Genericity of the existence of infinitely many solutions for a class of semilinear elliptic equations in RN

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Genericity of the existence of infinitely many solutions for a class of semilinear elliptic equations in $\mathbf{R}^N$

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Abstract. We show, by variational methods, that there exists a set $\mathcal{A}$ open and dense in $\{a \in L^\infty(\mathbf{R}^N) : \liminf_{|x|\to\infty} a(x) \geq 0\}$ such that if $a \in \mathcal{A}$ then the problem $-\Delta u + u = a(x)|u|^{p-1}u$, $u \in H^1(\mathbf{R}^N)$, with p subcritical (or more general nonlinearities), admits infinitely many solutions.

Key Words. Semilinear elliptic equations, locally compact case, minimax arguments, multiplicity of solutions, genericity.

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1 Introduction

In this paper we study the existence of infinitely many solutions for the semilinear elliptic problem

$$-\Delta u + u = a(x)f(u) \qquad u \in H^1(\mathbf{R}^N) \tag{P_a}$$

where we assume that $a \in L^\infty(\mathbf{R}^N)$, $\liminf_{|x|\to\infty} a(x) > 0$, and

($f1$) $f \in C^1(\mathbf{R})$,

($f2$) there exists $C > 0$ such that $|f(t)| \le C(1 + |t|^p)$ for any $t \in \mathbf{R}$, where $p \in (1, \frac{N+2}{N-2})$ if $N \ge 3$ and $p > 1$ if $N = 1, 2$,

($f3$) there exists $\theta > 2$ such that $0 < \theta F(t) \le f(t)t$ for any $t \ne 0$, where $F(t) = \int_0^t f(s)\, ds$,

($f4$) $\frac{f(t)}{t} < f'(t)$ for any $t \ne 0$.

Note that $f(t) = |t|^{p-1}t$ verifies ($f1$)–($f4$) whenever $p \in (1, \frac{N+2}{N-2})$ if $N \ge 3$ or $p > 1$ if $N = 1, 2$.

Such kind of problem has been studied in several papers and its main feature is given by a lack of compactness due to the unboundedness of the domain. Indeed the imbedding of $H^1(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ is not compact and the Palais Smale condition fails.

The existence of nontrivial solutions of (P_a) strongly depends on the behaviour of a. We refer to [18] where it is showed that if a is monotone (non costant) in one direction then the problem (P_a) has only the trivial solution.

On the other hand, if a is a positive constant or $a(x) \to a_\infty \ge 0$ as $|x| \to \infty$, the problem has been studied by using concentration-compactness arguments and several existence results are known. See, e.g., [30], [23], [17], [31], [7], [8], [11] and [20].

When a is periodic, the invariance under translations permits to prove existence, [27], and also multiplicity results, as in [6], [15], [1], [24], where infinitely many solutions (distinct up to translations) are found. In fact, in this case, the noncompactness of the problem can be exploited to set up a new minimax argument, in the spirit of the works [14] and [29], and then to exhibit a rich structure of the set of solutions.

Multiplicity results have been obtained also without periodicity or asymptotic assumptions on a, in some "perturbative" settings, where concentration phenomena occur and a localization procedure can be used to get some compactness in the problem. A first result in this direction is the paper [21] concerning the prescribing scalar curvature problem on $\mathbf{S}^3$ and $\mathbf{S}^4$. We also mention [28], [4], [5], [16], [19], [22] and the references therein, for the case of a nonlinear stationary Schrödinger equation $-\epsilon^2\Delta u + V(x)u = f(u)$ with $\epsilon > 0$ small and $V \in C^1(\mathbf{R}^N)$, $V(x) \geq V_0 > 0$ in $\mathbf{R}^N$, having local maxima or minima or other topologically stable critical points. Similar concentration phenomena occur also considering the equation $-\Delta u + \lambda u = a(x)f(u)$ for $\lambda > 0$ large enough (see [13]) or $-\Delta u + u = a(x)|u|^{p-1}u$ with $p = \frac{N+2}{N-2} - \epsilon$, $\epsilon > 0$ small, and $N \geq 3$, where a blow-up analysis can be done (see [25]).

In this paper, motivated by [3], we adopt a quite different viewpoint from the ones followed in the above quoted works and we show that the existence of infinitely many solutions for the problem (P_a) is a generic property with respect to $a \in L^\infty(\mathbf{R}^N)$, with $\liminf_{|x|\to\infty} a(x) \geq 0$. Precisely we prove

Theorem 1.1 *Let $f : \mathbf{R} \to \mathbf{R}$ satisfy $(f1)$–$(f4)$. Then there exists a set $\mathcal{A}$ open and dense in $\{a \in L^\infty(\mathbf{R}^N) : \liminf_{|x|\to\infty} a(x) \geq 0\}$ such that for every $a \in \mathcal{A}$ the problem (P_a) admits infinitely many solutions.*

In fact, given any $a \in L^\infty(\mathbf{R}^N)$ with $\liminf_{|x|\to\infty} a(x) > 0$, for all $\bar\alpha > 0$ we are able to construct a function $\alpha \in C(\mathbf{R}^N)$, $0 \leq \alpha(x) \leq \bar\alpha$ in $\mathbf{R}^N$, such that the problem $(P_{a+\alpha})$ admits infinitely many solutions. Moreover we show that this class of solutions is stable with respect to small L^∞ perturbations of the function $a + \alpha$.

The function α is obtained in a constructive way that can be roughly described as follows. First, we introduce the variational setting and we make a careful analysis of the functionals "at infinity" corresponding to the equations $-\Delta u + u = b(x)f(u)$ where $b \in H_\infty(a)$, i.e., the set of the w^*-L^∞ limits of the sequences $a(\cdot + x_j)$ with $(x_j) \subset \mathbf{R}^N$, $|x_j| \to \infty$. All the functionals at infinity have a mountain pass geometry and, called $c(b)$ the mountain pass level associated to the problem (P_b), we can show that there exists $a_\infty \in H_\infty(a)$ such that $c(a_\infty) \leq c(b)$ for any $b \in H_\infty(a)$ and the corresponding problem (P_{a_∞}) admits a solution characterized as mountain pass critical point. Then, following a suitable sequence $(x_j) \subset \mathbf{R}^N$ such that $a(\cdot + x_j) \to a_\infty$ w^*-L^∞, we construct α by perturbing a in neighborhoods of x_j in order to get local compactness and local minimax classes for the perturbed functional which allow us to prove existence of infinitely many critical points localized around any point x_j.

We note that, by a standard argument (taking $\bar f$ instead of f, defined by $\bar f(t) = 0$ for $t \leq 0$ and $\bar f(t) = f(t)$ for $t > 0$), it is possible to show the existence of infinitely many positive classical solutions of the problem (P_a) for any $a \in \mathcal{A}$, a smooth.

Finally we want to point out some possible easy extensions of our result. We observe firstly that with minor change, our argument can be used to prove an analogous result for the class of the nonlinear Schrödinger equations $-\Delta u + b(x)u = a(x)f(u)$ with $b \in L^\infty(\mathbf{R}^N)$, $b(x) \geq b_0 > 0$ for a. e. $x \in \mathbf{R}^N$, and a and f as above. Moreover, we point out that in proving Theorem 1.1 we never use comparison theorems based on the maximum principle. Then our argument can be repeated exactly in the same way to study systems of the form $-\Delta u + u = a(x)\nabla F(u)$ where $F \in C^2(\mathbf{R}^N, \mathbf{R}^M)$ satisfies properties analogous to $(f2)$, $(f3)$, $(f4)$. In particular the result can be established in the framework of the homoclinic problem for second order Hamiltonian systems in $\mathbf{R}^M$ (see [3] and the references therein).

Secondly we remark that the solutions we find satisfy suitable stability properties. These can be used to prove that in fact the perturbed problem $(P_{a+\alpha})$ admits multibump type solutions (see [29]) with bumps located around the points x_j. We refer in particular to [2] for a proof that can be adapted in this setting.

Finally we mention also the fact that if a is assumed to be positive and almost periodic (see [10]) then it is not known whether or not the problem (P_a) admits solutions. Following [3] it is possible to show that in this case one can construct a perturbation α almost periodic and with L^∞ norm small as we want, in such a way that the problem $(P_{a+\alpha})$ admits infinitely many (actually multibump type) solutions. Then we get a genericity result (with respect to the property of existence of infinitely many solutions) for the class of problems (P_a) with $a \in C(\mathbf{R}^N)$ positive and almost periodic.

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2 Variational setting and preliminary results

In this section we study the class of problem (P_a) with $a \in \mathcal{F}_{m,M} = \{a \in L^\infty(\mathbf{R}^N) : m \leq a(x) \leq M$ a.e. in $\mathbf{R}^N\}$, $0 < m < M$, and f satisfying $(f1)$–$(f4)$.

Let $X = H^1(\mathbf{R}^N)$ be endowed with its standard norm $\|u\| = (\int_{\mathbf{R}^N} (|\nabla u|^2 +$

$u^2)dx)^{\frac{1}{2}}$ and consider the functional

$$\varphi_a(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} a(x)F(u(x))\,dx. \tag{2.1}$$

By $(f3)$ we have $F(0) = f(0) = f'(0) = 0$ and then, by $(f2)$,

$$\forall \varepsilon > 0,\ \exists A_\varepsilon > 0 \text{ such that } |f(t)| \le \varepsilon|t| + A_\varepsilon|t|^p\ \forall\, t \in \mathbf{R} \tag{2.2}$$

from which it is standard to derive that $\varphi_a \in C^1(X, \mathbf{R})$ for all $a \in \mathcal{F}_{m,M}$ with

$$\varphi_a'(u)v = \langle u, v\rangle - \int_{\mathbf{R}^N} a(x)f(u(x))v(x)\,dx$$

where $\langle u, v\rangle = \int_{\mathbf{R}^N}(\nabla u\nabla v + uv)\,dx$. The critical points of φ_a are solutions of the problem (P_a) and, in the sequel, we will denote $\mathcal{K}_a = \{u \in X : \varphi_a'(u) = 0\}$.

We firstly give a result which describes in particular the behavior of any functional φ_a at 0. For every $\Omega \subset \mathbf{R}^N$, let us denote $\|u\|_\Omega = (\int_\Omega(|\nabla u|^2 + |u|^2)dx)^{\frac{1}{2}}$ and $\langle u, v\rangle_\Omega = \int_\Omega(\nabla u\nabla v + uv)\,dx$ for all $u, v \in X$. Then we have

Lemma 2.1 *There exists $\bar\rho \in (0, 1)$ such that if $\sup\limits_{y\in\mathbf{R}^N} \|u\|_{B_1(y)} \le 2\bar\rho$ then*

$$\int_{\mathbf{R}^N} F(u)\,dx \le \tfrac{1}{4M}\|u\|^2 \quad \textit{and} \quad \int_{\mathbf{R}^N} |f(u)v|\,dx \le \tfrac{1}{2M}\|u\|\|v\|$$

for all $u, v \in X$.

Proof. Let $\{B_1(y_i)\}_{i\in\mathbf{N}}$ be a family of balls such that each point of $\mathbf{R}^N$ is contained in at least one and at most l of such balls.
Let $\varepsilon = \frac{1}{4lM}$ and let $A_1 > 0$ be such that $\|u\|_{L^{p+1}(B_1(y))} \le A_1\|u\|_{B_1(y)}$ for all $u \in X$, $y \in \mathbf{R}^N$. Then, by (2.2), for any $u, v \in X$ and $y \in \mathbf{R}^N$ we have

$$\begin{aligned}\int_{B_1(y)} |f(u)v|\,dx &\le \varepsilon\|u\|_{B_1(y)}\|v\| + A_\varepsilon\|u\|^p_{L^{p+1}(B_1(y))}\|v\|_{L^{p+1}(B_1(y))}\\ &\le (\tfrac{1}{4lM} + A_\varepsilon A_1^{p+1}\|u\|^{p-1}_{B_1(y)})\|u\|_{B_1(y)}\|v\|.\end{aligned}$$

Let $\bar\rho \in (0, 1)$ be such that $A_\varepsilon A_1^{p+1}(2\bar\rho)^{p-1} \le \frac{1}{4lM}$. Then, if $\sup\limits_{y\in\mathbf{R}^N} \|u\|_{B_1(y)} \le 2\bar\rho$, by the above estimate, we obtain

$$\textstyle\int_{\mathbf{R}^N}|f(u)v|\,dx \le \sum\limits_{i\in\mathbf{N}} \int_{B_1(y_i)}|f(u)v|\,dx \le \frac{1}{2lM}\sum\limits_{i\in\mathbf{N}} \|u\|_{B_1(y_i)}\|v\| \le \frac{1}{2M}\|u\|\|v\|.$$

With analogous computation it can be proved that if $\sup\limits_{y\in\mathbf{R}^N} \|u\|_{B_1(y)} \le 2\bar\rho$ then $\int_{\mathbf{R}^N} F(u)\,dx \le \frac{1}{4M}\|u\|^2$. $\square$

Remark 2.1 In particular, we have that if $\sup_{y\in\mathbf{R}^N} \|u\|_{B_1(y)} \leq 2\bar\rho$ then

$$\Big|\int_{\mathbf{R}^N} a(x)F(u)\,dx\Big| \leq \tfrac{1}{4}\|u\|^2 \quad \text{and} \quad |\int_{\mathbf{R}^N} a(x)f(u)v\,dx| \leq \tfrac{1}{2}\|u\|\|v\|$$

for all $a \in \mathcal{F}_{m,M}$. Moreover, note that proving Lemma 2.1 we showed that if $y \in \mathbf{R}^N$ and $\|u\|_{B_1(y)} \leq 2\bar\rho$ then $|\int_{B_1(y)} a(x)f(u)v\,dx| \leq \frac{1}{2}\|u\|_{B_1(y)}\|v\|$ and $|\int_{B_1(y)} a(x)F(u)\,dx| \leq \frac{1}{4}\|u\|^2_{B_1(u)}$. This can be done independently of y since the imbedding constant A_1 does not depend on y. By the same argument we can assume that $\bar\rho$ is such that if $\|u\|_\Omega \leq 2\bar\rho$ then

$$\int_\Omega a(x)F(u)\,dx \leq \tfrac{1}{4}\|u\|^2_\Omega \quad \text{and} \quad |\int_\Omega a(x)f(u)v\,dx| \leq \tfrac{1}{2}\|u\|_\Omega\|v\|_\Omega$$

for all $u, v \in X$ and for all $a \in \mathcal{F}_{m,M}$, whenever Ω is an open regular subset of $\mathbf{R}^N$, satisfying the uniform cone property with respect to the cone $C = \{x \in B_1(0) : x \cdot e_1 > \frac{1}{2}|x|\}$, where $e_1 = (1, 0, \ldots, 0)$.

By $(f3)$, $F(t) \geq F(\frac{t}{|t|})|t|^\theta$ for $|t| \geq 1$ and then for any $u \in X \setminus \{0\}$ there exists $s(u) = s(u,m) > 0$ such that $\varphi_a(s(u)u) < 0$ for every $a \in \mathcal{F}_{m,M}$. Hence, by Lemma 2.1, any functional φ_a with $a \in \mathcal{F}_{m,M}$ has the mountain pass geometry with mountain pass level

$$c(a) = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} \varphi_a(\gamma(s)) \tag{2.3}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \varphi_a(\gamma(1)) < 0\ \forall\, a \in \mathcal{F}_{m,M}\}$. By Lemma 2.1, we have that $c(a) \geq \bar\rho^2 > 0$ for every $a \in \mathcal{F}_{m,M}$ and, by the mountain pass Lemma, there exists a sequence $(u_n) \subset X$ such that $\varphi_a(u_n) \to c(a)$ and $\varphi_a'(u_n) \to 0$. We remark that $c(M) \leq c(a) \leq c(m)$ for any $a \in \mathcal{F}_{m,M}$.

Remark 2.2 By $(f4)$ for every $u \in X \setminus \{0\}$ there exists a unique $s_u > 0$ such that $\frac{d}{ds}\varphi_a(su)|_{s=s_u} = 0$ and hence $\varphi_a(s_u u) = \max_{s\geq 0}\varphi_a(su)$. Moreover, we have $c(a) = \inf_{\|u\|=1} \sup_{s\geq 0} \varphi_a(su)$ and $\inf_{\mathcal{K}_a\setminus\{0\}} \varphi_a \geq c(a)$ for any $a \in \mathcal{F}_{m,M}$.

Remark 2.3 The assumption $(f3)$ implies that for every $a \in \mathcal{F}_{m,M}$

$$(\tfrac{1}{2} - \tfrac{1}{\theta})\|u\|^2 \leq \varphi_a(u) + \tfrac{1}{\theta}\|\varphi_a'(u)\|\,\|u\| \quad \forall\, u \in X. \tag{2.4}$$

In particular $(\frac{1}{2} - \frac{1}{\theta})\|u\|^2 \leq \varphi_a(u)$ for any $u \in \mathcal{K}_a$ and $a \in \mathcal{F}_{m,M}$.

Letting $\bar{\lambda} = (1 - \frac{2}{\theta})\bar{\rho}^2$, by Lemma 2.1 and Remark 2.3 we plainly derive the following compactness property of the functionals φ_a.

Lemma 2.2 *Let $(a_n) \subset \mathcal{F}_{m,M}$ and $(u_n) \subset X$ be such that $\varphi_{a_n}(u_n) \to l$ and $\varphi'_{a_n}(u_n) \to 0$. Then $l \geq 0$ and if $l < \bar{\lambda}$ or $\limsup\limits_{n\to\infty} \sup\limits_{y \in \mathbf{R}^N} \|u_n\|_{B_1(y)} \leq 2\bar{\rho}$ then $u_n \to 0$.*

By the previous lemma we have that for any sequence $(u_n) \subset X$ such that $\varphi_{a_n}(u_n) \to l \geq \bar{\lambda}$ and $\varphi'_{a_n}(u_n) \to 0$, for some $(a_n) \subset \mathcal{F}_{m,M}$, there exists a sequence $(y_n) \subset \mathbf{R}^N$ such that $\liminf \|u_n\|_{B_1(y_n)} \geq \bar{\rho}$. Moreover, in the next lemma we will prove that the vanishing case (see [23]) does not occur

Lemma 2.3 *There exists $\bar{r} \in (0, \bar{\rho})$ for which if $(u_n) \subset X$ is such that $\varphi_{a_n}(u_n) \to l \in [\bar{\lambda}, c(m)]$ and $\varphi'_{a_n}(u_n) \to 0$ for some $(a_n) \subset \mathcal{F}_{m,M}$ then $\liminf\limits_{n\to\infty} \sup\limits_{y\in\mathbf{R}^N} \|u_n\|_{L^2(B_1(y))} \geq \bar{r}$.*

Proof. By contradiction, using a diagonal procedure, there exists a sequence (u_n) such that $\varphi_{a_n}(u_n) \to l \in [\bar{\lambda}, c(m)]$, $\varphi'_{a_n}(u_n) \to 0$, for some sequence $(a_n) \subset \mathcal{F}_{m,M}$, and $\sup_{y\in\mathbf{R}^N} \|u_n\|_{L^2(B_1(y))} \to 0$. Let $A > 0$ be such that $\|u\|_{L^{p+1}(\mathbf{R}^N)} \leq A\|u\|$ for all $u \in X$ and let $\varepsilon = \frac{1}{2M}$. By (2.2), we obtain

$$\|u_n\|^2 + o(1) = \int_{\mathbf{R}^N} a_n(x) f(u_n) u_n \, dx \leq (\tfrac{1}{2} + MA^2 A_\varepsilon \|u_n\|^{p-1}_{L^{p+1}(\mathbf{R}^N)}) \|u_n\|^2$$

as $n \to \infty$. Then, since $l \geq \bar{\lambda}$, by Lemma 2.2, for n sufficiently large, we have $\|u_n\| \geq \bar{\rho}$ and so, by the previous estimate, $\|u_n\|^{p-1}_{L^{p+1}(\mathbf{R}^N)} \geq \frac{1}{4MA^2A_\varepsilon}$. Since $\sup_{y\in\mathbf{R}^N} \|u_n\|_{L^2(B_1(y))} \to 0$ and, by (2.4), (u_n) is bounded, we have that $u_n \to 0$ in $L^q(\mathbf{R}^N)$ for all $q \in (2, \frac{2N}{N+2})$ (see, e.g., [15]), a contradiction. $\square$

Now we state a characterization of the sequences $(u_n) \subset X$ such that $\varphi_{a_n}(u_n) \to l$ and $\varphi'_{a_n}(u_n) \to 0$ for some sequence $(a_n) \subset \mathcal{F}_{m,M}$.

Lemma 2.4 *Let $(a_n) \subset \mathcal{F}_{m,M}$, $(u_n) \subset X$ and $(y_n) \subset \mathbf{R}^N$ be such that $\varphi_{a_n}(u_n) \to l$, $\varphi'_{a_n}(u_n) \to 0$ and $\liminf \|u_n\|_{B_1(y_n)} \geq \bar{\rho}$. Then there exists $u \in X$ with $\|u\|_{B_1(0)} \geq \bar{\rho}$ such that, up to a subsequence,*

(i) $u_n(\cdot + y_n) \to u$ weakly in X, $\varphi_a(u) \leq l$ and $\varphi'_a(u) = 0$, where $a = \lim a_n(\cdot + y_n)$ in the w^*-L^∞ topology,

(ii) $\varphi_{a_n}(u_n - u(\cdot - y_n)) \to l - \varphi_a(u)$ *and* $\varphi'_{a_n}(u_n - u(\cdot - y_n)) \to 0$.

Proof. By (2.4), the sequence (u_n) is bounded in X and then there exists $u \in X$ such that, up to a subsequence, $u_n(\cdot + y_n) \to u$ weakly in X.
Moreover, since $\mathcal{F}_{m,M}$ is compact for the w^*-L^∞ topology, passing again to a subsequence, we have that $a_n(\cdot + y_n) \to a$ w^*-L^∞ for some $a \in \mathcal{F}_{m,M}$. We claim that $\varphi'_a(u) = 0$.
Indeed, for every $v \in C_c^\infty(\mathbf{R}^N)$, since $\bar{a}_n = a_n(\cdot + y_n) \to a$ w^*-L^∞, we have $\varphi'_a(u)v - \varphi'_{\bar{a}_n}(u)v \to 0$. Moreover, since $u_n(\cdot + y_n) \to u$ weakly in X and $\|a\|_{L^\infty(\mathbf{R}^N)} \leq M$, we obtain $\varphi'_{\bar{a}_n}(u)v - \varphi'_{\bar{a}_n}(u_n(\cdot + y_n))v \to 0$. Then, since $\|\varphi'_{\bar{a}_n}(u_n(\cdot + y_n))\| = \|\varphi'_{a_n}(u_n)\| \to 0$, we conclude that $\varphi'_a(u)v = 0$ for any $v \in C_c^\infty(\mathbf{R}^N)$ and, by density, the claim is proved.
Now, since $u_n(\cdot + y_n) \to u$ weakly in X we have that

$$\begin{gathered}\textstyle\int_{\mathbf{R}^N} (F(u_n - u(\cdot - y_n)) - F(u_n) + F(u(\cdot - y_n)))\, dx \to 0, \\ \sup_{\|v\| \leq 1} \textstyle\int_{\mathbf{R}^N} (f(u_n - u(\cdot - y_n)) - f(u_n) + f(u(\cdot - y_n)))v\, dx \to 0\end{gathered}$$

and since $\|a_n\|_{L^\infty} \leq M$ we derive

$$\varphi_{a_n}(u_n - u(\cdot - y_n)) - \varphi_{a_n}(u_n) + \varphi_{a_n}(u(\cdot - y_n)) \to 0\,, \tag{2.5}$$

$$\varphi'_{a_n}(u_n - u(\cdot - y_n)) - \varphi'_{a_n}(u_n) + \varphi'_{a_n}(u(\cdot - y_n)) \to 0\,. \tag{2.6}$$

Then, since $a_n(\cdot + y_n) \to a$ w^*-L^∞, we obtain $\varphi_{a_n}(u(\cdot - y_n)) \to \varphi_a(u)$ and therefore, by (2.5), $\varphi_{a_n}(u_n - u(\cdot - y_n)) \to l - \varphi_a(u)$. Moreover, since $\varphi'_a(u) = 0$, we have

$$\varphi'_{a_n}(u(\cdot - y_n))v = \int_{\mathbf{R}^N} (a(x) - a_n(x + y_n)) f(u) v(\cdot + y_n)\; dx,$$

and then

$$\|\varphi'_{a_n}(u(\cdot - y_n))\| \to 0$$

because the set $\{f(u)v : v \in X,\ \|v\| \leq 1\}$ is compact in $L^1(\mathbf{R}^N)$. Therefore, since $\varphi'_{a_n}(u_n) \to 0$, by (2.6), we conclude that

$$\varphi'_{a_n}(u_n - u(\cdot - y_n)) \to 0. \tag{2.7}$$

Finally, to show that $\|u\|_{B_1(0)} \geq \bar{\rho}$ we prove that $u_n(\cdot + y_n) \to u$ in $H^1_{loc}(\mathbf{R}^N)$. To this aim for any $R > 0$ let $\chi_R \in C_c^\infty(\mathbf{R}^N)$ be such that $\chi_R(x) \geq 0$ for any $x \in \mathbf{R}^N$, $\chi_R(x) = 1$ if $x \in B_R(0)$, $\operatorname{supp} \chi_R \subset B_{2R}(0)$. Setting

$v_n = u_n(\cdot + y_n) - u$, we have $\|v_n\|^2_{B_R(0)} = \langle v_n, \chi_R v_n\rangle - \langle v_n, \chi_R v_n\rangle_{|x|\geq R}$.
We first observe that $\langle v_n, \chi_R v_n\rangle \to 0$. Indeed we have

$$\langle v_n, \chi_R v_n\rangle = \varphi'_{a_n}(v_n(\cdot - y_n))\chi_R(\cdot - y_n)v_n(\cdot - y_n) + \int_{\mathbf{R}^N} a_n(x+y_n)f(v_n)\chi_R v_n\, dx$$

and the first term goes to zero by (2.7), while the second one by the Lebesgue dominated convergence theorem, since $\operatorname{supp}\chi_R \subset B_{2R}(0)$.
Then, as $n \to \infty$, we have

$$\begin{aligned}
\|v_n\|^2_{B_R(0)} &= o(1) - \int_{|x|\geq R} \nabla\chi_R \nabla v_n v_n\, dx - \int_{|x|\geq R} \chi_R(|\nabla v_n|^2 + v_n^2)\, dx \\
&\leq o(1) + \Big(\int_{R\leq|x|\leq 2R} |\nabla\chi_R \nabla v_n|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{R\leq|x|\leq 2R} v_n^2\, dx\Big)^{\frac{1}{2}} \\
&= o(1)
\end{aligned}$$

since $v_n \to 0$ in $L^2(B_{2R}(0)\setminus B_R(0))$ and (v_n) is bounded in X. □

We conclude this section studying the problems "at infinity" associated to any functional φ_a. Given $a \in \mathcal{F}_{m,M}$, let us denote

$$H(a) = \{b \in L^\infty(\mathbf{R}^N) : \exists\ (y_n) \subset \mathbf{R}^N,\ a(\cdot + y_n) \to b\ w^*\text{-}L^\infty\},$$

$$H_\infty(a) = \{b \in L^\infty(\mathbf{R}^N) : \exists\ (y_n) \subset \mathbf{R}^N,\ |y_n| \to \infty,\ a(\cdot + y_n) \to b\ w^*\text{-}L^\infty\}.$$

Then, we have

Lemma 2.5 *$H_\infty(a)$ is sequentially closed with respect to the w^*-L^∞ topology. In particular, if $\bar a \in H_\infty(a)$ then $H(\bar a) \subseteq H_\infty(a)$.*

Proof. Let $(b_n) \subset H_\infty(a)$ and $b \in L^\infty(\mathbf{R}^N)$ be such that $b_n \to b\ w^*$-L^∞. We prove that $b \in H_\infty(a)$. By definition, for all $n \in \mathbf{N}$ there exists a sequence $(y_j^n)_j \subset \mathbf{R}^N$ such that, as $j \to +\infty$, $|y_j^n| \to \infty$ and $a(\cdot + y_j^n) \to b_n\ w^*$-$L^\infty$. Let $\{\phi_l\}_{l\in\mathbf{N}} \subset L^1(\mathbf{R}^N)$ be dense in $L^1(\mathbf{R}^N)$. For all $l, n \in \mathbf{N}$ we have $\int_{\mathbf{R}^N}(a(\cdot + y_j^n) - b_n)\phi_l\, dx \to 0$ as $j \to +\infty$. Then, by induction, we can construct an increasing sequence $(j_n) \subset \mathbf{N}$ such that $|y_{j_n}^n| > |y_{j_{n-1}}^{n-1}|$ and

$$\int_{\mathbf{R}^N}(a(\cdot + y_{j_n}^n) - b_n)\phi_l\, dx < \tfrac{1}{n} \quad \text{for any } l = 1, \dots, n.$$

Therefore, setting $\bar y_n = y_{j_n}^n$, it holds that $|\bar y_n| \to \infty$ and for every $l \in \mathbf{N}$

$$\int_{\mathbf{R}^N}(a(\cdot + \bar y_n) - b_n)\phi_l\, dx \to 0 \quad \text{as } n \to +\infty.$$

Then, since $\{\phi_l\}_{l\in\mathbf{N}}$ is dense in $L^1(\mathbf{R}^N)$, we conclude that $a(\cdot+\bar{y}_n)-b_n \to 0$ w^*-L^∞. Finally, since $b_n \to b$ w^*-L^∞, we obtain that $a(\cdot+\bar{y}_n) \to b$ w^*-L^∞, that is $b \in H_\infty(a)$. □

Given $a \in \mathcal{F}_{m,M}$, we have that $H_\infty(a) \subset \mathcal{F}_{m,M}$ and we denote

$$c_\infty(a) = \inf_{b\in H_\infty(a)} c(b)\,. \tag{2.8}$$

Such value turns out to be attained. More precisely

Lemma 2.6 *For every $a \in \mathcal{F}_{m,M}$, there exist $a_\infty \in H_\infty(a)$ and $u_\infty \in X\setminus\{0\}$ such that $\varphi_{a_\infty}(u_\infty) = c(a_\infty) = c_\infty(a)$ and $\varphi'_{a_\infty}(u_\infty) = 0$.*

Proof. Let $(b_j) \subset H_\infty(a)$ be such that $c(b_j) \to c_\infty(a)$. By the mountain pass Lemma, for all $j \in \mathbf{N}$ there exists a sequence $(u_n^j)_n \subset X$ such that $\varphi_{b_j}(u_n^j) \to c(b_j)$ and $\varphi'_{b_j}(u_n^j) \to 0$ as $n \to \infty$. By Lemma 2.2, for all $j \in \mathbf{N}$ there exists a sequence $(y_n^j)_n \subset \mathbf{R}^N$ such that $\liminf_{n\to\infty} \|u_n^j\|_{B_1(y_n^j)} \geq \bar{\rho}$. Therefore, by Lemma 2.4, for all $j \in \mathbf{N}$, there exist $u_j \in X$ and $a_j \in \mathcal{F}_{m,M}$ such that $\|u_j\|_{B_1(0)} \geq \bar{\rho}$ and, up to a subsequence, $u_n^j(\cdot+y_n^j) \to u_j$ weakly in X, $b_j(\cdot+y_n^j) \to a_j$ w^*-L^∞, $\varphi'_{a_j}(u_j) = 0$ and $\varphi_{a_j}(u_j) \leq c(b_j)$. By Remark 2.2, we have $\varphi_{a_j}(u_j) \geq c(a_j)$. Moreover, by Lemma 2.5, $a_j \in H(b_j) \subset H_\infty(a)$ and therefore, by (2.8), $c(a_j) \geq c_\infty(a)$. Hence, since $c(b_j) \to c_\infty(a)$, we obtain $\varphi_{a_j}(u_j) \to c_\infty(a)$. Applying Lemma 2.4 to the sequence (u_j), we obtain that there exist $u_\infty \in X$ and $a_\infty \in \mathcal{F}_{m,M}$ such that $\|u_\infty\|_{B_1(0)} \geq \bar{\rho}$ and, up to a subsequence, $u_j \to u$ weakly in X, $a_j \to a_\infty$ w^*-L^∞, $\varphi'_{a_\infty}(u_\infty) = 0$ and $\varphi_{a_\infty}(u_\infty) \leq c_\infty(a)$. Since, by Lemma 2.5, $a_\infty \in H_\infty(a)$ we conclude, by (2.8) and Remark 2.2, that $\varphi_{a_\infty}(u_\infty) = c(a_\infty) = c_\infty(a)$. □

Finally, the following monotonicity property of the mountain pass levels $c_\infty(a)$ holds

Lemma 2.7 *Let $a \in \mathcal{F}_{m,M}$ and $\mu > 0$. Then $c_\infty(a+\mu) < c_\infty(a)$.*

Proof. By Lemma 2.6, there exist $a_\infty \in H_\infty(a)$ and $u_\infty \in X$ such that $\varphi_{a_\infty}(u_\infty) = c_\infty(a)$ and $\varphi'_{a_\infty}(u_\infty) = 0$. Moreover there is $\gamma \in \Gamma$ such that $u_\infty = \gamma(\bar{s})$ for some $\bar{s} \in (0,1)$ and $\max_{s\in[0,1]} \varphi_{a_\infty}(\gamma(s)) = \varphi_{a_\infty}(u_\infty)$. Let $(y_j) \subset \mathbf{R}^N$ be such that $|y_j| \to \infty$ and $a(\cdot+y_j) \to a_\infty$ w^*-L^∞. Let $b = a_\infty+\mu$ and $s_0 \in [0,1]$ be such that $\varphi_b(\gamma(s_0)) = \max_{s\in[0,1]} \varphi_b(\gamma(s))$. Then $\gamma(s_0) \neq 0$ and

$$c_\infty(a) \geq \varphi_{a_\infty}(\gamma(s_0)) = \varphi_b(\gamma(s_0)) + \mu \int_{\mathbf{R}^N} F(\gamma(s_0))\,dx > c(b) \geq c_\infty(a+\mu)$$

since $b \in H^\infty(a+\mu)$. □

3 The perturbed problem

In this section we consider any $a \in L^\infty(\mathbf{R}^N)$, $a(x) \geq a_0 > 0$ a.e. in $\mathbf{R}^N$ and f satisfying $(f1)$–(f_4). For all $\overline{\alpha} > 0$ we will construct a family of functions $\alpha_\omega \in C(\mathbf{R}^N)$ $(\omega > 0)$ with $\|\alpha_\omega\|_{L^\infty} \leq \overline{\alpha}$ for which the problem $(P_{a+\alpha_\omega})$ admits infinitely many solutions if $\omega > 0$ is small enough.

Let $\overline{\alpha} > 0$. By Lemma 2.6, since $H_\infty(a+\bar{\alpha}) = H_\infty(a) + \bar{\alpha}$, we know that there exists $a_\infty \in H_\infty(a)$ such that $c_\infty(a+\bar{\alpha}) = c(a_\infty + \bar{\alpha})$. By definition, there exists a sequence $(x_j) \subset \mathbf{R}^N$ such that $a(\cdot + x_j) \to a_\infty$ w^*-L^∞ and $|x_{j+1}| - |x_j| \uparrow +\infty$. Then, for $\omega \in (0,1)$ we define $j(\omega) = \inf\{j \in \mathbf{N} : |x_j| - |x_{j-1}| \geq \frac{4}{\omega}\}$ and

$$\alpha_\omega(x) = \begin{cases} \bar{\alpha}(1 - \frac{\omega^2}{4}|x - x_j|^2) & \text{for } |x - x_j| \leq \frac{2}{\omega}\,,\ j \geq j(\omega) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Note that $\max_{x \in \mathbf{R}^N} \alpha_\omega(x) = \bar{\alpha} = \alpha(x_j)$ for all $j \geq j(\omega)$ and $\alpha_\omega(x) \leq \frac{15}{16}\bar{\alpha}$ if $x \in \mathbf{R}^N \setminus \bigcup_{j \in \mathbf{N}} B_{\frac{1}{2\omega}}(x_j)$.

Let us introduce some notation. For $\omega \in (0,1)$, we set

$$\varphi_\omega(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} (a(x) + \alpha_\omega(x))F(u(x))\,dx\,.$$

Moreover, for any $b \in H_\infty(a)$ and $\beta \in L^\infty(\mathbf{R}^N)$ with $0 \leq \beta \leq \bar{\alpha}$, let

$$\varphi_{b\beta}(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} (b(x) + \beta(x))F(u(x))\,dx\,.$$

We point out that for every $\omega \in (0,1)$, b and β as before, we have $a+\alpha_\omega, b+\beta \in \mathcal{F}_{m,M}$ where $m = \frac{a_0}{2}$ and $M = 2(\|a\|_{L^\infty(\mathbf{R}^N)} + \bar{\alpha})$.
We also set $\mathcal{K}_\omega = \mathcal{K}_{a+\alpha_\omega}$ and $c_{b\beta} = c(b+\beta)$, where, according to Section 2, $c(b+\beta)$ is the mountain pass levels of the functional $\varphi_{b\beta}$. Finally we denote $\varphi_\infty = \varphi_{a_\infty\bar{\alpha}}$ and $c_\infty = c_\infty(a+\bar{\alpha}) = c(a_\infty + \bar{\alpha})$.

Remark 3.1 By construction, following the general results stated in Section 2, there exists $u_\infty \in X$ such that $\varphi_\infty(u_\infty) = c_\infty$ and $\varphi'_\infty(u_\infty) = 0$. Moreover, by Remark 2.2, there exists $\gamma_\infty \in \Gamma$, of the form $\gamma_\infty(s) = s s_\infty u_\infty$ for which

(i) $\max_{s \in [0,1]} \varphi_\infty(\gamma_\infty(s)) = \varphi_\infty(u_\infty)$,

(ii) for every $r > 0$ there is $h_r > 0$ such that $\varphi_\infty(u) \leq c_\infty - h_r$ for any $u \in \operatorname{range}\gamma_\infty$ with $\|u - u_\infty\| \geq r$.

Remark 3.2 By definition of c_∞, if $b \in H_\infty(a)$ and $\beta \in L^\infty(\mathbf{R}^N)$ with $0 \le \beta \le \bar{\alpha}$ a.e. in $\mathbf{R}^N$, then $c_{b\beta} \ge c_{b\bar{\alpha}} \ge c_\infty$. Moreover, if $\beta \in (0, \bar{\alpha})$, by Lemma 2.7, $c_\infty(a+\beta) > c_\infty(a+\bar{\alpha}) = c_\infty$.

In the following Lemma, using Lemma 2.4 and the definition of c_∞, we give an estimate from below on the level of the sequences $(u_n) \subset X$ such that $\varphi'_{\omega_n}(u_n) \to 0$ and that "carry mass" at infinity, i.e., for which $\|u_n\|_{B_1(y_n)} \ge \bar{\rho}$ for some sequence $|y_n| \to \infty$.

Lemma 3.1 *Let $(\omega_n) \subset (0,1)$, $(u_n) \subset X$ and $(y_n) \subset \mathbf{R}^N$ be such that $\varphi'_{\omega_n}(u_n) \to 0$, $|y_n| \to \infty$ and $\|u_n\|_{B_1(y_n)} \ge \bar{\rho}$ for every $n \in \mathbf{N}$. Then $c_\infty \le \liminf \varphi_{\omega_n}(u_n)$.*

Proof. For a subsequence, we have $a(\cdot + y_n) \to b$ and $\alpha_{\omega_n}(\cdot + y_n) \to \beta$, w^*-L^∞ where $b \in H_\infty(a)$ and $\beta \in L^\infty(\mathbf{R}^N)$ with $0 \le \beta \le \bar{\alpha}$ a.e. in $\mathbf{R}^N$. By Lemma 2.4, there exists $u \in X \setminus \{0\}$ such that, up to a subsequence, $u_n(\cdot + y_n) \to u$ weakly in X, $\varphi'_{b\beta}(u) = 0$ and $\liminf \varphi_{\omega_n}(u_n) \ge \varphi_{b\beta}(u)$. Then, using Remarks 2.2 and 3.2, the Lemma follows. $\square$

In particular, as immediate consequence of Lemma 3.1, since $j(\omega) \to \infty$ as $\omega \to 0$ we get an estimate from below on the level of critical points of φ_ω with a "mass" in $\overline{B}_{\frac{1}{\omega}}(x_j)$ with $j \ge j(\omega)$ and $\omega > 0$ small enough. More precisely, we have

Lemma 3.2 *For every $h > 0$ there exists $\omega_h \in (0,1)$ such that for all $\omega \in (0, \omega_h]$ if $u \in \mathcal{K}_\omega$ and $\|u\|_{B_1(y)} \ge \bar{\rho}$ for some $y \in \overline{B}_{\frac{1}{\omega}}(x_j)$, with $j \ge j(\omega)$, then $\varphi_\omega(u) \ge c_\infty - h$.*

Now, we can prove a compactness result for the sequences $(u_n) \subset X$ such that $\varphi'_{\omega_n}(u_n) \to 0$ with a "mass" located in $\overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$.

Lemma 3.3 *There exist $h_0 > 0$ and $\omega_0 \in (0,1)$ such that if $(\omega_n) \subset (0, \omega_0]$, $(u_n) \subset X$ and $(y_n) \subset \mathbf{R}^N$ satisfy $\varphi'_{\omega_n}(u_n) \to 0$, $\|u_n\|_{B_1(y_n)} \ge \bar{\rho}$, $y_n \in \overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$ with $j_n \ge j(\omega_n)$, and $\limsup \varphi_{\omega_n}(u_n) \le c_\infty + h_0$, then $(u_n(\cdot + y_n))$ is precompact in X.*

Proof. Let $h_0 \in (0, \frac{\bar{\lambda}}{2})$ and $\omega_0 = \omega_{h_0}$ be fixed according to Lemma 3.2. By Lemma 2.4 there exists $u \in X$ such that, up to a subsequence, $u_n(\cdot + y_n) \to u$ weakly in X, $\|u\|_{B_1(0)} \ge \bar{\rho}$, $\varphi'_{\omega_n}(u_n - u(\cdot - y_n)) \to 0$ and $\limsup \varphi_{\omega_n}(u_n - u(\cdot -$

$y_n)) \le \limsup \varphi_{\omega_n}(u_n) - \liminf \varphi_{\omega_n}(u(\cdot - y_n)) \le c_\infty + h_0 - \liminf \varphi_{\omega_n}(u(\cdot - y_n))$. By the choice of h_0, using Lemma 2.2, to get the thesis it is enough to check that $\liminf \varphi_{\omega_n}(u(\cdot - y_n)) \ge c_\infty - h_0$. We distinguish two cases:
(i) $|y_n| \to \infty$. In this case, since, up to a subsequence, $a(\cdot + y_n) \to b \in H_\infty(a)$ and $\alpha_{\omega_n}(\cdot + y_n) \to \beta \in L^\infty(\mathbf{R}^N)$ w^*-L^∞, $\beta \le \bar{\alpha}$ a.e. in $\mathbf{R}^N$, we have $\varphi'_{b\beta}(u) = 0$ and then, by Remark 3.2, $\liminf \varphi_{\omega_n}(u(\cdot - y_n)) = \varphi_{b\beta}(u) \ge c_{b\beta} \ge c_\infty$.
(ii) (y_n) is bounded. Then $\liminf \omega_n > 0$ and, up to a subsequence, $\omega_n \to \omega \in (0, \omega_0]$ and $y_n \to y$ for some $y \in \overline{B}_{\frac{1}{\omega}}(x_j)$ with $j \ge j(\omega)$. Hence, setting $v = u(\cdot - y)$ we have that $\varphi'_\omega(v) = 0$, $\|v\|_{B_1(y)} \ge \bar{\rho}$ and $\liminf \varphi_{\omega_n}(u(\cdot - y_n)) = \varphi_\omega(v) \ge c_\infty - h_0$, by Lemma 3.2. $\square$

Now, we have the following concentration result.

Lemma 3.4 *For every $\rho \in (0, \bar{\rho})$ there exist $\nu_\rho \in (0, \frac{\bar{\rho}}{8})$ and $R_\rho > 1$ such that for every $\omega \in (0, \omega_0)$, if $u \in X$ satisfies $\|\varphi'_\omega(u)\| \le \nu_\rho$, $\varphi_\omega(u) \le c_\infty + h_0$ and $\|u\|_{B_1(y)} \ge \bar{\rho}$ for some $y \in \overline{B}_{\frac{1}{\omega}}(x_j)$ with $j \ge j(\omega)$, then*

$$\|u\|_{\mathbf{R}^N \setminus B_{R_\rho}(y)} < \rho.$$

Proof. By contradiction, there exist $\rho \in (0, \bar{\rho})$, $R_n \subset (1, +\infty)$, $(\omega_n) \subset (0, \omega_0)$ and $(u_n) \subset X$ such that $R_n \to +\infty$, $\varphi'_{\omega_n}(u_n) \to 0$ and, for every $n \in \mathbf{N}$, $\varphi_{\omega_n}(u_n) \le c_\infty + h_0$, $\|u_n\|_{B_1(y_n)} \ge \bar{\rho}$ for some $y_n \in \overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$ with $j_n \ge j(\omega_n)$ and $\|u_n\|_{\mathbf{R}^N \setminus B_{R_n}(y_n)} \ge \rho$. This contradicts the fact that, by Lemma 3.3, the sequence $(u_n(\cdot + y_n))$ is precompact in X. $\square$

Using Remark 3.2 and Lemma 3.4, we will select infinitely many disjoint regions in X in which the Palais Smale condition holds. Precisely, for every $\omega \in (0,1)$, $h > 0$, $\nu > 0$ and $j \ge j(\omega)$, we consider the set

$$\mathcal{A}_j(\omega, h, \nu) = \{u \in X \;:\; \varphi_\omega(u) \le c_\infty + h,\ \|\varphi'_\omega(u)\| \le \nu \ \textit{and} \sup_{y \in \overline{B}_{\frac{1}{\omega}}(x_j)} \|u\|_{B_1(y)} \ge \bar{\rho}\}.$$

Setting $\rho_0 = \frac{\bar{\rho}}{8}$, let $\nu_0 = \nu_{2\rho_0}$ and $R_0 = R_{2\rho_0}$ be given by Lemma 3.4. Then, by the previous result, we obtain that the elements of $\mathcal{A}_j(\omega, h, \nu)$ concentrate in $B_{\frac{1}{2\omega}}(x_j)$ for ω, h, ν small enough.

Lemma 3.5 *There exist $\bar{\omega} \in (0, \omega_0)$, $\bar{h} \in (0, h_0)$ and $\bar{\nu} \in (0, \nu_0)$ such that if $u \in \mathcal{A}_j(\omega, \bar{h}, \bar{\nu})$ for some $\omega \in (0, \bar{\omega})$ and $j \ge j(\omega)$, then*

$$\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega} - 1}(x_j)} < 2\rho_0.$$

Proof. Arguing by contradiction, we have that there exist $(\omega_n) \subset (0, \omega_0)$ and $(u_n) \subset X$ such that $\omega_n \to 0$, $\varphi_{\omega_n}(u_n) \to l \leq c_\infty$, $\varphi'_{\omega_n}(u_n) \to 0$, $\|u_n\|_{B_1(y_n)} \geq \bar{\rho}$ for some $y_n \in \overline{B}_{\frac{1}{\omega_n}}(x_{j_n})$ and $\|u_n\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega_n}-1}(x_{j_n})} \geq 2\rho_0$, with $j_n \geq j(\omega_n)$. By Lemma 3.4 we have that $\|u_n\|_{\mathbf{R}^N \setminus B_{R_0}(y_n)} < 2\rho_0$. Therefore $\frac{1}{2\omega_n} - 1 - R_0 \leq |y_n - x_{j_n}| \leq \frac{1}{\omega_n}$ and then $\alpha(\cdot + y_n) \to \beta \in (0, \overline{\alpha})$ in L^∞_{loc}. By Lemma 2.4, up to a subsequence, we have $u_n(\cdot + y_n) \to u \neq 0$ weakly in X, $a(\cdot + y_n) \to b \in H_\infty(a)$ w^*-L^∞, $\varphi'_{b\beta}(u) = 0$ and $l \geq \varphi_{b\beta}(u)$. By Remarks 2.2 and 3.2, $\varphi_{b\beta}(u) \geq c_\infty(a + \beta) > c_\infty(a + \overline{\alpha}) = c_\infty$, contrary to $l \leq c_\infty$. □

¿From now on, we will denote $\mathcal{A}_j(\omega) = \mathcal{A}_j(\omega, \bar{h}, \bar{\nu})$ and $B_\rho(\mathcal{A}_j(\omega)) = \{u \in X : \inf_{v \in \mathcal{A}_j(\omega)} \|u - v\| < \rho\}$ for any $\rho > 0$. Then we have

Lemma 3.6 *Let $\omega \in (0, \bar{\omega})$ and $j \geq j(\omega)$.*
(i) If $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$, then $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} < \bar{\rho}$.
(ii) If $u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus \mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \bar{h}\}$, then $\|\varphi'_\omega(u)\| > \bar{\nu}$.

Proof. (i) By Lemma 3.5, if $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$, then

$$\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} \leq 2\rho_0 + 4\rho_0 < \bar{\rho}\,. \tag{3.2}$$

(ii) If $\sup_{y \in \overline{B}_{\frac{1}{\omega}}(x_j)} \|u\|_{B_1(y)} \geq \bar{\rho}$, since $u \notin \mathcal{A}_j(\omega)$, by definition of $\mathcal{A}_j(\omega)$, we obtain $\|\varphi'_\omega(u)\| > \bar{\nu}$. On the other hand, if $\sup_{y \in \overline{B}_{\frac{1}{\omega}}(x_j)} \|u\|_{B_1(y)} < \bar{\rho}$, by (3.2), we obtain $\sup_{y \in \mathbf{R}^N} \|u\|_{B_1(y)} < \bar{\rho}$. Therefore, by Lemma 2.1, we get $\|\varphi'_\omega(u)\| \geq \frac{1}{2}\|u\|$ and since $\inf_{v \in \mathcal{A}_j(\omega)} \|v\| \geq \bar{\rho}$ and $\bar{\nu} < \frac{\bar{\rho}}{8}$, we obtain $\|\varphi'_\omega(u)\| \geq \frac{1}{2}(\bar{\rho} - 4\rho_0) > \bar{\nu}$. □

Remark 3.3 By Lemma 3.3, for all $\omega \in (0, \bar{\omega})$ and $j \geq j(\omega)$, the Palais Smale condition holds in $\mathcal{A}_j(\omega)$ and then, by Lemma 3.6, in $B_{4\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \bar{h}\}$.

Now, we will prove the existence of infinitely many critical points of the functional φ_ω provided $\omega > 0$ is sufficientely small. More precisely, by the above stated properties of the sets $\mathcal{A}_j(\omega)$, we are able to show the existence of a pseudogradient flow which leaves invariant suitable localized minimax classes. This allows us to show the existence of critical points of φ_ω in $\mathcal{A}_j(\omega)$ whenever ω is small enough.

First, by Lemma 3.6 and Remark 3.3, we prove the existence of a pseudogradient vector field acting in $\mathcal{A}_j(\omega)$. We set $\bar{\mu} = \frac{1}{32}\min\{\bar{\nu}, \frac{\rho_0^2}{16}\}$ and $\bar{\varepsilon} = \frac{1}{2}\min\{\bar{h}, \frac{\rho_0}{4}, \frac{\bar{\nu}}{2}\}$.

Lemma 3.7 *For any $\varepsilon \in (0,\bar{\varepsilon})$ there exists $\omega_\varepsilon \in (0,\bar{\omega})$ for which if $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$ for some $\omega \in (0,\omega_\varepsilon)$ and $j \geq j(\omega)$, then there exist $\mu_{j\omega} > 0$ and a locally Lipschitz continuous function $V_{j\omega} : X \to X$ verifying:*

(i) $\|V_{j\omega}(u)\| \leq 1$, $\varphi'_\omega(u)V_{j\omega}(u) \geq 0$ *for all* $u \in X$ *and* $V_{j\omega}(u) = 0$ *for all* $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(ii) $\varphi'_\omega(u)V_{j\omega}(u) \geq \mu_{j\omega}$ *if* $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iii) $\varphi'_\omega(u)V_{j\omega}(u) \geq \bar{\mu}$ *if* $u \in (B_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iv) $\langle u, V_{j\omega}(u)\rangle_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq 0$ *if* $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq \varepsilon$.

Proof. By Lemma 3.6 (i), we know that $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} < \bar{\rho}$ for every $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$ with $\omega \in (0,\bar{\omega})$ and $j \geq j(\omega)$. Therefore, given $\varepsilon \in (0,\bar{\varepsilon})$ there exists $\omega_\varepsilon \in (0,\bar{\omega})$ such that if $\omega \in (0,\omega_\varepsilon)$ and $j \geq j(\omega)$ then for all $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$ there exists $i_u \in \{[\frac{1}{2\omega}], \ldots, [\frac{1}{\omega}] - 1\}$ (where $[r]$ is the integer part of $r \in \mathbf{R}$) for which

$$\|u\|_{B_{i_u+1}(x_j) \setminus B_{i_u}(x_j)} \leq \frac{\varepsilon}{4}\,. \tag{3.3}$$

Let $\omega \in (0,\omega_\varepsilon)$ and $j \geq j(\omega)$. For all $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$ we denote $\tilde{E}_u = \mathbf{R}^N \setminus B_{i_u}(x_j)$ and $E_u = \mathbf{R}^N \setminus B_{i_u+1}(x_j)$ and we define

$$\beta_u(x) = \begin{cases} 1 & \text{if } x \in \mathbf{R}^N \setminus \tilde{E}_u, \\ i_u + 1 - |x - x_j| & \text{if } x \in \tilde{E}_u \setminus E_u, \\ 0 & \text{if } x \in E_u. \end{cases}$$

Note that $\beta_u \in X$, $0 \leq \beta_u(x) \leq 1$ and $|\nabla\beta_u(x)| \leq 1$ for a.e. $x \in \mathbf{R}^N$. Moreover, if $\beta \in \{\beta_u, 1-\beta_u\}$, by (f_4), we have $|f(\beta u)| \leq |f(u)|$ and $F(\beta u) \leq F(u)$. Therefore, since $\|u\|_{\tilde{E}_u} < \bar{\rho}$, by Remark 2.1, if $E \in \{\tilde{E}_u \setminus E_u, E_u\}$, $\beta_1, \beta_2 \in \{1, \beta_u, 1-\beta_u\}$ and $v \in X$, then

$$\int_E F(\beta_1 u)\,dx \leq \frac{1}{4M}\|u\|_E^2 \text{ and } \int_E |f(\beta_1 u)\beta_2 v|\,dx \leq \frac{1}{2M}\|u\|_E\|v\|_E. \tag{3.4}$$

Using (3.3) and (3.4), by direct estimates, for all $u \in B_{4\rho_0}(\mathcal{A}_j(\omega))$ and $v \in X$ with $\|v\| \leq 1$ we obtain

(1) $\varphi'_\omega(u)(1-\beta_u)u \geq \frac{1}{2}\|u\|^2_{E_u} + \langle u, (1-\beta_u)u\rangle_{\tilde{E}_u\setminus E_u} - \frac{1}{2}\|u\|^2_{\tilde{E}_u\setminus E_u} \geq \frac{1}{2}(\|u\|^2_{E_u} - \frac{\varepsilon^2}{8})$,

(2) $\varphi_\omega(\beta_u u) \leq \frac{1}{2}\|u\|^2_{\mathbf{R}^N\setminus\tilde{E}_u} - \int_{\mathbf{R}^N\setminus\tilde{E}_u}(a(x)+\alpha_\omega(x))F(u)\ dx + \|u\|^2_{\tilde{E}_u\setminus E_u} \leq \varphi_\omega(u) + \frac{\varepsilon^2}{16}$,

(3) $|\varphi'_\omega(\beta_u u)v - \varphi'_\omega(u)\beta_u v| \leq \int_{\tilde{E}_u\setminus E_u}|\nabla\beta_u||u\nabla v - v\nabla u|\ dx + \|u\|_{\tilde{E}_u\setminus E_u} \leq 2\|u\|_{\tilde{E}_u\setminus E_u} \leq \frac{\varepsilon}{2}$.

Let us consider $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$ or $u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{2\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$. We distinguish the two alternative cases: $\|u\|_{E_u} \geq \frac{\varepsilon}{2}$ or $\|u\|_{E_u} < \frac{\varepsilon}{2}$. In the first case setting $V_u = (1-\beta_u)u$, we obtain $\|V_u\| \leq 1$, $\langle u, V_u\rangle_{R^N\setminus\overline{B}_{\frac{1}{\omega}}(x_j)} = \|u\|^2_{R^N\setminus\overline{B}_{\frac{1}{\omega}}(x_j)}$ and, by (1),

$$\varphi'_\omega(u)V_u \geq \tfrac{1}{2}(\tfrac{\varepsilon^2}{4} - \tfrac{\varepsilon^2}{8}) = \tfrac{\varepsilon^2}{16}.$$

In the second case, since $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$, by Remark 3.3, there exists $\mu_{j\omega} \in (0, \frac{\varepsilon^2}{32})$ such that $\|\varphi'_\omega(u)\| \geq 4\mu_{j\omega}$ for all $u \in B_{4\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$. Then, there exists $V_u \in X$ with $\|V_u\| \leq 1$ such that $\varphi'_\omega(u)V_u \geq 2\mu_{j\omega}$.

Now, let $u \in (\overline{B}_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$. We distinguish the cases: $\|u\|_{E_u} \geq \frac{\rho_0}{4}$ or $\|u\|_{E_u} < \frac{\rho_0}{4}$. In the first case we set $V_u = (1-\beta_u)u$. Then, we obtain $\|V_u\| \leq 1$, $\langle u, V_u\rangle_{R^N\setminus\overline{B}_{\frac{1}{\omega}}(x_j)} = \|u\|^2_{R^N\setminus\overline{B}_{\frac{1}{\omega}}(x_j)}$ and, since $\bar{\varepsilon} < \frac{\rho_0}{8}$, by (1), we get

$$\varphi'_\omega(u)V_u \geq \tfrac{1}{2}(\tfrac{\rho_0^2}{16} - \tfrac{\varepsilon^2}{8}) \geq \tfrac{\rho_0^2}{64}.$$

In the second case, note that since $\bar{\varepsilon} < \bar{h}$, by (2), we have $\varphi_\omega(\beta_u u) \leq \varphi_\omega(u) + \frac{\varepsilon^2}{16} \leq c_\infty + \bar{h}$. Moreover, since $\|(1-\beta_u)u\|^2 = \|u\|^2_{E_u} + \|(1-\beta_u)u\|^2_{\tilde{E}_u\setminus E_u} \leq \frac{\rho_0^2}{16} + \frac{\varepsilon^2}{8} \leq \frac{\rho_0^2}{4}$, we have

$$\inf_{v\in\mathcal{A}_j(\omega)} \|\beta_u u - v\| \leq \inf_{v\in\mathcal{A}_j(\omega)} \|u - v\| + \|(1-\beta_u)u\| \leq 2\rho_0 + \tfrac{\rho_0}{2} < 4\rho_0,$$

$$\inf_{v\in\mathcal{A}_j(\omega)} \|\beta_u u - v\| \geq \inf_{v\in\mathcal{A}_j(\omega)} \|u - v\| - \|(1-\beta_u)u\| \geq \rho_0 - \tfrac{\rho_0}{2} > 0.$$

Then $\beta_u u \in B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus \mathcal{A}_j(\omega)$ and $\varphi_\omega(\beta_u u) \leq c_\infty + \bar{h}$. By Lemma 3.6 (ii), there exists $W_u \in X$ with $\|W_u\| \leq 1$ such that $\varphi'_\omega(\beta_u u)W_u \geq \frac{\bar{\nu}}{2}$. By (3), since $\bar{\varepsilon} < \frac{\bar{\nu}}{4}$, we have $\varphi'_\omega(u)\beta_u W_u \geq \frac{\bar{\nu}}{4}$ and setting $V_u = \frac{1}{2}(\beta_u W_u + (1-\beta_u)u)$ we have $\|V_u\| \leq 1$, $\langle u, V_u\rangle_{R^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} = \|u\|^2_{R^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)}$ and, by (1),

$$\varphi'_\omega(u)V_u \geq \frac{\bar{\nu}}{8} - \frac{\varepsilon^2}{32} \geq \frac{\bar{\nu}}{16}.$$

In conclusion, for all $u \in B_{4\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$ we have shown that there exists $V_u \in X$ which verifies the following properties:

(i) $\|V_u\| \leq 1$,

(ii) $\varphi'_\omega(u)V_u \geq 2\mu_{j\omega}$ if $u \in B_{4\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iii) $\varphi'_\omega(u)V_u \geq 2\bar{\mu} = \frac{1}{16}\min\{\bar{\nu}, \frac{\rho_0^2}{16}\}$ if $u \in (\overline{B}_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iv) $\langle u, V_u\rangle_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} = \|u\|^2_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)}$ if $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq \frac{\varepsilon}{2}$.

Then the lemma follows by the use of a suitable partition of the unity and a suitable cut-off function as in the classical pseudogradient construction. □

By the previous Lemma, considering the Cauchy problem

$$\begin{cases} \frac{d\eta(s,u)}{ds} = -V_{j\omega}(\eta(s,u)) & s \geq 0 \\ \eta(0,u) = u & u \in X \end{cases}$$

and setting $\mathcal{E}_j(\omega) = \{u \in X : \|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \leq \varepsilon\}$, one can easily obtain the following deformation result:

Lemma 3.8 *For all $\varepsilon \in (0, \bar{\varepsilon})$ there exists $\omega_\varepsilon \in (0, \bar{\omega})$ such that if $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$ for some $\omega \in (0, \omega_\varepsilon)$ and $j \geq j(\omega)$, then there exists a continuous function $\eta_{j\omega} : X \to X$ which verifies:*

(i) $\eta_{j\omega}(u) = u$ *for all* $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(ii) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u)$ *for all* $u \in X$,

(iii) $\varphi_\omega(\eta_{j\omega}(u)) \leq \varphi_\omega(u) - \bar{\mu}\rho_0$ *if* $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iv) $\eta_{j\omega}(\mathcal{E}_j(\omega)) \subset \mathcal{E}_j(\omega)$.

By Remark 2.4 there exists $N > 0$ such that $\sup_{u \in B_{4\rho_0}(\mathcal{A}_j(\omega))} \|u\| \leq N$ for all $\omega \in (0, \bar{\omega})$, $j \geq j(\omega)$. Considered the path γ_∞ defined in Remark 3.1 we can choose $N > 0$ so large that $\max_{s \in [0,1]} \|\gamma_\infty(s)\| \leq N$. Let $\hat{\varepsilon} > 0$ be such that $\hat{\varepsilon} < \frac{1}{8} \min\{\bar{\varepsilon}, h_{\rho_0}, \bar{\mu}\rho_0\}$ where h_{ρ_0} is defined in Remark 3.1 (ii) and $\bar{\mu}$ and $\bar{\varepsilon}$ in Lemma 3.7. We define a family $(\Gamma_j(\omega))$ of local mountain pass classes by setting

$$\Gamma_j(\omega) = \{\gamma \in \Gamma : \|\gamma(s)\| \leq N \text{ and } \|\gamma(s)\|_{\mathbf{R}^N \setminus B_{\frac{1}{\omega}}(x_j)} \leq \hat{\varepsilon} \ \ \forall s \in [0,1]\}.$$

We have the following properties:

Lemma 3.9 *There exists $\hat{\omega} \in (0, \omega_{\hat{\varepsilon}})$ such that for all $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$, setting $\gamma_j(s) = \gamma_\infty(s)(\cdot - x_j)$ for all $s \in [0,1]$, there results:*

(i) $\gamma_j \in \Gamma_j(\omega)$,

(ii) $\max_{s \in [0,1]} \varphi_\omega(\gamma_j(s)) \leq c_\infty + \hat{\varepsilon}$,

(iii) *if* $\gamma_j(s) \notin B_{\rho_0}(\mathcal{A}_j(\omega))$ *then* $\varphi_\omega(\gamma_j(s)) \leq c_\infty - \frac{1}{2} h_{\rho_0}$.

Proof. Since $\gamma_\infty([0,1])$ is compact in X, for ω small enough we have that $\|\gamma_\infty(s)\|_{\mathbf{R}^N \setminus B_{\frac{1}{\omega}}(0)} < \hat{\varepsilon}$ for all $s \in [0,1]$ and (i) follows.
Moreover, since $a(\cdot + x_j) + \alpha_\omega(\cdot + x_j) \to a_\infty + \overline{\alpha}$ w^*-L^∞ as $\omega \to 0$, $j \geq j(\omega)$, we have

$$\sup_{j \geq j(\omega)} \max_{s \in [0,1]} |\varphi_\omega(\gamma_j(s)) - \varphi_\infty(\gamma_\infty(s))| \to 0 \tag{3.5}$$

as $\omega \to 0$. Then (ii) follows by Remark 3.1 (i). Finally, note that $u_\infty(\cdot - x_j) \in \mathcal{A}_j(\omega)$ for $\omega > 0$ small enough, then, by (3.5) and Remark 3.1 (ii), we derive (iii). □

In particular it follows that for all $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$, $\Gamma_j(\omega) \neq \emptyset$ and therefore

$$c_j(\omega) = \inf_{\gamma \in \Gamma_j(\omega)} \max_{s \in [0,1]} \varphi_\omega(\gamma(s)) \in \mathbf{R}.$$

We have that these mountain pass levels are close to the mountain pass level c_∞ in the following sense

Lemma 3.10 *For all $\omega \in (0, \hat{\omega})$ there exists $\hat{\jmath}(\omega) \geq j(\omega)$ such that $|c_j(\omega) - c_\infty| \leq \hat{\varepsilon}$ for all $j \geq \hat{\jmath}(\omega)$.*

Proof. By the previous Lemma we already know that $c_j(\omega) \leq c_\infty + \hat{\varepsilon}$ for all $\omega \in (0, \hat{\omega})$ and $j \geq j(\omega)$. To prove that $c_j(\omega) \geq c_\infty - \hat{\varepsilon}$ for j large enough, consider any path $\gamma \in \Gamma_j(\omega)$. By definition of $\Gamma_j(\omega)$, using Remark 2.1, for all $s \in [0,1]$ we have

$$
\begin{aligned}
&\varphi_\infty(\gamma(s)(\cdot + x_j)) - \varphi_\omega(\gamma(s)) = \\
&= \int_{\mathbf{R}^N} (a(x + x_j) + \alpha_\omega(x + x_j) - a_\infty(x) - \overline{\alpha}) F(\gamma(s)(\cdot + x_j))\, dx \\
&\leq \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(\gamma(s)(\cdot + x_j))\ dx + \\
&+ \int_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(\gamma(s)(\cdot + x_j))\ dx \\
&\leq \sup_{\|u\| \leq N} \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(u)\ dx + \tfrac{1}{4} \|\gamma(s)\|^2_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \\
&\leq \sup_{\|u\| \leq N} \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(u)\ dx + \tfrac{\hat{\varepsilon}^2}{4}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\inf_{\gamma \in \Gamma_j(\omega)} \max_{s \in [0,1]} \varphi_\infty(\gamma(s)(\cdot + x_j)) \\
&\leq \quad c_j(\omega) + \sup_{\|u\| \leq N} \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(u)\, dx + \tfrac{\hat{\varepsilon}^2}{4}.
\end{aligned}
$$

Since $\varphi_\infty(\gamma(s)(\cdot + x_j)) = \varphi_{b\overline{\alpha}}(\gamma(s))$ with $h = a_\infty(\cdot - x_j) \in H_\infty(a)$ and $\Gamma_j(\omega) \subset \Gamma$, by definition of c_∞, we obtain

$$
c_\infty \leq c_j(\omega) + \sup_{\|u\| \leq N} \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(u)\ dx + \tfrac{\hat{\varepsilon}^2}{4}.
$$

Finally, since $\{\chi_{\overline{B}_{\frac{1}{\omega}}(0)} F(u) : \|u\| \leq N\}$ is precompact in $L^1(\mathbf{R}^N)$ and $a(\cdot + x_j) \to a_\infty$ w^*-L^∞ as $j \to \infty$, we have that for all $\omega \in (0, \hat{\omega})$ there exists $\hat{j}(\omega) \geq j(\omega)$ such that for all $j \geq \hat{j}(\omega)$

$$
\sup_{\|u\| \leq N} \int_{\overline{B}_{\frac{1}{\omega}}(0)} (a(x + x_j) - a_\infty(x)) F(u)\ dx \leq \tfrac{\hat{\varepsilon}^2}{2}
$$

and therefore $c_\infty \leq c_j(\omega) + \hat{\varepsilon}$. □

Now, using Lemmas 3.8, 3.9 and 3.10, we can prove the existence of infinitely many solutions of the perturbed problem $(P_{a+\alpha_\omega})$ provided $\omega > 0$ is sufficiently small.

Theorem 3.1 *If $\omega \in (0, \hat{\omega})$ then $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega \neq \emptyset$ for every $j \geq \hat{\jmath}(\omega)$.*

Proof. Arguing by contradiction, suppose that there exist $\omega \in (0, \hat{\omega})$ and $j \geq \hat{\jmath}(\omega)$ such that $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega = \emptyset$. Let $\eta_{j\omega}: X \to X$ be the function given by Lemma 3.8 and $\gamma_j \in \Gamma_j(\omega)$ defined as in Lemma 3.9. Let $\hat{\gamma}_j(s) = \eta_{j\omega}(\gamma_j(s))$ for all $s \in [0,1]$. By Lemma 3.8 (i) and (iv), the class $\Gamma_j(\omega)$ is invariant under the deformation $\eta_{j\omega}$ and then $\hat{\gamma}_j \in \Gamma_j(\omega)$. We claim that $\max_{s\in[0,1]} \varphi_\omega(\hat{\gamma}_j(s)) \leq c_j(\omega) - \hat{\varepsilon}$ and therefore we get a contradiction with the definition of $c_j(\omega)$. Indeed, if $\gamma_j(s) \notin B_{\rho_0}(\mathcal{A}_j(\omega))$, by Lemma 3.9 (iii), we have $\varphi_\omega(\gamma_j(s)) \leq c_\infty - \frac{1}{2}h_{\rho_0} \leq c_\infty - 2\hat{\varepsilon}$, since $\hat{\varepsilon} < \frac{1}{4}h_{\rho_0}$. Then, by Lemma 3.8 (ii), $\varphi_\omega(\hat{\gamma}_j(s)) \leq \varphi_\omega(\gamma_j(s)) \leq c_\infty - 2\hat{\varepsilon}$. On the other hand, if $\gamma_j(s) \in B_{\rho_0}(\mathcal{A}_j(\omega))$, by Lemma 3.8 (iii) and Lemma 3.9 (ii) , we have $\varphi_\omega(\hat{\gamma}_j(s)) \leq \varphi_\omega(\gamma_j(s)) - \bar{\mu}\rho_0 \leq c_\infty + \hat{\varepsilon} - \bar{\mu}\rho_0 \leq c_\infty - 2\hat{\varepsilon}$, since $\hat{\varepsilon} \leq \frac{\bar{\mu}\rho_0}{9}$. Therefore, by Lemma 3.10, for all $s \in [0,1]$ we conclude $\varphi_\omega(\hat{\gamma}_j(s)) \leq c_\infty - 2\hat{\varepsilon} \leq c_j(\omega) - \hat{\varepsilon}$. □

4 Proof of the main Theorem

In this section we consider an arbitrary $a \in L^\infty(\mathbf{R}^N)$ with $\liminf_{|x|\to\infty} a(x) = a_0 > 0$. Given $\bar{\alpha} > 0$ let $\tilde{\alpha} = \frac{1}{2}\min\{a_0, \bar{\alpha}\}$ and $\tilde{a}(x) = \max\{a(x), \tilde{\alpha}\}$. Since the results proved in the previous section can be applied to $\tilde{a}$, defining α_ω according to (3.1), by Theorem 3.1, there is $\hat{\omega} > 0$ such that the problem $(P_{\tilde{a}+\alpha_\omega})$ admits infinitely many solutions whenever $\omega \in (0, \hat{\omega})$. More precisely, denoting by

$$\varphi_\omega(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} (\tilde{a}(x) + \alpha_\omega(x))F(u)\, dx$$

and by $\mathcal{A}_j(\omega)$ the set constructed in section 3 corresponding to $\tilde{a}$, if $\omega \in (0, \hat{\omega})$ then $\mathcal{A}_j(\omega) \cap \mathcal{K}_\omega \neq \emptyset$ for every $j \geq \hat{\jmath}(\omega)$, where $\mathcal{K}_\omega$ is the critical set of φ_ω. We point out that $\hat{\jmath}(\omega)$ can be choosen large enough in order that $\{x : \alpha_\omega(x) \neq 0\} \subset \{x : a(x) = \tilde{a}(x)\}$. Hence α_ω is in fact a perturbation of a.

Next goal is to prove that the critical points of φ_ω are "stable" under perturbations which are "small at infinity". In particular we can take as admissible perturbation the function $a - \tilde{a}$, so that we obtain the existence of infinitely many solutions for the perturbed problem $(P_{a+\alpha_\omega})$.

Given any $\beta \in L^\infty(\mathbf{R}^N)$, let us introduce the functionals

$$\varphi_{\omega\beta}(u) = \varphi_\omega(u) - \int_{\mathbf{R}^N} \beta(x) F(u)\, dx$$

and let us denote by $\mathcal{K}_\beta$ the corresponding critical sets.

First, we state a local compactness property satisfied by $\varphi_{\omega\beta}$. Let us fix $M = 2(\|a\|_{L^\infty} + \bar{\alpha})$.

Lemma 4.1 *If $\beta \in L^\infty(\mathbf{R}^N)$ is such that $\|\beta\|_{L^\infty} \le \frac{M}{2}$, then $\varphi_{\omega\beta}$ satisfies the Palais Smale condition in $B_{4\rho_0}(\mathcal{A}_j(\omega))$ for every $j \ge \hat{\jmath}(\omega)$ and $\omega \in (0, \hat{\omega})$.*

Proof. Let $(u_n) \subset B_{4\rho_0}(\mathcal{A}_j(\omega))$ be a PS sequence for $\varphi_{\omega\beta}$. Since $B_{4\rho_0}(\mathcal{A}_j(\omega))$ is bounded, there exists $u \in X$ such that, up to a subsequence, $u_n \to u$ weakly in X. Moreover, arguing as in the proof of Lemma 2.4, one can check that $\varphi'_{\omega\beta}(u) = 0$, $(u_n - u)$ is again a PS sequence for $\varphi_{\omega\beta}$ and $u_n \to u$ strongly in $H^1_{loc}(\mathbf{R}^N)$. Since $(u_n) \subset B_{4\rho_0}(\mathcal{A}_j(\omega))$, by Lemma 3.6 (i), $\|u_n\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} < \bar{\rho}$ for every $n \in \mathbf{N}$ and then, $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{2\omega}-1}(x_j)} \le \bar{\rho}$, too, being $\bar{\rho}$ given by Lemma 2.1. Hence $\limsup \|u_n - u\| \le \sqrt{2}\bar{\rho}$. Therefore, since $\|a + \alpha_\omega + \beta\|_{L^\infty} \le M$, Lemma 2.1 yields $\|u_n - u\|^2 \le \|\varphi'_{\omega\beta}(u_n - u)\|\, \|u_n - u\| + \frac{1}{2}\|u_n - u\|^2$ and then $\|u_n - u\| \to 0$. $\square$

Now the existence of infinitely many critical points for $\varphi_{\omega\beta}$ with $\beta \in L^\infty(\mathbf{R}^N)$ "small at infinity" can be stated in the following way.

Lemma 4.2 *There exists $\hat{\beta} > 0$ such that for all $\omega \in (0, \hat{\omega})$ and $\beta \in L^\infty(\mathbf{R}^N)$ with $\|\beta\|_{L^\infty(\mathbf{R}^N)} \le \frac{M}{2}$ and $\limsup_{|x|\to\infty} |\beta(x)| \le \hat{\beta}$, there exists $\hat{\jmath}_\beta(\omega) \ge \hat{\jmath}(\omega)$ such that $\mathcal{K}_\beta \cap \mathcal{A}_j(\omega) \ne \emptyset$ for every $j \ge \hat{\jmath}_\beta(\omega)$.*

Proof. Letting N be the constant fixed after Lemma 3.8, there exists $C = C(N) > 0$ such that for any $R > 0$ we have

$$\sup_{\|u\| \le N} \left| \int_{\mathbf{R}^N \setminus B_R(0)} \beta(x) F(u)\, dx \right| \le C \|\beta\|_{L^\infty(\mathbf{R}^N \setminus B_R(0))}\,, \tag{4.1}$$

$$\sup_{\|u\| \le N,\ \|v\| \le 1} \left| \int_{\mathbf{R}^N \setminus B_R(0)} \beta(x) f(u) v\, dx \right| \le C \|\beta\|_{L^\infty(\mathbf{R}^N \setminus B_R(0))}\,. \tag{4.2}$$

Let $\hat{\beta} \le \frac{1}{4} \min\{\bar{\alpha}, \frac{\hat{\varepsilon}}{C}\}$, being $\hat{\varepsilon} > 0$ fixed after Lemma 3.8. Since $\limsup\limits_{|x|\to\infty} |\beta(x)| \le \hat{\beta}$, there exists $R > 0$ such that $\|\beta\|_{L^\infty(\mathbf{R}^N \setminus B_R(0))} \le 2\hat{\beta}$. Moreover, let us fix $\hat{\jmath}_\beta(\omega) \geq \hat{\jmath}(\omega)$ such that $B_{\frac{1}{\omega}}(x_j) \subset \mathbf{R}^N \setminus B_R(0)$ for all $j \geq \hat{\jmath}_\beta(\omega)$. By contradiction, assume that $\mathcal{K}_\beta \cap \mathcal{A}_j(\omega) = \emptyset$ for some $\omega \in (0, \hat{\omega})$ and $j \geq \hat{\jmath}_\beta(\omega)$. We firstly note that, since $\mathcal{A}_j(\omega) \subset \overline{B}_N(0)$, using (4.1) and Lemma 4.1,

(1) there exists $\nu_j > 0$ such that $\|\varphi'_{\omega\beta}(u)\| \geq \nu_j$ for all $u \in \mathcal{A}_j(\omega) \cap \{\varphi_\omega \leq c_\infty + \bar{h}\}$.

Now, using (1) and Lemma 2.1 (note that $\|\tilde{a} + \alpha_\omega + \beta\|_{L^\infty} \leq M$), we can repeat the argument of the proof of Lemma 3.7 to construct a pseudogradient vector field for $\varphi_{\omega\beta}$ acting on the set $\mathcal{A}_j(\omega)$. Precisely, we have that there exist $\mu_j > 0$ and a locally Lipschitz continuous function $\tilde{V}_j \colon X \to X$ verifying:

(i) $\|\tilde{V}_j(u)\| \leq 1$, $\varphi'_{\omega\beta}(u)\tilde{V}_j(u) \geq 0$ for all $u \in X$ and $\tilde{V}_j(u) = 0$ for all $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(ii) $\varphi'_{\omega\beta}(u)\tilde{V}_j(u) \geq \mu_j$ if $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iii) $\varphi'_{\omega\beta}(u)\tilde{V}_j(u) \geq \frac{\bar{\mu}}{2}$ if $u \in (B_{2\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iv) $\langle u, \tilde{V}_j(u)\rangle_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq 0$ if $\|u\|_{\mathbf{R}^N \setminus \overline{B}_{\frac{1}{\omega}}(x_j)} \geq \hat{\varepsilon}$.

Let us remark that the only difference with respect to the proof of Lemma 3.7 concerns the case $u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$, because now we cannot use directly Lemma 3.6 *(ii)*. In this case, using the same notation of the proof of Lemma 3.7, $\beta_u u \in (B_{4\rho_0}(\mathcal{A}_j(\omega)) \setminus B_{\rho_0}(\mathcal{A}_j(\omega))) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$ always holds. Moreover we can obtain again

$$\|\varphi'_{\omega\beta}(\beta_u u)\| \geq \|\varphi'_\omega(\beta_u u)\| - \|\varphi'_\omega(\beta_u u) - \varphi'_{\omega\beta}(\beta_u u)\| \geq \bar{\nu} - \frac{\bar{\varepsilon}}{4} \geq \frac{2}{3}\bar{\nu}$$

because supp $\beta_u u \subset \mathbf{R}^N \setminus B_R(0)$ and we can use (4.2) and Lemma 3.6 *(ii)*. Considering the flow associated to the field $\tilde{V}_j$, we obtain the existence of a continuous function $\eta_j : X \to X$ which verifies:

(i)' $\eta_j(u) = u$ for all $u \in X \setminus B_{4\rho_0}(\mathcal{A}_j(\omega))$,

(ii)' $\varphi_{\omega\beta}(\eta_j(u)) \leq \varphi_{\omega\beta}(u)$ for all $u \in X$,

(iii)' $\varphi_{\omega\beta}(\eta_j(u)) \leq \varphi_{\omega\beta}(u) - \frac{\bar{\mu}\rho_0}{2}$ if $u \in B_{\rho_0}(\mathcal{A}_j(\omega)) \cap \{\varphi_\omega \leq c_\infty + \frac{\bar{h}}{2}\}$,

(iv)' $\eta_j(\mathcal{E}_j(\omega)) \subset \mathcal{E}_j(\omega)$.

Then, considering the path $\tilde\gamma_j = \eta_j(\gamma_j)$, where γ_j is given by Lemma 3.9, by $(i)'$ and $(iv)'$, we obtain $\tilde\gamma_j \in \Gamma_j(\omega)$. Furthermore, for every $u \in$ range $\gamma_j \cup$ range $\tilde\gamma_j$ we have

$$|\varphi_\omega(u) - \varphi_{\omega\beta}(u)| \le \Big|\int_{\mathbf{R}^N\setminus B_R(0)} \beta(x)F(u)\ dx\Big| + \Big|\int_{B_R(0)} \beta(x)F(u)\ dx\Big|$$

with

$$\Big|\int_{\mathbf{R}^N\setminus B_R(0)} \beta(x)F(u)\ dx\Big| \le 2C\hat\beta \le \tfrac{\hat\varepsilon}{2},$$

because of (4.1) and the choice of $\hat\beta$, and

$$\Big|\int_{B_R(0)} \beta(x)F(u)\ dx\Big| \le \tfrac14\|u\|^2_{B_R(0)} \le \tfrac14\|u\|^2_{\mathbf{R}^N\setminus B_{\frac{1}{\omega}}(x_j)} \le \tfrac{\hat\varepsilon^2}{4}$$

because of Remark 2.1 and the definition of $\Gamma_j(\omega)$ (in fact $\|u\|_{B_R(0)} \le \frac{1}{64}\bar\rho$). Hence

$$\max\nolimits_{s\in[0,1]} \varphi_\omega(\tilde\gamma_j(s)) \le \max\nolimits_{s\in[0,1]} \varphi_{\omega\beta}(\tilde\gamma_j(s)) + \tfrac{\hat\varepsilon}{2} + \tfrac{\hat\varepsilon^2}{4}, \tag{4.3}$$

$$\max\nolimits_{s\in[0,1]} \varphi_{\omega\beta}(\gamma_j(s)) \le \max\nolimits_{s\in[0,1]} \varphi_\omega(\gamma_j(s)) + \tfrac{\hat\varepsilon}{2} + \tfrac{\hat\varepsilon^2}{4}. \tag{4.4}$$

Finally, by (4.3), (4.4), $(ii)'$ and $(iii)'$, since $\hat\varepsilon < \frac18 \min\{h_{\rho_0}, \bar\mu\rho_0\}$, using Lemma 3.10, we have

$$\begin{aligned}\max\nolimits_{s\in[0,1]}\ \varphi_\omega(\tilde\gamma_j(s)) &\le \max_{s\in[0,1]} \varphi_{\omega\beta}(\tilde\gamma_j(s)) + \tfrac{\hat\varepsilon}{2} + \tfrac{\hat\varepsilon^2}{4} \\ &\le \max\{c_\infty - \tfrac12 h_{\rho_0} + \hat\varepsilon + \tfrac{\hat\varepsilon^2}{2}, c_\infty - \tfrac12\bar\mu\rho_0 + 2\hat\varepsilon + \tfrac{\hat\varepsilon^2}{2}\} < c_j(\omega),\end{aligned}$$

a contradiction with the definition of $c_j(\omega)$. □

Lastly, the following result completes the proof of Theorem 1.1.

Theorem 4.1 *If $\omega \in (0,\hat\omega)$ then the problem $(P_{a+\alpha_\omega})$ admits infinitely many solutions. In addition, there exists $\beta_0 > 0$ such that for all $\omega \in (0,\hat\omega)$ and $\beta \in L^\infty(\mathbf{R}^N)$ with $\|\beta\|_{L^\infty(\mathbf{R}^N)} \le \beta_0$, also the problem $(P_{a+\alpha_\omega+\beta})$ admits infinitely many solutions.*

Proof. The first part follows by Lemma 4.2, taking $\beta = a - \tilde a$. Indeed in this case $\beta(x) = 0$ for $|x|$ large enough, and $\|\beta\|_{L^\infty} \le \|a\|_{L^\infty} + \tilde\alpha \le \frac{M}{2}$ since $\tilde\alpha < \bar\alpha$. The second part is again a consequence of Lemma 4.2. Indeed, fixed $\beta_0 = \min\{\bar\alpha - \tilde\alpha, \hat\beta\}$, where $\hat\beta$ is given by Lemma 4.2, for any $\beta \in L^\infty(\mathbf{R}^N)$ with $\|\beta\|_{L^\infty} \le \beta_0$, we can write $a+\alpha_\omega+\beta = \tilde a+\alpha_\omega+\tilde\beta$ where $\tilde\beta = a-\tilde a+\beta$ satisfies the assumptions of Lemma 4.2. Indeed $\|\tilde\beta\|_{L^\infty} \le \|a\|_{L^\infty} + \tilde\alpha + \|\beta\|_{L^\infty} \le \frac{M}{2}$, since $\|\beta\|_{L^\infty(\mathbf{R}^N)} \le \bar\alpha - \tilde\alpha$, and $\limsup_{|x|\to\infty} |\tilde\beta(x)| = \limsup_{|x|\to\infty} |\beta(x)| \le \beta_0 \le \hat\beta$. □

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