},$$

where $\delta > 0$ depends on the quadrature rule considered (see [32] and [33, Theorem 4.1] for the Reichel algorithm). More precisely, the speed of convergence of $||H(\cdot, q) - H_m(\cdot, q)||_{L^\infty(\Omega_T)}$ to zero can be determined by using results on the speed of convergence of the integration rule when it is applied to the integral $\int_{\Omega_T} K(\cdot, y) dy$, as discussed in [4, Chapter 4]. Thus, when Nystrom–Gaussian quadrature rules are considered, δ depends on the regularity of K and thus on the transition density function f_Z.

6. Numerical results.

All the numerical experiments were performed in MATLAB R2007a using a personal computer equipped with 4GB of RAM and an Intel Core 2 Quad Q6600 (2400MHz) processor. Table 1 lists the parametric Lévy processes used in our numerical experiments and their associated characteristic exponents.

The Gaussian model (G) is the benchmark assumption: The ensuing process is a purely diffusive Brownian motion, which gives rise to the geometric Brownian motion process for the price of the underlying asset. The jump diffusion (JD) model, introduced by Merton [28], and the double exponential (DE) model, introduced by Kou [25], are jump diffusion processes that account for the presence of fat tails in the empirical distribution of the underlying asset. The normal-inverse Gaussian (NIG) and the CGMY models, introduced in [5] and [8], respectively, are pure jump processes with finite variation that can display both finite and infinite activity. They are subordinate Brownian motions; in other words, they can be interpreted as Brownian motions subject to a stochastic time change that is related to the level of activity in the market. For further details, see [35].

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Table 1

Characteristic exponents of some parametric Lévy processes.

Model	$\psi(\omega)$
CGMY	$C\Gamma(-Y)\left((M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\right)$
DE	$-\frac{1}{2}\sigma^2\omega^2 + \lambda\left(\frac{(1-p)\eta_2}{\eta_2+i\omega} + \frac{p\eta_1}{\eta_1+i\omega} - 1\right)$
G	$-\frac{\sigma^2}{2}\omega^2$
NIG	$-\delta\left(\sqrt{\alpha^2-(\beta+i\omega)^2} - \sqrt{\alpha^2-\beta^2}\right)$
JD	$-\frac{1}{2}\sigma^2\omega^2 + \lambda\left(e^{i\omega\alpha-\frac{1}{2}\omega^2\delta^2} - 1\right)$

Table 2

Fixed strike Asian options: A comparison with Černý and Kyriakou [11]. Parameters: $r = 0.04$, $d = 0$, $T = 1$, $S_0 = 100$, $N = 50$, and $K = 100$. $\gamma = 1$ in (3.2).

Model	Parameters	m	l	Price (D)	Price (P)	Benchmark
G	$\sigma = 0.3$	1000	100	7.69860	7.69826	7.69859
		2000	159	7.69860	7.69851	
		3000	209	7.69859	7.69860	
		4000	252	7.69859	7.69859	
CGMY	$C = 0.6509$	1000	100	7.32360	7.34134	7.34742
	$G = 5.853$	2000	159	7.34754	7.34765	
	$M = 18.27$	3000	209	7.34747	7.34783	
	$Y = 0.8$	4000	252	7.34745	7.34741	
NIG	$\delta = 0.7543$	1000	100	7.33836	7.34182	7.34265
	$\alpha = 12.3407$	2000	159	7.34268	7.34305	
	$\beta = -5.8831$	3000	209	7.34265	7.34262	
		4000	252	7.34265	7.34265	

6.1. Accuracy. First of all, in Table 2, with reference to the fixed strike Asian option, we compare our pricing procedures, considering as a benchmark the numerical results in [11, Table 7]. We denote with P and D the cases in which recursions (3.7) (or (3.8) for the floating case) and (3.9) are considered, respectively. The additional numerical integration (3.10) and the integral in (4.3) were performed with a Gauss–Legendre quadrature formula with $m+1000$ and $m+2500$ nodes, respectively. The values of the function at the quadrature nodes are computed using cubic spline interpolation, which is favored with respect to linear interpolation because it captures the curvature of the integrand function.

In order to show the convergence of the Greeks (see Remark 1), in Table 3 we report the call price, as well as the Greeks Delta and Gamma, for different values of the volatility. The Greeks for the D algorithm are computed according to Remark 1, while for the method based on the pricing recursion (P) we use a finite difference approximation. As expected, the algorithm performs better for high volatilities (see also Table 2); however, convergence for low volatilities is still acceptable.

In Table 4 we consider as a benchmark[3] the results provided in [26] for floating strike call options, where a recursive integration, in the Gaussian framework, is proposed. Moreover,

[3]Notice that the benchmark is provided by the author with only the first three decimal digits.

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Table 3

Fixed strike Asian options. Parameters as in Černý and Kyriakou [11]: $r = 0.04$, $d = 0$, $T = 1$, $S_0 = 100$, $N = 50$, *and* $K = 100$. $\gamma = 1$ *in* (3.2).

Model	Parameters	m	Price (D)	Delta	Gamma	Price (P)	Delta	Gamma
G	$\sigma=0.1$	1000	3.33938	0.63256	0.063895	3.33782	0.57794	0.052021
		2000	3.33779	0.63253	0.063895	3.33684	0.61364	0.060358
		3000	3.33790	0.63253	0.063895	3.33768	0.62368	0.062318
		4000	3.33801	0.63253	0.063895	3.33792	0.62740	0.063028
NIG	$\delta=0.2515$	1000	3.11511	0.58111	0.045751	3.34075	0.58858	0.052682
	$\alpha=37.0242$	2000	3.26557	0.66207	0.062430	3.31039	0.63889	0.060243
	$\beta=-17.6537$	3000	3.31746	0.67366	0.063385	3.31642	0.65749	0.062172
		4000	3.32034	0.67390	0.063379	3.31986	0.66424	0.062825
G	$\sigma=0.5$	1000	12.09153	0.56140	0.013326	12.09101	0.55798	0.013221
		2000	12.09153	0.56140	0.013326	12.09140	0.56053	0.013300
		3000	12.09153	0.56140	0.013326	12.09147	0.56101	0.013315
		4000	12.09153	0.56140	0.013326	12.09152	0.56138	0.013320
NIG	$\delta=0.12573$	1000	11.23577	0.59073	0.014782	11.23612	0.58439	0.014672
	$\alpha=7.4046$	2000	11.23576	0.59072	0.014782	11.23606	0.58911	0.014756
	$\beta=-3.5302$	3000	11.23576	0.59072	0.014782	11.23606	0.59001	0.014771
		4000	11.23576	0.59072	0.014782	11.23580	0.59032	0.014776

Table 4

Floating strike Asian options: A comparison with Lim [26]. *Parameters:* $\sigma = 0.2$, $r = 0.1$, $d = 0$, $\lambda = 1$, $T = \frac{182}{265}$, $S_0 = 100$, $N = 91$, *and* $K = 100$. $\alpha = \gamma = 1$ *in* (3.3).

Model	m	Price (P)	Benchmark
G	1000	3.91980	4.565
	2000	4.56533	
	3000	4.56517	
	4000	4.56516	

Table 5 shows the behavior of our algorithm considering different monitoring dates under a jump diffusion distribution with the parameters proposed in [20]. Notice that in this table we denote with P-CM the P algorithm implemented, considering the Carr–Madan formula in the right-hand side of (4.2) (see Remark 2), computed with 2^{16} points. All the price estimates fall into the confidence interval estimated using the 1,000,000 control variate Monte Carlo simulations reported in [20].

From these numerical results it seems that our algorithm with the density recursion (D) provides price estimates for the fixed strike case with an accuracy of

- at least three decimal digits if $m = 2000$,
- at least four decimal digits if $m = 3000$,

while the algorithm based on price recursions (3.8) and (3.7) seems to provide a slower convergence to the true price. This speed of convergence could be slower for low volatility levels, as shown in Table 3. Moreover, considering the Carr–Madan formula as the initial condition to eliminate the nonsmoothness of the payoff (see Remark 2) does not seem to provide great advantages in terms of accuracy and computational cost. Only for the floating strike case, with few monitoring dates, do the numerical experiments not reported here show better accuracy if the Carr–Madan formula is used.

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Table 5

Fixed and floating strike Asian options: Jump diffusion distribution. Parameters: $r = 0.0367$, $d = 0$, $T\Delta = 1$, $S_0 = 100$, $\sigma = 0.126349$, $\lambda = 0.174814$, $\alpha = -0.390078$, *and* $\delta = 0.338796$. $\alpha = \gamma = 1$ *in* (3.2) *and* (3.3).

N	m	Fixed strike (D)	Fixed strike (P-CM)	Floating strike (P-CM)
12	1000	5.01130	5.01118	5.11719
	2000	5.01129	5.01127	5.11719
	3000	5.01129	5.01130	5.11719
	4000	5.01129	5.01129	5.11719
25	1000	5.03920	5.03922	5.14707
	2000	5.03916	5.03907	5.14755
	3000	5.03916	5.03909	5.14755
	4000	5.03916	5.03912	5.14755
50	1000	5.05306	5.05326	5.14780
	2000	5.05299	5.05270	5.16267
	3000	5.05299	5.05307	5.16246
	4000	5.05299	5.05304	5.16246
100	1000	5.06017	5.05989	3.52697
	2000	5.06015	5.05997	5.17561
	3000	5.06015	5.06025	5.17013
	4000	5.06015	5.06015	5.17011

Table 6

Fixed strike Asian options: A comparison with Fusai and Meucci [20]. *Parameters:* $r = 0.0367$, $d = 0$, $T = 1$, $S_0 = 100$, $N = 250$, *and* $K = 100$. $\gamma = 1$ *in* (3.2).

Model	Parameters	m	Price (D)	Price (P)	CI
G	$\sigma = 0.17801$	1000	4.95265	4.97341	4.783-5.121
		2000	4.95212	4.95072	
		3000	4.95212	4.95266	
		4000	4.95212	4.95242	
DE	$\sigma = 0.120381$	1000	4.73740	4.80054	4.837-5.301
	$\lambda = 0.330966$	2000	5.07019	5.06934	
	$p = 0.2071$	3000	5.07019	5.07130	
	$\eta_1 = 9.65997$, $\eta_2 = 3.13868$	4000	5.07019	5.07013	
JD	$\sigma = 0.126349$	1000	4.82410	4.91492	4.820-5.308
	$\lambda = 0.174814$	2000	5.06452	5.06308	
	$\alpha = -0.390078$	3000	5.06452	5.06514	
	$\delta = 0.338796$	4000	5.06452	5.06493	

In order to investigate the convergence of our pricing procedure when a large number of monitoring dates is considered, in Table 6 we report the price estimates obtained with 250 monitoring dates and the same parameters as Fusai and Meucci [20]. Our price estimates fall within the Monte Carlo confidence intervals (CI) provided therein.

Finally, in Figure 1 we plot the error for the algorithm based on the density recursion. The reference values are computed with 8000 quadrature nodes and solving the linear systems by the GMRes algorithm. From this figure, it is clear that the error due to the numerical solution of the integral equations by the Reichel algorithm (see section 5.3.3) rapidly reaches the 10^{-8} accuracy. This is the maximum accuracy we can achieve by using the z-transform

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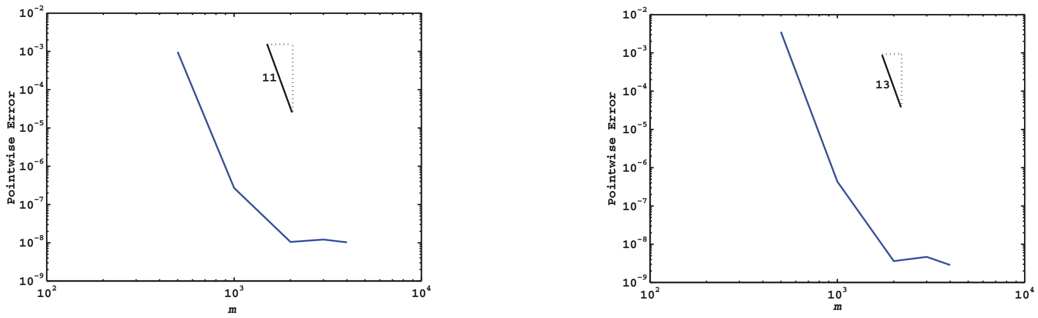

Figure 1. *Pricing error in loglog scale for double exponential (left) and jump diffusion (right) distributions. Parameters as in Fusai and Meucci* [20]*:* $r = 0.0367$*,* $d = 0$*,* $T = 1$*,* $S_0 = K = 1$*, and* $N = 100$*.*

Table 7

Fixed strike Asian option—CPU time (in seconds). Parameters as in Černý and Kyriakou [11]*.* $\gamma = 1$ *in* (3.2).

	CPU time (D)				CPU time (P)			
N	$m =$ 1000	2000	3000	4000	$m =$ 1000	2000	3000	4000
12	2.76	11.26	30.40	65.95	4.17	14.78	35.85	72.65
50	5.15	17.59	41.98	83.56	5.81	20.15	47.50	89.43
100	7.59	22.51	48.50	94.32	7.73	22.50	48.92	91.50
250	14.45	38.32	73.32	121.65	10.81	31.93	53.39	95.80

(see section 5.3.1). If the algorithm based on the price recursion is considered, we obtain slower convergence rates, as is clear from the above tables.

6.2. Computational cost. This section deals with the computational cost of our algorithm. Tables 7–8 report the CPU time necessary for pricing fixed and floating strike options, respectively, and the Gaussian model with parameters as in [11].

These tables and Figure 2 show that the computational cost of our algorithm grows less than linearly with respect to the number of monitoring dates, since the number of linear systems to be solved is bounded by $n_e + m_e + 1$ (see section 5.2), while all existing numerical algorithms have a cost at least linear. Moreover, if the number of monitoring dates is greater than $n_e + m_e$, the increase in the cost depends only on the number of iterations necessary for the convergence of the Reichel algorithm (see Table 9). We recall that the recursion methods such as the ones proposed in [11] and [20] have a computational cost linear with respect to the number of monitoring dates. Notice that when the number of monitoring dates increases, the pricing procedure (D), that is, the one with the recursion on the density, is the slowest one. This is due to the spectral properties of the matrix $\mathbf{I}_m - \mathbf{K}_m\mathbf{D}_m$ in (5.1) (see Table 10).[4]

In order to appreciate the benefits of the Reichel algorithm, in Table 11 we compare it with the option price estimates obtained by solving the linear systems with Gaussian elimination and the GMRes iterative method. We consider as a test case the CGMY model with the parameters proposed in [11] (see also Table 2). The Reichel algorithm appears to be the

[4]We recall that the number of iterations necessary for the convergence of an iterative method depends on the condition number of the linear system matrix.

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Table 8

Floating strike Asian option and CPU time (in seconds). Parameters as in Černý and Kyriakou [11]. $\alpha = \gamma = 1$ *in* (3.3).

N	$m =$ 1000	2000	3000	4000
12	2.56	12.18	34.46	70.76
50	3.60	15.70	41.62	81.84
100	4.07	16.70	43.21	82.14
250	5.90	18.59	47.17	87.54

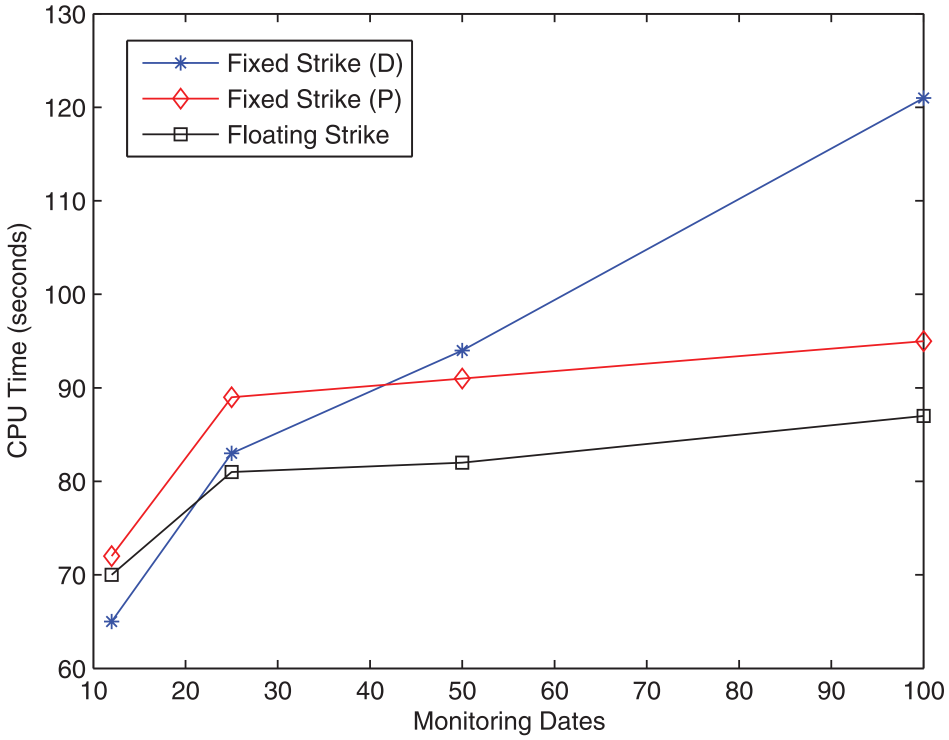

Figure 2. *CPU time and the number of monitoring dates. Parameters as in Černý and Kyriakou [11].* $m = 4000$. $\alpha = \gamma = 1$ *in* (3.2) *and* (3.3).

Table 9

Maximum number of iterations for the solution of a linear system using the Reichel procedure. Parameters as in Černý and Kyriakou [11]. $\alpha = \gamma = 1$ *in* (3.2) *and* (3.3).

	Fixed strike (D)				Floating strike (P)			
N	$m =$ 1000	2000	3000	4000	$m =$ 1000	2000	3000	4000
12	3	2	1	1	2	2	1	1
50	12	6	4	3	3	2	2	2
100	24	13	8	7	6	3	3	3
250	60	35	24	17	16	7	6	5

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Table 10

Condition number of the matrix $\mathbf{I}_m - \mathbf{K}_m\mathbf{D}_m$, computed in MATLAB with the `condest` command. Parameters as in Černý and Kyriakou [11].

	Fixed strike (D)			Floating strike (P)		
N	$m=$ 2000	3000	4000	$m=$ 2000	3000	4000
12	334	335	335	205	204	204
50	1374	1379	1381	604	611	617
100	3540	2805	2788	1344	1363	1346

Table 11

Fixed strike Asian option price and CPU time. Parameters as in Černý and Kyriakou [11]. $\gamma = 1$ in (3.2). Benchmark price: 7.34742. $T = 50$.

	Reichel		GMRes		Gaussian elimination	
m	Price (D)	CPU	Price (D)	CPU	Price (D)	CPU
1000	7.32360	7.76	7.30692	21.76	7.30692	14.87
2000	7.34754	22.40	7.34747	61.07	7.34747	93.68
3000	7.34747	51.96	7.34746	122.71	7.34746	298.00
4000	7.34745	99.67	7.34745	205.93	7.34745	680.06

Table 12

Floating strike Asian option and CPU time (in seconds) using grid computing. Parameters as in Černý and Kyriakou [11]. $\alpha = \gamma = 1$ in (3.3).

N	$m = 1000$	$m = 2000$	$m = 3000$	$m = 4000$
100	2.33	8.44	20.11	45.42
250	2.45	8.66	20.59	45.97

fastest one; moreover, increasing m, the differences between the prices computed with the three different solvers become negligible.

Finally, since the integral equations parametrized by q can be solved independently, our pricing method is suitable for parallelization. More precisely, we can exploit a grid architecture[5] composed of a set of six personal computers (the grid nodes), each equipped with 4GB of RAM and an Intel Core 2 Quad Q6600 (2400MHz) processor. Table 12 reports the maximum CPU time recorded by the grid nodes for the floating strike case, considering the same distribution and parameters as in Table 8. From this table it is clear that the benefits of our procedure are maximized using a distributed grid environment. Notice that we cannot obtain greater advantages from our grid, since the construction of the matrices and vectors necessary for the Reichel procedure cannot be parallelized (see Remark 5).

7. Conclusions. This paper has shown how to price discretely monitored Asian options in an exponential Lévy setting, introducing a new method based on the randomization of the option expiry. The starting point of our method comprises recursive formulas for floating and fixed strike options. Our procedure transforms the pricing problem into a set of integral equations that are solved using appropriate quadrature rules and linear system solvers. In

[5]We acknowledge the support of Avanade Italy and especially Roberto Chinelli, Laura Mariano, Raffaele Sgherri, and Ljubomir Konjic for kindly and freely providing the Department SEMeQ with the grid AGA (Avanade Grid Architecture).

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particular, the Reichel algorithm for the fast solution of the one-dimensional Fredholm integral equation performs well. By applying Euler summation, numerical experiments show that, without losing accuracy, our method has a cost that is nearly independent of the number of monitoring dates, different from other existing algorithms, if a recursion on the price is considered, while it grows less than linearly if our algorithm is applied to the recursion on the density, limitedly to the fixed strike case. In addition, the algorithm is suitable for parallelization.

Appendix A. Proof of expression (3.6).

Theorem A.1. *Assume that $x \geq 0$, $c \leq 0$, and $\lambda_n > 0$ for $n = 1, \dots, N$. Then it holds that*

$$v_n(x) = e^{-r(N-n)\Delta}\left[x + \sum_{i=0}^{N-n-1} \lambda_{N-i} e^{(N-n-i)(r-d)\Delta} - ce^{(N-n)(r-d)\Delta}\right]$$

for $n = 0, \dots, N$.

Proof. If $x \geq 0$, i.e., $\frac{I_n}{S_n} \geq 0$, $\lambda_n > 0$ for $n = 1, \dots, N$, and $c \leq 0$, the payoff is equal to

$$(I_N - cS_N)^+ = I_N - cS_N = \sum_{i=0}^{N} \lambda_i S_i - cS_N.$$

Thus it holds that

$$\begin{aligned} V(S_{N-1}, I_{N-1}; N-1) &= e^{-r\Delta}\mathbb{E}_{N-1}\left[\sum_{i=0}^{N-1} \lambda_i S_i + (\lambda_N - c)S_N\right] \\ &= e^{-r\Delta}\left[\sum_{i=0}^{N-1} \lambda_i S_i + \lambda_N S_{N-1} e^{(r-d)\Delta} - cS_{N-1}e^{(r-d)\Delta}\right], \end{aligned}$$

and, in general,

$$\begin{aligned} &V(S_{N-n}, I_{N-n}; N-n) \\ &\quad = e^{-nr\Delta}\left[\sum_{i=0}^{N-n} \lambda_i S_i + S_{N-n}\left(\sum_{i=0}^{n-1} \lambda_{N-i} e^{(n-i)(r-d)\Delta} - ce^{n(r-d)\Delta}\right)\right] \\ &\quad = e^{-nr\Delta} S_{N-n}\left[\frac{\sum_{i=0}^{N-n} \lambda_i S_i}{S_{N-n}} + \sum_{i=0}^{n-1} \lambda_{N-i} e^{(n-i)(r-d)\Delta} - ce^{n(r-d)\Delta}\right]. \end{aligned}$$

Since, by definition, it holds that

$$V(S_{N-n}, I_{N-n}; N-n) = S_{N-n}\ \ v_{N-n}\left(\frac{I_{N-n}}{S_{N-n}}\right),$$

we have

$$v_{N-n}(x) = e^{-rn\Delta}\left[x + \sum_{i=0}^{n-1} \lambda_{N-i} e^{(n-i)(r-d)\Delta} - ce^{n(r-d)\Delta}\right],$$

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which can be rewritten as

$$v_n(x) = e^{-r(N-n)\Delta} \left[x + \sum_{i=0}^{N-n-1} \lambda_{N-i} e^{(N-n-i)(r-d)\Delta} - c e^{(N-n)(r-d)\Delta} \right]. \qquad \blacksquare$$

Acknowledgments. We would like to thank the two anonymous referees for constructive suggestions on the subject.

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