Measure properties of the set of initial data yielding non uniqueness for a class of differential inclusions

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Measure Properties of the Set of Initial Data yielding Non Uniqueness for a class of Differential Inclusions

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Abstract

We study uniqueness property for the Cauchy problem $x' \in \partial V(x)$, $x(0) = \xi$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous, quasiconvex function (i.e. the sublevel sets $\{V \leq c\}$ are convex) and $\partial V(x)$ is the generalized gradient of V at x. We prove that if $0 \notin \partial V(x)$ for $V(x) \geq b$, then the set of initial data $\xi \in \{V = b\}$ yielding non uniqueness of solution in a geometric sense has $(n–1)$-dimensional Hausdorff measure zero in $\{V = b\}$.

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1. Introduction and Statement of the Result

In this work we study uniqueness property for Cauchy problems of the type

$$(\mathcal{P}_\xi) \qquad \begin{cases} x'(t) \in \partial V(x(t)) \\ x(0) = \xi. \end{cases}$$

When $V : \mathbb{R}^n \to \mathbb{R}$ is convex and locally Lipschitz continuous, $\partial V(x)$ is the subdifferential in the sense of convex analysis and the multifunction $x \mapsto \partial V(x)$ is a maximal monotone operator in $\mathbb{R}^n$. This implies that for every $\xi \in \mathbb{R}^n$ the problem $(\mathcal{P}_\xi)$ admits at least a solution, which is unique in the past. Moreover, as proved by Cellina [C] in a more general setting, for almost every $\xi \in \mathbb{R}^n$ the problem $(\mathcal{P}_\xi)$ admits a unique solution.

The interest in this subject arises from some questions in the calculus of variations. In this framework V would be a solution to a minimization problem and the hypothesis of convexity for V seems to be too strong.

It is more natural to consider the case in which $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and its sublevel sets are convex. In this case $\partial V(x)$ is the generalized gradient, introduced by Clarke [Cl]. Since the multifunction $x \mapsto \partial V(x)$ is upper semicontinuous with compact convex values, for any $\xi \in \mathbb{R}^n$ the problem $(\mathcal{P}_\xi)$ admits at least a local solution.

We point out that in general the operator $\partial V(\cdot)$ is not monotone (actually it is monotone if and only if V is convex, see [Cl, Proposition 2.2.9]) and therefore nothing can be said about uniqueness of solutions of $(\mathcal{P}_\xi)$. Hence it is natural to consider uniqueness property in a geometrical sense. Precisely, two solutions to $(\mathcal{P}_\xi)$ are geometrically distinct in the future if, for positive times they draw two different curves in $\mathbb{R}^n$.

We prove the following result.

Theorem. *Let $V : \mathbb{R}^n \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $b > a$, satisfy:*
(V_1) *V is locally Lipschitz continuous;*
(V_2) *the sets $\{V \le c\}$ are convex for every $c \ge a$;*
(V_3) *$0 \notin \partial V(x)$ for every $x \in \{V \ge b\}$.*
Let Ω be the set of the points $\xi \in \mathbb{R}^n$ such that the problem $(\mathcal{P}_\xi)$ admits (at least) two geometrically distinct solutions in the future. Then the $(n-1)$–dimensional Hausdorff measure of the set $\{V = b\} \cap \Omega$ is zero.

As immediate consequence of the theorem we get the following result.

Corollary. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function whose sublevel sets are convex and $0 \notin \partial V(x)$ for $V(x) > \inf V$. Then the n–dimensional Lebesgue measure of Ω is zero.*

Remarks. (i) We point out that, thanks to (V_1)–(V_3), for any $c \ge b$ the level set $\{V = c\}$ is covered by a countable collection of $(n-1)$–dimensional rectifiable sets.

(ii) When V is convex, the generalized gradient $\partial V(x)$ coincides with the standard subdifferential in the sense of convex analysis. In addition two different solutions to $(\mathcal{P}_\xi)$ are geometrically distinct in the future.

(iii) The hypothesis (V_3) cannot be omitted, as we emphasize in the example below.

2. Notations and Preliminary Results

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. By Rademacher's Theorem, the function V is differentiable almost everywhere (in the sense of Lebesgue measure) on $\mathbb{R}^n$. Moreover the generalized gradient $\partial V(x)$ can be defined as the set

$$\partial V(x) = \overline{co}\,\{\, lim\, \nabla V(x_i) \,:\, (x_i) \subset D\,,\; lim\, x_i = x\,,\; \exists\, lim\, \nabla V(x_i)\,\} \tag{2.1}$$

where $D = \{\, x \in \mathbb{R}^n \,:\, \exists\, \nabla V(x)\,\}$.

A first useful property that we need concerns the relationship between the generalized gradient $\partial V(x)$ and the cone of normals to $\{V \le V(x)\}$ at the point x.

To state it, we introduce some standard definitions, following [AC] and [Cl]. Given a convex closed nonempty subset C of $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we set $d_C(x) = inf\,\{\, |x-y| \,:\, y \in C\,\}$. It holds that:

(2.2) the function $x \mapsto d_C(x)$ is convex and $|d_C(x) - d_C(y)| \le |x-y|$ for any $x, y \in \mathbb{R}^n$;

(2.3) for any $x, v \in \mathbb{R}^n$ there exists $\displaystyle \lim_{t\to 0+} \frac{1}{t}(d_C(x+tv) - d_C(x)) = d'_C(x;v)$.

Then we define the tangent cone to C at x as

$$T_C(x) = \{\, v \in \mathbb{R}^n \,:\, d'_C(x;v) = 0\,\} \tag{2.4}$$

and the normal cone to C at x as

$$N_C(x) = \{\, y \in \mathbb{R}^n \,:\, \langle y, v\rangle \le 0\ \forall\, v \in T_C(x)\,\}. \tag{2.5}$$

The sets $T_C(x)$ and $N_C(x)$ are closed convex cones in $\mathbb{R}^n$ and $T_C(x) \cap N_C(x) = \{0\}$. In addition, since C is convex, $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis, namely

$$N_C(x) = \{\, \xi \in \mathbb{R}^n \,:\, \langle \xi, x-y\rangle \ge 0\ \forall y \in C\,\} \tag{2.6}$$

(see [Cl, proposition 2.4.4]).

Lemma 1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}$ satisfy (V_1) and (V_2). Then $\partial V(x) \subset N_{\{V \le V(x)\}}(x)$ for every $x \in \{V > a\}$.*

Proof. Let $C = \{V \le V(x)\}$. By (2.1), since C is a closed convex subset of $\mathbb{R}^n$, it is enough to prove that $\{\, lim\, \nabla V(x_i) \,:\, (x_i) \subset D\,,\; lim\, x_i = x\,,\; \exists\, lim\, \nabla V(x_i)\,\} \subset N_C(x)$.

Arguing by contradiction, let $(x_i) \subset D$ be such that $lim\, x_i = x$ and $lim\, \nabla V(x_i) = \xi$ with $\xi \notin N_C(x)$. Then, by (2.6), there exists $y \in C$ such that $\langle \xi, y - x\rangle > 0$. Since $x, y \in C$ and C is convex, $d_C(x+t(y-x)) = 0$ for all $0 \le t \le 1$ and so $d'_C(x; y-x) = 0$, that is $y - x \in T_C(x)$.
Hence, setting $v = y - x$ we have

(2.7) $\quad v \in T_C(x)$;

(2.8) $\quad x + v \in C$;

(2.9) $\quad \langle \xi, v\rangle > 0$.

Now we build a sequence $(v_i) \subset \mathbb{R}^n$ in the following way: setting $C_i = \{V \le V(x_i)\}$, we define $v_i = v$ if $v \in T_{C_i}(x_i)$. Otherwise we take $v_i \in \mathbb{R}^n$ such that $|v - v_i| \le d_{C_i}(x_i + v) + \frac{1}{i}$ and $x_i + v_i \in C_i$. In any case, by the convexity of C_i, we have that

(2.10) $\quad v_i \in T_{C_i}(x_i)$.

Noting that $d_{C_i}(x_i + v) \le d_{C_i}(x + v) + |x_i - x|$, since $lim\, x_i = x$, we can conclude that

(2.11) $\quad lim\, v_i = v$

provided that we prove that $lim\, d_{C_i}(x + v) = 0$. But this is true, because otherwise there is $\epsilon > 0$ and a sequence $(i_k) \subset \mathbb{N}$ such that $B_{2\epsilon}(x + v) \cap C_{i_k} = \emptyset$ for every $k \in \mathbb{N}$. Without loss of generality, we can also assume that the sequence $(V(x_{i_k}))$ is monotone, so that $\{V < V(x)\} \subseteq \bigcup_{k\ge1} \bigcap_{h\ge k} C_{i_h}$. Hence $B_{2\epsilon}(x+v) \cap \{V < V(x)\} = \emptyset$ and consequently $B_\epsilon(x + v) \cap C = \emptyset$ in contradiction with (2.8).
Since $\nabla V(x_i) \in N_{C_i}(x_i)$, by (2.10) and (2.6) we get $\langle \nabla V(x_i), v_i\rangle \le 0$ for every $i \in \mathbb{N}$. Passing to the limit $i \to \infty$, by (2.11) we conclude that $\langle \xi, v\rangle \le 0$ in contradiction with (2.9). □

Now we discuss some properties about the Cauchy problem $(\mathcal{P}_\xi)$. First of all we point out that the mapping $x \mapsto \partial V(x)$ is an upper semicontinuous compact convex multifunction defined on $\mathbb{R}^n$ (see [Cl]). This implies that for any $\xi \in \mathbb{R}^n$, problem $(\mathcal{P}_\xi)$ admits at least a solution, namely a function $x(\cdot, \xi) \in AC_{loc}(I, \mathbb{R}^n)$, where I is an interval containing 0, such that $x(0, \xi) = \xi$ and $x'(t, \xi) \in \partial V(x(t, \xi))$ for almost every $t \in I$. Moreover, maximal solutions are defined on $\mathbb{R}$ and the reacheable set at time t, denoted by $R(t, \xi)$, is connected (see [AC]). In addition the following properties hold.

Proposition 2. *Let $V : \mathbb{R}^n \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$, satisfy (V_1)–(V_3).*

(i) *If $x(\cdot, \xi) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$ is a solution of the problem $(\mathcal{P}_\xi)$ for $\xi \in \{V \ge b\}$, then $x(t, \xi) \in \{V > b\}$ for every $t > 0$, the function $t \mapsto V(x(t, \xi))$ is increasing and $\frac{d}{dt} V(x(t, \xi)) > 0$ for almost every $t \in \mathbb{R}$ such that $x(t, \xi) \in \{V \ge b\}$.*

(ii) *If $V(\xi_1) = V(\xi_2) \ge b$ and $t_1, t_2 > 0$ are such that $V(x(t_1, \xi_1)) = V(x(t_2, \xi_2))$ then $|x(t_1, \xi_1) - x(t_2, \xi_2)| \ge |\xi_1 - \xi_2|$.*

(iii) *For any $\xi \in \{V \ge b\}$ there is a unique $t \le 0$ such that $x(t, \xi) \in \{V = b\}$.*

Proof. (i) Let $x(\cdot,\xi) \in AC_{loc}(\mathbb{R},\mathbb{R}^n)$ be a solution of $(\mathcal{P}_\xi)$. The function $t \mapsto V(x(t,\xi))$ is locally Lipschitz continuous and then differentiable almost everywhere on $\mathbb{R}$. First of all we prove that

$$\frac{d}{dt}V(x(t,\xi)) \geq 0 \text{ for almost every } t \in \mathbb{R} \text{ such that } x(t,\xi) \in \{V \geq b\}. \tag{2.12}$$

Let $t \in \mathbb{R}$ be such that $x(t,\xi) \in \{V \geq b\}$, $x(\cdot,\xi)$ and $V(x(\cdot,\xi))$ are differentiable at t and $\frac{d}{dt}V(x(t,\xi)) < 0$. Then there is $\delta > 0$ such that $V(x(t+h,\xi)) < V(x(t,\xi))$ for $0 < h < \delta$. Hence, setting $C_t = \{V \leq V(x(t,\xi))\}$ we have $x'(t,\xi) \in T_{C_t}(x(t,\xi))$ because the tangent cone is closed. On the other hand, by Lemma 1, $x'(t,\xi) \in N_{C_t}(x(t,\xi))$ and this implies that $x'(t,\xi) = 0$, in contradiction with (V_3). Therefore (2.12) is proved. Now we check that the function $t \mapsto V(x(t,\xi))$ is strictly increasing. By the contrary, let us suppose that there exists an interval $[t_1,t_2]$ in which the function is constant, let us say equal to c. For any $t \in [t_1,t_2]$ and h such that $t+h \in [t_1,t_2]$, by a convexity argument we have $x(t+h,\xi) - x(t,\xi) \in T_{C_t}(x(t,\xi))$ and again a contradiction, as before. Now we prove that $x(t,\xi) \in \{V > b\}$ for every $t > 0$. By the upper semicontinuity of $\partial V(\cdot)$, there is $\epsilon > 0$ and $\delta > 0$ such that $|x'(t,\xi)| \geq \delta$ for almost any t for which $V(\xi) - \epsilon < V(x(t,\xi)) \leq V(\xi)$. Using this remark the conclusion follows arguing as above.

(ii) Let $\bar{b} = V(\xi_1) = V(\xi_2)$ and $c = V(x(t_1,\xi_1)) = V(x(t_2,\xi_2))$. We denote by τ_i the inverse function of $V(x(\cdot,\xi_i))$, defined by $V(x(\tau_i(\alpha),\xi_i)) = \alpha$ for $\alpha \in [\bar{b},c]$. It is an absolutely continuous increasing function and

$$\begin{aligned}
&\frac{d}{d\alpha}\frac{1}{2}\left|x(\tau_1(\alpha),\xi_1) - x(\tau_2(\alpha),\xi_2)\right|^2 \\
&\quad = \langle x(\tau_1(\alpha),\xi_1) - x(\tau_2(\alpha),\xi_2)\,,\, x'(\tau_1(\alpha),\xi_1)\,\tau_1'(\alpha) - x'(\tau_2(\alpha),\xi_2)\,\tau_2'(\alpha)\rangle \\
&\quad = \langle x(\tau_1(\alpha),\xi_1) - x(\tau_2(\alpha),\xi_2), x'(\tau_1(\alpha),\xi_1)\rangle\,\tau_1'(\alpha) \\
&\qquad + \langle x(\tau_2(\alpha),\xi_2) - x(\tau_1(\alpha),\xi_1), x'(\tau_2(\alpha),\xi_2)\rangle\,\tau_2'(\alpha).
\end{aligned} \tag{2.13}$$

Now we remark that $\tau_i'(\alpha) = (\frac{d}{dt}V(x(\tau_i(\alpha),\xi)))^{-1} > 0$ for almost every $\alpha \in [\bar{b},c]$. In addition, by lemma 1, $x'(\tau_i(\alpha),\xi_i)$ belongs to the normal cone to $\{V \leq V(x(\tau_i(\alpha),\xi_i)\}$ at the point $x(\tau_i(\alpha),\xi_i)$. Then, by (2.6), the last term in (2.13) is non negative and this proves (ii).

(iii) Arguing by contradiction, let $\xi \in \{V > b\}$ be such that $x(t,\xi) \in \{V > b\}$ for all $t \leq 0$. Then, by the part (i), the function $t \mapsto V(x(t,\xi))$ is strictly increasing on $\mathbb{R}$ and there exists $\lim\limits_{t\to-\infty} V(x(t,\xi)) = l \in [\,b, V(\xi)\,)$. Firstly we prove that

$$\text{the set } \{\, x(t,\xi)\ :\ t \leq 0\,\} \text{ is bounded.} \tag{2.14}$$

Indeed, let us take $\bar{\xi} \in \{V = l\}$. By the part (i), there exists $\bar{s} > 0$ such that $l < V(x(\bar{s},\bar{\xi})) \leq V(\xi)$. In addition there exists $\bar{t} \leq 0$ such that $V(x(\bar{t},\xi)) = V(x(\bar{s},\bar{\xi}))$. Moreover, since $V(x(t,\xi))$ decreases to l as $t \to -\infty$, for any $t \leq \bar{t}$ there is $s \in [0,\bar{s}]$ such that $V(x(t,\xi)) = V(x(s,\bar{\xi}))$. Hence, by the part (ii), we get $|x(t,\xi) - x(s,\bar{\xi})| \leq |x(\bar{t},\xi) - x(\bar{s},\bar{\xi})|$. Setting $r = |x(\bar{t},\xi) - x(\bar{s},\bar{\xi})|$ and $K = \{\, y \in \mathbb{R}^n\ :\ |y - x(s,\bar{\xi})| \leq r\,,\ s \in [0,\bar{s}]\,\}$, it holds that K is compact and $x(t,\xi) \in K$ for all $t \leq \bar{t}$. Thus (2.14) is proved.

By (2.14), we can take a sequence $(t_i) \subset \mathbb{R}$ such that $t_i \to -\infty$, $x(t_i, \xi) \to \bar{\xi}$ and $x(t_i + 1, \xi) \to \zeta$ for some $\bar{\xi}, \zeta \in \{V = l\}$. Then, setting $\xi_i = x(t_i, \xi)$, it holds that $x(t, \xi_i) = x(t_i + t, \xi)$ for every $t \in [0, 1]$ and, by (2.14), $\|x'(\cdot, \xi_i)\|_{L^\infty(0,1)} \leq const$. Therefore, up to a subsequence, $x(\cdot, \xi_i)$ converges uniformly on $[0, 1]$ to some $y(\cdot) \in AC([0, 1], \mathbb{R}^n)$ and $x'(\cdot, \xi_i) \to y'(\cdot)$ in the weak* topology of $L^\infty(0, 1)$. By [AC, Theorem 1, pag. 104] the function $y(\cdot)$ is a solution to the problem $(\mathcal{P}_{\bar{\xi}})$ on the interval $[0, 1]$ and $y(1) = x(1, \bar{\xi}) = \zeta$. In addition $l \leq V(y(t)) = lim\, V(x(t + t_i, \bar{\xi})) \leq lim\, V(x(t_i + 1, \bar{\xi})) = l$, that is $x(t, \bar{\xi}) \in \{V = l\}$ for all $t \in [0, 1]$, which contradicts the part (i). Thus we proved that for any $\xi \in \{V \geq b\}$ there exists at least a value $t \leq 0$ such that $x(t, \xi) \in \{V = b\}$. The uniqueness of such a value t follows from the part (i). $\square$

We now introduce the notion of geometrically distinct solutions for the Cauchy problem $(\mathcal{P}_\xi)$.

Definition. *Given $\xi \in \mathbb{R}^n$ two solutions $x_i(\cdot, \xi) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$, $i = 1, 2$ to the problem $(\mathcal{P}_\xi)$ are geometrically distinct in the future (respectively in the past) if the sets $\{x_i(t, \xi) : t \geq 0\}$ (resp. $\{x_i(t, \xi) : t \leq 0\}$) are different. We say that the problem $(\mathcal{P}_\xi)$ has uniqueness of solution in the future (resp. in the past) if there do not exist two solutions geometrically distinct in the future (resp. in the past).*

Remark 1. Property (ii) of Proposition 2 implies that, for any $\xi \in \mathbb{R}^n$, problem $(\mathcal{P}_\xi)$ has uniqueness of solution in the past.

Remark 2. For any $c \geq b$ we can define a function $f_c : \{V = c\} \to \{V = b\}$ in the following way: $f_c(y) = \xi$, where ξ is such that there exist a solution $x(\cdot, \xi)$ of problem $(\mathcal{P}_\xi)$ and a value $t \geq 0$ satisfying $x(t, \xi) = y$. Property (iii) of Proposition 2 guarantees that this function is well defined and, thanks to (ii), it satisfies the global Lipschitz condition $|f_c(x) - f_c(y)| \leq |x - y|$ for any $x, y \in \{V = c\}$.

Example. When hypothesis (V_3) is not satisfied it may happen that some level set $\{V = c\}$ has Hausdorff dimension greater than $n - 1$. Even assuming that any level set $\{V = c\}$ has Hausdorff dimension $n - 1$, the hypothesis (V_3) plays a fundamental role. Indeed, let us consider the function $V : \mathbb{R} \to \mathbb{R}$ defined by

$$V(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ \frac{2}{3}\sqrt{(x-1)^3} & \text{if } x > 1 \\ \frac{2}{3}\sqrt{(-x-1)^3} & \text{if } x < -1. \end{cases}$$

It is easy to check that $0 \in \partial V(x)$ if $x \in \{V = 1\}$. For any $0 < \xi \leq 1$ we have that

$$x_1(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ 1 & \text{if } t > 1 - \xi \end{cases}$$

and

$$x_2(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ \frac{1}{4}(t-1)^2 + 1 & \text{if } t > 1 - \xi \end{cases}$$

are two geometrically distinct solutions of the problem $(\mathcal{P}_\xi)$.

3. Proof of the Theorem

Let Ω be the set of initial data such that the Cauchy problem $(\mathcal{P}_\xi)$ has at least two solutions geometrically distinct in the future. We denote by $\mathcal{H}^s$ the s–dimensional Hausdorff measure and by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$.

We want to prove that $\mathcal{H}^{n-1}(\Omega \cap \{V = b\}) = 0$. For $\xi \in \{V = b\}$ and $c \geq b$ we denote $R_c(\xi) = \{\, y \in \{V = c\} \;:\; f_c(y) = \xi \,\}$ where f_c is given as in Remark 2. We devide the proof in the following steps.

Step 1. For every $\xi \in \Omega \cap \{V = b\}$ there is $c > b$, $c \in \mathbb{Q}$ such that $\mathcal{H}^0(R_c(\xi)) = \infty$.

Let (C_j) be a countable collection of compact sets that cover $\mathbb{R}^n$.

Step 2. For every $\xi \in \Omega \cap \{V = b\}$ there exists $c > b$, $c \in \mathbb{Q}$ and $\bar{j} \in \mathbb{N}$ such that $\mathcal{H}^0(R_c(\xi) \cap C_{\bar{j}}) = \infty$.

Step 3. Application of the coarea formula.

Step 1. Let $\xi \in \{V = b\}$ be such that there exist two solutions $x_i(\cdot, \xi)$ $(i = 1, 2)$ of the problem $(\mathcal{P}_\xi)$ geometrically distinct in the future. We can say that there are a rational $c > b$ and $t_1, t_2 > 0$ such that $x_i(t_i, \xi) \in \{V = c\}$ $(i = 1, 2)$ and $x_1(t_1, \xi) \neq x_2(t_2, \xi)$. We define

$\gamma_i = \{x_i(t, \xi) : t \geq 0\}$,

$\bar{t}_2 = sup\{t \in [0, t_2] : d(x_2(t, \xi), \gamma_1) = 0\}$

$\xi_1 = x_2(\bar{t}_2, \xi)$.

We remark that $\xi_1 \in \gamma_1 \cap \gamma_2$, there exists $\bar{t}_1$ such that $\xi_1 = x_1(\bar{t}_1, \xi)$ and $x_i(t - \bar{t}_i, \xi)$ are solutions of problem $(\mathcal{P}_{\xi_1})$. Moreover, if $y(\cdot, \xi_1)$ is a solution of $(\mathcal{P}_{\xi_1})$ in $[0, +\infty)$, then

$$x(t) = \begin{cases} x_i(t, \xi) & \text{for } t \in [0, \bar{t}_i] \\ y(t - \bar{t}_i, \xi_1) & \text{for } t > \bar{t}_i \end{cases}$$

is a solution of $(\mathcal{P}_\xi)$. Then $R(t, \xi_1) \subset R(t + \bar{t}_i, \xi)$ for any $t > 0$ and, by Proposition 2, part (i), $R(t, \xi_1) \subset \{V > V(\xi_1)\}$ for any $t > 0$. We have also $R(t, \xi_1) \cap \gamma_i \neq \emptyset$ and $R(t, \xi_1) \cap \gamma_1 \cap \gamma_2 = \emptyset$. We can suppose that $R(t, \xi_1) \subset \{V(\xi_1) < V \leq c\}$ if t is sufficiently small. In fact, if it is not the case we can substitute $x_2(\cdot, \xi)$ with an other solution, denoted again by $x_2(\cdot, \xi)$, in such a way the above inclusion holds. We state the following

Claim: there is a solution $x_3(\cdot, \xi)$ of the problem $(\mathcal{P}_\xi)$ and a value $t_3 > 0$ such that $x_3(t_3, \xi) \in \{V = c\}$ and $x_3(t_3, \xi) \neq x_i(t_i, \xi)$ for $i = 1, 2$.

Indeed for any $\alpha \in \{V(y) : y \in R(t, \xi_1)\}$ and $i = 1, 2$ there is a unique $z^i_\alpha \in \gamma_i \cap \{V = \alpha\}$ and $z^1_\alpha \neq z^2_\alpha$. Moreover there exists $\alpha \in \{V(y) : y \in R(t, \xi_1)\}$ for which the set $\{V = \alpha\} \cap R(t, \xi_1)$ contains a third point $z^3_\alpha \neq z^i_\alpha$, $i = 1, 2$. Let $y_3(\cdot, \xi)$ be a solution to the problem $(\mathcal{P}_\xi)$ such that, for a suitable t, it is $y_3(t, \xi) = z^3_\alpha$.

If $lim_{t\to+\infty} V(y_3(t,\xi)) > c$ then there exists $t_3 > 0$ such that $V(y_3(t_3,\xi)) = c$ and $y_3(t_3,\xi) \neq x_i(t_i,\xi)$ for $i = 1, 2$. In this case we put $x_3(\cdot,\xi) = y_3(\cdot,\xi)$.

If $lim_{t\to+\infty} V(y_3(t,\xi)) \leq c$, by Proposition 2, part (i) $V(y_3(t,\xi)) < c$ for any $t > 0$. Considering the problem $(\mathcal{P}_{x_1(t_1,\xi)})$, we have that for any $t > 0$ $\{V > c\} \cap R(t+t_1,\xi) \supset R(t, x_1(t_1,\xi)) \neq \emptyset$. Moreover $R(t+t_1,\xi) \cap \{V < c\} \neq \emptyset$. Then for any $t > 0$ there exists $z \in R(t+t_1,\xi) \cap \{V = c\}$. We want to prove that $z \neq x_i(t_i,\xi)$. Let us suppose that $z = x_1(t_1,\xi)$. Then $x_1(t_1,\xi) \in R(t+t_1,\xi)$ for any $t > 0$. In other words, for any $t > 0$ we can find a solution $y(\cdot)$ to the problem $(\mathcal{P}_\xi)$ such that $y(t+t_1) = x_1(t_1,\xi)$ and for any $\tau \in [0, t+t_1]$ there exists $s \in [0, t_1]$ such that $y(\tau) = x_1(s,\xi)$. For $s \in [0, t_1]$ the curve described by $x_1(s,\xi)$ (the same described by $y(\tau)$ for $\tau \in [0, t+t_1]$) is contained in a compact set $K \subset \{b \leq V \leq c\}$, so there exist two positive constants m, M such that for any $v \in \partial V(x)$, $x \in K$ it is $m \leq |v| \leq M$. Computing the lenght of this curve we have

$$m\,(t+t_1) \leq \int_0^{t+t_1} |y'(\tau)|\,d\tau = \int_0^{t_1} |x_1'(s,\xi)|ds \leq Mt_1$$

and we get that $t \leq t_1\left(\frac{M}{m} - 1\right)$, contradicting the fact that t is an arbitrary positive value. Then repeating the same argument for $x_2(t_2,\xi)$, and choosing t sufficiently large, we can denote by $x_3(\cdot,\xi)$ a solution to the problem $(\mathcal{P}_\xi)$ for which there exists t_3 such that $x_3(t_3,\xi) = z$. Hence the claim is proved.

Let now $\gamma_3 = \{\, x_3(t,\xi) \,:\, t \geq 0 \,\}$. We can suppose without restriction that $\gamma_1 \cap \gamma_3 \subset \gamma_2 \cap \gamma_3$. Let $\bar{t}_3 = sup\{t \in [0, t_3] : d(x_3(t,\xi), \gamma_2) = 0\}$ and $\xi_2 = x_3(\bar{t}_3,\xi)$. Now, we can repeat the same argument to construct a sequence $(x_i(\cdot,\xi))$ of solutions of the problem $(\mathcal{P}_\xi)$ and a corresponding sequence $(t_i) \subset \mathbb{R}^+$ such that $x_i(t_i,\xi) \in \{V = c\}$ and $x_i(t_i,\xi) \neq x_j(t_j,\xi)$ for $i \neq j$. Hence we have that $x_i(t_i,\xi) \in R_c(\xi)$ for any $i \in \mathbb{N}$ and then $\mathcal{H}^0(R_c(\xi)) = \infty$, being $\mathcal{H}^0$ the counting measure.

Step 2. Arguing by contradiction, let us suppose that for any $\epsilon > 0$ and $k > 0$ there exists $i_{\epsilon,k} \in \mathbb{N}$ such that for any $i > i_{\epsilon,k}$ it is $|x_1(t_1(b+\epsilon),\xi) - x_i(t_i(b+\epsilon),\xi)| > k$. Hence $lim_{\epsilon\to 0^+} |x_1(t_1(b+\epsilon),\xi) - x_i(t_i(b+\epsilon),\xi)| \geq k$ and this contradicts the fact that for any $i \in \mathbb{N}$ $lim_{\epsilon\to 0^+} x_i(t_i(b+\epsilon)) = \xi$.

Step 3. By the step 2, we have

$$\{V = b\} \cap \Omega = \bigcup_{j\in\mathbb{N}} \bigcup_{c\in\mathbb{Q}, c>b} \left\{\xi : \mathcal{H}^0(R_c(\xi) \cap C_j) = \infty\right\}.$$

We now apply the coarea formula (see [M, Theorem 3.13])

$$\int_{\{V=c\}\cap C_j} Jf_c(x)\, d\mathcal{H}^{n-1}(x) = \int_{\{V=b\}} \mathcal{H}^0(R_c(\xi) \cap C_j)\, d\mathcal{H}^{n-1}(\xi).$$

The left hand side is finite because f_c is Lipschitzean and $\mathcal{H}^{n-1}(\{V = c\} \cap C_j) < \infty$. Then for any $c \in \mathbb{Q}$, $c > b$ and for any $j \in \mathbb{N}$

$$\mathcal{H}^{n-1}\left(\{V = b\} \cap \left\{\xi : \mathcal{H}^0(R_c(\xi) \cap C_j) = \infty\right\}\right) = 0.$$

□

We conclude this section proving the Corollary.

Proof of the Corollary. We can apply the Theorem with $a = inf\, V$. Then $\mathcal{H}^{n-1}(\{V = b\} \cap \Omega) = 0$ for any $b > a$ and using again the coarea formula, we get

$$\int_{\Omega \cap C_j} JV(x)\, d\mathcal{H}^n(x) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\{V = b\} \cap \Omega \cap C_j)\, d\mathcal{H}^1(b) = 0$$

where (C_j) is a covering of $\mathbb{R}^n$ of compact sets. By (V_3), $JV(x) > 0$ for $\mathcal{H}^n$–almost every $x \in \mathbb{R}^n \setminus \{V = a\}$. Moreover, if $\{V = a\} \neq \emptyset$, the set $\Omega \cap \{V = a\}$ is contained in the boundary of the convex set $\{V = a\}$ in $\mathbb{R}^n$, denoted $\partial\{V = a\}$. Indeed if $\xi \in \mathbb{R}^n$ is an interior point of $\{V = a\}$ then the only solution to $(\mathcal{P}_\xi)$ is the constant function $x(t, \xi) = \xi$. Since the Hausdorff dimension of $\partial\{V = a\}$ is strictly less that n, $JV(x) > 0$ for $\mathcal{H}^n$–almost every $x \in \Omega$. Therefore $\mathcal{H}^n(\Omega \cap C_j) = 0$ for any $j \in \mathbb{N}$. Since $\mathcal{H}^n$ coincides with the Lebesgue measure $\mathcal{L}^n$, we have that Ω is Lebesgue measurable and $\mathcal{L}^n(\Omega) = 0$.

□

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