Delta-Gamma hedging of mortality and interest rate risk

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Abstract

One of the major concerns of life insurers and pension funds is the increasing longevity of their beneficiaries. This paper studies the hedging problem of annuity cash flows when mortality and interest rates are stochastic. We first propose a Delta-Gamma hedging technique for mortality risk. The risk factor against which to hedge is the difference between the actual mortality intensity in the future and its "forecast" today, the forward intensity. We specialize the hedging technique first to the case in which mortality intensities are affine, then to Ornstein-Uhlenbeck and Feller processes, providing actuarial justifications for this selection. We show that, without imposing no arbitrage, we can get equivalent probability measures under which the HJM condition for no arbitrage is satisfied. Last, we extend our results to the presence of both interest rate and mortality risk. We provide a UK calibrated example of Delta-Gamma Hedging of both mortality and interest rate risk.

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1 Introduction

One of the major concerns of life insurers and pension funds is the increasing longevity of their beneficiaries. In this paper we study the hedging problem of annuity cash flows in the presence of longevity risk. This means that we assume a stochastic mortality. In recent years, the literature has focused on the stochastic modeling of mortality rates, in order to deal with unexpected changes in the life expectancy of policyholders or members of pension funds. This kind of risk, due to the stochastic nature of death intensities, is referred to as systematic mortality risk. In the present paper we deal with this risk, as well as with financial risk, which originates from the stochastic nature of interest rates.

The problem of hedging life insurance or pension funds liabilities in the presence of systematic mortality risk has attracted much attention in recent years. It has been addressed either via risk-minimizing and mean-variance indifference hedging strategies, or through the creation of mortality-linked derivatives and securitization. The first approach has been taken by Dahl and Møller (2006) and Barbarin (2008). The second approach was discussed by Dahl (2004) and Cairns, Blake, Dowd, and MacMinn (2006) and has witnessed a lively debate, see f.i. Blake, De Waegenaere, MacMinn, and Nijman (2010) and references therein.

We study Delta-Gamma hedging for insurance companies or annuity providers. This requires choosing a specific change of measure, but has two main advantages with respect to risk-minimizing and mean-variance indifference strategies. On the one side it represents systematic mortality risk in a very intuitive way, namely as the difference between the actual mortality intensity in the future and its "forecast" today. On the other side, Delta-Gamma hedging is easily implementable and adaptable to self-financing constraints. Indeed, it ends up in solving a linear system of equations.

The Delta-Gamma hedging complements the securitization approach strongly supported by most academics and industry leaders. Securitization aims at one-to-one hedging or replication, while we push hedging one step further through local, but less costly, coverage.

Following a well established stream of actuarial literature, we represent death arrival as the first jump time of a doubly stochastic process. To enhance analytical tractability, we assume a pure diffusion of the affine type for the spot mortality intensity. Namely, the process has drift and instantaneous variance linear in the intensity itself. In this setting, Cairns, Blake, and Dowd (2006) point out that an HJM condition for forward death intensities – similar to the one of the financial market – can be imposed. We show that for two non-mean reverting processes for the spot intensity there exists an infinity

of probability measures – equivalent to the historical one – in which forward death intensities satisfy an HJM condition. No arbitrage holds under all of these measures, even though it is not imposed a priori. These processes belong to the Ornstein-Uhlenbeck and the Feller class.
As a consequence, we start by introducing spot mortality intensities, focus on the Ornstein-Uhlenbeck and the Feller class, discuss their soundness as descriptors of the actual – or historical – mortality dynamics, derive the corresponding forward death intensities and tackle the change of measure issue. Among the possible changes, we select the minimal one – which permits to remain in the Ornstein-Uhlenbeck and Feller class – and parametrize it by assuming that the risk premium for mortality risk is constant. In this way, we can focus on Delta-Gamma hedging.
For the sake of simplicity we assume that the market of interest rate bonds is not only arbitrage-free but also complete. First, we consider hedging in the presence of systematic mortality risk only. Then, under independence of mortality and financial risks, we provide an extension of the hedging strategy to both these risks.
To keep the treatment simple, we build Delta-Gamma coverage on pure endowments. These are the single cash flows in which an annuity or a life insurance contract can be decomposed. We use as hedging tools either pure endowments or zero-coupon survival bonds for mortality risk and zero-coupon-bonds for interest rate risk. Since all these assets can be understood as Arrow-Debreu securities in the insurance/annuity and fixed income market, the Delta-Gamma hedge could be extended to the entire annuity or to more complex insurance contracts.
The final calibration of the strategies – which uses UK mortality rates for the male generation born in 1945 and the Hull and White interest rates on the UK market – shows that

1. the unhedged effect of a sudden change on mortality rate is remarkable, especially for long time horizons;

2. the corresponding Delta and Gamma coefficients are quite different if one takes into consideration or ignores the stochastic nature of the death intensity;

3. the hedging strategies are easy to implement and customize to self-financing constraints;

4. Delta and Gamma coefficients are bigger for mortality than for financial risk.

The paper is structured as follows: Section 2 recalls the doubly stochastic approach to mortality modelling and introduces the two intensity processes considered in the paper. Section 3 presents the notion of forward death intensity. Section 4 describes the standard financial assumptions on the market for interest rates. Section 5 derives the dynamics of forward intensities and survival probabilities, after the appropriate change of measure. Section 6 shows that the HJM restriction is satisfied without imposing no arbitrage a priori. In Section 7 we discuss the hedging technique for mortality risk. Section 8 addresses mortality and financial risk. Section 9 presents the application to a UK population sample. Section 10 summarizes and concludes.

2 Cox modelling of mortality risk

This section introduces mortality modelling by specifying the so-called spot mortality intensity (mortality intensity for short). Section 2.1 describes the general framework, while Section 2.2 studies two specific processes which will be considered throughout the paper.

2.1 Spot death intensity

Mortality has been recently described by means of Cox or doubly stochastic counting processes, as studied by Brémaud (1981). The modelling technique has been drawn from the financial domain and in particular from the reduced form models of the credit risk literature, where the time-to-default is described as the first stopping time of a Cox process.[1] Mortality modelling via Cox processes has been introduced by Milevsky and Promislow (2001) and Dahl (2004). Intuitively, the time of death - analogously to the time-to-default in finance - is supposed to be the first jump time of a Poisson process with stochastic intensity. The existence of a stochastic mortality intensity generates systematic mortality risk. The intensity process may be either a pure diffusion or may present jumps. If in addition it is an affine process, then the survival function can be derived in closed form.
Let us introduce a filtered probability space $(\Omega, \mathbf{I}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ which satisfies the usual properties of right-continuity and completeness. On this space, let us consider a non-negative, predictable process $\lambda_x(t)$, which represents the mortality intensity of an individual or head belonging to generation x at (calendar) time t. We introduce the following

[1] See the seminal paper Lando (1998).

Assumption 1 *The mortality intensity λ_x follows a process of the type:*

$$d\lambda_x(t) = a(t, \lambda_x(t))dt + \sigma(t, \lambda_x(t))dW_x(t) + dJ_x(t), \tag{1}$$

where J is a pure jump process whose jump intensity is η, W_x is a standard one-dimensional Brownian motion[2] and the regularity properties for ensuring the existence of a strong solution of equation (1) are satisfied for any given initial condition $\lambda_x(0) = \lambda_0 > 0$.

Given this assumption on the dynamics of the death intensity, let τ be the time to death of an individual of generation x. We define the survival probability from t to $T \geqslant t$, $S_x(t,T)$, as the survival function of τ under the probability measure $\mathbb{P}$, conditional on the survival up to time t:

$$S_x(t,T) := \mathbb{P}\left(\tau \geqslant T \mid \tau > t\right).$$

It is known since Brémaud (1981) that - under the previous assumption - the survival probability $S_x(t,T)$ can be represented as

$$S_x(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) \mid \mathcal{F}_t\right], \tag{2}$$

where the expectation is computed under $\mathbb{P}$. When the evaluation date is zero ($t = 0$), we simply write $S_x(T)$ instead of $S_x(0,T)$.

In this paper, we suppose in addition that

Assumption 2 *The drift $a(t, \lambda_x(t))$, the instantaneous variance $\sigma^2(t, \lambda_x(t))$ and the jump intensity η associated with J have affine dependence on $\lambda_x(t)$.*

Hence, we assume that these coefficients are of the form:

$$a(t, \lambda_x(t)) = b + c\lambda_x(t),$$

$$\sigma^2(t, \lambda_x(t)) = d \cdot \lambda_x(t),$$

$$\eta(t, \lambda_x(t)) = l_0 + l_1\lambda_x(t),$$

where $b, c, d, l_0, l_1 \in \mathbb{R}$. Under Assumption 2 standard results on functionals of affine processes allow us to provide a closed form for the survival probability

$$S_x(t,T) = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)},$$

[2]The extension of the mortality intensity definition to a multidimensional Brownian motion is straightforward.

where $\alpha(\cdot)$ and $\beta(\cdot)$ solve the following system of Riccati differential equations (see for instance Duffie, Pan, and Singleton (2000)):

$$\begin{cases} \beta'(t) = \beta(t)c + \frac{1}{2}\beta(t)^2 d^2 + l_1 \left[\int_{\mathbb{R}} e^{\beta(t)z} d\nu(z) - 1\right], \\ \alpha'(t) = \beta(t)b + l_0 \left[\int_{\mathbb{R}} e^{\beta(t)z} d\nu(z) - 1\right], \end{cases} \tag{3}$$

where ν is the distribution function of the jumps of J. The boundary conditions are $\alpha(0) = 0$ and $\beta(0) = 0$.

2.2 Ornstein-Uhlenbeck and Feller processes

In this paper we focus on two intensity processes, which are natural stochastic generalizations of the Gompertz model for the force of mortality and are thus easy to interpret in the light of traditional actuarial practice. They belong to the affine class and are purely diffusive. These processes, together with the solutions $\alpha(\cdot)$ and $\beta(\cdot)$ of the associated Riccati ODEs, are:

— Ornstein-Uhlenbeck (OU) process without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma dW_x(t), \tag{4}$$

$$\alpha(t) = \frac{\sigma^2}{2a^2}t - \frac{\sigma^2}{a^3}e^{at} + \frac{\sigma^2}{4a^3}e^{2at} + \frac{3\sigma^2}{4a^3}, \tag{5}$$

$$\beta(t) = \frac{1}{a}(1 - e^{at}). \tag{6}$$

— Feller process (FEL) without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t), \tag{7}$$

$$\alpha(t) = 0, \tag{8}$$

$$\beta(t) = \frac{1 - e^{bt}}{c + de^{bt}}. \tag{9}$$

with $b = -\sqrt{a^2 + 2\sigma^2}, c = \frac{b+a}{2}, d = \frac{b-a}{2}$.
In both processes we assume $a > 0, \sigma \geq 0$.

A process, in order to describe human survivorship realistically, has to be "biologically reasonable", i.e. it has to satisfy two technical features: the intensity must never be negative and the survival function has to be decreasing in time T.
Indeed, in the OU case, λ_x can turn negative with positive probability:

$$\mathbb{P}(\lambda_x(t) \leqslant 0) = \phi\left(-\frac{\lambda_x(0)e^{at}}{\sigma\sqrt{\frac{e^{2at}-1}{2a}}}\right), \tag{10}$$

where ϕ denotes the distribution function of a standard normal random variable. The survival function is decreasing as long as T is below the level T^* defined as:

$$T^* = \frac{1}{a} \ln \left[1 + \frac{a^2 \lambda_x(0)}{\sigma^2} \left(1 + \sqrt{1 + \frac{2\sigma^2}{a^2 \lambda_x(0)}} \right) \right]. \tag{11}$$

In practical applications (Section 9) we verify that the probability (10) is negligible and that the length of the time horizon we consider (the duration of a human life) never exceeds T^*.
For the FEL process, instead, the intensity can never turn strictly negative and the survival function is guaranteed to be decreasing in T if and only if the following condition holds:

$$e^{bt}(\sigma^2 + 2d^2) > \sigma^2 - 2dc. \tag{12}$$

In spite of the technical restrictions, Luciano and Vigna (2008) suggest the appropriateness of these processes for describing the intensity of human mortality. In fact, they show that these models meet all but one of the criteria - motivated by Cairns, Blake, and Dowd (2006) - that a good mortality model should meet:

1. the model should be consistent with historical data: the calibrations of Luciano and Vigna (2008) show that the models meet this criterium;

2. the intensity of mortality should keep strictly positive: the model's intensities do not stay strictly positive with probability 1, but for practical applications it can be shown that they turn negative with negligible probability, once the model is calibrated;

3. long-term future dynamics of the model should be biologically reasonable: the models meet this criterium, as the calibrated parameters satisfy conditions (11) and (12) above;

4. long-term deviations in mortality improvements should not be mean-reverting to a pre-determined target, even if the target is time-dependent: the models meet this criterium by construction;

5. the model should be comprehensive enough to deal appropriately with pricing, valuation and hedging problems: indeed, showing that the models meet this criterium is the scope of the present paper;

6. it should be possible to value mortality linked derivatives using analytical methods or fast numerical methods: these models meet this criterium, as they produce survival probabilities in closed form and with a very small number of parameters.

Cairns, Blake, and Dowd (2006) add that no one of the previous criteria dominates the others. Consistently with their view, we claim the validity of the proposed models, which meet at least five criteria out of six. In addition, the fact that survival functions are given in closed form and depend on a very small number of parameters simplifies the calibration procedure enormously. An application of the FEL model to a sample of Canadian insured couples can be found in Luciano, Spreeuw, and Vigna (2008).
These processes (and especially the first one, the OU) turn out to be significantly suitable for the points 5 and 6 above. In fact, in Sections 6, 7 and 8 we will show that the Delta and Gamma OU-coefficients can be expressed in a very simple closed form. Thus, the Delta-Gamma Hedging technique – widely used in the financial context to hedge purely financial assets – turns out to be straightforward to apply also in the actuarial-financial context. The Delta and Gamma FEL coefficients are more complicated to find, but the technique is still applicable.

3 Forward death intensities

This section aims at shifting from mortality intensities to their forward counterparts, both for the general affine case and for the OU and FEL processes. We first provide the motivation and then give the definitions.

The notion of forward instantaneous intensity for counting processes representing firm defaults has been introduced by Duffie (1998) and Duffie and Singleton (1999), following a discrete-time definition in Litterman and Iben (1991). Stochastic modelling of forward intensities has been extensively studied in the financial domain. Indeed, in credit risk modelling this notion is very helpful, since it allows to determine the change of measure or the intensity dynamics useful for pricing and hedging defaultable bonds. The characterization is obtained under a no-arbitrage assumption for the financial market and is unique when the market is also complete.
Suppose that arbitrages are ruled out, that the recovery rate is null and $\lambda_x(t)$ in (1) represents the default intensity of a firm whose debt is traded in a complete market. Then, we have the following HJM restriction under the

(unique) risk-neutral measure corresponding to $\mathbb{P}$:

$$a(t, \lambda_x(t)) = \sigma(t, \lambda_x(t)) \int_0^t \sigma(u, \lambda_x(u))du. \tag{13}$$

Once the HJM restriction is in place, we can proceed to pricing and hedging. In the actuarial domain, forward death intensities have already been introduced by Dahl (2004) and Cairns, Blake, and Dowd (2006), paralleling the financial definition. In Section 5 we prove that, even though restriction (13) can be violated by death intensities in general, it holds true for the OU and FEL intensity processes, even without imposing no arbitrage, but simply restricting the measure change so that the intensity remains OU or FEL under the new measures.

Let us start from the forward death rate over the period $(t, t + \Delta t)$, evaluated at time zero, as the ratio between the conditional probability of death between t and $t + \Delta t$ and the time span Δt, for a head belonging to generation x, conditional on the event of survival until time t. The forward death intensity is its instantaneous version:

$$f_x(0, t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(1 - \frac{S_x(t + \Delta t)}{S_x(t)} \right).$$

It is evident from the definition that - if it exists - the forward death intensity is the logarithmic derivative of the (unconditional) survival probability, as implied by the process λ_x:

$$f_x(0, t) = -\frac{\partial}{\partial t} \ln (S_x(t)).$$

The similarity of this definition with the force of mortality is quite strong and in fact the two coincide when the diffusion coefficient of the intensity process is null. Similarly, one can define the forward death intensity for the tenor T, as evaluated at time $t < T$, starting from the survival probability $S_x(t, T)$:

$$f_x(t, T) = -\frac{\partial}{\partial T} \ln (S_x(t, T)). \tag{14}$$

The forward death intensity[3] $f_x(t, T)$ represents the intensity of mortality which will apply instantaneously at time T, implied by the knowledge of

[3]Please notice also that the forward death intensity definition, and consequently its expression for the affine case, is analogous to the one of forward instantaneous interest rates, the latter being defined starting from discount factors rather than survival probabilities. As in the case of forward instantaneous interest rates, it can be shown that forward intensities, for given t, can be increasing, decreasing or humped functions of the application date T.

the process λ_x up to t (or under the filtration $\mathcal{F}_t$). In other words, the quantity $f_x(t,T)\Delta t$ approximates the probability of dying right after time T, conditional on survival from t to T.
The forward death intensity can be interpreted as the "best forecast" of the actual mortality intensity, since it coincides with the latter when $T = t$:

$$f_x(t,t) = \lambda_x(t).$$

It follows from the above definition that the survival probabilities from t to T can be written as integrals of (deterministic) forward death probabilities:

$$S_x(t,T) = \exp\left(-\int_t^T f_x(t,s)ds\right), \tag{15}$$

and not only as expectations wrt the intensity process λ_x , as in (2) above.[4]

Let us turn now to the affine case. It can be easily shown from (15) that, when λ_x is an affine process, the initial forward intensity depends on the functions $\alpha(\cdot)$ and $\beta(\cdot)$, as well as on the initial spot intensity as follows:

$$f_x(0,t) = -\alpha'(t) - \beta'(t)\lambda_x(0) = -\alpha'(t) - \beta'(t)f_x(0,0). \tag{16}$$

At any time $0 \le t \le T$:

$$f_x(t,T) = -\alpha'(T-t) - \beta'(T-t)\lambda_x(t) = -\alpha'(T-t) - \beta'(T-t)f_x(t,t).$$

For the processes defined by equations (4) and (7), the instantaneous forward intensities can be computed as:

$$OU \qquad f_x(t,T) = \lambda_x(t)e^{a(T-t)} - \frac{\sigma^2}{2a^2}(e^{a(T-t)}-1)^2, \tag{17}$$

$$FEL \qquad f_x(t,T) = \frac{4\lambda_x(t)b^2e^{b(T-t)}}{[(a+b)+(b-a)e^{b(T-t)}]^2}.$$

[4]Notice that, at any initial time t, forward death intensities can be interpreted as the (inhomogeneous) Poisson arrival rates implied in the current Cox process. Indeed, it is quite natural, especially if one wants a description of survivorship without the mathematical complexity of Cox processes, to try to describe mortality via the equivalent survival probability in a simpler (inhomogeneous) Poisson model. Once a λ_x process has been fixed, and therefore survival probabilities have been computed, according to (2), one can wonder: what would be the intensity of an inhomogeneous Poisson death arrival process, that would produce the same survival probabilities? Recalling that in the Poisson case survival probabilities are of the type (15), one can interpret – and use – $f(t,T)$ exactly as the survival intensity of an (inhomogeneous) Poisson model equivalent to the given, Cox one.

It can be shown that in both cases $f_x(t,T)$ is a decreasing function of σ. This underlines from an analytical point of view the difference with the deterministic case, $\sigma = 0$. Everything else being equal, a positive σ implies a lower death intensity, hence greater longevity than in the deterministic framework. Increasing σ further improves longevity.

4 Financial risk

In order to introduce a valuation framework for insurance policies, we need to provide a description of the financial environment. In addition to mortality risk, we assume the existence of a financial risk, in the sense that the interest rate is described by a stochastic process. While in the mortality domain we started from (spot) intensities – for which we were able to motivate specific modelling choices – and then we went to their forward counterpart, here we follow a well established bulk of literature – starting from Heath, Jarrow, and Morton (1992) – and model directly the instantaneous forward rate $F(t,T)$, i.e. the date-t rate which applies instantaneously at T .

Assumption 3 *The process for the forward interest rate $F(t,T)$, defined on the probability space $(\Omega, \mathbf{I}, \mathbb{P})$, is:*

$$dF(t,T) = A(t,T)dt + \Sigma(t,T)dW_F(t), \tag{18}$$

where the real functions $A(t,T)$ and $\Sigma(t,T)$ satisfy the usual assumptions for the existence of a strong solution to (18), and W_F is a univariate Brownian motion[5] independent of W_x for all x.

The independence between the Brownian motions means, loosely speaking, independence between mortality and financial risk.[6]
Let us also denote as $\{\mathcal{H}_t : 0 \leq t \leq T\}$ the filtration generated by the interest rate process.
As a particular subcase of the forward rate, obtained when $t = T$, one obtains the short rate process, which we will denote as $r(t)$:

$$F(t,t) := r(t). \tag{19}$$

We also introduce the following

[5]We assume a single Brownian motion for the forward rate dynamics, since we have reduced the discussion of mortality risk to a single risk source too: however, the extension to a multidimensional Brownian motion is immediate.

[6]This assumption is common in the literature and seems to be intuitively appropriate.

Assumption 4 *The financial market admits no arbitrages and is complete.*

It is well known that – if Assumption 4 holds – there is a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ - which we characterize in the next section - under which the zero-coupon-bond price for the maturity T, evaluated at time t, $B(t,T)$, is

$$B(t,T) = \exp\left(-\int_t^T F(t,u)du\right) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right]. \quad (20)$$

We provide a specific choice for the forward interest rate only at a later stage. We will select a very popular model in Finance, the constant-parameter Hull and White (Hull and White (1990)).

5 Change of measure and insurance valuations

This section discusses the change of measure that allows us to evaluate policies subject to mortality risk in a fashion analogous to (20). First, let us embed our valuation in a more general setting which includes both systematic and idiosyncratic mortality risk. The latter arises from the randomness of death occurrence inside the portfolio of insured people. To formalize it, let us define the process of death occurrence in the portfolio as follows. Let $\tau_1, \tau_2, ...\tau_N$ be the lifetimes of the N insured in the cohort x, assumed to be i.i.d. with distribution function $S_x(t,T)$ in (2). Let $D(x,t)$ be the pure-jump process which counts the number of deaths in such an insurance portfolio:

$$D(x,t) := \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \le t\}},$$

where $\mathbf{1}_{\{.\}}$ is the indicator function. We define a filtration on $(\Omega, \mathbf{I}, \mathbb{P})$ whose σ -algebras $\{\mathcal{G}_t : 0 \le t \le T\}$ are generated by $\mathcal{F}_t$ and $\{D(x,s) : 0 \le s \le t\}$. This filtration intuitively collects the information on both the past mortality intensity and on actual death occurrence in the portfolio. Let us consider, on the probability space $(\Omega, \mathbf{I}, \mathbb{P})$, the sigma algebras $\mathcal{I}_t := \mathcal{G}_t \vee \mathcal{H}_t$ generated by unions of the type $\mathcal{G}_t \cup \mathcal{H}_t$, where the σ-algebra $\mathcal{G}_t$ collects information on mortality risk, while $\mathcal{H}_t$, which is independent of $\mathcal{G}_t$, reflects information on financial risk, namely on the forward rate process. In other words, $\mathbf{I} = \{\mathcal{I}_t : 0 \le t \le T\}$. The filtration $\mathcal{I}_t$ represents all the available information on both financial and mortality risk. In order to perform insurance policies evaluations in $(\Omega, \mathbf{I}, \mathbb{P})$, equipped with such a filtration, we need to characterize at least one equivalent measure. This can be done using a version of Girsanov's theorem, as described in Jacod and Shiryaev (1987) and applied in the actuarial setting in Dahl and Møller (2006):

Theorem 5.1 *Let the bi-dimensional process* $\theta(t) := [\theta_x(t) \quad \theta_F(t)]$ *and the univariate, positive one* $\varphi(t)$ *be* $\mathcal{I}_t$ *-predictable, with*

$$\begin{aligned} \int_0^T \theta_i^2(t)dt &< \infty, \qquad i = x, F, \\ \int_0^T |\varphi(t)| \lambda_x(t)dt &< \infty. \end{aligned}$$

Define the likelihood process $L(t)$ *by*

$$\left\{ \frac{dL(t)}{L(t-)} = \theta_x(t)dW_x(t) + \theta_F(t)dW_F(t) + \varphi(t)dD(x,t), \right.$$

and assume $L(0) = 1, \mathbb{E}^{\mathbb{P}}[L(t)] = 1, t \leq T$. *Then there exists a probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$, *such that the restrictions of* $\mathbb{P}$ *and* $\mathbb{Q}$ *to* $\mathcal{I}_t$, $\mathbb{P}_t := \mathbb{P} \mid \mathcal{I}_t$, $\mathbb{Q}_t := \mathbb{Q} \mid \mathcal{I}_t$, *have Radon-Nykodim derivative* $L(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}$. *The mortality indicator process has intensity* $(1 + \varphi(t))\lambda_x(t)$ *under* $\mathbb{Q}$ *and*

$$dW_i' := dW_i - \theta_i(t)dt, \qquad i = x, F,$$

define $\mathbb{Q}-$*Brownian motions. All the probability measures equivalent to* $\mathbb{P}$ *can be characterized this way.*

Actually, the previous theorem characterizes an infinity of equivalent measures, depending on the choices of the processes $\theta_x(t), \theta_F(t)$ and $\varphi(t)$. These processes represent the prices - or premia - given to the three different sources of risk we model. The first source of risk, the systematic mortality one, is represented by $\theta_x(t)$. This source of risk is not diversifiable, since it originates from the randomness of death intensity. We have no standard choices to apply in the choice of $\theta_x(t)$: see for instance the extensive discussion in Biffis (2005) and Cairns, Blake, Dowd, and MacMinn (2006). For the sake of analytical tractability, we focus on purely diffusive intensities and we select $\theta_x(t)$ so that the risk-neutral intensity is still affine, as in Dahl and Møller (2006). Therefore, we replace Assumptions 1 and 2 with the following:

Assumption 5 *The intensity process under* $\mathbb{P}$ *is purely diffusive and affine. The systematic mortality risk premium is such as to leave it affine under* $\mathbb{Q}$*:*

$$\theta_x(t) := \frac{p(t) + q(t)\lambda_x(t)}{\sigma(t, \lambda_x(t))},$$

with $p(t)$ *and* $q(t)$ *continuous functions of time.*

Indeed, with such a risk premium, the intensity process under $\mathbb{Q}$ is

$$d\lambda_x(t) = [a(t, \lambda_x(t)) + p(t) + q(t)\lambda_x(t)]\, dt + \sigma(t, \lambda_x(t))dW'_x, \tag{21}$$

which is still affine. This choice boils down to selecting the so-called minimal martingale measure.
For the OU and FEL processes we choose the functions $p = 0$ and q constant, so that we have the same type of process under $\mathbb{P}$ and $\mathbb{Q}$, with the coefficient a in equations (4) and (7) replaced by $a' := a + q$. We therefore introduce the following specification of Assumption 5:

Assumption 5' *In the OU and FEL case* $\theta_x(t) = \frac{q\lambda_x(t)}{\sigma}$ *with* $q \in \mathbb{R}$.

The second source of risk, the financial one, originates from the stochastic nature of interest rates. The process $\theta_F(t)$ represents the so-called premium for financial risk. Under Assumption 4 the only choice consistent with no arbitrage is

$$\theta_F(t) := -A(t, T)\Sigma^{-1}(t, T) + \int_t^T \Sigma(t, u)du.$$

Indeed, under this premium the drift coefficient of the forward rates $A'(t, T)$ is tied to the diffusion by an HJM relationship:

$$A'(t, T) = \Sigma(t, T) \int_t^T \Sigma(t, u)du. \tag{22}$$

It follows that, under the measure $\mathbb{Q}$,

$$dF(t, T) = \left[\Sigma(t, T) \int_t^T \Sigma(t, u)du\right] dt + \Sigma(t, T)dW'_F(t). \tag{23}$$

The third source of risk is the non-systematic mortality one. In the presence of well diversified insurance portfolios, insurance companies are uninterested in hedging this idiosyncratic component of mortality risk, since the law of large numbers is expected to apply. Hence, we assume that the market gives no value to it and we make the following assumption for $\varphi(t)$:

Assumption 6 *The premium for idiosyncratic mortality risk is null:* $\varphi(t) = 0$ *for every* t.

The reserves of life insurance policies can be computed as expected values under the measure $\mathbb{Q}$. Independence between financial and mortality risk – which we assumed under the physical measure $\mathbb{P}$ – is preserved under the change of measure since the filtration $\mathbf{H} = \{\mathcal{H}_t : 0 \leq t \leq T\}$ is generated by a Brownian motion, see Bielecki, Jeanblanc, and Rutkowski (2008).

Consider the case of a pure endowment contract starting at time 0 and paying one unit of account if the head x is alive at time T. The fair value of such an insurance policy at time $t \geq 0$, given the independence between financial and actuarial risk is:

$$\begin{aligned} P(t,T) &= S_x(t,T)B(t,T) \qquad (24) \\ &= E_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right)|\mathcal{G}_t\right] E_{\mathbb{Q}}\left[-\exp\left(\int_t^T r(u)du\right)|\mathcal{H}_t\right] \\ &= e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)} E_{\mathbb{Q}}\left[-\exp\left(\int_t^T r(u)du\right)|\mathcal{H}_t\right], \end{aligned}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$, due to Assumption 5', solve the system (3) with drift $a' = a + q$ instead of a.

If we assume a single premium, paid at the policy issue, this is the time-t reserve for the policy, which the insurance company will be interested in hedging.

6 HJM restriction on forward death intensities

In this section we provide a necessary and sufficient condition for affine intensities to satisfy an HJM-like restriction on drift and diffusion. We also show that the condition is met in the OU and FEL case if Assumption 5' holds. This is important, since proving that the HJM condition holds means showing that no arbitrage holds, without having assumed it a priori. We keep the head x fixed, and in the notation we drop the dependence on x.

Forward death intensities, being defined as log derivatives of survival probabilities, follow a stochastic process. This process can be derived starting from the one of the survival probabilities themselves, recalling that under Assumption 5 the process λ_x is given by (21). Ito's lemma implies that the functional S follows the process:

$$dS(t,T) = S(t,T)m(t,T)dt + S(t,T)n(t,T)dW'(t),$$

where

$$m(t,T) = \frac{1}{S}\left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \lambda_x}\left[a(t,\lambda_x) + p(t) + q(t)\lambda_x(t)\right] + \frac{1}{2}\frac{\partial^2 S}{\partial \lambda_x^2}\sigma^2(t,\lambda_x)\right], \quad (25)$$

$$n(t,T) = \frac{1}{S}\frac{\partial S}{\partial \lambda_x}\sigma(t,\lambda_x). \quad (26)$$

The forward death intensity $f(t,T)$ can then be shown to follow the dynamics:

$$df(t,T) = v(t,T)dt + w(t,T)dW'(t), \quad (27)$$

where the drift and diffusion coefficients are:

$$v(t,T) = \frac{\partial n(t,T)}{\partial T}n(t,T) - \frac{\partial m(t,T)}{\partial T}, \quad (28)$$

$$w(t,T) = -\frac{\partial n(t,T)}{\partial T}. \quad (29)$$

In general, the forward dynamics then depends on the drift and diffusion coefficients of the mortality intensity and on the properties of the solutions of the Riccati equations. One can wonder whether - starting from a given spot intensity - an HJM-like condition, which works on the forward survival intensities,

$$v(t,T) = w(t,T)\int_t^T w(t,s)ds, \quad (30)$$

is satisfied. The Appendix proves the following:

Proposition 6.1 *Let λ_x be a purely diffusive process which satisfies Assumption 5. Then, the HJM condition (30) is satisfied if and only if:*

$$\frac{\partial m(t,T)}{\partial T} = n(t,t)\frac{\partial n(t,T)}{\partial T}, \quad (31)$$

where $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$ are defined as in (25) *and* (26).[7]

It is natural to wonder whether the Ornstein-Uhlenbeck and the Feller processes satisfy the HJM condition. The following Proposition – proven in the Appendix – shows that they do.

Proposition 6.2 *If Assumption 5' holds, the Ornstein-Uhlenbeck process (4) and the Feller process (7) satisfy the HJM condition* (30).

[7] Notice that a similar condition on the drift and diffusion of spot interest rates is in Shreve (2004).

The HJM condition is a characterizing feature of some models for interest rates such as Vasicek (1977), Hull and White (1990), CIR (Cox, Ingersoll, and Ross (1985)). It is well known that the HJM condition (30), applied to the coefficients of the interest rate process, as in (22), is equivalent to the absence of arbitrage. In our case, since we have shown that - under Assumption 5' - the OU and FEL processes satisfy the HJM condition, arbitrage in the market for mortality risk is ruled out without being imposed.

7 Mortality risk hedging

In order to study the hedging problem of a portfolio of pure endowment contracts, we assume first that the interest rate is deterministic and, without loss of generality, equal to zero. This allows us to focus in this section on the hedging of systematic mortality risk only. At a later stage, we will introduce again financial risk (Section 8) and study the problem of hedging both mortality and financial risk simultaneously.
Once the risk-neutral measure $\mathbb{Q}$ has been defined, in order to introduce a hedging technique for systematic mortality risk we need to derive the dynamics of the reserve, which represents the value of the policy for the issuer. We do this, for the sake of simplicity, assuming an OU behavior for the intensity.

7.1 Dynamics and sensitivity of the reserve

7.1.1 Affine intensity

Let us integrate (27), to obtain the forward death probability:

$$f(t,T) = f(0,T) + \int_0^t \left[v(u,T)du + w(u,T)dW'(u)\right]. \tag{32}$$

Substituting it into the survival probability (15) and recalling that we write $S(u)$ for $S(0,u)$, we obtain an expression for the future survival probability $S(t,T)$ in terms of the time-zero ones:

$$P(t,T) = S(t,T) = \frac{S(T)}{S(t)} \exp\left(-\int_t^T \int_0^z \left[v(u,T)du + w(u,T)dW'(u)\right] dz\right).$$

Considering the expressions for v and w under Assumption 5, we have:

$$\begin{aligned} P(t,T) = S(t,T) &= \tfrac{S(T)}{S(t)} \exp\Big\{-\textstyle\int_t^T \int_0^z \{\alpha''(T-u) + \beta''(T-u)\lambda_x(u) + \\ &- \beta'(T-u)\left[a(u,\lambda_x) + p(u) + q(u)\lambda_x(u)\right] du + \\ &- \beta'(T-u)\sigma(u,\lambda_x)dW'(u)\}dz\}. \end{aligned} \tag{33}$$

This is the representation of the time-t reserve. It depends on the reserves that apply at time 0^+ for pure endowments of maturity t and T ($S(t)$ and $S(T)$), on the functions $\alpha''(\cdot)$, $\beta'(\cdot)$ and $\beta''(\cdot)$ – which stem from the Riccati equations of each λ_x-process – and on the coefficients of the risk-neutralized intensity, $a + p + q\lambda_x$ and σ. The presence of a stochastic integral in this representation not only prevents us from giving a general, closed-form expression, but also points out once more the stochastic nature of the insurance company's obligations, at any future point in time t.

7.1.2 OU intensity

We focus now on the OU intensity. The dynamics of the forward intensity under $\mathbb{Q}$ is

$$df(t,T) = \frac{\sigma^2}{a'} e^{a'(T-t)} \left(e^{a'(T-t)} - 1 \right) dt + \sigma e^{a'(T-t)} dW'(t). \tag{34}$$

Integrating this dynamics we derive the expression for the forward survival probabilities:

$$\begin{aligned} f(t,T) &= f(0,T) - \frac{\sigma^2}{2a'^2} \{ e^{2a'T} \left[e^{-2a't} - 1 \right] - 2e^{a'T} \left[e^{-a't} - 1 \right] \} \\ &+ \sigma \int_0^t e^{a'(T-s)} dW(s). \end{aligned}$$

Following Jarrow and Turnbull (1994), the reserve can be written simply as

$$P(t,T) = S(t,T) = \frac{S(T)}{S(t)} \exp\left[-X(t,T)I(t) - Y(t,T) \right], \tag{35}$$

where[8]

$$\begin{aligned} X(t,T) &= \frac{\exp(a'(T-t)) - 1}{a'}, \\ Y(t,T) &= -\sigma^2 [1 - e^{2a't}] X(t,T)^2 / (4a'), \\ I(t) &:= \lambda_x(t) - f(0,t). \end{aligned}$$

We have therefore provided an expression for the future survival probabilities - and reserves - in terms of deterministic quantities (X, Y) and of a stochastic term $I(t)$, defined as the difference between the actual mortality intensity at time t and its forecast today, $f(0,t)$. $I(t)$, therefore, represents the systematic mortality risk factor. Let us notice that the risk factor is

[8] Notice that $-X(t,T) = \beta$ as soon as $a = a'$.

unique for all the survival probabilities from t onwards, no matter which horizon $T-t$ they cover. Applying Ito's lemma to the reserves, considered as functions of time and the risk factor, we can write their differential in terms of the change in t and I:

$$dP = dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial I}dI + \frac{1}{2}\frac{\partial^2 S}{\partial I^2}(dI)^2 .$$

It follows that the hedging coefficients for mortality risk are

$$\frac{\partial S}{\partial I} = -S(t,T)X(t,T) \leq 0, \tag{36}$$

$$\frac{\partial^2 S}{\partial I^2} = S(t,T)X^2(t,T) \geq 0. \tag{37}$$

For given t, we can also write the percentage change in the reserve as a function of the risk factor ones. The differential expression is

$$\frac{dP(t,T)}{P(t,T)} = -X(t,T)dI + \frac{1}{2}X(t,T)^2(dI)^2.$$

We denote (36) as Delta(Δ^M) and (37) as Gamma (Γ^M), where the superscript M indicates that the coefficient refers to mortality risk. These factors allow us to hedge mortality risk up to first and second order effects. They are the analogous of the duration and convexity terms in classical financial hedging of zero-coupon-bonds, and they actually collapse into them when $\sigma(t,\lambda_x) = \sigma = 0$.[9]

It is straightforward to compute the sensitivity of any pure endowment policy portfolio with respect to mortality risk. Evidently, this must be done for each generation separately. Let the portfolio be made up of n_i policies with maturity T_i, $i = 1,..n$, issued to individuals belonging to the same generation. If each one has value $S(t,T_i)$, the overall portfolio value Π is $\sum_{i=1}^n n_i S(t,T_i)$. Its change is

$$d\Pi = \sum_{i=1}^n n_i \frac{\partial S}{\partial t}dt + \sum_{i=1}^n n_i \frac{\partial S}{\partial I}dI + \frac{1}{2}\sum_{i=1}^n n_i^2 \frac{\partial^2 S}{\partial I^2}(dI)^2 .$$

7.2 Hedging

In order to hedge the reserve we assume that the insurer can use either other pure endowments – with different maturities – or zero-coupon longevity

[9] In this case, we have $Y(t,T) = 0$. Hence, Delta and Gamma coefficients are functions of a' only: $\Delta^{\sigma=0} = \frac{S(T)}{S(t)}X(t,T)$ and $\Gamma^{\sigma=0} = \frac{S(T)}{S(t)}X^2(t,T)$.

bonds on the same generation. These, by definition, are products whose payout is linked to a survival index, which in our case should be a generation index. They pay one unit of account for each survivor in the predefined group or generation.[10] Indeed, since we do not price idiosyncratic mortality risk, the price/value of a zero-coupon longevity bond is equal to the pure endowment one. The difference, from the standpoint of an insurance company, is that it can sell endowments – or reduce its exposure through reinsurance – and buy longevity bonds, while, at least in principle, it cannot do the converse.[11] We could use a number of other instruments to cover the initial pure endowment, starting from life assurances or death bonds, which pay the benefit in case of death of the insured individual. We restrict the attention to pure endowments and longevity bonds for the sake of simplicity. Let us recall also that – together with the life assurance and death bonds – they represent the Arrow-Debreu securities of the insurance market. Once hedging is provided for them, it can be extended to every more complicated instrument.

Suppose for instance that, in order to hedge n endowments with maturity T, it is possible to choose the number of endowments/longevity bonds with maturity T_1 and T_2: call them n_1 and n_2. Since the insurance company is short on endowment, n must be negative. Any negative choice for n_i has to be interpreted as the sale of n_i endowments of maturity T_i. Any positive solution for n_i has to be interpreted as the purchase of n_i longevity bonds of maturity T_i. The value of a portfolio made up of the three assets is

$$\Pi(t) = nS(t,T) + n_1 S(t,T_1) + n_2 S(t,T_2).$$

Its Delta and Gamma coefficients are respectively

$$\begin{aligned}
\Delta_{\Pi}^{M}(t) = \quad & n\frac{\partial S}{\partial I}(t,T) + n_1\frac{\partial S}{\partial I}(t,T_1) + n_2\frac{\partial S}{\partial I}(t,T_2), \\
\Gamma_{\Pi}^{M}(t) = \quad & n\frac{\partial^2 S}{\partial I^2}(t,T) + n_1\frac{\partial^2 S}{\partial I^2}(t,T_1) + n_2\frac{\partial^2 S}{\partial I^2}(t,T_2).
\end{aligned}$$

We can set these Delta and Gamma coefficients to zero (or some other precise value) by choosing the appropriate quantities n_1 and n_2. One can easily solve the system of two equations in two unknowns and obtain hedged portfolios:

$$\begin{cases} \Delta_{\Pi}^{M} = 0, \\ \Gamma_{\Pi}^{M} = 0. \end{cases}$$

[10] If there is no longevity bond for a specific generation, basis risk arises: see for instance Cairns, Blake, Dowd, and MacMinn (2006).

[11] Reinsurance companies have less constraints in this respect. For instance, they can swap pure endowments or issue longevity bonds: see for instance Cowley and Cummins (2005).

The cost of setting up the covered portfolio can be paid using the pure endowment premium received by the policyholder. Alternatively, the problem can be extended so as to make the hedged portfolio self-financing.[12] Self-financing can be guaranteed by endogenizing n and solving simultaneously the equations $\Pi = 0, \Delta_\Pi^M = 0$ and $\Gamma_\Pi^M = 0$ for n, n_1, n_2. As an alternative, if n is fixed, a third pure endowment/bond with maturity T_3 can be issued or purchased, so that the portfolio made up of $S(t,T), S(t,T_1), S(t,T_2)$ and $S(t,T_3)$ is self-financing and Delta-Gamma hedged. Our application in Section 9 will cover both the non self-financing and self-financing possibilities.[13]

8 Mortality and financial risk hedging

Let us consider now the case in which both mortality and financial risk exist. Again we develop the technique assuming a OU intensity. We also select a constant-parameter Hull and White model for the interest rate under the risk-neutral measure:

$$\Sigma(t,T) = \Sigma \exp(-g(T-t)), \tag{38}$$

with $\Sigma, g \in \mathbb{R}^+$. Substituting in (23) we have the following expression for the spot interest rate

$$r(t) = F(0,t) + \frac{1}{2}\frac{\Sigma^2}{g^2}(1-e^{-gt})^2 + \Sigma \int_0^t e^{-g(t-s)} dW_F'(s).$$

This allows us to derive an expression for the zcb $B(t,T)$ analogous to the one we had for survival probabilities:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right],$$

where

$$\bar{X}(t,T) := \frac{1-\exp(-g(T-t))}{g},$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g}\left[1-\exp(-2gt)\right]\bar{X}^2(t,T),$$

[12]The self-financing condition makes our strategy an asset-liability management one (or a partial ALM one, if the actual premium is greater than the fair one). In the absence of self-financing our strategy can be classified among the liability management.

[13]Notice that we are not considering re-adjustments of the hedging positions after time t. In that case, a money market account - or equivalent - must be added.

and K is the financial risk factor, measured by the difference between the short and forward rate:

$$K(t) := r(t) - F(0,t).$$

The pure endowment reserve at time t, according to (24), is $P(t,T) = S(t,T)B(t,T)$.

Given the independence stated in Assumption 3, we can apply Ito's lemma and obtain the dynamics of the reserve $P(t,T)$ as

$$\begin{aligned} dP &= BdS + PdB = B\left[\frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial I}dI + \frac{1}{2}\frac{\partial^2 S}{\partial I^2}(dI)^2\right] + \\ &+ S\left[\frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial K}dK + \frac{1}{2}\frac{\partial^2 B}{\partial K^2}(dK)^2\right], \end{aligned}$$

where both the changes in the mortality and interest rate risk factor - as well as time - appear. As in the mortality case, we can compute the derivatives of the zcb price with respect to the financial risk factor:

$$\begin{aligned} \frac{\partial B(t,T)}{\partial K} &= -B(t,T)\bar{X}(t,T) \leq 0, \\ \frac{\partial^2 B(t,T)}{\partial K^2} &= B(t,T)\bar{X}^2(t,T) \geq 0. \end{aligned}$$

It follows that, for given t, the percentage change in the reserve can be written in terms of the changes in the risk factors up to the second order:

$$\frac{dP(t,T)}{P(t,T)} = -X(t,T)dI + \frac{1}{2}X(t,T)^2(dI)^2 - \bar{X}^2(t,T)dK + \frac{1}{2}\bar{X}^2(t,T)(dK)^2.$$

Hedging of the reserve is again possible by a proper selection of pure endowment/longevity bond contracts with different maturities and/or zero-coupon-bonds with different maturities. Here we consider the case in which the hedge against mortality and financial risk is obtained either issuing (purchasing) pure endowments (longevity bonds) or using also bonds.
Consider first using mortality linked contracts only. We can see that Delta-Gamma hedging of both the mortality and financial risk of n endowments with maturity T can be obtained via a mix of n_1, n_2, n_3, n_4 endowments/longevity bonds with maturities ranging from T_1 to T_4, by solving simultaneously the following hedging equations:

$$\begin{cases} \Delta_{\Pi}^{M} = 0, \\ \Gamma_{\Pi}^{M} = 0, \\ \Delta_{\Pi}^{F} = 0, \\ \Gamma_{\Pi}^{F} = 0. \end{cases} \tag{39}$$

Indeed, this means solving the system of equations

$$\begin{cases} \Delta_{\Pi}^{M} = nBSX + n_1B_1S_1X_1 + n_2B_2S_2X_2 + n_3B_3S_3X_3 + n_4B_4S_4X_4 = 0, \\ \Gamma_{\Pi}^{M} = nBSX^2 + n_1B_1S_1X_1^2 + n_2B_2S_2X_2^2 + n_3B_3S_3X_3^2 + n_4B_4S_4X_4^2 = 0, \\ \Delta_{\Pi}^{F} = nBS\bar{X} + n_1B_1S_1\bar{X}_1 + n_2B_2S_2\bar{X}_2 + n_3B_3S_3\bar{X}_3 + n_4B_4S_4\bar{X}_4 = 0, \\ \Gamma_{\Pi}^{F} = nBS\bar{X}^2 + n_1B_1S_1\bar{X}_1{}^2 + n_2B_2S_2\bar{X}_2{}^2 + n_3B_3S_3\bar{X}_3{}^2 + n_4B_4S_4\bar{X}_4{}^2 = 0. \end{cases} \tag{40}$$

where B denotes $B(t,T)$ and $B_i, X_i, \bar{X}_i$ denote $B(t,T_i), X(t,T_i), \bar{X}(t,T_i)$ for $i = 1, ..., 4$.

Consider now using both mortality-linked contracts and zero-coupon-bonds. In this case, the hedging equations (39) become:

$$\begin{cases} \Delta_{\Pi}^{M} = nBSX + n_1B_1S_1X_1 + n_2B_2S_2X_2 = 0, \\ \Gamma_{\Pi}^{M} = nBSX^2 + n_1B_1S_1X_1^2 + n_2B_2S_2X_2^2 = 0, \\ \Delta_{\Pi}^{F} = nBS\bar{X} + n_1B_1S_1\bar{X}_1 + n_2B_2S_2\bar{X}_2 + n_3B_3\bar{X}_3 + n_4B_4\bar{X}_4 = 0, \\ \Gamma_{\Pi}^{F} = nBS\bar{X}^2 + n_1B_1S_1\bar{X}_1{}^2 + n_2B_2S_2\bar{X}_2{}^2 + n_3B_3\bar{X}_3{}^2 + n_4B_4\bar{X}_4{}^2 = 0. \end{cases} \tag{41}$$

These equations can be solved either all together or sequentially (the first two with respect to n_1, n_2, the others with respect to n_3 and n_4), covering mortality risk at the first step and financial risk at the second step.
Both problems outlined in (40) and (41) can be extended to self-financing considerations. In both cases the value of the hedged portfolio is given by

$$\Pi(t) = nBS + n_1B_1S_1 + n_2B_2S_2 + n_3B_3S_3 + n_4B_4S_4.$$

It is self-financing if $\Pi(0) = 0$ or if an additional contract is inserted, so that the enlarged portfolio value is null. In our applications we will explore both possibilities.

9 Application to a UK sample

In this section we present an application of our hedging model to a sample of UK insured people. We exploit our minimal change of measure, which preserves the biological and historically reasonable behaviour of the intensity. We also assume that $a' = a$, i.e. that the risk premium on mortality risk is null. This assumption could be easily removed by calibrating the model parameters to actual insurance products, most likely derivatives. We take the view that their market is not liquid enough to permit such calibration (see also Biffis (2005), Cairns, Blake, Dowd, and MacMinn (2006), to mention a few). We therefore calibrate the mortality parameters to historical data, the

IML tables, that are projected tables for English annuitants. We assume also - at first - that the interest rate is constant and, without loss of generality, null - as in Section 7. We derive pure-endowments prices or reserves, we analyze the effect of mortality shocks and the values of the Delta and Gamma coefficients. Afterwards, we introduce also a stochastic interest rate.

9.1 Mortality risk hedging

We keep the head fixed, considering contracts written on the lives of male individuals who were 65 years old on 31/12/2010. We select the OU model and calibrate it to the generation of individuals that were born in 1945. The value of the parameters, considering $t = 0$, are: $a_{65} = 10.94\%$, $\sigma_{65} = 0.07\%$, $\lambda_{65}(0) = -\ln p_{65} = 0.885\%$.[14]

First of all, we analyze the effect of a shock of one standard deviation on the Wiener driving the intensity process. Figure 1 shows graphically the impact of an upward and downward shock of one standard deviation on the forward intensity at $t = 1$ for different time horizons T. The forward mortality structure is derived from (17) using (4). The Figure clearly highlights that the effect becomes more and more evident – the trumpet opens up – as soon as the time horizon of the forward mortality becomes longer. Please notice that the behaviour is – as it should, from the economic point of view – opposite to the one of the corresponding Hull and White interest rates. Indeed, in the rates case the trumpet is reversed, since short-term forward rates are affected more than longer ones. This remarkable difference is certainly due to the fact that short-term interest rates dynamics are mean reverting, while mortality intensity dynamics are not.

Table 1 reports the pure endowment prices or reserves. It compares the Delta and Gamma coefficients associated with contracts of different maturity in the stochastic case with the deterministic ones. It appears clearly from Table 1 that for long maturities the model gives hedging coefficients for mortality-linked contracts which are remarkably different from the deterministic ones. For instance, the Δ^M and Γ^M hedging coefficients for a contract with maturity 30 years are respectively 6% smaller and larger than their deterministic counterparts. Contracts with long maturities are clearly very interesting from an insurer's point of view and hence their proper hedging is important.

As an example, imagine that an insurer has issued a pure endowment contract with maturity 15 years. Suppose that he wants to Delta-Gamma hedge this position using as cover instruments mortality-linked contracts with

[14] We refer the reader to Luciano and Vigna (2008) for a full description of the data set and the calibration procedure. The calibrated parameters satisfy the sufficient condition for biological reasonableness for the OU model (11).

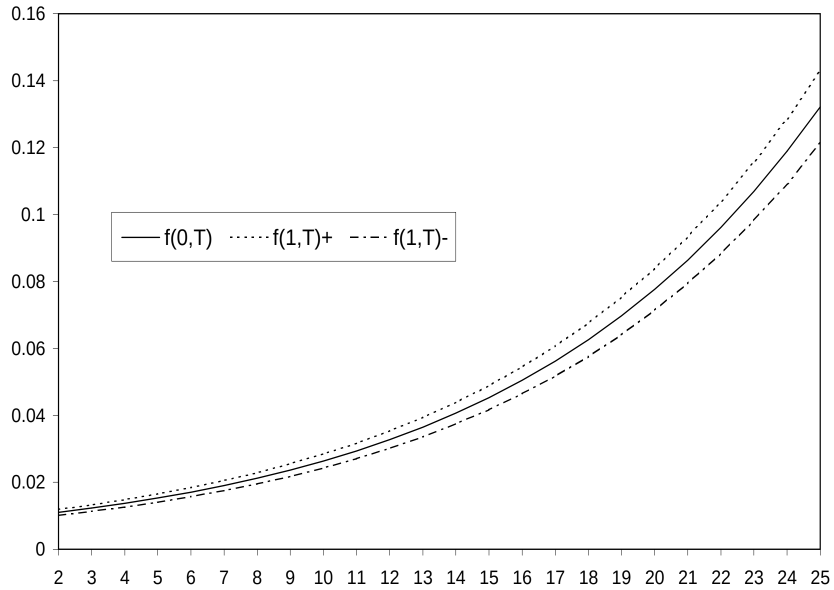

Figure 1: This figure shows the effect on the forward death intensity $f(1,T)$ of a shock equal to one standard deviation as a function of T. The central – solid – line represents the initial forward mortality intensity curve $f(0,T)$

Table 1: Stochastic vs. deterministic hedging coefficients

	Stochastic hedge			Deterministic hedge	
Maturity	$S(t,T)$	Δ^M	Γ^M	$\Delta^{\sigma=0}$	$\Gamma^{\sigma=0}$
1	0.99069	-1.04691	1.10633	-1.04691	1.10633
2	0.98041	-2.19187	4.90030	-2.19187	4.90030
5	0.94282	-6.27449	41.75698	-6.27439	41.75633
7	0.91116	-9.58396	100.80807	-9.58347	100.80284
10	0.85174	-15.46366	280.74803	-15.46053	280.69129
12	0.80306	-19.94108	495.16678	-19.93255	494.95501
15	0.71505	-27.19228	1034.08392	-27.16108	1032.89754
18	0.60899	-34.31821	1933.91002	-34.22325	1928.55907
20	0.52957	-38.32543	2773.64051	-38.14219	2760.37929
25	0.31713	-41.77104	5501.91988	-41.05700	5407.86868
27	0.23633	-39.27090	6525.53620	-38.18393	6344.91753
30	0.13319	-31.20142	7309.51024	-29.46466	6902.64225
35	0.03144	-12.93603	5322.98669	-10.78469	4437.74408

maturity 10 and 20 years. At a cost of 0.37, the insurer can instantaneously Delta-Gamma hedge its portfolio, by purchasing, respectively, 1.11 and 0.26 zero-coupon longevity bonds on these maturities. Having the possibility of using contracts with a maturity of 30 years on the same population, a self-financing Delta-Gamma hedging strategy can be implemented by purchasing 0.48 and 0.60 longevity bonds with maturity respectively 10 and 20 years, and issuing 0.10 pure endowments with maturity 30 years.

9.2 Mortality and financial risk hedging

As shown in Section 8, the same procedure can be followed to hedge simultaneously the risks deriving from both stochastic mortality intensities and interest rates. Notice that, if we consider that the interest rate is stochastic (or at least different from zero), prices of pure endowment contracts no longer coincide with survival probabilities. Nonetheless, their Δ^M and Γ^M, the factors associated to mortality risk, remain unchanged when we introduce financial risk (see Section 8). Once one has estimated the coefficients underlying the interest rate process, we can easily derive the values of Δ^F and Γ^F, the factors associated to the financial risk, and the prices $P(t,T)$ of pure endowment/longevity bond contracts.

We calibrate our constant-parameter Hull and White model for forward inter-

Table 2: Hedging coefficients for stochastic financial risk

Maturity	$P(t,T)$	Δ^F	Γ^F
1	0.98395	-0.9798	0.9666
2	0.96214	-1.9103	3.7185
5	0.86696	-4.2988	20.0963
7	0.78430	-5.4865	34.9707
10	0.64372	-6.6170	57.9341
12	0.54597	-6.9606	71.2657
15	0.40404	-6.9596	85.7216
20	0.20649	-6.0149	92.7836
25	0.07972	-4.5599	82.7129
27	0.04902	-3.9667	75.8645
30	0.02037	-3.1366	64.3246
35	0.00278	-1.9995	45.1377

est rates to the observed zero-coupon UK government bonds at 31/12/2010.[15] Table 2 shows prices/reserves and financial risk hedging factors of pure endowment contracts subject to both financial and mortality risks. Notice that the absolute values of the factors related to the financial market are smaller than the ones related to the mortality risk. In fact, $|\Delta^M| > |\Delta^F|$ and $\Gamma^M > \Gamma^F$. However, under realistic hypothesis on the shocks ΔI and ΔK, the effects of mortality and financial risk have the same order of magnitude, i.e.

$$\Delta^M \Delta I \simeq \Delta^F \Delta K \qquad \text{and} \qquad \Delta^M \Delta I + \frac{1}{2}\Gamma^M \Delta I^2 \simeq \Delta^F \Delta K + \frac{1}{2}\Gamma^F \Delta K^2.$$

Take for instance $T = 25$, $\Delta I = -5$ bp, $\Delta K = -50$ bp. Then,

$$\Delta^M \Delta I = 0.0209, \qquad \Delta^F \Delta K = 0.0228.$$

Adding the Γ components we obtain 0.0216 and 0.0238, respectively.

These factors Δ^F and Γ^F, together with their mortality risk counterparts, Δ^M and Γ^M, allow us to hedge pure endowment contracts from both financial and mortality risk by setting up a portfolio - even self-financing - which instantaneously presents null values of all the Delta and Gamma factors.
As an example, consider again the hedging of a pure endowment with maturity 15 years. In order to Delta-Gamma hedge against both risks, we need to use four instruments (five if we want to self-finance the strategy). We

[15] The parameter g is 2.72%, while the diffusion parameter Σ is calibrated to 0.65%.

can either use four pure endowments/longevity bonds written on the lives of the 65 year-old individuals or two mortality-linked contracts and two zero-coupon-bonds. In the first case, imagine to use contracts with maturity 10, 20, 25 and 30 years. The hedging strategy consists then in purchasing 0.35 longevity bonds with maturity 10 years, 1.27 with maturity 20 years and 0.30 with maturity 30 years, while issuing 0.87 pure endowment policies with maturity 25 years. In the second case, imagine the hedging instruments are mortality contracts with maturities 10 and 20 years and two zero-coupon-bonds with maturities 5 and 20 years. The strategy consists in purchasing 1.11 longevity bonds with maturity 10 years and 0.26 with maturity 20 years and in taking a short position on 0.60 zero-coupon-bonds with maturity 5 years and a long one on 0.10 zero-coupon-bonds with maturity 20 years. A self-financing hedge can be easily obtained by adding an instrument to the portfolio. For example, such a self-financing hedge can be obtained by purchasing 0.41 longevity bonds with maturity 10 years, 0.98 with maturity 20 years and 0.22 with maturity 35 years and issuing 0.38 pure endowments with maturity 25 years and 0.13 with maturity 30 years.

10 Summary and conclusions

This paper develops a Delta-Gamma hedging framework for mortality and interest rate risk and applies it to a UK calibrated example.
We have shown that, consistently with the interest rate market, when the spot intensity of stochastic mortality follows an OU or FEL process, an HJM condition on its drift holds for every constant risk premium, without assuming no arbitrage. Hence, it is possible to hedge systematic mortality risk in a way which is identical to the Delta-Gamma hedging approach in the HJM framework for interest rates. Delta and Gamma coefficients are very easy to compute, at least in the OU case. The hedging quantities are easily obtained as solutions to linear systems of equations. This means that the hedging model can be very attractive for practical applications.
Adding financial risk is a straightforward extension in terms of insurance pricing, if the bond market is assumed to be without arbitrages (and complete, so that the financial change of measure is unique). Delta-Gamma hedging is straightforward too if - as in the examples - the risk-neutral dynamics of the forward interest rate follows the familiar constant-parameter Hull and White.
Our application shows that the unhedged effect of a sudden change on the mortality rate is remarkable and the stochastic and deterministic Delta and Gamma coefficients are quite different, especially for long time horizons. The

calibrated Delta and Gamma coefficients are bigger for mortality than for financial risk. However, since typical mortality shocks are much smaller than interest rate shocks, we find that the effects of mortality and financial risk on the policy reserve have the same order of magnitude. This result seems relevant, considering that traditionally mortality risk has always been considered of lower importance than interest rate risk.

Future research is dictated mainly by the following considerations. Models for stochastic mortality in continuous-time, although very useful and effective for the treatment of one generation at a time, are not capable to catch the entire mortality surface. Indeed, correlations among intensities of different generations are not described. The next step is then to extend the model to encompass different generations. This essential improvement should allow the description of the whole mortality surface, and, not least, would allow the life office to adopt hedging strategies in the whole portfolio, rather than for each single generation. Namely, the creation of a so-called age-period-cohort continuous-time model, the discrete-time versions of which have already been developed, is in the agenda for future research.

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Appendix

Proof of Proposition 6.1

Using (29), we get the r.h.s. of the HJM condition (30) in terms of the coefficients of the survival probability:

$$w(t,T)\int_t^T w(t,s)ds = \frac{\partial n(t,T)}{\partial T}(n(t,T)-n(t,t)).$$

The l.h.s. is given in the same terms by (28). Combining the two we get (31). □

Proof of Proposition 6.2

For any intensity process of the affine class, the drift and diffusion of the survival probabilities are

$$\begin{aligned} m(t,T) &= -\alpha'(T-t)-\beta'(T-t)\lambda_x(t)+ \\ &+ \left[a(t,\lambda_x)+p(t)+q(t)\lambda_x(t)\right]\beta(T-t)+\frac{1}{2}\sigma^2(t,\lambda_x)\beta^2(T-t), \quad (42) \\ n(t,T) &= \sigma(t,\lambda_x)\beta(T-t), \quad (43) \end{aligned}$$

while the forward intensity process coefficients are:

$$\begin{aligned} v(t,T) &= \alpha''(T-t)+\beta''(T-t)\lambda_x(t)-\left[a(t,\lambda_x)+p(t)+q(t)\lambda_x\right]\beta'(T-t), \quad (44) \\ w(t,T) &= -\sigma(t,\lambda_x)\beta'(T-t). \quad (45) \end{aligned}$$

Due to Assumption 5', the functions $\alpha(\cdot)$ and $\beta(\cdot)$ of the OU process satisfy the system of ODEs':

$$\begin{cases} \beta'(t) = -1 + a'\beta(t), \\ \alpha'(t) = \frac{1}{2}\sigma^2\beta^2(t), \end{cases} \tag{46}$$

with the boundary conditions $\alpha(0) = 0$ and $\beta(0)$=0. It follows from (42),(43) and (46) that Proposition 6.1 is satisfied.
An alternative proof is the following: from (44),(45) and (46) that

$$\begin{aligned} v(t,T) &= \alpha''(T-t) + \beta''(T-t)\lambda_x(t) - \beta'(T-t)a'\lambda_x(t) \\ &= \sigma^2\beta(T-t)\beta'(T-t), \\ w(t,T) &= -\sigma\beta'(T-t), \end{aligned}$$

and property (30) is satisfied.
Consider now the Feller process. Under Assumption 5' its $\alpha(\cdot)$ and $\beta(\cdot)$ solve the Riccati ODE:

$$\begin{cases} \alpha'(t) = 0, \\ \beta'(t) = -1 + a'\beta(t) + \frac{1}{2}\sigma^2\beta^2(t), \end{cases}$$

with the boundary conditions $\alpha(0) = 0$ and $\beta(0)$=0. Again, we can easily show that the condition of Proposition 6.1 is true or that condition (30) is satisfied, since

$$v(t,T) = \beta''(T-t)\lambda_x(t) - a'\lambda_x(t)\beta'(T-t) = \sigma^2\beta(T-t)\beta'(T-t)\lambda_x(t),$$

$$w(t,T) = -\sigma(t,\lambda_x)\beta'(T-t) = -\sigma\sqrt{\lambda_x(t)}\beta'(T-t).$$

$\square$