A Simple, Microfounded Lucas Island Model

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Abstract

This paper provides a version of the Lucas Island model which is both completely microfounded and suitable for teaching. By replacing the original overlapping generation structure with the producer-shopper distinction within the household and using the classical perfect information case as the benchmark of the analysis, the model presented here has two key distinctive features. First, it shows in a critical manner how the imperfect information problem actually arises. Second, it avoids a shortcut approach to modelling aggregate demand by adding a money market to the analysis.

Key words: new classical macroeconomics, imperfect information
JEL codes: E32, E52

1 Introduction

Lucas' s aggregate supply function presented in his 1972 paper (Lucas 1972) is regarded by many as a mile stone in the construction of modern macroeconomics because of its emphasis on explicit microeconomic foundations and for this reason is still widely taught in both undergraduate and graduate courses. In the meantime, however, in view of the fact that in the literature there is also a growing emphasis on the limitations of Lucas's model as an account of business cycles (see *e.g.*, Zarnowitz 1992, Romer 2005) there is a need to address this topic in a critical manner also at the classroom level. One way to do this is to go beyond most textbooks' simplified presentations which paradoxically do not have explicit or complete microfoundations. For example, an advantage of a full microfounded model is that it allows students to grasp how the information imperfections which characterize Lucas's model actually arise. Ever since Lucas's original articles many important contribution have sought to remedy this flaw (see e.g., Azariadis 1981, Bénassy 1999, Bull and Friedman 1983) but turn out difficult to teach. Indeed a gap still exists between these relatively advanced contributions and classroom presentations.

This aim of this paper is to fill this gap by proposing a version of Lucas's original model which both has complete microfoundations and is suitable for teaching. This version presents two distinctive features. First, it shows how the imperfect information problem arises by assuming from the start – that is, in the specification of the perfect information case which represents the benchmark of the analysis – that the household consists of two individuals, a " producer " and a " shopper " in a two-island context. As noted for example by Romer (2005), this assumption represents an alternative way with respect to Lucas's original overlapping generation structure to account for imperfect information since it makes possible the lack of communication between the two individuals and thus the confusion between relative and absolute prices changes needed for deriving the positively sloped aggregate supply. It can be argued that this approach holds at least two distinctive pedagogic advantages with respect to Lucas's and other presentations which are currently available in the literature: it is both more realistic and increases students' intuitive understanding of the limitations of his model. In particular, it shows why in modern economies the kind of information imperfections he underlines are rather implausible. Second, instead of a shortcut approach to modelling aggregate demand such as Romer's, it adds a money market. While it is true, as Romer says in his textbook, that there is little point in modelling aggregate demand more fully in view of the key focus on aggregate supply, yet we believe that this adds pedagogical value to the presentation as it allows students to think in terms of a more familiar aggregate demand and supply structure.

The paper is organized as follows. Section 2 presents the basic model and analyzes the role of money in the perfect information context. Section 3 introduces imperfect information and derives the Lucas model.

2 The Lucas Island model with perfect information: the microfondations of the classical model

2.1 The features of the economy

The economy is geographically bipartite in two islands, in which there are $L+L$ identical households that live one period (hereafter, we will label them as *household* 1 and *household* 2, respectively). Two different perishable goods, produced in island 1 and island 2 respectively, exist together with money, which is the only available medium of exchange issued by the publiic sector. Each island works in a decentralized way and has a *local* goods market as well as a *local* money market. Thus, by considering both islands, two aggregate markets (for goods and money) are otained. The geographic bipartition allows each household of a given island to go to the other one for buying goods. Finally, two shocks hit the economy: a real, idiosyncratic, local shock (households migrate from one island to the other), and a nominal aggregate shock (a change in the money supply) affecting both islands uniformly. Each household is made up of two individuals: the *producer* who utilizes its labor endowment to produce consumption goods that he/she sells to receive revenue and the *shopper* that spends this revenue to buy consumption goods.

2.2 The goods market

In the economy there are $L+L$ nomadic households that on *average* bipartite between the two islands. Because of nomadism, the economy is hit by a local demographic idiosyncratic shock continuosly: in each period households 1 can increase or decrease depending on inward or outward flows. Therefore, at period t, the number of househols 1 is equal to:

$$L_1 = (1+\Lambda)L$$

where $-1<\Lambda<1$ is the rate of change of households 1 at period t: if $\Lambda>0$ ($\Lambda<0$) a positive (negative) migratory flow towards island 1 occurs. Given the economic bipartition, the demographic shock implies that the number of households 2 is equal to:

$$L_2 = (1-\Lambda)L$$

where $L_1 + L_2 = L + L$. Taking logarithms of the number and of the rate of change of households:

$$\begin{cases} l = \log L \\ \lambda = \log(1+\Lambda) \\ -\lambda = \log(1-\Lambda) \end{cases} \tag{1}$$

where lower-case letters denote logarithms of the corresponding upper-case variables. From (1) it follows that

$$\begin{cases} l_1 = \log L_1 = \log\left[(1+\Lambda)L\right] & \xrightarrow{\text{therefore}} & l_1 = l + \lambda \\ l_2 = \log L_2 = \log\left[(1-\Lambda)L\right] & \xrightarrow{\text{therefore}} & l_2 = l - \lambda \\ \log L_1 + \log L_2 = 2\log L & \xrightarrow{\text{therefore}} & l_1 + l_2 = 2l \end{cases} \tag{2}$$

By assumption, Λ is a stochastic variable that follows a lognormal distribution with mean 0 and variance σ_λ^2:

$$\lambda \sim N\left(0,\, \sigma_\lambda^2\right)$$

The variance is a parameter that measures the intensity of the local shock and captures the "fundamentals" of the economy: it is higher when the local shock is stronger and the migratory flows between islands are greater.
The representative household 1 at period t has to make three decisions on[1]:

- its labor supply and, as a consequence, output supply;
- its demand for goods;
- its demand for money.

First, the household produces goods that are sold on the local market to households 2[2]. In carrying on the production process, the household uses the following production function:

$$X_1 = A\,N_1 \tag{3}$$

where X_1 is the level of output, A is the constant labor productivity level (average and marginal) and finally N_1 is the household's labor supply. Second, once having chosen its labor supply and the level of output, the household sells the goods at the current price in order to get the revenue it needs to buy goods on island 2. The household's preferences on goods and labor are represented by the following utility function:

$$U_1(C_{1,2}, N_1) = C_{1,2} - \frac{1}{2} N_1^2 \tag{4}$$

where $C_{1,2}$ is the household's consumption of the goods produced by households 2; N_1 is the number of labor units (e.g., hours) employed by the household to produce goods. Utility depends positvely on consumption and negatively on the amount worked: the marginal utility of consumption is constant whereas the marginal disutility of labor is increasing. The household's choice has to satisfy the following budget constraint:

$$P_1 A N_1 \equiv P_2 C_{1,2} \tag{5}$$

that is, the household's revenue must be equal to its demand for goods produced and sold in island 2. The rational household, therefore, has to solve the following utility maximation problem:

$$\begin{array}{cc} \max\limits_{\text{respect to } C_{1,2},\, N_1} & U_1 = C_{1,2} - \frac{1}{2} N_1^2 \\ \text{subject to} & P_2^{\mathrm{E}} C_{1,2} = P_1 A N_1 \end{array}$$

In order to solve the utility maximization problem, the household has to form expectations on the price of good 2, P_2^{E}. In this section, we assume that such expectations are perfect because the household has perfect information about this price. In particular, we assume that, thanks to the use of technological devices, such as cellular phones, the *representative shopper* 1 is able to transmit immediately to the producer of his/her household the current price of good 2. Thus it follows that:

$$P_2^{\mathrm{E}} = P_2 \tag{6}$$

The household derives his *labor supply function* from the above utility maximization solution:

$$N_1 = A\left(\frac{P_1}{P_2}\right)$$

By taking logarithms this reduces to

$$n_1 = \overline{n} + (p_1 - p_2) \tag{7}$$

where $\overline{n} = a = \log A$, $p_1 = \log P_1$ e $p_2 = \log P_2$. Two remarks on equation (7) are in order.

- First, since the household is both a producer and a consumer, equation (7) is a *sui generis* labor supply function because a proper labor market does not actually exist in the economy.

- Second, the labor supply is an increasing function of the relative price of good 1 in terms of good 2[3]. Furthermore, because of the double nature of the household, the relative price of goods is equivalent to *real wage* on island 1: an increase of p_1, given p_2, implies an increase in the household 's revenue and, in turn, an increase in its labor supply.

By substituting the production function (in logarithm terms *i.e.*, $x_1 = a + n_1$) in equation (7) we obtain the household's *supply function*:

$$x_1 = \bar{x} + (p_1 - p_2) \tag{8}$$

where $\bar{x} = 2a$. Equation (8) shows that supply is an increasing function of the relative price. By substituting (8) in the budget constraint (in logarithm terms *i.e.*, $c_{1,2} + p_2 = p_1 + x_1$) we obtain the household's *demand function of good* 2:

$$c_{1,2} = \bar{x} + 2(p_1 - p_2) \tag{9}$$

which shows that demand depends positively on the relative price. By repeating the same procedure with respect to household 2, we obtain its supply and demand functions:

$$x_2 = \bar{x} - (p_1 - p_2) \tag{10}$$

$$c_{2,1} = \bar{x} - 2(p_1 - p_2) \tag{11}$$

which depend negatively on the relative price.

2.2.1 The equilibrium in the local good market

The equilibrium in the local good market of island 1 requires L_1 households supply to be equal to L_2 households demand, *i.e.*, $L_1 X_1 = L_2 C_{2,1}$ (in real terms). Taking logarithms of the latter expression:

$$(l + \lambda) + \bar{x} + (p_1 - p_2) = (l - \lambda) + \bar{x} - 2(p_1 - p_2)$$

Rearranging the equilibrium equation and setting $a_1 = (2/3) < 1$, we obtain the equilibrium value of the relative price:

$$p_1 - p_2 = -a_1 \lambda \tag{12}$$

that follows a normal distribution with mean 0 and variance σ_λ^2 *i.e.*:

$$(p_1 - p_2) \sim N(0, a_1^2 \sigma_\lambda^2) \tag{13}$$

By subsituting (12) into the (8) we obtain the equilibrium value of output:

$$x_1 = \bar{x} - a_1 \lambda \tag{14}$$

that follows a normal distribution with mean $\bar{x}$ and variance $a_1^2 \sigma_\lambda^2$ *i.e.*, $x_1 \sim N$ (0, $a_1^2 \sigma_\lambda^2$). Equations (12) and (14) show that the equilibrium value of the relative price and output depends on the real local shock. Fig. 1 shows that if the local real shock is absent *i.e.*, $\lambda = 0$, the relative price is equal to zero: the demand and supply curves intersect at point A. If we assume that $\lambda > 0$, a real negative shock hits island 1. Because of the increase in households 1 the supply curve shifts to the right, while the demand curve shifts to the left due to the fall in households 2. The new equilibrium is thus reached at point B where the relative price is negative and output is less than $\bar{x}$.

By replicating the procedure with respect to island 2, we obtain the equilibrium values of both the relative price of good 2 in terms of good 1 and the supply of good 2:

$$p_2 - p_1 = a_1 \lambda \tag{15}$$

$$x_2 = \bar{x} + a_1 \lambda \tag{16}$$

Because of its idisyncratic nature, a negative real local shock on island 1 generates a positive shock on island 2. Therefore, it brings about an increase in both the relative price of good 2 in terms of good 1 and in the supply of good 2.

2.2.2 The equilibrium in the aggregate goods market

Equilibrium in the goods market occurs when aggregate demand and aggregate supply in real terms are equal. **The aggregate supply** is equal to $(LX + LX) = (L_1 X_1 + L_2 X_2)$ where $X = (X_1 + X_2)/2$ is the supply per household on average. Taking logarithms one obtains $2l + 2x = l_1 + l_2 + x_1 + x_2$. Given (2), (14) and (16), the supply per-household is equal to:

$$x = x^{\mathrm{P}} \tag{1.a}$$

where $x^{\mathrm{P}} = (2\bar{x})/2 = \bar{x}$. The supply per household on average is equal to permanent output, x^{P} that, in turn, is equal to the arithmetic average of the permanent supply per household on the two islands. Therefore, the supply is independent of the price level p as well as the real local shock[4]. **Aggregate demand** is equal to $(LC + LC) = (L_1 C_{1,2} + L_2 C_{2,1})$ where $C = (C_1 + C_2)/2$ is the average of demand per household. Taking logarithms, given (9) and (11), it follows:

$$c = x^{\mathrm{P}} \tag{2.a}$$

the average of demand per household is equal to permanent output per household and is independent of both the relative price and the real local shock.

Combining the previous equations, we can then represent equilibrium in the goods market as follows: $L_1X_1 + L_2X_2 = L_1C_{1,2} + L_2C_{2,1}$. Taking logarithms and given (1.a) and (2.a) it is straightforward to get:

$$x = c \tag{3.a}$$

Equation (3.a) is always satisfied regardless of the price level and local shocks. Therefore, *Say's law* holds: aggregate supply generates aggregate demand in such a way that a general glut is ruled out.

Fig. 2 shows the *AS* curve which is the graphical representation of the equilibrium equation of the goods market and is given by the combinations of output and the price level that ensure the equality between aggregate demand and supply.
Because both demand and supply functions are independent of the price level, the two respective curves are perpendicular at x-axes at point $x = x^{\mathrm{P}}$; furthermore, because of Say's law, the two curves overlap.

2.3 The money market

Money is the only medium of exchange. Hence, the nominal demand for money per household (of island 1 and 2 respectively) is equal to nominal consumption and, in turn, to nominal revenue: $M_1^{\mathrm{D}} = P_2C_{1,2} = P_1X_1$ and $M_2^{\mathrm{D}} = P_1C_{2,1} = P_2X_2$. **The aggregate demand for money**, therefore, is $L_1M_1^{\mathrm{D}} + L_2M_2^{\mathrm{D}} = L_1P_1X_1 + L_2P_2X_2$, Taking logarithms, this becomes

$$m_1^{\mathrm{D}} + m_2^{\mathrm{D}} = p_1 + p_2 + x_1 + x_2 \tag{4.a}$$

The **aggregate supply** is $LM + LM$ where M is the nominal money supply per household that splits up evenly between the two islands so that the nominal money supply in a given island is equal to LM. The supply per household is made up of two components: the predictable one, $\overline{M}$, and the unpredictable one, Z, so that $M = Z\overline{M}$ where $Z > 0$. The variable Z represents the *nominal shock* that hits both islands uniformly: if $Z > 1$ ($Z < 1$) the public sector implements an expansionary (restrictive) monetary policy through the unpredictable component. By assumption Z is a stochastic variable that follows a log-normal distribution with mean 0 and variance σ_z^2

$$z \sim N\left(0, \sigma_z^2\right)$$

The variance σ_z^2 measures the intensity of the aggregate nominal shock; it is a *monetary policy parameter* rather than a structural one: it is higher when the nominal aggregate shock is stronger and the economy more perturbed. Taking logarithms, the aggregate nominal money supply thus becomes

$$m = z + \overline{m} \tag{5.a}$$

Given the output level, the equilibrium in the two local money markets are respectively $LM = L_1 M_1^{\mathrm{D}} = L_1 P_1 X_1$ and $LM = L_2 M_2^{\mathrm{D}} = P_2 X$ where, implicitly, it is assumed that money velocity is constant and equal to 1. In logarithms terms, monetary equilibrium on island 1 can be described as follows:

$$l_1 + p_1 + x_1 = l + z + \overline{m}$$

Solving for p_1 we obtain the equilibrium value of the absolute price of good 1 :

$$p_1 = \overline{m} - x_1 - (\lambda - z) \tag{17}$$

Replicating the procedure with respect to island 2 we obtain

$$p_2 = \overline{m} - x_2 + (\lambda + z) \tag{18}$$

Fig. 3 represents the graphical solution of the equilibrium in the money market in island 1. Given an arbitrary value of x_1 and given l_1^1, it is clear that the nominal demand for money per household $m_1^{\mathrm{D}} = \left(l_1^1 + x_1\right) + p_1$ is an increasing function of p_1; while the exogenous nominal money supply $m_1 = l + z_1 + \overline{m}_1$ is perpendicular to the x-axes. Therefore, for m_1 and the given output value, the equilibrium is reached at point $A = (m_1, p_1^1)$. Furthermore, fig. 3 shows how a local shock greater than the initial one (*i.e.*, $l_1^2 > l_1^1$) produces, *ceteris paribus*, a reduction in the equilibrium value of the absolute price of good 1 (point $B = (m_1, p_1^2)$) and, how an increase in the nominal money supply from m_1 to m_2 – either in the predictable or unpredictable component – given l_1^1, produces, on the contrary, an increase in the equilibrium values of p_1 (point $C = (m_2, p_1^3)$).

Let us now assume that the general price level is measured by a geometric average of the absolute price of the two goods $P = (P_1 P_2)^{1/2}$. Taking logarithms, it follows that $p = (p_1 + p_2)/2$. The equilibrium in the money market requires the equality between aggregate demand and supply $L_1 P_1 X_1 + L_2 P_2 X_2 = LZ\overline{M} + LZ\overline{M}$. Taking logarithms, we first obtain $(p_1 + p_2) + (x_1 + x_2) = 2\overline{m}^{\mathrm{S}} + 2z$ and, subsequently, after expressing the equilibrium equation in average terms per household and recalling that $x = (x_1 + x_2)/2$ e $p = (p_1 + p_2)/2$, we finally obtain

$$p + x = \overline{m} + z \tag{6.a}$$

Equation (6.a) shows that, once output is known, the money market determines the equilibrium value of the general price level p. On the other hand, the (6.a) can be seen as an equation in two variables *i.e.*, the price level and output, whose combinations realize market equilibrium.

In fig. 4 we derive graphically the *AD* curve which represents the equilibrium on the money market and is given by the combinations of output and price level that ensure the equality between aggregate supply and demand for money.

Equation (6.a) shows the decreasing relationship between otuput and the price level ensuring money market equilibrium. In graph (a) we represent the money market equilibrium at point A, where the demand curve (drawn for an arbitrary value of output, x_1) intersects the supply curve $m_1 = (m_1, p_1)$. In graph (b) we draw a price level/output combination that establishes, at point A', an equilibrium in the money market, in the plane (x, p). In graph (a) we see that a *ceteris paribus* increase in output from x_1 to x_2 generates a rightward shift of the money demand curve and a lower price level, which is needed to re-establish equilibrium in the goods market. The negative slope of the *AD* curve emerges from graph (b) where point B' shows the new price level/output combination that establishes money market equilibrium (x_2, p_2).

2.4 General macroeconomic equilibrium with perfect information

The economy considered in this paper is described by the system of six equation (1.a)-(6.a) in six unkwown variables. The reduced form of the model is given by two equations in two unknown variables that is p and x:

$$\begin{cases} x = x^{\mathrm{P}} & (1.b) \\ p + x = \overline{m} + z & (2.b) \end{cases}$$

Based on this, we are able to define the general equilibrium with perfect information as a set of values (x, p) that satisfies the equation system (1.b)-(2.b), given the monetary policy parameters $\overline{m}$ and z, and the local shock λ.

The solution of the model is straightforward: the goods market determines equilibrium ouput, x^{P}, whereas the money market determines the equilibrium price level, p. Indeed, by substituting (1.b) into the (2.b) and by putting $\overline{p} = (\overline{m} - x^{\mathrm{P}})$ we finally obtain

$$p = \left(\overline{m} - x^{\mathrm{P}}\right) + z = \overline{p} + z \tag{19}$$

The general price level turns out to be a stochastic variable that follows a normal distribution with mean $\overline{p}$ and variance σ_z^2

$$p \sim N(\overline{p}, \sigma_z^2) \tag{20}$$

On the other hand, recalling that $x_1 = x_2 = x^{\mathrm{P}}$, from (17) and (18) it follows that the equilibrium values of the prices of the two goods are respectively

$$p_1 = \overline{p} - (\lambda - z) \tag{21}$$
$$p_2 = \overline{p} + (\lambda + z) \tag{22}$$

Such prices, then, are stochastic variables that follow a normal distribution with mean $\overline{p} = (\overline{m} - x^{\mathrm{P}})$ and variance $\sigma_z^2 + \sigma_\lambda^2$.

$$p_j \sim N(\overline{p}, \sigma_z^2 + \sigma_\lambda^2) \text{ where} j = 1,2 \tag{23}$$

From (19) it follows the neutrality of money propoisition, according to which, in the perfect information case, monetary policy is completely ineffective, both in the predictable and the unpredictable component, since it produces only nominal effects a change in absolute prices.

Fig. 5 represents the general macroeconomic equilibrium and shows the effects of a monetary policy implemented by a change in the predictable component only. First, given $m = m_1$, the equilibrium is reached at point $A = (x^{\mathrm{P}}, p_1)$ where the *AS* and *AD* curves intersect. Second, an increase in the money supply from $\overline{m}_1$ to $\overline{m}_2$ shifts the *AD* curve to the right and produces a permanent increase in the price level from p_1 to p_2. The new equilibrium at point $B = (x^{\mathrm{P}}, p_2)$ shows the neutrality of the predictable component of money. Fig. 6 shows instead that even the unpredictable component of money is neutral. Given $m = \overline{m}$, an expansionary monetary policy implemented by rising the unpredictable component to $z > 0$ determines a temporary increase in the price level from p_1 to $p_1 + z$.

To summarize, the Lucas Island model with perfect information generates the following basic conclusions of the standard classical benchmark:

- Dichotomy *i.e.*, the equilibrium value of the real variables (*i.e.*, real consumption, output and the relative price) are determined by the real side of the economy (*i.e.*, the labor and goods markets) whereas the equilibrium values of the nominal variables are determined by the nominal side of the economy (*i.e.*, the monetary market represented by the *AD* equation).
- Money is neutral.
- Say's law *i.e.*, supply creates its own demand.
- The occurrence of shocks is not sufficient to explain output fluctuations: it merely produces price level oscillations.

3 The Lucas Island model with imperfect information and rational expectations

3.1 Imperfect information and rational expectations

In the previous section, the existence of a simple technological device allows households to get perfect information about the key relative price, despite the geographical distance between the two islands. In this section, we present instead a simple version of the original Lucas Island model where this simple device does not exist and an information imperfection thus arises. In particular, the *shopper* of island 1 is unable to transmitt immediately to the producer of his/her household the price of good 2. In consequence, to choose its labor and output supply, the household 1 needs to form expectations about such a price. The solution of the expected utility maximization problem under uncertainty can be simplified by assuming that the household adopts the *certainty equivalence principle*. This means proceeding in two steps:

- first, the househlold forms its expectation p_2^{E} and, subsequently, assumes that this expectation is fulfilled for certain;
- second, given the certain expectation, the household solves its utility maximization problem in the same way as in the previous section.

This means that, given the certainty equivalence behavoir and the expectation on p_2, the household maximizes $U_1 = C_{1,2} - \frac{1}{2}N_1^2$ subject to the budget constraint $P_2^{\mathrm{E}}C_{1,2} = P_1 X_1$. Taking logarithms, we first obtain the household's labor supply:

$$n_1 = \overline{n} + (p_1 - p_2^{\mathrm{E}}) \tag{24}$$

and then its output supply of good 1

$$x_1 = \overline{x} + (p_1 - p_2^{\mathrm{E}}) \tag{25}$$

To complete the picture, like Lucas we assume that the household forms rational expectations on p_2.

3.2 The information transmission structure

In order to form rational expectations, households have to gather information on both the structural relations of the economy and the values of current variables. In the model considered here, households have both a general information about the overall economy and a local information about the markets where they sell their goods. These two kinds of information differ both in terms of completeness and the timing of information acquisition. On the one hand, the general information $I_t^{\mathrm{G}}(j)$ at disposal of household $j = 1,2$ is complete but delayed. On the other, the local information $I_t^{\mathrm{L}}(j)$ at disposal of household $j = 1,2$ is incomplete but immediate

At time t, the household 1 has past general information on the economy, that is relatively at period $t-1$, I_{t-1}^{G}. In particular, as already noted, it knows the structural relations, the parameters of the model and the first moments of the probability distributions of the relative price, the price level, and absolute prices of the two goods:

- ***The relative price:***

$$p_2 - p_1 = a_1\lambda \qquad p_2 - p_1 \sim N\left(0, a_2^2\sigma_\lambda^2\right) \tag{26}$$

- ***The price level:***

$$p = \overline{p} + z \qquad p \sim N\left(\overline{p}, \sigma_z^2\right) \tag{27}$$

- ***Absolute prices of the two goods:***

$$\begin{aligned} p_1 &= \overline{p} - (\lambda - z) \qquad p_1 \sim N\left(\overline{p}, \sigma_\lambda^2 + \sigma_z^2\right) &(28)\\ p_2 &= \overline{p} + (\lambda + z) \qquad p_2 \sim N\left(\overline{p}, \sigma_\lambda^2 + \sigma_z^2\right) &(29)\end{aligned}$$

where $\lambda \sim N\left(0, \sigma_\lambda^2\right)$ e $z \sim N\left(0, \sigma_z^2\right)$.

In conclusion, the set of general information is given by

$$I_{t-1}^{\mathrm{G}} = \{\overline{p}, \sigma_z^2, \sigma_\lambda^2, [26], [27], [28], [29]\} \tag{30}$$

As for local information, at time t the household knows the equilibrium value of p_1 but does not know p_2, so that it has to form a rational expectation on this price to make its decisions. However, the household can cumulate its current information about p_1 with the information available on the probability distribution of the absolute price. Indeed from (28), given p_1, the household realizes that:

$$\lambda - z = \overline{p} - p_1 \tag{31}$$

As a consequence, the local information $I_t^{\mathrm{L}}(1)$ at disposal of the household is given by the current price of good 1 and the algebric sum of real and nominal shocks

$$I_t^{\mathrm{L}}(1) = \{p_1, \lambda - z\} \tag{32}$$

Summing up these two information sets, the household is able to define the set of overall information, $I_t^{\mathrm{C}}(1)$ which is at its disposal at time t:

$$I_t^{\mathrm{C}}(1) = \left\{p_1; \lambda - z, \overline{p}, \sigma_z^2, \sigma_\lambda^2, [26], [27], [28], [29]\right\} \tag{33}$$

3.3 Signal extraction from current information

However, this is not the end of the story. To form a rational expectation on p_2, the household has to take another step: exploit all available information efficiently.

One possibility is to use the *general information set* and form the rational expectation on p_2 conditional on I^{G}_{t-1} so as to obtain

$$p_2^{\mathrm{E}}(I^{\mathrm{G}}_{t-1}) = E(p_1 \mid I^{\mathrm{G}}_{t-1}) = \overline{p} \tag{34}$$

By using solely past information, the household regards the deviation of p_1 from $\overline{p}$ (*i.e.*, $(\lambda - z)$) as due to the real shock only and thus as signalling a relative price change which leads it to modify its output supply. One obvious limitation of this approach is that the household does not exploit the information available at time t, which includes the sum of the real and nominal shocks. However, once the household considers this information, the issue of how it manages to decompose this sum of shocks in order to single out the relative price change and form rational expectations more efficiently *i.e.*, the signal extraction problem, cannot be avoided. In principle, if the household were able to exploit the value of λ exactly by using the [26], it might determine the current relative price with certainty. In two extreme cases, the signal coming from $\lambda - z$ can be easily decoded:

- **the real local shock is absent**: $\boldsymbol{\sigma^2_\lambda = 0}$. In this case, the household knows for sure that the deviation of absolute prices from their respective mean is due to the nominal shock only, so that the relative price does not change;
- **the nominal aggregate shock is absent**: $\boldsymbol{\sigma^2_z = 0}$. In this case, the household knows instead that the deviation of p_1 from $\overline{p}$ is totally due to the real local shock: the occurrence of $\lambda - z$ thus signals a relative price change.

Clearly, the signal extraction problem becomes significant when both shocks are present, that is when both variances differ from 0. The solution calls for five steps.

• **First**, the household calculates the expected value of p_2 through the (26) conditional on the overall information set. Since the household knows p_1 (so that $E(p_1 \mid I^{\mathrm{C}}_t) = p_1$) and assigns to the real local shock at least a fraction of the observed sum of shocks, we have

$$p_2^{\mathrm{E}}(I^{\mathrm{C}}_t) = p_1 + a_1 E(\lambda \mid I^{\mathrm{C}}_t) \tag{35}$$

• **Second**, the household estimates the expected value of the real local shock *i.e.*, $E(\lambda \mid I^{\mathrm{C}}_t)$ quite naively, that is by regarding it simply as a fraction β of $(\lambda - z)$ plus an error term, ε, following a normal distribution with mean 0:

$$\lambda = \beta(\lambda - z) + \varepsilon \tag{36}$$

In this case, the signal extraction factor (*i.e.*, the coefficient β) is equal to the ratio between the real local shock and the observed shock plus the random error term $\varepsilon_1 = \varepsilon/(\lambda - z)$)

$$\beta = \frac{\lambda}{\lambda - z} + \varepsilon_1$$

• **Third**, the household calculates the expected value of the real shock conditional on overall information by using the (36).Since $\lambda - z$ is a constant, it obtains:

$$E\left(\lambda \mid I_t^{\mathrm{C}}\right) = \beta E\left(\lambda - z\right) = \beta\left(\lambda - z\right) \tag{37}$$

• **Fourth**, the household computes the value of β by using the least square method, which minimizes the sum of squared errors *i.e.*, $\varepsilon^2 = \left[\lambda - \beta\left(\lambda - z\right)\right]^2$

$$\varepsilon^2 = \left[(1-\beta)\lambda + \beta z)\right]^2 = \left[(1-\beta)^2\lambda^2 + \beta^2 z^2 + 2(1-\beta)\beta z\lambda\right]$$

By solving the following minimization problem

$$\min_{\text{with respect to } \beta} \quad E\left(\varepsilon^2\right) = E\left[(1-\beta)^2\lambda^2 + \beta^2 z^2 + 2(1-\beta)\,\beta\, z\,\lambda\right]$$

and therefore differentiating with respect to β, it gets

$$\frac{d}{d\beta}E\varepsilon^2 = E\left[-2(1-\beta)\lambda^2 + 2\beta z^2 + 2\beta z\lambda + 4\beta z\lambda\right] = 0$$

By assuming $\mathrm{cov}(z,\lambda) = 0$ and solving for β, it finally obtains

$$\beta = \frac{\sigma_\lambda^2}{\sigma_z^2 + \sigma_\lambda^2} \tag{38}$$

• **Fifth**, the household calculates the rational expectation of p_2 conditional on the overall information set. First of all, from the (37), given the (28), it obtains:

$$E(\lambda \mid I_t^{\mathrm{C}}) = \beta(\lambda - z) = \beta\left(\overline{p} - p_1\right) \tag{39}$$

By setting $\beta_1 = a_1\beta = \left[(2/3)/\beta\right] < 1$ and substituing the [39] into the [35] it gets

$$\begin{aligned}
p_2^{\mathrm{E}}(I_t^{\mathrm{C}}) &= p_1 + a_1 E(\lambda \mid I_t^{\mathrm{C}}) \\
p_2^{\mathrm{E}}(I_t^{\mathrm{C}}) &= p_1 + a_1\beta\left(\overline{p} - p_1\right) \\
p_2^{\mathrm{E}}(I_t^{\mathrm{C}}) &= (1-\beta_1)p_1 - \beta_1 E(p_1 \mid I_{t-1}^{\mathrm{G}}) \\
p_2^{\mathrm{E}}(I_t^{\mathrm{C}}) &= (1-\beta_1)p_1 + \beta_1\overline{p}
\end{aligned} \tag{40}$$

The rational expectation of p_2 at time t conditional on the overall information set at time t is equal to the weighted arithmetic average of the absolute price of good 1 (*the current information*) and the mean of the probability distribution of the general price level *i.e.*, $\overline{p}$ (*the past information*) where the weights depend on β. Three cases can be distinguished:

• $\sigma_z^2 \approx 0 \Rightarrow \beta \approx 1 \Rightarrow \beta_1 \approx a_1 \Rightarrow p_2^{\mathrm{A}} \approx p_1 + a_1(\overline{p} - p_1) = p_1 + a_1(\lambda - z)$. In this case, β indicates that only a real local shock occurs.

• $\sigma_\lambda^2 \approx 0 \ \Rightarrow \ \beta \approx 0 \ \Rightarrow \beta_1 \approx 0 \Rightarrow p_2^{\mathrm{E}} \approx p_1$. In this case instead only a nominal shock occurs.

• $\sigma_\lambda^2 \neq 0 \ \sigma_z^2 \neq 0 \Rightarrow 0 < \beta < 0 \Rightarrow 0 < \beta_1 < 0 \Rightarrow p_2^{\mathrm{E}} - p_1 = \beta_1(\overline{p} - p_1)$. In this case, both shocks occur. In consequence, the relative price changes by a fraction of the difference between the price of good 1 and the mean of the probability distribution of the general price level.

3.4 The local and aggregate supply functions

3.4.1 The local supply functions

From (25), given (40) it follows:

$$x_1 = x^{\mathrm{P}} + \left[p_1 - p_2^{\mathrm{E}}(I_t^{\mathrm{C}})\right] \tag{41}$$

$$x_1 = x^{\mathrm{P}} + [p_1 - (1-\beta_1)p_1 - \beta_1\overline{p}]$$

$$x_1 = x^{\mathrm{P}} + \beta_1\left[p_1 - p_2^{\mathrm{E}}(I_{t-1}^{\mathrm{G}})\right] \tag{42}$$

$$x_1 = x^{\mathrm{P}} + \beta_1(p_1 - \overline{p})$$

$$x_1 = x^{\mathrm{P}} + \beta_1(z - \lambda) \tag{43}$$

The local supply function permits three interpretations. Equation (41) shows that the deviation of the supply of good 1 from its potential level depends on the expected relative price conditional on the overall information set. Equation (42) indicates that this deviation is equal to a fraction of the difference between the price of good 1 and the mean of the probability distribution of the price level. Finally, equation (43) shows that the supply of good 1 can be seen as the sum of a constant and a cyclical component, equal to a fraction of the shocks that hit the economy.

The good 1 supply function (43) cab be rearranged as follows $(p_1 - \overline{p}) = -x^{\mathrm{P}}/\beta_1 + x_1/\beta_1$. As for the slope of the curve, which depends upon β, one can distinguish three cases:

• if $\sigma_z^2 \approx 0 \Longrightarrow \beta \approx 1 \Longrightarrow \beta_1 \approx a_1$. The household holds that only the real local shock occurs so that he regards the change in price 2 as a relative price change.

• if $\sigma_\lambda^2 \approx 0 \Longrightarrow \beta \approx 0 \Longrightarrow \beta_1 \approx 0$. The household considers instead that the change in the price of good 2 is entirely due to the occurrence of a nominal shock. In this case, the model corresponds to the classical benchmark of the previous section.

• if $\sigma_\lambda^2 \neq 0$ and $\sigma_z^2 \neq 0 \Longrightarrow 0 < \beta < 1 \Longrightarrow 0 < \beta_1 < 1$. In this intermediate case, both shocks occur so that the household assigns a fraction of the observed shock to the real shock.

3.4.2 The aggregate good supply: the Lucas supply function

The aggregate supply of goods can be obtained by summing up the two local supplies: $2x = x_1 + x_2 = (x^{\mathrm{P}} + x^{\mathrm{P}}) + \beta_1 (p_1 + p_2 - 2\overline{p})$. Diving both sides of the equation by 2, we finally obtain the aggregate supply equation:

$$\begin{aligned} x &= x^{\mathrm{P}} + \beta_1 (p - \overline{p}) \\ x &= x^{\mathrm{P}} + \beta_1 z \end{aligned} \tag{44}$$

According to the Lucas supply function, the deviation of current supply from its permanent level depends on the unpredictable change in the general price level, which is equal to a fraction of the nominal aggregate shock. The Lucas function is also known as the “surprise” aggregate supply curve precisely because an unpredictable nominal shock takes agents by surprise and leads them to change their output with respect to its natural level. Equation (44) shows that the aggregate function, like its local counterpart, is formed by a constant and a cyclical component depending on the unpredictable nominal shock. In Fig. 7 we draw the aggregate supply curve (44), which is rearranged as follows $(p - \overline{p}) = -x^{\mathrm{P}} / (a_1 \beta) + [1/(a_1 \beta)]x$. While the same conclusions obtained for the local supply case also apply to the Lucas function, three further remarks are in order.

A first remark is that while in the local goods market output oscillations may be due to both types of shock, at the aggregate level instead such oscillations depend exclusively on the aggregate nominal shock. This result, however, is not general since it depends crucially on the assumption of an idiosyncratic real shock, according to which the latter hits the two islands in an opposite way so that the net aggregate effect is nil. Obviously, if local shocks hit both islands uniformously, then the supply function would incorporate an additional random variable so that aggregate output would depend on the unpredictable real shock as well.

A second remark is that imperfect information is not a sufficient condition for the occurence of output fluctuations. Indeed, if the real shock were absent, an aggregate nominal shock would not bring about output oscillations. It can generate this effect only if it is combined with a real shock. It is because both shocks occur that agents get confused and tend, through the signal extraction process, to form rational expectation on the relative price incorporating a random forecast error.

In the end, the slope of the aggregate supply function depends on both structural and policy parameters. Hence, the so-called Lucas critique follows: econometric models based on the assumption of structural parameters invariant to policy changes lead to biased conclusions on the effects of macroeconomic policies.

3.5 General equilibrium with imperfect information

The economy with imperfect information is described by a system of 7 equations (one of these, say equation (2.a), is redundant and can be removed) in 6 unknown variables x, c, m, m^{D}, p e p^{A}:

$$x = x^{\mathrm{P}} + \beta_1 \, (p - p^{\mathrm{E}}) \tag{1.a}$$

$$c = x \tag{2.a}$$

$$c = x \tag{3.a}$$

$$m^{\mathrm{D}} = x + p \tag{4.a}$$

$$m = \overline{m} + z \tag{5.a}$$

$$m^{\mathrm{D}} = m \tag{6.a}$$

$$p^{\mathrm{E}} = E(p \mid I_t^{\mathrm{C}}) \tag{7.a}$$

The first three equations represent the goods market: (1.a) is the supply function; (2.a) is the demand function that, given the validity of Say's law, is equal to supply and (3.a) is the market equilibrium condition. The successive three equations represent the money market: (4.a) is nominal demand that is equal to nominal labor income; (5.a) is the nominal supply, while (6.a) is the market equilibrium condition. In the end, (7.a) describes agents' rational expectations on the price level.

Let us make a digression on the timing of events and agents' choices in a given period t that is represented in Fig. 8.

- First, at the beginning of period t the public sector reveals the predictable component of the nominal money supply; subsequently, a real local shock (*i.e.*, nomadism) and a nominal aggregate shock (*i.e.*, a change in the unpredictable component of money supply) occur.
- Second, given the past general information set, households form rational expectations and define their demand and supply for goods and money demand.
- Third, the auctioneer fixes absolute prices on both islands.
- Fourth, households observe the absolute price on their respective islands and extract the signal from current information about relative price changes.
- Fifth, transactions occur in perfect markets and equilibrium prices and quantities are determined.

- Sixth, agents know the size of the relevant shocks and update the probability distributions of the key random variables needed to form rational expectations.

Let us return to the solution of the model. From the above equation system, we can derive the reduced form of the model:

$$x = x^{\mathrm{P}} + \beta_1 \left(p - p^{\mathrm{E}}\right) \tag{4.b}$$

$$x = \overline{m} - p + z \tag{5.b}$$

$$p^{\mathrm{E}} = E(p \mid I_t^{\mathrm{C}}) \tag{6.b}$$

This shows that, given the monetary policy parameters $\overline{m}$ and z, and the local shock λ, the general equilibrium with imperfect information is a set of values (x, p, p^{E}) that satisfies the equation system (4.b)-(6.b).

The solution of the model proceeds in two steps.

- **Calculation of the rational expectation of p_t.** Taking the expected value of (4.b) and of (5.b) given (6.b), and recalling the reiterative property of rational expectations, we obtain respectively

$$\begin{aligned} E(x) &= x^{\mathrm{P}} + \beta_1 \left(p^{\mathrm{E}} - p^{\mathrm{E}}\right) \\ E(x) &= \overline{m} - p^{\mathrm{E}} \end{aligned}$$

  Since the right-hand side of the expressions are equal we get:

$$p^{\mathrm{E}} = \overline{p} = \overline{m} - x^{\mathrm{P}} \tag{45}$$

  The rational expectation of p is equal to the mean of the probability distribution of the price level conditional on the general information set of period t that, in turn, is equal to the difference between the predictable component of the money supply and permanent output.

- **Calculation of equilibrium values of prices and quantities.** Making equal (4.b) e la (5.b), given (45), we obtain

$$\overline{m} - p + z = x^{\mathrm{P}} + \beta_1 \left(p - \overline{p}\right)$$

  Solving the previous expression with respect to p, we first obtain the equilibrium price level:

$$p = \overline{p} + \frac{1}{(1+\beta_1)}\, z \tag{46}$$

  By susbstituting (46) into (4.b) we finally obtain equilibrium output:

$$x = x^{\mathrm{P}} + \frac{\beta_1}{(1+\beta_1)}\, z \tag{47}$$

From (47) the invariance proposition follows: current output oscillates erratically around permanent output.

3.6 The effects of predictable and unpredictable monetary policies

Equations (47) and (46) show that the unpredictable component of monetary policy generates real effects: it changes current output. In Fig. 9 we draw the graphical solution of the reduced model in the plane (x, p) and analyze the effects of an expansionary monetary policy implemented through a change in the unpredictable component. With respect to initial equilibrium at point A, this change shifts the *AD* to the right, whereas the *AS* curve stays put (since its position depends solely on the predictable component of the money supply $\overline{m}$ *via* $p^{\mathrm{E}} = \overline{p}$). Households do not fully predict this change and erroneously regard it at least partly as a real local shock and thus change their output supply. The new point of equilibrium is reached at $B = (x_1, p_1)$ where $x_1 > x^{\mathrm{P}}$ e $p > p_1$.

If the public sector instead carries out a permanent change in the predictable component of the money supply, as figure 10 shows, only an inflationary bias occurs. Starting from the equilibrium point $A = (x^{\mathrm{P}}, p_0)$, this change produces two effects. First, it leads households (who fully observe this change) to update their rational expectation of p from p_0^{E} to $p_1^{\mathrm{A}} = \overline{m}_1 - x^{\mathrm{P}}$: the *AS* shifts upwards. Second, it generates a disequilibrium in the money market that, in turn, given the constant level of output, implies a price level rise: the *AD* curve shifts upwards. In consequence, the new equilibrium point is reached at $B = (x^{\mathrm{P}}, p_1)$ where output remains constant at the initial level and a permanent increase in the price level occurs.

4 Conclusions

This paper has sought to provide a presentation of the Lucas Island model which has some pedagogic advantages over alternative presentations which are currently available. In particular, it helps students to understand more clearly the existence of a kind of inconsistency in the information structure implied by this model. On the one hand, by making the rational expectations assumption, Lucas's model implies a very high degree of information efficiency, a feature which is made much more plausible by the existence of advanced technology such as computers. On the other, however, the model actually implies technological backwardness: the simple geographic distance between islands turns out to be sufficient to generate the lack of communication between the producer and the shopper which is responsible for agents' confusion between relative and absolute prices. As the perfect information case discussed in this paper shows, this "naturalistic" hurdle can be overcome by the existence of technological devices such as cellular phones which enable shoppers to communicate the prices on island 2 to producers on island 1.

Notes

1. The same demonstration applies to the representative household 2 because of household homogeneity.

2. Goods that are produced and consumed directly by households 1 do not flow into the local market. They can be considered like a minimum subsistence consumption level that allows households to carry on the production activity and, therefore, do not enter their utility function.

3. Since there are two relative prices (*i.e.*, the relative price of good 1 in terms of good 2 and the relative price of good 2 in terms of good 1), hereafter we refer only to the first one.

4. The hypothesis of local idiosyncratic shocks implies that output variations on two island cancel each other out.

References

Azariadis, C. 1981. A Reexamination of Natural Rate Theory. *The American Economic Review* 71 (5): 946-960.

Bénassy, J.P. 1999. Analytical solutions to a structural signal extraction model: Lucas 1972. *Journal of Monetary Economics* 40: 509-521.

Bull, C and R. Frydman 1983. The Derivation and Interpretation of the Lucas Supply Function. *Journal of Money, Credit and Banking* 15 (1): 82-95.

Lucas Jr., R. E. 1972. Expectations and the Neutrality of Money. *Journal of Economic Theory* 103 (4): 138-145.

Romer, D. 2005. *Macroeconomics*. 3th ed. New York: McGraw-Hill.

Zarnowitz, V. 1992. *Business Cycle. Theory, History, Indicators, and Forecasting*. Chigaco, USA: University of Chigaco Press.

Figures

Figure 1: The equilibrium in the local goods market on island 1. The equilibrium values of output and the relative price depends on the real local shock.

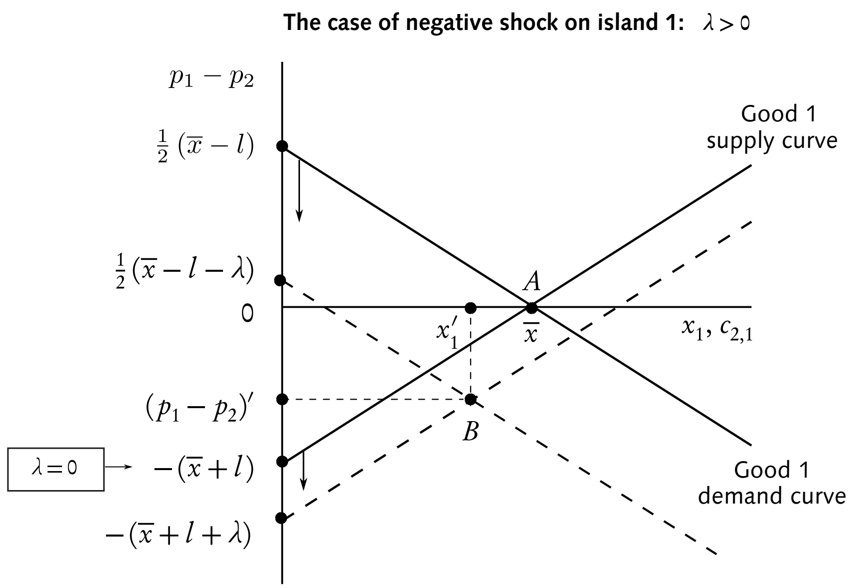

Figure 2: The equilibrium in the aggregate goods market. The equilibrium value of output does not depend on the price level and the real local shock.

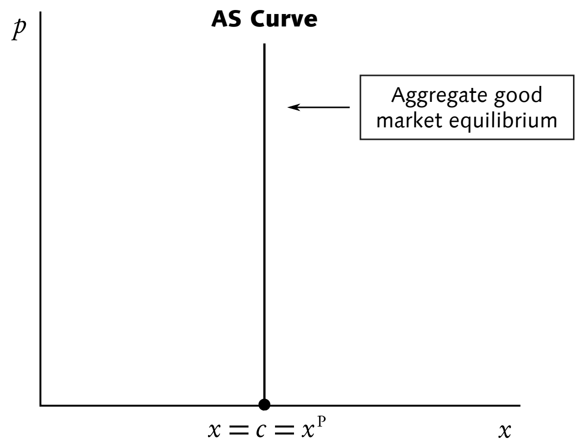

Figure 3: The partial equilibrium in the money market of island 1.

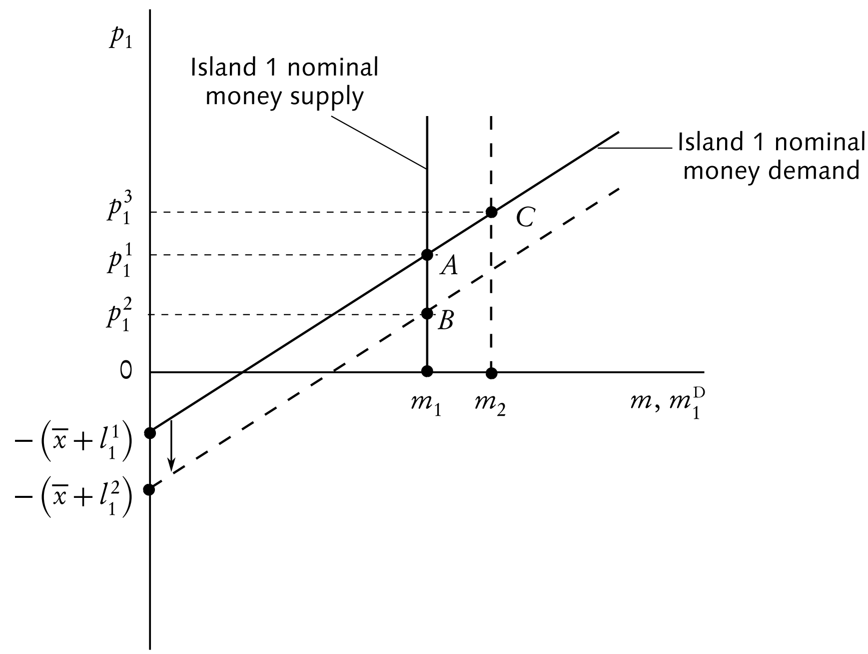

Figure 4: The graphical derivation of the *AD* curve.

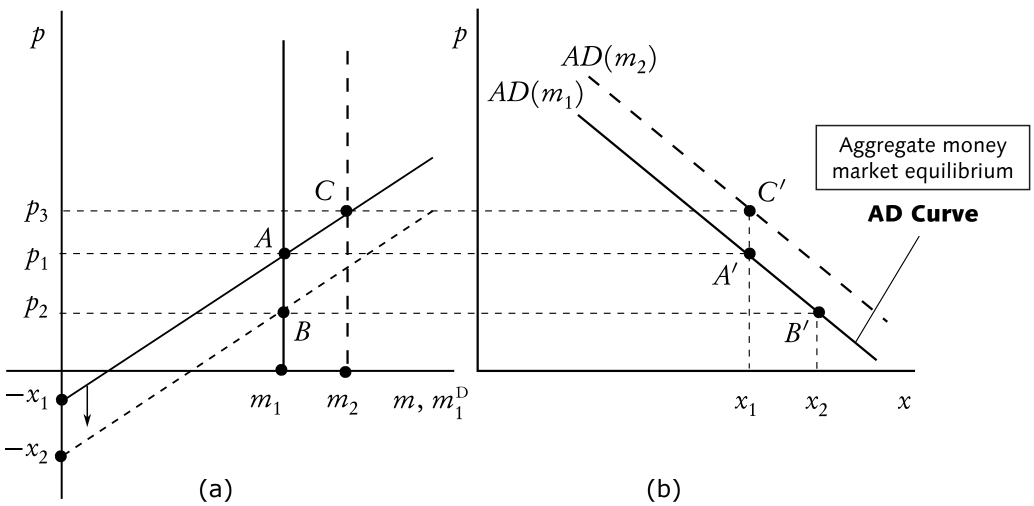

Figure 5: Macroeconomic general equilibrium in the Lucas Island model with perfect information: the neutrality of money in the predictable component.

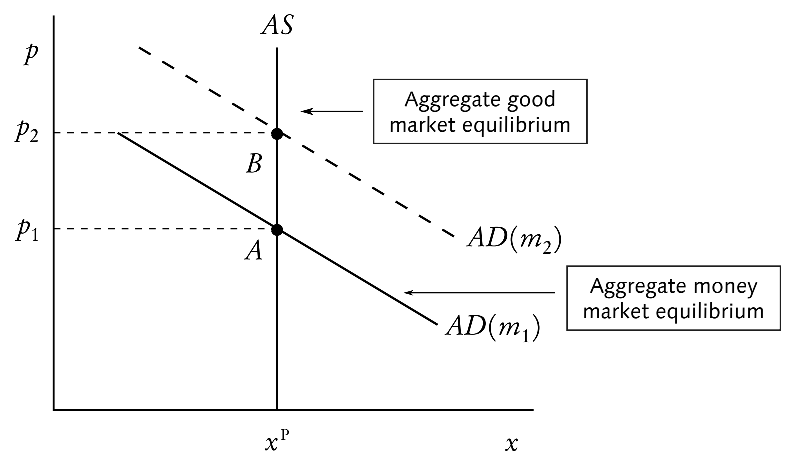

Figure 6: The neutrality of the unpredictable component of money.

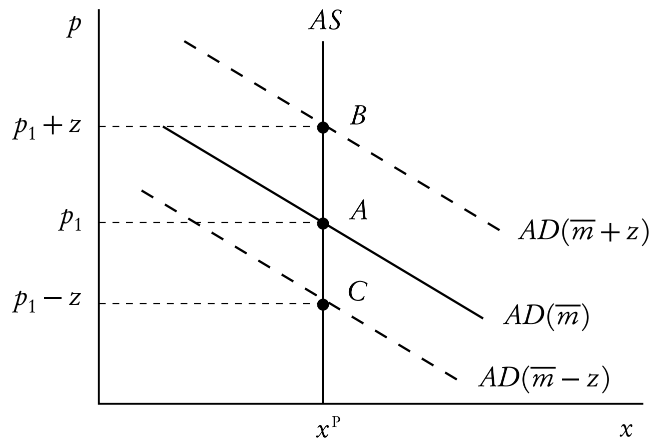

Figure 7: The aggregate good supply curve with information imperfection.

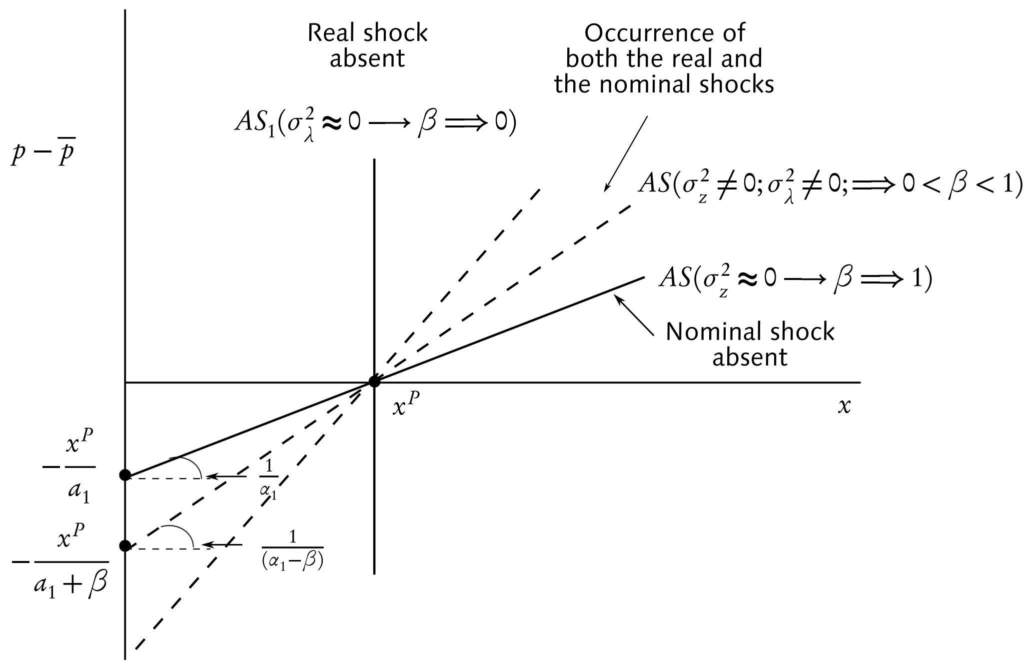

Figure 8: The timing of events and agents' choices.

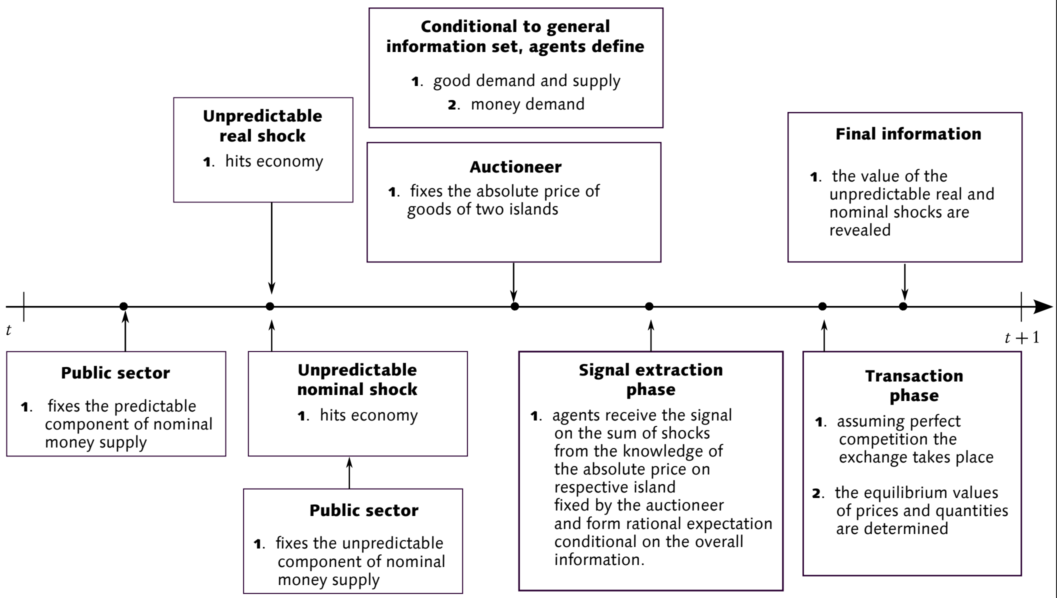

Figure 9: General equilibrium with imperfect information: the invariance proposition.

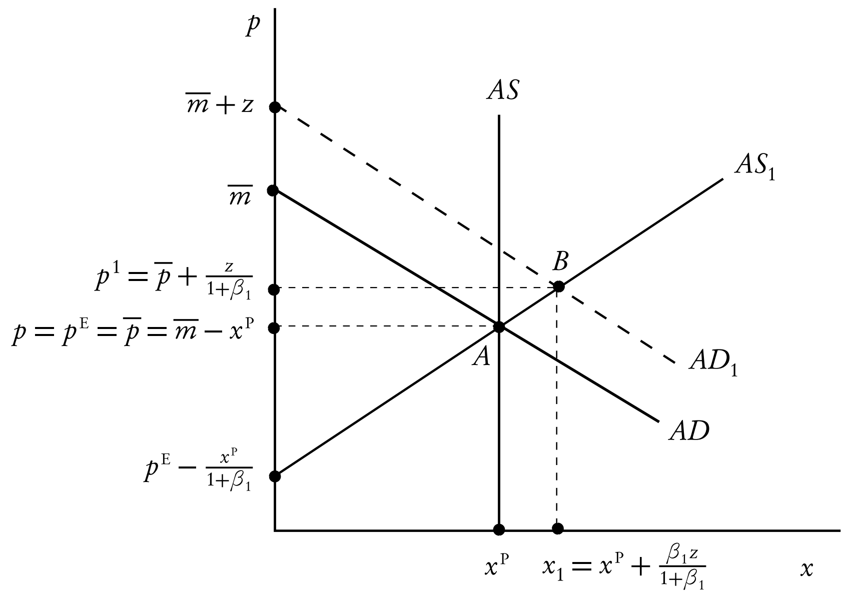

Figure 10: The effects of a permanent expansionary predictable monetary policy. A permanent increase in the predictable component of the money supply produces an inflationary bias.