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On the sub-optimality cost of immediate annuitization in DC pension funds*

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Abstract

We consider the position of a member of a defined contribution (DC) pension scheme having the possibility of taking programmed withdrawals at retirement. According to this option, she can defer annuitization of her fund to a propitious future time, that can be found to be optimal according to some criteria. This option, that adds remarkable flexibility in the choice of pension benefits, is not available in many countries, where immediate annuitization is compulsory at retirement. In this paper, we address and try to answer the questions: "Is immediate annuitization optimal? If it is not, what is the cost to be paid by the retiree obliged to annuitize at retirement?". In order to do this, we consider the model by [12] and extend it in two different ways.
In the first extension, we prove a theorem that provides necessary and sufficient conditions for immediate annuitization being always optimal. The not surprising result is that compulsory immediate annuitization turns out to be sub-optimal. We then quantify the extent of sub-optimality, by defining the *sub-optimality cost* as the loss of expected present value of consumption from retirement to death and measuring it in many typical situations. We find that it varies in relative terms between 6% and 40%, depending on the risk aversion of the member. In the second extension, we make extensive numerical investigations of the model and seek the optimal annuitization time. We find that the optimal annuitization time depends on personal factors such as the retiree's risk aversion and her subjective perception of remaining lifetime. It also depends on the financial market, via the Sharpe ratio of the risky asset. Optimal annuitization should occur a few years after retirement with high risk aversion, low Sharpe ratio and/or short remaining lifetime, and many years after retirement with low risk aversion, high Sharpe ratio and/or long remaining lifetime.
This paper supports the availability of programmed withdrawals as an option to retirees of DC pension schemes, by giving insight into the extent of loss in wealth suffered by a retiree who cannot choose programmed withdrawals, but is obliged to annuitize immediately on retirement.

J.E.L. classification: D91, G11, G23, J26.

Keywords: Defined contribution pension scheme, decumulation phase, optimal annuitization time, cost of sub-optimality.

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1 Introduction and motivation

The crisis of international Pay As You Go (PAYG) public pension system - where contributions of active workers are used to finance current pensions in payment - caused by the ageing population problem, is forcing governments of most countries to cut drastically pension benefits of future generations and to encourage the development of the so-called second pillar of a pension system, i.e. fully funded pension schemes. In a few decades future retirees will have to rely heavily on the provision of the second pillar, if they want to maintain the pre-retirement standard of life.

Pension funds can be either defined benefit (DB) or defined contribution (DC).
In DB pension schemes, benefits are based on a simple formula given in the rules and do not depend on the investment experience of the scheme. Benefits are often proportional to the length of active membership in the scheme and may be defined as an income (i.e. a pension) and/or a cash lump sum. They are usually linked in some direct or indirect way to the final salary or to the past salaries received during service. Since benefits at retirement are guaranteed by definition and do not depend on the investment conditions during the accumulation of the contributions, it follows that the contribution rate is adjusted regularly by the actuary, depending both on the experienced returns and on the final pension that has to be paid at retirement. Usually the member pays the same fixed percentage, while the employer pays the remaining aleatory part of the adjusted contribution rate. In other words, the employer bears the financial risk in DB schemes.
On the contrary, in DC schemes the contributions paid into the fund are based on a simple formula given in the scheme rules (typically as a percentage of the salary) and the level of the pension achieved at retirement depends on the performance of investment returns achieved during the active membership. That is, the member knows exactly in advance how much she will pay in the fund, but does not know the amount of pension she will receive at retirement. Thus, the member bears the financial risk in DC schemes.

Not surprisingly, the pension reforms actually occurring in most of industrialized countries give a preference towards the development of DC plans for the support of the second pillar. It is also clearly observable a continuous trend of replacement of DB schemes by DC plans occurring nowadays in all countries where the second pillar is already well developed. Thus, DC pension schemes will play a crucial role in the social pension systems and all research products providing optimal solutions to problems of DC plans will be more and more important.

In DC pension schemes the financial risk can be split into two parts: investment risk and annuity risk. The investment risk is the risk that lower than expected investment returns from the financial market in the accumulation phase lead to a lower than expected accumulated wealth at retirement, leading to lower than expected pension income. The annuity risk is the risk that lower than expected yields rates at retirement produce a higher than expected price of the annuity, leading to lower than expected pension income. The annuity risk can be reduced by the option available in many countries of deferring the annuitization of final wealth. Meanwhile, the fund is invested in financial assets and the pensioner withdraws periodic amounts until annuitization occurs (if ever). This option is named "income drawdown option" in UK, "phased withdrawals" or "programmed withdrawals" in US. By taking it, the risk is moved from the retirement date to the whole decumulation phase, that is the period post-retirement. To sum up, the investment risk is born during the accumulation phase, while the annuity risk is focused at retirement and in the decumulation phase, whenever phased withdrawals are available.

Basically, most of the actuarial literature on DC pension funds can be divided in two streams: the first one on the investment risk born in the accumulation phase, the second one on the annuity risk born at retirement and in the decumulation phase. This paper belongs to the second stream. As for the first stream, since in the accumulation phase the member has some freedom in choosing the portfolio composition, the investment risk can be approached by finding the best investment strategy according to some criterion. A rich stream of literature focuses on this aspect. Among many others, [3, 4, 6, 8, 9, 13].
As for the second stream, the annuity risk can be tackled by finding optimal investment-consumption-annuitization rules. Namely, a pensioner who takes the income drawdown option has three degrees of freedom:

1. she can decide what investment strategy to adopt in investing the fund at her disposal;
2. she can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any);
3. she can decide when to annuitize (if ever).

The first two choices represent a classical intertemporal decision-making problem, which can be dealt with using optimal control techniques in the typical Merton framework ([16]), whereas the third choice can be tackled by defining an optimal stopping time problem.

Examples of works dealing with these three choices via the formulation of a combined stochastic control and optimal stopping problem are [12, 18, 20, 24].

Differently from the other mentioned papers, in [12] the authors solve the problem of optimal annuitization in the presence of quadratic, target-depending loss functions. By so doing, they extend [10] and [11], where the optimal couple investment-consumption is found in the presence of quadratic loss function. The main novelty in the literature is that the optimal annuitization time depends on the level of the wealth, that is linked to the stochastic evolution of the market returns.

In this paper, we extend [12] by addressing some of the issues left open. The main motivation of the paper comes from the evidence that a number of countries do not provide programmed withdrawals for members of DC schemes. Countries where programmed withdrawals are an option to retirees of DC schemes include Argentina, Australia, Brazil, Canada, Chile, Denmark, El Salvador, Japan, Peru, UK, US. Countries where this option is not available include Austria, Bulgaria, Colombia, Germany, Hong Kong, India, Ireland, Italy, Luxembourg, Netherlands, Poland, Portugal, South Africa, Sweden, Switzerland.(1) Certainly, the introduction of such an option has a relevant cost for the management of the pension fund. Thus, the trade-off between increased management costs and increased flexibility for the pensioner has to be carefully analyzed. The first aim of this paper is to give mathematical tools for the exact quantification of this trade-off. As a consequence, the second ambitious goal would be to provide sufficient motivation for the inclusion of programmed withdrawals among the benefits offered to retirees.

From the theoretical point of view, we prove a theorem that provides necessary and sufficient conditions for immediate annuitization being always optimal. A first application of this theorem shows the not surprising result that compulsory immediate annuitization is sub-optimal within this model. The result is not surprising in that a bigger set of choices can only improve the potential reward of the decision-maker. However, what is really crucial for the individual behaviour and

(1) An exhaustive recent survey regarding the list of several forms of payment offered by pension schemes can be found in [2].

for the pension fund design is if it *does* improve her effective reward. Here we show that it does. Not only, we also quantify the extent of sub-optimality, by defining the *sub-optimality cost* as the loss of expected present value of consumption from retirement to death, and measuring it in many typical situations. We find that the sub-optimality cost varies in relative terms between 6% and 40%, depending on the risk aversion.

From the applicative point of view, we carry out a broad sensitivity analysis of the model. The main aim we have in mind is to provide a wide range of results regarding optimal annuitization time and consequent size of annuity in most common situations.

The remainder of the paper is organized as follows. Section 2 presents a review of the actuarial literature on the decumulation phase of DC pension funds. Section 3 presents briefly the model exploited. In Section 4 we prove a theorem that states the necessary and sufficient condition for optimality of immediate annuitization and we interpret this condition. The sub-optimality of compulsory immediate annuitization comes as a consequence. In Section 5 the practical implications of the theorem are illustrated: in Section 5.1 we present four typical scenarios which will be used in the simulations and we check the condition of sub-optimality; in Section 5.2 the cost of sub-optimality is defined and measured in those scenarios. Section 6 completes the analysis, by reporting the results of a broad sensitivity analysis of the model, i.e. optimal annuitization time and other related issues in the selected scenarios. Section 7 concludes.

2 Review of actuarial literature on decumulation phase

A number of authors have dealt with the problem of managing the financial resources of a pensioner who takes phased withdrawals after retirement. Notice that, whereas the income drawdown option adds flexibility in the choices of the pensioner, and gives her the hope of being able to buy later on a better annuity than the one purchasable at retirement, the main drawback is that with self-annuitization the member bears the longevity risk, i.e. the risk of outliving her own assets. Therefore, another relevant issue that arises when the income drawdown option is chosen is the ruin probability, i.e. the probability that the pensioner runs out of money when she is still alive. All the papers in the literature approach some or all of the four important issues related to the income drawdown option, namely, investment and consumption strategies, optimal annuitization time, ruin probability.
Papers that address mainly the ruin probability are, e.g., [1] and [19]. Papers that explore different investment strategies and/or different consumption paths, possibly analyzing also the ruin issue are, e.g., [7, 10, 11, 15, 25]. Papers that add to their analysis also investigations on the optimal annuitization time are, e.g., [5, 12, 14, 17, 18, 20, 24].

In this paper we focus on the optimal time of annuitization. It is then important to review briefly the other contributions to this relevant topic. In [14] and [17] the basic idea is that since the annuity price is calculated with the riskfree rate, in the first years after retirement the equity risk premium pays more than the mortality credits (due to annuitants who die earlier than average); therefore in the first years after retirement the individual should invest and consume, and should annuitize when the mortality credits become so large that it becomes worthwhile annuitize ("do-it-yourself-and-then-switch" strategy). According to their simulations, in UK the maximum annuitization age of 75 (which is even 10 years greater than NRA[(2)]) is at least 5 years too low, and a Canadian female aged 65 has 90% probability of beating the interest rate return until the age of 80. In [5] the authors find that the optimal annuitization age is sensitive to the degree of risk aversion and varies

[(2)]NRA stands for "normal retirement age".

from NRA (for very high risk aversion) to 79–80 (for very low risk aversion). [10] and [11] find that income drawdown option has to be preferred to immediate annuitization when risk aversion is not too high and the risky asset is sufficiently good compared to the riskfree one. In [20] the authors solve an optimization problem and find optimal investment-consumption strategies and optimal annuitization time using power utility function. They find that optimal annuitization age, which depends on the relative risk aversion and on the wealth status, is typically higher than 70. In [12] the authors solve a similar problem with quadratic loss functions. They find that optimal annuitization is mainly driven by risk profile of the retiree, level of the fund and market conditions, and in some typical situations should occur 6–7 years after retirement, but may occur also 10–15 years after it.

It is our opinion that the problem of finding optimal investment and consumption strategies and optimal annuitization time should be rigorously formulated as a combined stochastic control and optimal stopping problem. Up to our knowledge, in the literature this has been done only by [12, 18, 20, 24]. While in [18] the authors minimize the probability of financial ruin, in [24] the author finds optimal choices in a very general expected utility setting, distinguishing between utility pre-annuitization and utility post-annuitization and selecting as a special case the power utility function, and in [20] the authors maximize expected utility of lifetime consumption and bequest, with age-dependent force of mortality and power utility function. Differently from these papers, in [12] the authors solve the problem of optimal investment and consumption strategies as well as optimal annuitization time by selecting a quadratic loss function. We briefly present their model in the next Section.

3 The model

In [12], a retiree has a lump sum of size $x_0 > 0$ which can be invested either in a riskless asset paying interest at fixed rate $r > 0$ or in a risky asset, whose price evolves randomly following a geometric Brownian motion with diffusion $\lambda > r$ and diffusion $\sigma > 0$. We assume that the remaining lifetime of the pensioner is exponentially distributed with constant force of mortality. The retiree can choose the proportion of the fund to be invested in the risky asset and the withdrawal rate from the fund until the time of annuitization. She is also able to select the time of eventual annuitization. The size of the annuity purchasable with sum x is kx, where $k > r$. If the amount of money in the fund is ever exhausted, no further investment or withdrawal is permitted, that means that occurrence of ruin is prevented by the model's design.

We use this notation:

- $y(t)$ is the proportion of the fund invested in the risky asset at time t;
- $b(t)\,dt$ is the income withdrawn from the fund between time t and time $t + dt$.
- T is the time of annuitization;
- T_0 is the time when the fund goes below 0;
- T_D is the retiree's time of death, as measured from the time when the lump sum is received;
- $x(t)$ is the size of the fund at time t (where $t < \min(T, T_D, T_0)$).

This model investigates the problem using $y(\cdot)$ and $b(\cdot)$ as control variables, and T as stopping time. The proportion invested in the fund, the income withdrawn, and the annuitization time are chosen in such a way as to minimize the following quadratic cost criterion

$$J^{b,y,T}(x) = E_x\left[v\int_0^\tau e^{-(\rho+\delta)t}\left(b_0 - b(t)\right)^2\,dt + \frac{we^{-(\rho+\delta)\tau}}{\rho+\delta}\left(b_1 - kx(\tau)\right)^2\right], \tag{1}$$

where:

- $E_x(\cdot) = E(\cdot|x(0) = x)$, i.e. the expectation value is conditioned to the current size of the fund;
- $\tau = \min(T, T_0)$;
- v and w are positive weights which determine the relative importance in the cost criterion of the payment before and after annuitization, respectively;
- ρ is a subjective discount factor;
- δ is the force of mortality which is assumed to be constant;
- b_0 is the income target before purchasing the annuity;
- b_1 is the amount that represents the targeted income after ultimate annuitization;
- k is the amount of annuity which can be purchased with one unit of money.

The intertemporal wealth equation is

$$dx(t) = \left[x(t)\big(y(t)(\lambda - r) + r\big) - b(t)\right]dt + \sigma x(t)y(t)dB(t),$$

where $B(\cdot)$ is a standard Brownian motion representing market risk.

The amount b_0, the income target until the annuity is purchased, will in many cases be equal to kx_0, the size of the annuity which could have been purchased if the retiree had annuitized immediately on retirement. The process x evolves until either it is advantageous to annuitize or the fund falls to a negative value, in which case no further trading is permitted. The loss associated with annuitization when the level of the fund is $x \geq 0$, so that the annuity pays kx per unit time, is

$$K(x) = \frac{w}{\rho+\delta}(b_1 - kx)^2. \tag{2}$$

The ratio $\frac{b_1}{b_0}$ is a measure of risk propensity: the higher $\frac{b_1}{b_0}$, the higher the target, the lower the risk aversion and vice versa. It is natural to assume that the targeted annuity b_1 is greater than income withdrawn b_0, hence $\frac{b_1}{b_0} > 1$.

Should the fund hit $\frac{b_1}{k}$, the pensioner would be able to buy a lifetime annuity with income rate b_1 and the penalty from that moment on would be null, which is what we expect to be.

Clearly, if the fund is equal to $\frac{b_1}{k}$ the optimal decision is to annuitize. However, the main novelty of the model solved in [12] is that optimal annuitization occurs also with a fund size x^* that is *lower* than $\frac{b_1}{k}$. The intuition behind that is the following. When one reaches a certain level close enough to the desired target, it is better to stop the self-annuitization strategy and accept the low penalty given by (2), rather than keeping on investing and facing the risk of departing even more from the desired level.

The region where it is optimal the self-annuitization strategy turns out to be

$$U = [0, x^*) \cup \left(\frac{b_1}{k}, +\infty\right) \tag{3}$$

that in the optimal stopping theory terminology is called "continuation region". The name of the region U is intuitive: if the wealth x is in the continuation region, then the loss paid in the case of annuitization is higher than that paid in the case of continuation of the optimization program, so that it is optimal to keep playing the game; vice versa, if one's wealth is outside that region, then the loss paid in case of annuitization is lower than that paid playing the game, thus the game is over and the pensioner annuitizes.

For a complete analysis of this combined stochastic control and optimal stopping problem, we refer to the original paper ([12]), and for the reader's convenience we report a synthetic description of the optimal solution in the Appendix.

4 Sub-optimality of immediate annuitization

In this Section, we prove the main theoretical result of this paper. In Section 4.1, we state and prove a theorem that gives a necessary and sufficient condition under which immediate annuitization is optimal for every initial wealth $x_0 \in [0, b_1/k]$. In Section 4.2 we illustrate the applicability of the theorem and analyze the impact of different key parameters on the criterion provided by the theorem.

4.1 The criterion

Before stating the criterion, we notice that the region of interest for the initial wealth is $\left[0, \frac{b_1}{k}\right]$. Namely, for $x_0 < 0$ the problem is not defined; for $x_0 > \frac{b_1}{k}$ the optimal policy is to keep playing the game with consumption rate $b = b_0$ up until the fund falls to $\frac{b_1}{k}$. (3) What happens for the much more interesting case $x_0 \in \left[0, \frac{b_1}{k}\right]$ depends on the form of the continuation region (3) via the value of the model's parameters.

Theorem 1 *In the model outlined in Section 3, it is optimal to annuitize immediately at retirement for every initial wealth* $x \in \left[0, \frac{b_1}{k}\right]$ *if and only if*

$$\phi \leq 2rD\frac{k}{b_1}, \tag{4}$$

where $D = \frac{b_0}{r} - \frac{b_1}{k} > 0$, $\phi = \rho + \delta + \beta^2 - 2r + k^2 \frac{w}{v(\rho+\delta)}$, *and* $\beta = \frac{\lambda - r}{\delta}$ *is the Sharpe ratio of the risky asset.*

Proof
($\Leftarrow$) Denoting by T^* the optimal annuitization time, by application of the Corollary 2.4 of [12], it is $T^* = 0$ if $LK(x) \geq 0 \quad$ for all $x \in \left[0, \frac{b_1}{k}\right]$, where $K(x)$ is given by (2) and

$$\begin{aligned} LK(x) &= \inf_{b,y} \left\{ v\,(b_0 - b)^2 - (\rho+\delta)K + \left[-b + (\lambda + r)yx + rx\right] K' + \frac{1}{2}\sigma^2 y^2 x^2 K'' \right\} = \\ &= w(b_1 - kx)[2krD - \phi(b_1 - kx)]. \end{aligned}$$

(3) Indeed, with this simple strategy, the value function is identically null.

Let $\phi \leq 2rD\frac{k}{b_1}$. Then,

$$LK(x) \geq \frac{2wk^2rD}{b_1}(b_1 - kx)x \geq 0 \quad \text{for } 0 \leq x \leq \frac{b_1}{k}.$$

($\Rightarrow$) If $T^* = 0$ for all $x \in \left[0, \frac{b_1}{k}\right]$, then $V(x) = K(x)$, where V is the value function associated to the quadratic cost criterion (1) given by (15) and K is given by (2). Therefore, due to the variational inequalities (17), $LK(x) \geq 0$ for all $x \in \left[0, \frac{b_1}{k}\right]$. This, in turn implies that given the set U_0 defined as

$$U_0 = \{x \in \mathbb{R} \ : \ LK(x) < 0\},$$

we have

$$U_0 \cap \left[0, \frac{b_1}{k}\right] = \emptyset.$$

On the other hand, if $\phi > 2rD\frac{k}{b_1}$, then

$$U_0 \cap \left[0, \frac{b_1}{k}\right] = \left[0, \frac{b_1}{k} - \frac{2rD}{\phi}\right) \neq \emptyset.$$

Then, necessarily

$$\phi \leq 2rD\frac{k}{b_1}.$$

□

4.2 Some intuition and interpretation behind the criterion

We believe that the practical implications of Theorem 1 are worth exploring. In fact, the meaning of this theorem is the following.

When the condition (4) is satisfied, it is optimal to annuitize immediately *for every initial fund* $x_0 \in \left[0, \frac{b_1}{k}\right]$. This means that compulsory immediate annuitization turns out to be optimal, in the sense that retirees are obliged to behave optimally by the legislation.

When the criterion (4) is violated, it is not true that *for each initial fund* x_0 the optimal annuitization time is $T^* = 0$. This does *not* mean that $T^* = 0$ is *never* optimal, for there are cases in which $\phi > \frac{2krD}{b_1}$ and x_0 stays in the stopping region, leading to $T^* = 0$.[(4)] Clearly, if (4) is not satisfied, *compulsory* immediate annuitization is not optimal, for there are values of the initial wealth that fall in the continuation region, for which it is optimal to take programmed withdrawals.[(5)]

For this reason, from now on we will associate the condition (4) to the optimality of a pension system where immediate annuitization is compulsory. In other words, Theorem 1 can be stated equivalently by saying that *compulsory immediate annuitization is optimal if and only if* (4) *holds*. Thus, in a pension system where immediate full or partial annuitization is the only option, the lack of flexibility in choosing timing of annuitization can be shown to be sub-optimal by verification of violation of (4).

We can rewrite the condition (4) in several ways, in order to study the role and the impact of each key factor on the criterion.

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(4) This happens for all $x_0 \in \left[x^*, \frac{b_1}{k}\right]$.

(5) This happens for all $x_0 \in [0, x^*)$.

1. A first equivalent way is

$$\beta^2 \leq 2k\frac{b_0}{b_1} - \left(\rho + \delta + k^2 \frac{w}{v}\frac{1}{\rho+\delta}\right). \tag{5}$$

   Inequality (5) indicates that the value of the Sharpe ratio cannot be too high for having optimality of compulsory immediate annuitization. This is intuitive, as the choice of programmed withdrawals makes sense only in the hope of being better off than immediate annuitization. This is possible only in the presence of a sufficiently good risky asset. If the Sharpe ratio is too small, then immediate annuitization should be preferred, as the benefits of mortality credits provided by the insurance products would enhance the return, making immediate annuitization dominant with respect to investment in financial market. This is the classic result of [26], and has also been found in [10] and [21].

   If we now consider the expression of k as given by $k = \frac{(r+\delta)}{(1+L)}$, with L insurance loading factor, and we set the simplifying and not unreasonable values $\rho = r, L = 0$ (notice that the former is a typical assumption in this kind of literature, see e.g. [20]), we have

$$k = \rho + \delta \tag{6}$$

   and we obtain the simple condition

$$\beta^2 \leq (\rho+\delta)\left[2\frac{b_0}{b_1} - \left(1 + \frac{w}{v}\right)\right] \tag{7}$$

   from which we can gather useful information. Optimality of compulsory annuitization is not possible if $\frac{b_1}{b_0}$ is too high. In particular, if the targeted pension income b_1 is higher than twice the annuity b_0 purchasable at retirement, then the r.h.s. of (7) is negative, rendering the inequality impossible to hold. The factor $\frac{w}{v}$ also cannot be too high: considering that $\frac{b_0}{b_1} < 1$ for the problem to make sense, if $\frac{w}{v} \geq 1$, compulsory immediate annuitization can never be optimal. Interestingly, from (7) we can also notice that the factor $\rho + \delta$ is of secondary importance with respect to $\frac{b_0}{b_1}$ and $\frac{w}{v}$ whenever the term in squared brackets is negative: namely, in this case optimality of compulsory annuitization can never occur, no matter what the value of $\rho + \delta$ is. On the other hand, if the term in squared brackets is positive, then the value of $\rho + \delta$ can affect the validity of the criterion: in fact, a too low value of $\rho + \delta$ will make impossible the criterion to hold, while a very high value of it can lead to optimality of compulsory annuitization. This is consistent with intuition: $\rho + \delta$ is a measure of the intolerance towards future income (this will be explained clearly in the next Section). Thus, a very low value of $\rho + \delta$, that is associated to high tolerance for the future and high expectation of future remaining lifetime, is likely to render immediate annuitization sub-optimal.

2. A second equivalent way is

$$\frac{w}{v} \leq \frac{2b_0(\rho+\delta)}{kb_1} - \frac{(\rho+\delta+\beta^2)(\rho+\delta)}{k^2}. \tag{8}$$

   Inequality (8) says that in order to have optimality of compulsory immediate annuitization the weight w given to loss in case of annuitization needs to be sufficiently low with respect to that given to loss for running consumption v, or, equivalently, the weight given to loss in case of running consumption v needs to be sufficiently high with respect to that given to annuitization w.

If we make the simplifying assumption (6) on k, we obtain

$$\frac{w}{v} \leq \frac{2b_0}{b_1} - \left(1 + \frac{\beta^2}{\rho + \delta}\right). \tag{9}$$

Notice that, again, the risk aversion of the pensioner plays a role. As before, if the targeted pension income b_1 is higher than twice the annuity b_0 purchasable at retirement, then the r.h.s. of (9) is negative, rendering the above inequality impossible to hold. Again, also the financial market has an effect: with too high values of the Sharpe ratio β, the r.h.s. can become very small and even negative, and the same can happen with a too low value of $\rho+\delta$.

3. A third equivalent way is

$$\frac{b_0}{b_1} \geq \frac{\rho + \delta + \beta^2}{2k} + \frac{k}{2}\frac{w}{v}\frac{1}{\rho + \delta}. \tag{10}$$

Inequality (10) indicates that the ratio $\frac{b_0}{b_1}$ cannot be too low for compulsory immediate annuitization to be optimal. This is equivalent to say that the targeted pension b_1 cannot be too high with respect to the pension b_0 purchasable at retirement. This is again intuitive, as immediate annuitization can turn out to be optimal only if the retiree is not too much unsatisfied with it, and this can happen only if her desired income does not deviate too much from the pension provided at retirement.

If we make the simplifying assumption (6) on k, we obtain

$$\frac{b_0}{b_1} \geq \frac{1}{2} + \frac{1}{2}\left(\frac{\beta^2}{\rho + \delta} + \frac{w}{v}\right). \tag{11}$$

Consistently with the analysis made before, we see that β and $\frac{w}{v}$ cannot be too high and $\rho+\delta$ cannot be too low for the condition to hold. In particular, we see that if

$$\frac{\beta^2}{\rho + \delta} + \frac{w}{v} \geq 1,$$

then compulsory immediate annuitization cannot be optimal, given that $\frac{b_0}{b_1} < 1$.

4. A simple condition on $\rho + \delta$ is not easy to interpret.[6] However, if we apply the simplifying assumption (6) on k, we get again (7):

$$(\rho + \delta)\left[2\frac{b_0}{b_1} - \left(1 + \frac{w}{v}\right)\right] \geq \beta^2. \tag{12}$$

Expectedly, this implies that the factor $\rho + \delta$ has to be high enough to render possible optimality of compulsory immediate annuitization. In addition, as commented in point 1 above, the value of $\rho + \delta$ has no impact whenever the other factors do not behave nicely. In fact, if $\frac{w}{v}$ and/or $\frac{b_1}{b_0}$ are too high, then compulsory immediate annuitization is not optimal no matter what the value of $\rho + \delta$ is.

[6] The condition is $z_1 \leq \rho + \delta \leq z_2$, where, for $i = 1, 2$, we have

$$z_i = \frac{1}{2}\left[2k\frac{b_0}{b_1} - \beta^2 + (-1)^i\sqrt{\left(\beta^2 - 2k\frac{b_0}{b_1}\right)^2 - 4k^2\frac{w}{v}}\right]$$

.

Remark 2 *Notice that the occurrence of condition* (4) *is linked to the simultaneous occurrence of the following:*

- *sufficiently low value of* β*;*
- *sufficiently low value of* $\frac{w}{v}$*;*
- *sufficiently low value of* $\frac{b_1}{b_0}$*;*
- *sufficiently high value of* $\rho + \delta$*.*

Therefore, since these four key factors cannot be controlled – as some are linked to the financial market, some to mortality conditions, and some to personal preferences – it is evident that a pension system that imposes compulsory immediate annuitization to the whole universe of retirees is bound to be sub-optimal, according to this model. This simple conclusion evidently is not a great discovery. Clearly, giving more flexibility to the decision maker has the effect of increasing her individual utility, and this holds in every context. However, here we state that, even if immediate annuitization might turn out to be optimal for the single retiree, it cannot be optimal for the universe of retirees in its globality. More importantly, this model does provide what is the optimal annuitization time (see [12]), and therefore it enables us to quantify *sub-optimality of compulsory immediate annuitization, by defining and computing the* cost of being sub-optimal. *This is the topic of the next Section.*

5 Sub-optimality of compulsory immediate annuitization: practical implications

In this Section, we illustrate the practical implications of Theorem 1. In Section 5.1 we present four typical scenarios which will be used in the simulations and, in each of them, check the violation of criterion (4); in Section 5.2 the cost of sub-optimality is defined and measured in those scenarios.

5.1 Basic assumptions and scenario generation

We consider a male retiree aged 60 and time horizon equal to $T = 30$. In fact, accordingly to the existing actuarial literature (see Section 2), ages of optimal annuitization range typically between 70 and 80, and only rarely up to 85-90. It is therefore reasonable to assume that if the pensioner has not annuitized at the age of 90, he will not do it later. His initial wealth is $x_0 = 100$. The riskfree rate in each scenario will be chosen at 3%. For the annuity calculation, we make use of an Italian projected mortality table (RG48). This set of assumptions corresponds to an immediate annuity value equal to $b_0 = 6.22$, and the conversion factor from lump sum into annuity is $k = 0.085$. The parameters of the problem are

$$r, \qquad \lambda, \qquad \sigma, \qquad \delta, \qquad k, \qquad v, \qquad w, \qquad b_1, \qquad \rho. \tag{13}$$

In a realistic setting r, λ, σ characterize the investment opportunities and depend on the financial market, δ depends on the demographic assumptions, and k depends on both the financial and demographic hypotheses.

Parameters that can be chosen are the weights given to penalty for running consumption, v, and to penalty for final annuitization, w, although it turns out that the relevant quantity is the ratio of these weights, $\frac{w}{v}$. Another parameter chosen by the retiree is the targeted level of annuity, b_1, while it is reasonable to assume that the level of interim consumption b_0 is given and depends on the size of the fund at retirement. A typical choice for b_0 is the size of annuity purchasable at retirement with the initial fund x_0. Thus, typically b_1 is multiple of b_0, and the relevant quantity is $\frac{b_1}{b_0} > 1$.

A parameter that is somehow arbitrary and somehow given is ρ, the intertemporal discount factor: although subjective by its own nature, in typical situations cannot differ too much from the riskfree rate of return r and in all our simulations it will be assumed $\rho = r$. We notice that the discount factor ρ is always associated to the mortality rate δ. This is intuitive: $\rho + \delta$ is the global discount factor of the individual, that takes into account the subjective tolerance towards future income, ρ, and the expected remaining lifetime, driven by δ. In this way the relevant quantity measuring the patience of the retiree for future events, the value of time, is given by the sum $\rho + \delta$.

In [12] the authors show that, expectedly, what really counts in the applications, are some relevant ratios of these values, that are

$$\beta, \qquad \frac{w}{v}, \qquad \frac{b_1}{b_0}, \qquad \rho + \delta. \tag{14}$$

This is indeed consistent with the fact that, as illustrated in Section 4.2, the validity of the criterion (4) only depends on the value of these key quantities. This allows us in the following to fix some of the values (13) and change some others to get different values of the relevant quantities (14). In fact, consistently with the results summarized in Remark 2, [12] find that equal sign variations of the first three quantities and opposite sign variation of the fourth one produce equal qualitative variations to some relevant features of the problem solution. In particular, they find that everything else being equal, the ratio $\frac{x^*}{(\frac{b_1}{k})}$, i.e. the width of the continuation region:

1. increases by increasing β;
2. increases by increasing $\frac{w}{v}$;
3. increases by increasing $\frac{b_1}{b_0}$;
4. generally slightly decreases by increasing $\rho + \delta$.

Indeed, it is reasonable to accept that a high Sharpe ratio can well be coupled with low risk aversion, and also with high penalty in case of annuitization w.r.t. that paid in case of running consumption. Moreover, it is natural to expect that a low $\rho + \delta$ is consistent with a high $\frac{w}{v}$, because these choices are both led by high tolerance toward future income. And vice versa. Notice finally that these relationships between the key parameters and the width of the continuation region are consistent with the analysis made in Section 4.2. In fact, we have commented above that $T^* = 0$ for every initial wealth, meaning $x^*/(\frac{b_1}{k}) = 0$, is possible only with low enough values of $\beta, \frac{w}{v}$ and $\frac{b_1}{b_0}$ and with high enough value of $\rho + \delta$.

In a first set of simulations (the methodology of which is described in the next Section), here not reported, we let all the relevant quantities vary accordingly to the description. We have found out that the results do not differ very much when the ratio $\frac{w}{v}$ increases. Results have turned out to be more sensible to the choice of $\rho + \delta$ that, as mentioned, measures the weight given to future and present flows, i.e. the time value of money for the retiree. For this reason, we have finally fixed the ratio $\frac{w}{v}$ equal to 1.

We fix four different scenarios, by starting with low values of β and $\frac{b_1}{b_0}$ and high value of $\rho + \delta$, and then progressively augmenting the first two and reducing the third one, at the same time. The values of the relevant quantities of the four scenarios, which we call A, B, C, and D, are reported in Table 1.(7)

	A	B	C	D
β	0.25	0.33	0.40	0.50
$\frac{b_1}{b_0}$	1.50	1.75	2.00	2.25
$\rho + \delta$	0.09	0.07	0.05	0.03

Table 1: Value of key parameters in scenarios A, B, C, D.

Although, clearly, the set of combinations of different values of the relevant ratios is potentially unlimited, here we focus on these four scenarios, which we find representative for four different kinds of individuals:

- Scenario A would be suitable for individuals quite risk averse, who desire a pension target only 50% higher than b_0. They do not expose themselves too much to financial risk, and gain a small value of β on the market. These individuals strongly prefer current income to future income, probably due to a high estimate of the subjective mortality rate.
- Scenario B reports preferences for individuals moderately risk averse, who target a final pension 75% higher than b_0. They find a $\beta = 0.33$ on the market, and give more weight to the present rather than the future.
- Scenario C would be suitable for individuals with low risk aversion, who aim to double their final annuity via programmed withdrawals. They are able to gain $\beta = 0.40$ on the financial market and give approximately the same importance to future and present income, taking into account also the mortality rate.
- Scenario D would be suitable to risk lovers, who aim to more than double the immediate annuity b_0, and get a very high value of β on the market. They put substantial weight to future income, compared to present one, either due to low estimate of their own mortality rate or due to strict preference for the future.

Straight application of the definitions allows us to check that, as expected, in the four scenarios chosen the criterion (4) is not satisfied.

In each of the four scenarios, we have also tried to see if it was possible to obtain the validity of the criterion (4) just by changing one parameter out of the four relevant ones (that are β^2, $\frac{w}{v}$, $\frac{b_1}{b_0}$, $\rho+\delta$), leaving the other three fixed. This turns out to be impossible: in all cases the value of the key parameter that allows optimality of compulsory immediate annuitization is not admissible. In particular, in each of the four scenarios selected, and leaving unchanged the value of the other three key parameters:

- the value of β^2 such that $\phi = \frac{2krD}{b_1}$ is negative,

(7) In particular, we have set in all scenarios: $r = \rho = 0.03$, $k = 0.085$, $w = v = 1$, $b_0 = 6.22$; in scenario A: $\lambda = 0.06$, $\sigma = 0.12$, $\delta = 0.06$, $b_1 = 9.33$; in scenario B: $\lambda = 0.08$, $\sigma = 0.15$, $\delta = 0.04$, $b_1 = 10.80$; in scenario C: $\lambda = 0.102$, $\sigma = 0.18$, $\delta = 0.02$, $b_1 = 12.44$; in scenario D: $\lambda = 0.13$, $\sigma = 0.20$, $\delta = 0.005$, $b_1 = 14.00$.

- the value of $\rho + \delta$ such that $\phi = \frac{2krD}{b_1}$ is negative,
- the value of $\frac{w}{v}$ such that $\phi = \frac{2krD}{b_1}$ is negative,
- the value of $\frac{b_1}{b_0}$ such that $\phi = \frac{2krD}{b_1}$ is lower than 1.

This is further evidence of the fact observed in Remark 2 that if the preferences and needs of pensioners can be represented by the model exploited, then a system where immediate annuitization is compulsory and programmed withdrawals are not an option is sub-optimal.

The next Section is devoted to measuring the extent of sub-optimality for the single individual in the four scenarios selected previously.

5.2 Sub-optimality of immediate annuitization: cost of sub-optimality

In this Section, we define two measures – one in absolute and the other in relative terms – for the cost of sub-optimality of immediate annuitization in terms of loss of expected present value (EPV) of consumption stream from retirement up to death. In particular, we calculate the EPV of consumption in the two cases of optimal annuitization (EPV_{OA}) and immediate annuitization (EPV_{IA}) and make the difference.[(8)] Henceforth, we will call the difference $EPV_{OA} - EPV_{IA}$ *cost of sub-optimality*, or *sub-optimality cost* (SC). In other words:

$$SC = EPV_{OA} - EPV_{IA} = \text{sub-optimality cost}.$$

To give a better idea of the improvement that can be achieved in case of optimal annuitization, we introduce a new quantity, called *relative sub-optimality cost* (RSC). This is defined as the ratio between the cost of sub-optimality and the EPV of immediate annuitization:

$$RSC = \frac{SC}{EPV_{IA}} = \text{relative sub-optimality cost}.$$

This quantity indicates by how much in percentage the pensioner can increase his reward (measured in terms of EPV of consumption) by adopting programmed withdrawals and annuitizing optimally with respect to immediate annuitization.

In order to have an idea of the extent of the cost of sub-optimality in real applications, we perform numerical simulations for each scenario selected in Section 5.1. The methodology of the simulations is the following. Firstly, with the help of a Perl program that finds the solution with the methodology described in Section 3.6 of [12], we have found the wealth level that triggers annuitization (x^*) in the four scenarios introduced above. Secondly, we have implemented the numerical tests using a MATLAB code. In each scenario we have simulated the behaviour of the risky asset, by means of Monte Carlo simulations. For each scenario, we run 1000 Monte Carlo simulations for the risky asset. Across the four scenarios, we have fixed the 1000 trajectories of the Brownian motion. For each trajectory we have found T^* as the first time that the fund hits x^* and have calculated the annuity value A^* purchasable with fund x^* and with the age at time T^*. A^* has been calculated each time through the actuarial fairness principle, using the RG48 mortality table and the interest rate $r = 0.03$.

It is important to underline that in order to calculate the sub-optimality cost, we focus only on those trajectories for which optimal annuitization has actually occurred between retirement and

[(8)] By expected present value of a future stream we mean the actuarial value at retirement of the future stream.

the time-horizon of 30 years. The trajectories where the optimal fund fails to reach the boundary of the continuation region x^* within the time-frame of 30 years have been assigned a 0 value to the SC, because the model does not require a finite time-horizon and, more importantly, because optimal annuitization at a later age would produce different results in terms of SC. Assignment of $SC = 0$ to all trajectories where optimal annuitization does not occur is the explanation for a bar around 0 in the histograms. However, in the statistics of SC (Table 2) we will not consider the zeroes and will present the statistics only for the relevant cases in which $T^* < 30$.

In each scenario and for each trajectory, the EPV of consumption stream has been calculated (again according to actuarial principles) with the interest rate $r = 3\%$ for the financial basis, and with the mortality table RG48 for the demographic basis. For the immediate annuitization option the flow to be discounted is equal to b_0 at any time from retirement up until age 110, that is the extreme age of the adopted mortality table. For the optimal annuitization case, the flow to be discounted is equal to the optimal consumption from retirement until time of optimal annuitization T^*, to the actual pension rate A^* achieved from that time up to age 110.

Figures (1a), (1b), (1c), and (1d) report the histograms of the distribution of SC in scenarios A, B, C and D, respectively.

Table 2 is made by two parts. In what follows, by Prob(E) we mean the frequency over the 1000 simulations of the event E.

1. The first part reports for each scenario the probability that optimal annuitization does not occur in the time-frame of 30 years, the probability that the sub-optimality cost SC is positive, the mean of the relative sub-optimality cost RSC, the mean of the optimal annuitization time T^*, and the mean of the final annuity A^* received upon annuitization at time T^* (more detailed statistics of T^* and A^* are provided in the next Section).

2. The second part reports for each scenario some statistics of the sub-optimality cost.

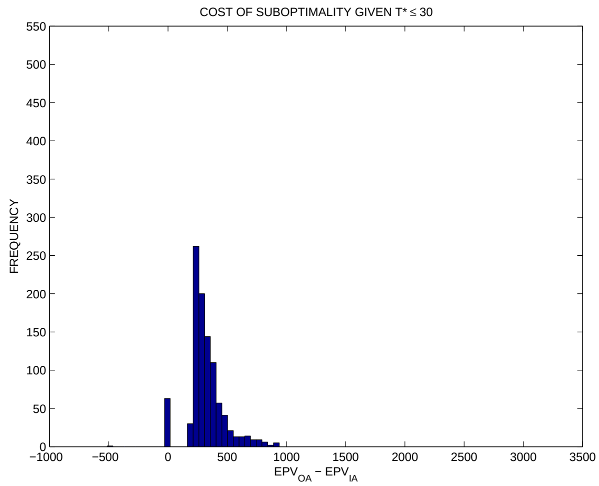
Figures (1a), (1b), (1c), (1d), and Table 2 allow a broad comparison among all scenarios regarding all types of quantities analyzed throughout the paper. Let us recall that the scenarios A, B, C and D are descriptive of different risk aversion attitudes, with the A scenario representing the highest and the D scenario reporting the lowest. The general comment is that with high risk aversion optimal annuitization occurs almost always within the predefined time-frame of 30 years and is associated to a small SC. On the contrary, with low risk aversion, optimal annuitization occurs less frequently within the time-frame and is associated to higher SC.

More into details, we can gather the following information:

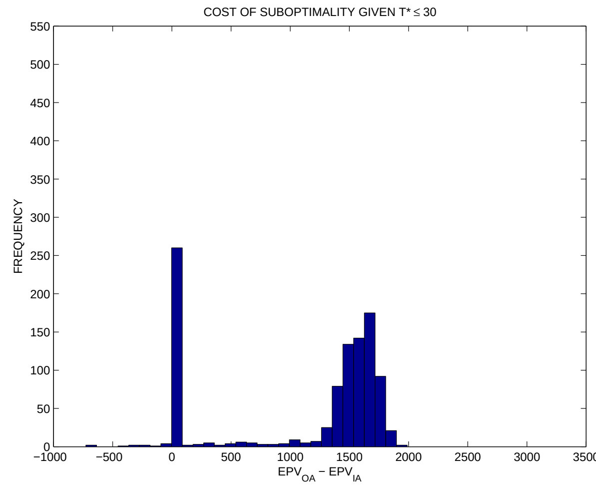
- considering that the sum of the number of cases in which optimal annuitization does not occur and number of cases characterized by $SC > 0$ (sum of the first two lines of Table 2) is almost approximately 99%, for each scenario we find that in all but 10 cases in which optimal annuitization occurs, the sub-optimality cost is positive, meaning that the pensioner is better off when programmed withdrawals are adopted;

- the few cases when optimal annuitization does occur but $SC < 0$ are motivated by the fact that the dynamic programming approach minimizes the expected loss (or maximizes expected utility) and fails to capture very rare unfavorable scenarios (e.g. extreme events);

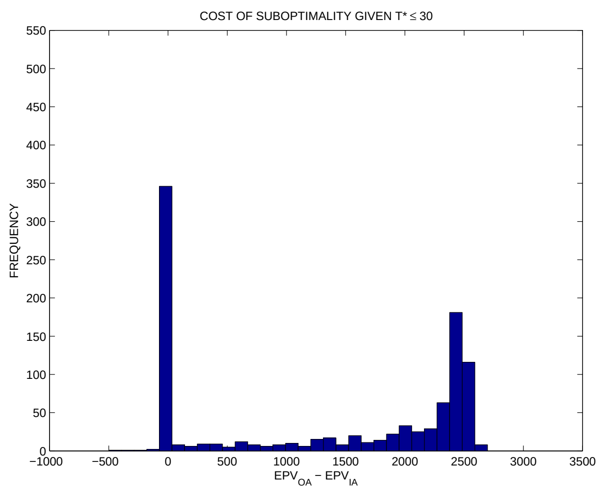
- the extent of improvement with optimal annuitization (or the extent of loss with immediate annuitization) is measured by the mean RSC, in the third line of the table: it amounts to



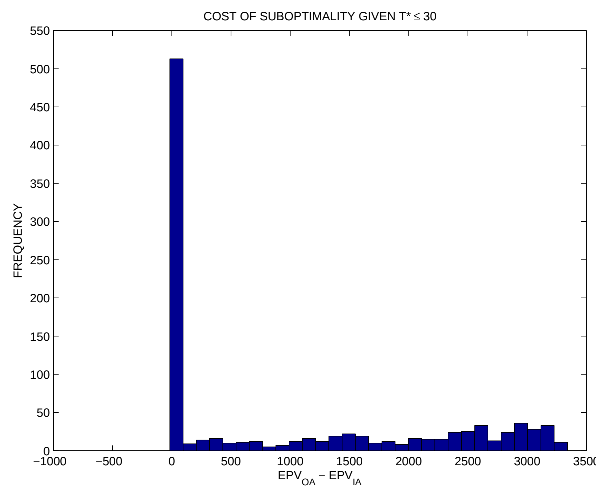
(a) Scenario A.



(b) Scenario B.



(c) Scenario C.



(d) Scenario D.

Figure 1: The sub-optimality cost deriving from all scenarios.

only 7% for scenario A, to 29% for scenario B, to 39% for scenarios C and D. As expected, the margin of improvement increases when risk aversion of the individual decreases;

- the probability of unsuccessful use of this model, meaning absence of annuitization within 30 years from retirement, also increases when decreasing the risk aversion;
- the optimal annuitization time increases when decreasing the risk aversion;
- the size of annuity achieved upon optimal annuitization also increases on average when decreasing the risk aversion.

6 The optimal annuitization time: performance analysis

In order to make the analysis more complete, in this Section we provide more detailed information gathered from the simulations, about the optimal annuitization time T^* and the final annuity size A^* achieved at time T^*.

	Scenario A	Scenario B	Scenario C	Scenario D
Prob($T^* > 30$)	6.30%	25.30%	34.10%	50.90%
Prob($SC > 0$)	93.60%	73.50%	65.00%	49.00%
Mean RSC	6.62%	28.91%	38.80%	38.82%
Mean T^*	1.0312	8.1532	14.1807	19.6263
Mean A^*	6.8397	11.6771	18.0114	26.8969
Mean SC	338.9556	1,479.9263	1,985.7943	1,987.0308
Standard deviation SC	134.4206	378.5419	700.8871	935.0963
Minimum SC	-514.0536	-726.6107	-499.4168	-16.5548
Maximum SC	938.7281	1,988.1095	2,697.1238	3,340.5489
5th percentile SC	214.8806	603.5453	352.3035	301.9541
25th percentile SC	248.7041	1,451.6475	1,724.5226	1,254.0524
50th percentile SC	303.6802	1,568.8207	2,350.9992	2,200.6371
75th percentile SC	383.6738	1,675.8389	2,458.8257	2,841.1091
95th percentile SC	642.5026	1,789.9402	2,561.7912	3,183.1867

Table 2: Statistics of the cost of sub-optimality and of all relevant quantities deriving from all scenarios.

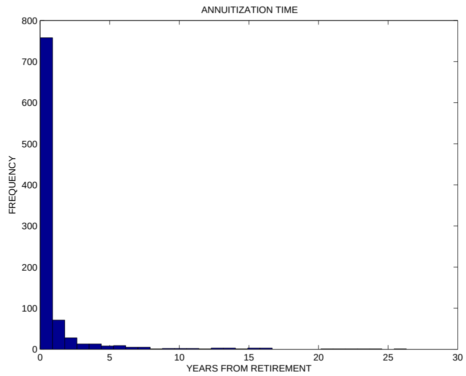
In particular, for each scenario we provide the following (here, again, by Prob(E) we mean the frequency over the 1000 simulations of the event E):

- the histogram with the distribution of the optimal annuitization time T^*, measured in years from retirement (Figures (2a), (2c), (2e) and (2g));
- the histogram with the distribution of the final annuity A^* achieved upon optimal annuitization (Figures (2b), (2d), (2f) and (2h));
- statistics of the optimal annuitization time T^* and of the size of annuity A^* upon optimal annuitization, with consequent comparison with that achievable on immediate annuitization (Table 3; note that the mean of T^* and of A^* are already reported in Table 2);
- some relevant information relative to extreme cases (Table 3) such as:
  - probability of optimal annuitization within one year from retirement, Prob($T^* < 1$),
  - probability of optimal annuitization after the time-frame considered, Prob($T^* > 30$),
  - probability that the final annuity A^* is lower or equal than b_1, Prob($A^* \leq b_1$);
  - probability of negative optimal consumption, Prob($b^* < 0$), and average time of negative consumption, given that there is negative consumption.[9]

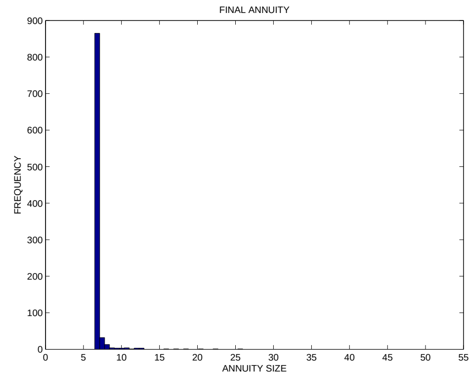
[9] Notice that negative consumption, though not desirable, corresponds to paying money into the fund, rather than withdrawing from it.

The statistics of optimal annuitization time and final annuity reported in Table 3 are conditional on $T^* \leq 30$. When optimal annuitization does not occur within the 30-years time-frame, T^* is set equal to 0, but its statistics is not reported into the mentioned Table, and the same for the statistics of the final annuity. Similarly, in the histograms of distribution of T^* and final annuity A^* reported in Figure 2, the cases when optimal annuitization does not occur have not been reported.

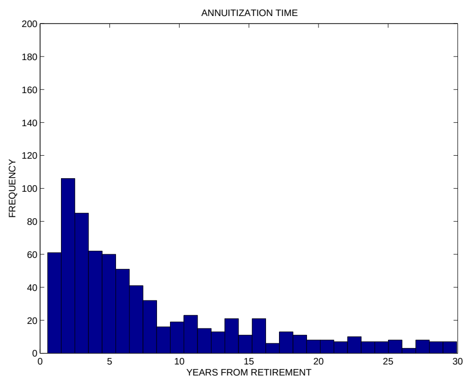
In order to improve readability, notice that scenarios B, C and D are characterized by the same scale in all Figures ((2c), (2d), (2e), (2f), (2g) and (2h)), whereas Figures (2a) and (2b), relative to scenario A, differ in the scale from the others. This is due to the much higher concentration of values around the mode of the distribution in scenario A than in the other scenarios.



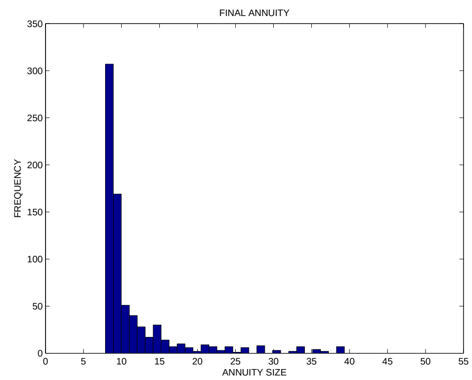
(a) Optimal annuitization time of scenario A.



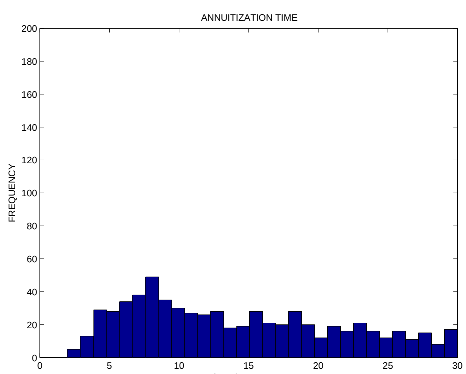
(b) Annuity of scenario A.



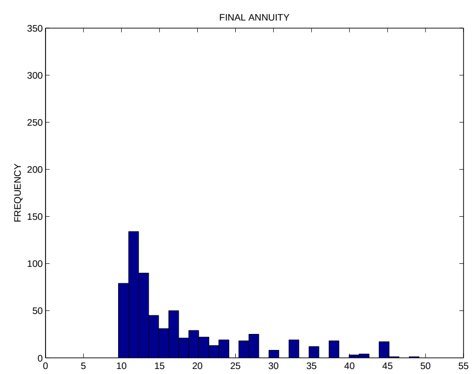
(c) Optimal annuitization time of scenario B.



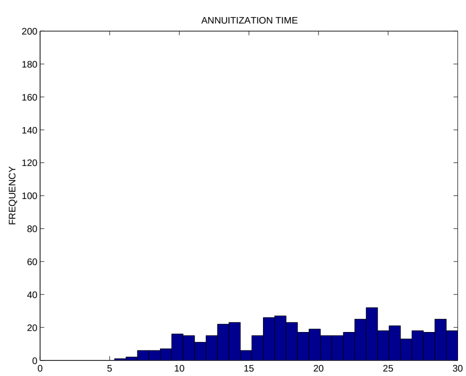
(d) Annuity of scenario B.



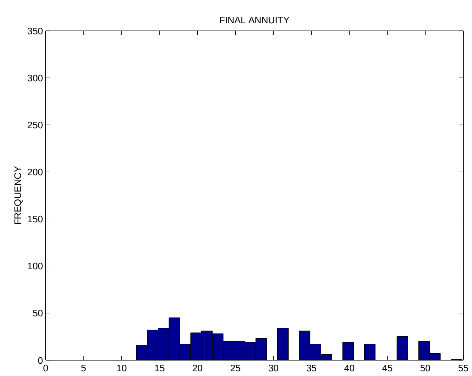
(e) Optimal annuitization time of scenario C.



(f) Annuity of scenario C.



(g) Optimal annuitization time of scenario D.



(h) Annuity of scenario D.

Figure 2: Optimal annuitization time and size of final annuity in scenarios A, B, C and D.

	Scenario A	Scenario B	Scenario C	Scenario D
Mean T^*	1.0312	8.1532	14.1807	19.6263
Standard deviation T^*	2.8087	7.2171	7.3754	6.1493
Mean A^*	6.8397	11.6771	18.0114	26.8969
Standard deviation A^*	1.2638	5.9555	28.8639	10.9322
Prob($T^* < 1$)	77.00%	2.20%	0.00%	0.00%
Prob($T^* > 30$)	6.30%	25.30%	34.10%	50.09%
Prob($A^* \geq b_1$)	2.24%	29.58%	63.13%	93.48%
Prob($b^* < 0$)	2.40%	5.30%	3.20%	1.10%
Mean time of $b^* < 0$ (given $b^* < 0$)	4.40 yrs	2.84 yrs	2.01 yrs	2.05 yrs

Table 3: Statistics of the optimal annuitization time, statistics of the size of annuity, and relevant information on extreme cases in all scenarios.

Figure 2 and Table 3 provide results that are intuitive, expected and in line with the analysis conducted in Section 5.1 on the width of the continuation region. The most relevant observable trend is that with a high risk aversion and a poor risky asset, optimal annuitization occurs soon after retirement and provides a final annuity that is only slightly better than that achievable at retirement. When the risk aversion decreases and the risk premium improves, the individual defers for longer time ultimate annuitization and the level of final annuity increases. If the risk tolerance and the Sharpe ratio of the risky asset become too high, optimal annuitization may not happen in a reasonable time-frame (e.g. 30 years), and it can be questioned whether in this case this model should be used (see Section 6.1).

More detailed information has been grouped according to the following classification of relevant issues.

1. Timing of optimal annuitization, T^*.
    - In scenario A, the most striking feature that can be observed is that, in most of the cases, T^* occurs only 1-2 years after retirement. In particular, in 770 cases out of 1000 optimal annuitization occurs within one year from retirement; in 63 cases out of 1000 optimal annuitization does not occur within the time-frame of 30 years after retirement.
    - In scenario B, T^* is much more spread out than in scenario A: on average, T^* occurs 8 years after retirement, and in 50% of the cases it occurs after 5 years. In only 22 cases out of 1000 optimal annuitization occurs within one year from retirement and in 25% of the cases it occurs at a date later than 11 years after retirement; in 253 cases out of 1000 optimal annuitization does not occur within the time-frame of 30 years after retirement.
    - In scenario C, the distribution of T^* has more or less the same dispersion than in scenario B, but the mean is much higher: on average T^* occurs 14 years after retirement, and in 50% of the cases it occurs after 13 years. Optimal annuitization never occurs within one year from retirement and the minimum T^* is equal to 2 years. In 25% of the cases it occurs at a date later than 20 years after retirement; in 341 cases out of 1000 optimal

annuitization does not occur within the time-frame of 30 years after retirement; this too high percentage of individuals not annuitizing before age 90 is the price to be paid when the target aimed is chosen to be very high.

- In scenario D, the most striking feature is the probability that optimal annuitization does not occur in the 30 years time-frame: indeed, it is as high as 50.09%. We believe that with such a high probability of failure to annuitizing in a reasonable time horizon, this model should not be used ; when $T^* < 30$, it occurs on average after 19 years from retirement, with a moderately high dispersion; in 70% of the cases it occurs between 10 and 25 years from retirement. Moreover, optimal annuitization never occurs within 5 years from retirement. In 25% of the cases it occurs at a date later than 25 years after retirement.

2. Size of final annuity A^* upon optimal annuitization and its comparison with the pension achievable on immediate annuitization b_0.

   - In scenario A, A^* is always higher than b_0; however, the improvement does not seem to be particularly significant, since in 75% of the cases the final annuity A^* lies between 6.48 and 6.71, vs $b_0 = 6.22$. This is due to the fact that annuitization occurs too early and the price of annuity at that age is still too high, compared to the value of k chosen, and this results into a low pension rate.
   - In scenario B, A^* is always higher than b_0; this time, the improvement is more significant than in scenario A, since in 75% of the cases the final annuity A^* lies between 7.88 and 11.77, vs $b_0 = 6.22$. This is due to the fact that now optimal annuitization occurs at a later age than in scenario A and the price of annuity at that age is sufficiently low to guarantee a mode than adequate improvement in the pension rate.
   - In scenario C, A^* is always significantly higher than b_0; the improvement with respect to b_0 is now remarkable: in 75% of the cases the final annuity A^* lies between 9.62 and 20.88, vs $b_0 = 6.22$. This is effect of the two combined facts that x^* is much higher than before and that optimal annuitization occurs at a much later age, with consequent very low price of the annuity.
   - In scenario D, A^* is dramatically higher than b_0: in case of optimal annuitization the minimum pension rate achieved is 11.94, that is almost the double than $b_0 = 6.22$, and the size of annuity between the fifth and the seventy-fifth percentiles ranges between 14 and 33, while in 25% of the cases it ranges between 33 and 50. This is mainly due to the too high selected b_1, that leads to an extremely high value of the boundary of the continuation region x^*: in the most favorable scenarios, when the wealth reaches x^*, the annuity size is extremely high.

3. Size of final annuity A^* upon optimal annuitization and its comparison with the targeted pension b_1.

   - In scenario A, in 98% of the cases the annuity value A^* turns out to be lower than b_1. On the contrary, in very few cases (2.24%) the annuity value turns out to be higher than $b_1 = 9.33$: this is due to annuitization occurring 20-25 years after retirement, when the old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^*.
   - In scenario B, in 30% of the cases the annuity value A^* turns out to be higher than $b_1 = 10.89$: as before, this is due to annuitization occurring 15-25 years after retirement,

when the relatively old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^*.

- In scenario C, in 63% of the cases the annuity value A^* turns out to be higher than $b_1 = 12.44$: as before, this is due to annuitization occurring 15-25 years after retirement, when the relatively old age of the pensioner pushes downwards the price of the annuity, leading to a high annuity value to be purchased with the wealth available at T^*.
- In scenario D, in 93% of the cases the annuity value A^* turns out to be higher than $b_1 = 14$. This is due to the high value of x^* coupled with the high value of T^*, hence the older age of the pensioner.

4. Occurrence of optimal negative consumption.

   - In scenario A, optimal negative consumption occurs in 24 cases out of 1000, and on average consumption remains negative for 4 years.
   - In scenario B, optimal negative consumption occurs in 53 cases out of 1000, and on average consumption remains negative for 2.8 years.
   - In scenario C, optimal negative consumption occurs in 32 cases out of 1000, and on average consumption remains negative for 2 years.
   - In scenario D, optimal negative consumption occurs in 11 cases out of 1000, and on average consumption remains negative for 2 years.

6.1 Possible application of the model as controlling tool for post-retirement decisions

The richness of outcomes found in the previous Section encourages us to regard this model as a decision-making tool sufficiently flexible to allow for a number of personal situations. Regarding its applicability by pension fund advisors – in countries where programmed withdrawals are an option – we would like to make two considerations.

First, in order to help retirees to make conscious and optimal choices, pension fund advisors should not forget to show them clearly the strict correspondence between the choice of the model's parameters and the statistics of the final outcome. One could be tempted to conclude that the higher the propensity towards risk, the higher the reward by adopting programmed withdrawals. This is partially true. In fact, one should not forget that the probability of failure in achieving the wealth level x^* that triggers annuitization increases remarkably when risk aversion decreases, that is somehow expected. Indeed, the probability of failing to achieve the wealth level before age 90 is 34% with low risk aversion, but can be as high as 50% with very low risk aversion.

Second, it seems particularly important to be able to control the probability of success of adoption of this model, for three main reasons:

1. the improvement in EPV turns out to be positive only in those cases in which optimal annuitization occurs; for the other cases there is evidence suggesting that it would be negative;
2. it is likely that after a certain maximum age (here set equal to 90, but could be even lower) the pensioner would not be willing to keep on adopting the "do-it-yourself" strategy, and would prefer to pass financial and longevity risk to an insurance company;

3. the legislation may ask a pensioner to annuitize her remaining wealth at a certain limit age (as in UK, where annuitization is compulsory at the age of 75).

Therefore, we think that the model should be used to help a pensioner deciding about her post-retirement optimal decisions only when the probability of success is high enough, according to her needs and, possibly, to legislation constraints. However, we believe that the great flexibility of this model allows several attractive combinations of the parameters, characterized by high enough probability of success within a reasonable time-horizon.

7 Conclusions

In this paper, we have considered the position of a retiree member of a DC pension scheme and have investigated the optimal annuitization time by application of the model introduced by [12]. In particular, we have focused on the appropriateness of immediate annuitization, that is compulsory in many countries where the option of programmed withdrawals is not available.

In a first part of the paper, we have assessed a necessary and sufficient condition for compulsory immediate annuitization to be optimal. We have found that the validity of this condition depends on some key factors, that are linked to the financial market (via the Sharpe ratio of the risky asset) to the demographic conditions (via the force of mortality of the retiree) and to individual preferences (via the risk aversion of the individual and the different weights given to consumption pre- and post-annuitization). In particular, we have found that the Sharpe ratio, the risk tolerance and the weight given to loss for consumption post-annuitization, relative to that pre-annuitization, have to be sufficiently low in order to have sub-optimality of compulsory immediate annuitization. On the contrary, the intolerance towards future income as well as the force of mortality have to be sufficiently high for the condition to be satisfied. These are intuitive results. As a consequence, we have found that, since these key factors cannot be controlled a priori, a pension system that imposes compulsory immediate annuitization is bound to be sub-optimal.

In a second part of the paper, we have measured the extent of sub-optimality, in terms of loss of expected present value of consumption. We have defined the sub-optimality cost both in absolute and in relative terms as the difference between expected present value of consumption from retirement to death in the case of optimal annuitization and in the case of immediate annuitization. We have run extensive simulations to find the sub-optimality cost in different representative scenarios. With a very few exceptions, the cost of sub-optimality turns out to be positive in all cases in which optimal annuitization occurs in the time-frame of 30 years from retirement considered. Quite remarkably, in relative terms the sub-optimality cost varies from 6% with high risk aversion up to $20-30\%$ with medium risk aversion, and can be as high as 40% with low risk aversion. This clearly indicates by how much a pensioner gives up part of her future wealth when she annuitizes immediately, with respect to undertaking programmed withdrawals followed by optimal annuitization.

Finally, in order to complete the analysis, from the simulations we have collected and provided useful information regarding other relevant issues, like optimal annuitization time and size of annuity upon optimal annuitization. We have found the intuitive result that the optimal annuitization time decreases with the risk aversion. In particular, we find that it should occur on average 1-2 years after retirement when risk aversion of the retiree is very high, 8-9 years after retirement with medium risk aversion, 14-15 years after retirement when risk aversion is very low and should occur after 20 years or may not happen at all if the individual is strongly risk lover. Finally, we find the quite obvious result that the size of final annuity upon optimal annuitization is always higher than that obtainable on immediate annuitization. Expectedly, the size of annuity on average increases when the optimal annuitization time increases: in other words, it is seemingly worth to wait in order to obtain a higher reward.

We think that the main contributions of this paper are twofold.
The first novelty is the proof of sub-optimality of a pension system where immediate annuitization is mandatory. A priori this result may appear trivial, as a bigger set of choices can only improve the potential reward of the decision-maker. However, what is really crucial for the individual behaviour and for the pension fund design is if it *does* improve her effective reward. Here we show that it does.
The second novelty is the introduction of the sub-optimality cost, that is a quantification tool of the degree of sub-optimality of immediate annuitization. At the practical level, the sub-optimality cost can be used to make judgements on the opportunity of introducing the income drawdown option when it is not available, by facilitating the analysis of the trade-off between increased management costs and increased flexibility for pensioners.

It is our opinion that the striking figures of the sub-optimality cost in relative terms suggest the suitability of programmed withdrawals for retirees of DC pension schemes. One of the main concerns of each legislator should be the welfare of its pensioners, and this is obviously strictly linked to the affordability of a proper consumption stream. Therefore, the ambitious aim behind the evidence provided by this paper would be to hint appropriate adjustment to the legislation of those countries where programmed withdrawals are not available. We believe that to some extent this option should be offered to retirees, so as to increase flexibility in choosing a proper timing of benefits after retirement, and – not least – allow optimal choices for pensioners.

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Appendix

Solution of the model by Gerrard, Højgaard and Vigna (2011)

In this Section, we present briefly the solution of the model introduced by [12]. However, for a deeper analysis of the mathematical model, we refer the reader to the above mentioned paper.

The HJB equation and the continuation region

Let $V(x)$ denote the value function, i.e. the inferior of the expected loss given by (1). Therefore

$$V(x) = \inf_{b,y,T} J^{y,b,T}(x). \tag{15}$$

It is well known from the theory of optimal stopping time combined with stochastic control (see for instance [22] or [23] that the value function must satisfy the following Hamilton-Jacobi-Bellman equation

$$V(x) = \min\left\{ \frac{w\,(b_1 - kx(\tau))^2}{\rho+\delta}, V(x) + \inf_{b,y}\left[v\,(b_0 - b(t))^2 - (\rho+\delta)V + \mathcal{L}^{b,y}V \right] \right\}, \tag{16}$$

where

$$\mathcal{L}^{b,y}V = [-b + rx + (\lambda - r)xy]V'(x) + \frac{1}{2}\sigma^2 x^2 y^2 V''(x).$$

The HJB equation (16) in turn is equivalent to the two following variational inequalities

$$\inf_{b,y}[v(b_0 - b)^2 - (\rho+\delta)V(x) + \mathcal{L}^{b,y}V(x)] = 0 \qquad \text{and} \quad V(x) \le K(x) \qquad \text{for } x \in U \tag{17a}$$

$$\inf_{b,y}[v(b_0 - b)^2 - (\rho+\delta)V(x) + \mathcal{L}^{b,y}V(x)] \ge 0 \qquad \text{and} \quad V(x) = K(x) \qquad \text{for } x \in U^c \tag{17b}$$

where the region U is defined by

$$U = \{x \in \mathbb{R} : V(x) < K(x)\}.$$

Remark 3 In the continuation region (see (17a)), the HJB equation is exactly the same that one would have in a standard stochastic control problem, without optimal exit from the optimization program. This is consistent with the fact that as long as one lives in U, she behaves optimally as though there were no exit time.

The dual problem and the boundary conditions

In the continuation region, the value function satisfies (see (17a))

$$\frac{1}{2}\beta^2 \frac{(V')^2}{V''} + \frac{1}{4v}(V')^2 + (b_0 - rx)V' + (\rho+\delta)V = 0, \tag{18}$$

where $\beta = \frac{\lambda - r}{\sigma}$ is the Sharpe ratio of the risky asset. Given the high-nonlinearity of the PDE (18), [12] make use of a methodology largely used in stochastic control problems hard to solve with the guessing techniques. They transform the original problem into a dual one, by introducing a new function $X(z)$ to be the inverse of $-V'$

$$V'\big(X(z)\big) = -z,$$

the HJB equation changes into

$$-\frac{1}{2}\beta^2 z^2 X'(z) + \frac{1}{4v}z^2 - \big(b_0 - rX(z)\big)z + (\rho+\delta)V\big(X(z)\big) = 0,$$

which differentiated with respect to z becomes

$$-\frac{1}{2}\beta^2 z^2 X''(z) - (\rho+\delta+\beta^2-r)zX'(z) + rX(z) = b_0 - \frac{z}{2v}.$$

The general solution of this second-order Euler ODE is

$$X(z) = \frac{b_0}{r} - \frac{z}{2v(r-\gamma)} + C_1 z^{\alpha_1} + C_2 z^{\alpha_2}, \tag{19}$$

where C_1 and C_2 are constants to be determined by the boundary conditions, γ is a shorthand notation and is given by

$$\gamma = \rho + \delta + \beta^2 - r$$

and α_1 and α_2 are the two zeros of the quadratic

$$P(\alpha) = \frac{1}{2}\beta^2\alpha^2 + (\gamma - \frac{1}{2}\beta^2)\alpha - r, \tag{20}$$

so that

$$\alpha_1, \alpha_2 = \frac{-(\gamma - \frac{1}{2}\beta^2) \pm \sqrt{(\gamma - \frac{1}{2}\beta^2)^2 + 2r\beta^2}}{\beta^2}. \tag{21}$$

The corresponding value function is

$$V\big(X(z)\big) = \frac{z^2}{4v(r-\gamma)} - \frac{1}{\rho+\delta}\left[A_1 C_1 z^{1+\alpha_1} + A_2 C_2 z^{1+\alpha_2}\right], \tag{22}$$

where

$$A_1 = r - \frac{1}{2}\beta^2\alpha_1, \qquad A_2 = r - \frac{1}{2}\beta^2\alpha_2. \tag{23}$$

Notice that the coefficients A_1 and A_2 are both positive. In fact, the polynomial P given by (20) satisfies $P(2r/\beta^2) > 0$, so that $\alpha_i < 2r/\beta^2$ for $i = 1, 2$, thus $A_i = r - \frac{1}{2}\beta^2\alpha_i > 0$ for both i.

The optimal control functions can then be written as

$$y^*(X(z)) = -\frac{\beta}{\sigma}\frac{zX'(z)}{X(z)} \tag{24}$$

$$b^*(X(z)) = b_0 - \frac{z}{2v}. \tag{25}$$

[12] find that the continuation region is given by

$$U = [0, x^*) \cup \left(\frac{b_1}{k}, +\infty\right), \tag{26}$$

where

$$x^* \geq \frac{b_1}{k} - \frac{2rD}{\phi}, \tag{27}$$

with[10]

$$D = \frac{b_0}{r} - \frac{b_1}{k} > 0 \qquad \text{and} \qquad \phi = \rho + \delta + \beta^2 - 2r + k^2 \frac{w}{v(\rho+\delta)}. \tag{28}$$

Clearly, the difficult and crucial task – typical of this kind of problems – is to find x^*. To this aim, one needs the boundary conditions. We also notice that this is a boundary value problem, and therefore existence of solution is not guaranteed. [12] show that there are indeed three possible cases:

1. the parameters' values are such that a certain set of boundary conditions are satisfied, and we have a RP (Ruin Possibility) solution;
2. the parameters' values are such that a certain set of boundary conditions are satisfied, and we have a NR (No Ruin) solution;
3. the parameters' values are such that there is no solution to the problem.

Since explaining the technicalities of the problem is beyond the scope of the present paper, we here present the two sets of boundary conditions, without providing explanations. Both sets contain the obvious boundary conditions on x^*. These are continuity and smoothness (see [23]) of the value function at x^*

$$V(x^*) = K(x^*); \tag{29}$$

and

$$V'(x^*) = K'(x^*). \tag{30}$$

If we define z_* by $z_* = -V'(x^*)$, so that $X(z_*) = x^*$, then these two boundary conditions (29) and (30) can be written in the form

$$\begin{aligned} -z_* &= -\tfrac{2kw}{\rho+\delta}(b_1 - kx^*) \\ \tfrac{w}{\rho+\delta}(b_1 - kx^*)^2 &= \tfrac{z_*^2}{4v(r-\gamma)} - \tfrac{1}{\rho+\delta}\left[A_1 C_1 z_*^{1+\alpha_1} + A_2 C_2 z_*^{1+\alpha_2}\right] \\ x^* &= \tfrac{b_0}{r} - \tfrac{z_*}{2v(r-\gamma)} + C_1 z_*^{\alpha_1} + C_2 z_*^{\alpha_2}. \end{aligned} \tag{31}$$

Moreover, defining $z_0 = \inf\{z > 0 : X(z) = 0\}$ another common boundary condition is:

$$\frac{b_0}{r} - \frac{z_0}{2v(r-\gamma)} + C_1 z_0^{\alpha_1} + C_2 z_0^{\alpha_2} = 0. \tag{32}$$

What makes the difference between RP and NR solution is the boundary condition at z_0. The boundary condition at z_0 corresponding to a RP solution, i.e. $V(0) = K(0)$, is

$$\frac{z_0^2}{4v(r-\gamma)} - \frac{1}{\rho+\delta}\left[A_1 C_1 z_0^{1+\alpha_1} + A_2 C_2 z_0^{1+\alpha_2}\right] = \frac{wb_1^2}{\rho+\delta}. \tag{33}$$

[10] *The fact that $D > 0$ is not due to mathematical constraints, but to the formulation of the problem. In fact, it can be argued that a problem with $D \leq 0$ is not interesting from an applicative point of view. See [12].*

A NR solution is characterized by the fact that $V(0) \leq K(0)$, and we require that when the fund approaches 0, the fund is invested in the riskless asset only. This corresponds to: $X'(z_0) = 0$. These conditions are translated into

$$\begin{aligned} \alpha_1 C_1 z_0^{\alpha_1 - 1} + \alpha_2 C_2 z_0^{\alpha_2 - 1} &= \tfrac{1}{2v(r-\gamma)} \\ \tfrac{z_0^2}{4v(r-\gamma)} - \tfrac{1}{\rho+\delta}\left[A_1 C_1 z_0^{1+\alpha_1} + A_2 C_2 z_0^{1+\alpha_2}\right] &\leq \tfrac{wb_1^2}{\rho+\delta}. \end{aligned} \tag{34}$$

The set of boundary conditions that characterizes a RP solution consists in (31), (32) and (33), while the one that characterizes a NR solution consists in (31), (32) and (34). If none of the specified sets of boundary conditions can be satisfied, there is no solution to the problem.

The main theorem and its application

[12] prove the following theorem.

Theorem 4 *Assume that $D > 0$ and that $\phi \geq 2krD/b_1$. Suppose that there exist constants C_1, C_2, z_0 and z_* with $0 < z_* < z_0 < \infty$, such that the function $X(z)$ given by* (19) *satisfies the boundary conditions (31), (32) and either (33) or (34).*

Then:

(i) For each $z \in (z_*, z_0)$ there is a corresponding $x \in (0, x^*)$ such that $X(z) = x$;

(ii) the function V given by

$$\begin{aligned} &V(x) = 0, && \text{for } x \geq \tfrac{b_1}{k} \\ &V(x) = K(x), && \text{for } x^* \leq x \leq \tfrac{b_1}{k} \\ &V\big(X(z)\big) = \tfrac{z^2}{4v(r-\gamma)} - \tfrac{1}{\rho+\delta}\left[A_1 C_1 z^{1+\alpha_1} + A_2 C_2 z^{1+\alpha_2}\right] && \text{for } z_* \leq z \leq z_0 \end{aligned}$$

is the optimal value function;

(iii) the optimal time to annuitize is $\tau^* = \inf\{t : x(t) \in U^c\}$, where the continuation set U is given by

$$U = [0, x^*) \cup \left(\frac{b_1}{k}, \infty\right);$$

(iv) for values of x belonging to $[0, x^*)$, the optimal controls are given by

$$y^*(t) = -\frac{\lambda - r}{\sigma^2} \cdot \frac{V'\big(x(t)\big)}{x(t)\, V''\big(x(t)\big)}, \qquad b^*(t) = b_0 + \frac{1}{2v} V'\big(x(t)\big).$$

The hard task of solving the problem consists in finding constants that satisfy the assumptions of Theorem 4. [12] show a method of construction of a solution starting from the parameters the problem. They state an algorithm for numerical solutions and show that it leads either to RP, or to NR solution, or to no solution.