On Primitively Generated Braided Bialgebras

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ON PRIMITIVELY GENERATED BRAIDED BIALGEBRAS

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Abstract. The main aim of this paper is to investigate the structure of primitively generated braided bialgebras A with respect to the braided vector space P consisting of their primitive elements. When the Nichols algebra of P is obtained dividing out the tensor algebra $T(P)$ by the two-sided ideal generated by its primitive elements of degree at least two, we show that A can be recovered as a sort of universal enveloping algebra of P. One of the main applications of our construction is the description, in terms of universal enveloping algebras, of braided bialgebras whose associated graded coalgebra is a quadratic algebra.

Contents

1. Introduction

Let K be a fixed field, let H be a pointed Hopf algebra over K (this means that all its simple subcoalgebras are one dimensional). Denote by G the set of grouplike elements in H. It is well known that the graded coalgebra $\mathrm{gr}H$, associated to the coradical filtration of H, is a Hopf algebra itself and can be described as a Radford-Majid bosonization by KG of a suitable graded connected braided bialgebra R, called diagram of H, in the braided monoidal category of Yetter-Drinfeld modules over KG. This is the starting point of the so called *lifting method* for the classification of finite dimensional pointed Hopf algebras, introduced by N. Andruskiewitsch and H.J. Schneider, see e.g. [ASc1]. Accordingly to this method, first one has to describe R by generators and relations, then to lift the informations obtained to H. It is worth noticing that in the finite dimensional case there is a conjecture asserting that, in characteristic zero, R is always primitively generated [ASc3, Conjecture 2.7] (see also [ASc2, Conjecture 1.4]).

Therefore, in many cases, for proving certain properties of Hopf algebras, it is enough to do it in the connected case. The price that one has to pay is to work with Hopf algebras in a braided category, and not with ordinary Hopf algebras. Actually in our case it is more convenient to work with braided bialgebras, that were introduced in [Ta].

Motivated by these observations, in this paper we will investigate the structure of primitively generated (whence connected) braided bialgebras. Let A be such a bialgebra. Then A is in

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particular a braided vector space i.e. it is endowed with a map $c : A \otimes A \to A \otimes A$, called braiding, fulfilling the quantum Yang-Baxter equation (1). It is remarkable that c induces a braiding $c_P : P \otimes P \to P \otimes P$ on the space P of primitive elements in A. The braided vector space (P, c_P) will be called the infinitesimal part of the braided bialgebra A. The tensor algebra $T(P)$ carries a braided bialgebra structure depending on c_P, that will be denoted by $T(P, c_P)$, and there is a unique surjective braided bialgebra map $\varphi : T(P, c_P) \to A$ that lifts the inclusion $P \subseteq A$. Let $E(P, c_P)$ be the space spanned by the homogeneous primitive elements in $T(P, c_P)$ of degree at least two. Since φ preserves primitive elements, we can define the map $b_P : E(P, c_P) \to P, b(z) = \varphi(z)$ and consider the quotient

$$U(P, c_P, b_P) := \frac{T(P, c_P)}{((\mathrm{Id} - b_P)[E(P, c_P)])}.$$

Now φ quotients to a surjective braided bialgebra map $\overline{\varphi} : U(P, c_P, b_P) \to A$. It is worth noticing that the canonical K-linear map $i_U : P \to U(P, c_P, b_P)$ is injective as $\overline{\varphi} \circ i_U$ is the inclusion $P \subseteq A$. It is also notable that $E(P, c_P)$ may contain homogeneous elements of arbitrary degree.

By a famous result due to Heyneman and Radford (see [Mo, Theorem 5.3.1]), $\overline{\varphi}$ is an isomorphism if the space of primitive elements in $U(P, c_P, b_P)$ identifies with P via i_U (the converse is trivial).

We prove that this property holds whenever P belongs to a large class $\mathcal{S}$ of braided vector spaces. It stems from our construction that $\mathcal{S}$ can be taken to be the class of braided vector spaces (V, c) such that the Nichols algebra $\mathcal{B}(V, c)$, i.e. the image of the canonical graded braided bialgebra homomorphism from the tensor algebra $T(V, c)$ into the quantum shuffle algebra $T^c(V, c)$, is obtained dividing out $T(V, c)$ by the two-sided ideal generated by $E(V, c)$. As mentioned, although not all braided vector spaces belongs to it, the class $\mathcal{S}$ is quite large. Meaningful examples can be found in [Ar]. An application of our construction is that $A \simeq U(P, c_P, b_P)$ whenever A is a braided bialgebra such that $\mathrm{gr}(A)$ is a quadratic algebra with respect to its natural braided bialgebra structure.

We point out that the structure and properties of $U(P, c_P, b_P)$ are encoded in the datum (P, c_P, b_P). Indeed this leads to the introduction of what will be called a braided Lie algebra (V, c, b) for any braided vector space (V, c) and of the related universal enveloping algebra $U(V, c, b)$. When c is a symmetry, i.e. $c^2 = \mathrm{Id}_{V \otimes V}$, and the characteristic of K is zero, our enveloping algebra reduces to the one introduced in [Gu1]. In this case, the crucial property, mentioned above, that the space of primitive elements in $U(V, c, b)$ identifies with V via i_U was proved in [Kh5, Lemma 6.2] and it was applied to obtain a Cartier-Kostant-Milnor-Moore type theorem for connected braided Hopf algebras with involutive braidings [Kh5, Theorem 6.1]. Other notions of Lie algebra and enveloping algebra, extending the ones in [Gu1] to the non symmetric case, appeared in the literature. Let us mention some of them.

- Lie algebras for braided vector spaces (V, c) where c is a braiding of Hecke type i.e. $(c + \mathrm{Id}_{V \otimes V})(c - q\mathrm{Id}_{V \otimes V}) = 0$ for some $q \in K \setminus \{0\}$ which is called the mark of c [Wa, Definition 7.1]. See [AMS1, Theorem 5.5] for a strong version of Cartier-Kostant-Milnor-Moore theorem in this setting.
- Lie algebras for braided vector spaces (V, c) where c is not of Hecke-type but constructed by means of braidings of Hecke type [Gu2, Definition 1]. There a suitable quadratic algebra is required to be Koszul.
- Lie algebras for objects in the braided monoidal category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode [Pa, Definition 4.1] (here called Pareigis-Lie algebras - see Definition 8.5). In the present paper we also stress the relation between our notion of Lie algebra and the one by Pareigis, which involves partially defined n-ary bracket multiplications as well.
- Lie algebras defined by considering quantum operations (see [Kh1, Definition 2.2]) as primitive polynomials in the tensor algebra. When the underline braided vector space is an object in the category of Yetter-Drinfeld modules over some group algebra, to any Lie algebra of this kind, the author associates a universal enveloping algebra which is not connected (see [Kh2]). Note that the universal enveloping algebra we introduce in the present

paper is connected as the scope here is to investigate the structure of primitively generated (whence connected) braided bialgebras over K.

Finally, many authors have tried to characterize quantized enveloping algebras $U_q(\mathfrak{g})$ of Drinfeld and Jimbo or other quantum groups as bialgebras generated by a finite-dimensional Lie algebra like object via some kind of enveloping algebra construction (see the introduction of [Maj]). One idea is to attempt to build this on $\mathfrak{g}$ itself but with some kind of deformed bracket obeying suitable new axioms. Alternatively one can consider some subspace of $U_q(\mathfrak{g})$ endowed with some kind of 'quantum Lie bracket' based on the quantum adjoint action. Contributions to these problems can be found e.g. in [Wo, LS, DGG, GM, GS] and in the references therein.

Now, since $U_q(\mathfrak{g})$ is not connected but just pointed, we have no hope to describe it directly as an enveloping algebra of our kind (see [Kh3, Section 2] for a different approach). Furthermore, it is remarkable that, by [Mas, Theorem 7.8], $U_q(\mathfrak{g})$ can be viewed as a cocycle deformation of $\mathrm{gr}\left(U_q(\mathfrak{g})\right)$. By the foregoing, $\mathrm{gr}\left(U_q(\mathfrak{g})\right)$ is the Radford-Majid bosonization of the diagram Q of $U_q(\mathfrak{g})$ by KG, where G denotes the set of grouplike elements in $U_q(\mathfrak{g})$. One could believe that Q is an enveloping algebra of our kind as it is a primitively generated connected braided bialgebra. The point is that Q is much more. It is a Nichols algebra whence defined as a quotient of the tensor algebra $T\left(P\left(Q\right)\right)$ by homogeneous relations. Thus no enveloping algebra is required to describe Q.

Methodology. We proceed as follows. Let (V,c) be a braided vector space and denote by $E\left(V,c\right)$ the space generated by homogeneous primitive elements in $T\left(V,c\right)$ of degree at least two. A bracket on (V,c) is a K-linear map $b:E\left(V,c\right)\rightarrow V$ that commutes with the braiding of $T(V,c)$ in the sense that (10) is satisfied. If b is a bracket on (V,c), then the *universal enveloping algebra* of (V,c) is defined to be

$$U\left(V,c,b\right):=\frac{T\left(V,c\right)}{\left(\left(\mathrm{Id}-b\right)\left[E\left(V,c\right)\right]\right)}.$$

Thus $U\left(V,c,b\right)$ carries a unique braided bialgebra structure such that the canonical projection $\pi_U:T\left(V,c\right)\rightarrow U\left(V,c,b\right)$ is a braided bialgebra homomorphism (Theorem 3.6). We say that (V,c,b) is a *braided Lie algebra* whenever (V,c) is a braided vector space, $b:E\left(V,c\right)\rightarrow V$ is a bracket on (V,c) and the canonical K-linear map $i_U:V\rightarrow U\left(V,c,b\right)$ is injective (Definition 4.1). Natural examples of braided Lie algebras arise as follows:

1) If (V,c) is a braided vector space, then $(V,c,0)$ is a braided Lie algebra (Proposition 4.2). The corresponding universal enveloping algebra is the *symmetric algebra* $S\left(V,c\right)$.
2) If (A,c) is a braided algebra, then (A,c,b) is a braided Lie algebra, where b acts on the t-th graded component of $E\left(A,c\right)$ as the restriction of the iterated multiplication of A (Proposition 4.5).
3) If (A,c) is a braided bialgebra and $P=P\left(A\right)$ is the space of primitive elements of A, then P forms a braided Lie algebra (P,c_P,b_P) with structures induced by (A,c,b) as in 2), see Lemma 4.7. (P,c_P,b_P) will be called the *infinitesimal braided Lie algebra of* A.

In Theorem 5.4 we prove a PBW type theorem, asserting that, when (V,c) belongs to the class $\mathcal{S}$, then a braided Lie algebra (V,c,b) is of PBW type in the sense of Definition 4.14.

Furthermore, in Corollary 5.5 we get that $i_U:V\rightarrow U\left(V,c,b\right)$ induces an isomorphism between V and $P\left(U\left(V,c,b\right)\right)$.

The main consequence of this property, which really justifies our construction, is Theorem 5.7 concerning primitively generated braided bialgebras A. If $(P,c_P)\in\mathcal{S}$, then A is isomorphic to $U\left(P,c_P,b_P\right)$ as a braided bialgebra, where (P,c_P,b_P) is the infinitesimal braided Lie algebra of A. The proof of this fact uses the universal property of the universal enveloping algebra (Theorem 4.9) to obtain a bialgebra projection of $U\left(P,c_P,b_P\right)$ onto A; this projection comes out to be injective as $P\left(U\left(V,c,b\right)\right)$ identifies with V via i_U.

The main applications and examples we are interested in are given in the last Sections 9, 6, 7 and 8.

Explicitly, in Section 6 we deal with the class of braided vector spaces (V,c) such that c is a braiding of Hecke type i.e. it satisfies the equation $(c+\mathrm{Id}_{V\otimes V})\,(c-q\mathrm{Id}_{V\otimes V})=0$ for some regular

$q \in K \setminus \{0\}$ (see Definition 6.1) called mark of c. In Theorem 6.3, using results in [AMS1, AMS2, Ar], we show that the universal enveloping algebra of (V, c, b) reduces to

$$U(V, c, b) = \frac{T(V, c)}{(c(z) - qz - [z]_b \mid z \in V \otimes V)}.$$

Here $[-]_b : V \otimes V \to V$ is given by $[z]_b = b(c(z) - qz)$ and it is zero if $q \neq 1$ and $\mathrm{char}(K) \neq 2$.

Section 7, is devoted to study the class of braided vector spaces (V, c) such that the Nichols algebra $\mathcal{B}(V, c)$ is a quadratic algebra (see Definition 7.1). In the literature there is plenty of examples of braided vector spaces of this kind, see Remark 7.3. The main result of the section is Theorem 7.2 concerning braided bialgebras A such that the graded coalgebra $\mathrm{gr}(A)$ associated to the coradical filtration of A is a quadratic algebra with respect to its natural braided bialgebra structure. Such an A is proved to be isomorphic as a braided bialgebra to the enveloping algebra of its infinitesimal braided Lie algebra. The proof relies on the fact that the Nichols algebra associated to this braided Lie algebra results to be quadratic. Braided Lie algebras with this property will be further investigated in [ASt].

In Section 8, we stress the relation between our notions of Lie algebra and universal enveloping algebra, and those introduced by Pareigis in [Pa]. The main result of the section is Theorem 8.7 establishing that a Pareigis-Lie algebra $(V, c, [-])$ is associated to any braided Lie algebra (V, c, b). Moreover there is a canonical braided bialgebra projection $p : U_P(V, c, [-]) \to U(V, c, b)$ where $U_P(V, c, [-])$ denotes the universal enveloping algebra of $(V, c, [-])$ (see Definition 8.5). We prove this projection to be an isomorphism if condition (21) holds.

Note that, if (V, c) lies in $\mathcal{S}$, then p is an isomorphism if and only if $P(U_P(V, c, [-]))$ identifies with V through the canonical map $i_{U_P} : V \to U_P(V, c, [-])$, see Remark 8.8.

In Section 9, we collect general results on braided vector spaces of diagonal type. In particular, in Theorem 9.5, we prove that such a braided vector space is endowed with a bracket only if it fulfills suitable identities. This result can be used to establish conditions on the bracket. As an application, we prove that both the infinitesimal part of a quantum linear space and a braided vector space with braiding of Drinfeld-Jimbo type admit only trivial brackets. We also prove that the only primitively generated braided bialgebras having one of these braided vector spaces as infinitesimal part are the corresponding Nichols algebras.

2. Preliminaries

Throughout this paper K will denote a field. All vector spaces will be defined over K and the tensor product over K will be denoted by $\otimes$.

In this section we define the main notions that we will deal with in the paper.

Definition 2.1. Let V be a vector space over a field K. A K-linear map $c = c_V : V \otimes V \to V \otimes V$ is called a **braiding** if it satisfies the quantum Yang-Baxter equation

$$c_1 c_2 c_1 = c_2 c_1 c_2 \tag{1}$$

on $V \otimes V \otimes V$, where we set $c_1 := c \otimes V$ and $c_2 := V \otimes c$. The pair (V, c) will be called a **braided vector space**. A morphism of braided vector spaces (V, c_V) and (W, c_W) is a K-linear map $f : V \to W$ such that $c_W(f \otimes f) = (f \otimes f)c_V$.

Remark 2.2. Note that, for every braided vector space (V, c) and every $k \in K$, the pair (V, kc) is a braided vector space too.

A general method for producing braided vector spaces is to take an arbitrary braided category $(\mathcal{M}, \otimes, K, a, l, r, c)$, which is a monoidal subcategory of the category of K-vector spaces (here a, l, r denote the associativity, the left and the right unit constraints respectively). Hence any object $V \in \mathcal{M}$ can be regarded as a braided vector space with respect to $c := c_{V,V}$, where $c_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the braiding in $\mathcal{M}$, for all $X, Y \in \mathcal{M}$.

Let $\mathcal{N}$ be either the category of comodules over a coquasitriangular Hopf algebra or the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode. Then the forgetful functor F from $\mathcal{N}$ into the category of K-vector spaces is a strict monoidal functor. Hence $\mathcal{M} = \mathrm{Im}F$ is an example of a category as above.

DEFINITION 2.3 (Baez, [Ba]). A quadruple (A, m, u, c) is called a **braided algebra** if

- (A, m, u) is an associative unital algebra;
- (A, c) is a braided vector space;
- m and u commute with c, that is the following conditions hold:

$$c(m \otimes A) = (A \otimes m)(c \otimes A)(A \otimes c), \tag{2}$$

$$c(A \otimes m) = (m \otimes A)\,(A \otimes c)\,(c \otimes A), \tag{3}$$

$$c(u \otimes A) = A \otimes u, \qquad c(A \otimes u) = u \otimes A. \tag{4}$$

A morphism of braided algebras is, by definition, a morphism of ordinary algebras which, in addition, is a morphism of braided vector spaces.

A quadruple $(C, \Delta, \varepsilon, c)$ is called a **braided coalgebra** if

- (C, Δ, ε) is a coassociative counital coalgebra;
- (C, c) is a braided vector space;
- Δ and ε commute with c, that is the following relations hold:

$$(\Delta \otimes C)c = (C \otimes c)(c \otimes C)(C \otimes \Delta), \tag{5}$$

$$(C \otimes \Delta)c = (c \otimes C)(C \otimes c)(\Delta \otimes C), \tag{6}$$

$$(\varepsilon \otimes C)c = C \otimes \varepsilon, \qquad (C \otimes \varepsilon)c = \varepsilon \otimes C. \tag{7}$$

A morphism of braided coalgebras is, by definition, a morphism of ordinary coalgebras which, in addition, is a morphism of braided vector spaces.

[Ta, Definition 5.1] A sextuple $(B, m, u, \Delta, \varepsilon, c)$ is a called a **braided bialgebra** if

- (B, m, u, c) is a braided algebra
- $(B, \Delta, \varepsilon, c)$ is a braided coalgebra
- the following relations hold:

$$\Delta_B m = (m \otimes m)(B \otimes c \otimes B)(\Delta \otimes \Delta). \tag{8}$$

Examples of the notions above are algebras, coalgebras and bialgebras in any braided category $\mathcal{M}$ which is a monoidal subcategory of the category of K-vector spaces. The notion of braided bialgebra admits a graded counterpart which is called graded braided bialgebra. For further results on this topic the reader is refereed to [Ar].

EXAMPLE 2.4. Let (V, c) be a braided vector space. Consider the tensor algebra $T = T(V)$ and let m_T and u_T denote its multiplication and unit respectively. This is a graded braided algebra with n-th graded component $T^n(V) = V^{\otimes n}$. The braiding c_T on T is defined using the the braiding of V :

Now $T \otimes T$ becomes itself an algebra with multiplication $m_{T\otimes T} := (m_T \otimes m_T) \circ (T \otimes c_T \otimes T)$. This algebra is denoted by $T \otimes_c T$. The universal property of the tensor algebra yields two algebra homomorphisms $\Delta_T : T \to T \otimes_c T$ and $\varepsilon_T : T \to K$. It is straightforward to check that $(T, m_T, u_T, \Delta_T, \varepsilon_T, c_T)$ is a graded braided bialgebra.

Definition 2.5. The graded braided bialgebra described in Example 2.4 is called the **braided tensor algebra** and is denoted by $T(V,c)$.

Note that Δ_T really depends on c. For instance, one has $\Delta_T(z) = z \otimes 1 + 1 \otimes z + (c + \mathrm{Id})(z)$, for all $z \in V \otimes V$.

2.6. Recall that a coalgebra C is called **connected** if the coradical C_0 of C (i.e the sum of all simple subcoalgebras of C) is one dimensional. In this case there is a unique group-like element $1_C \in C$ such that $C_0 = K1_C$. A morphism of connected coalgebras is just a coalgebra homomorphisms (clearly it preserves the grouplike element).

By definition, a braided coalgebra (C,c) is **connected** if the underlying coalgebra is connected and, for any $x \in C$, $c(x \otimes 1_C) = 1_C \otimes x$ and $c(1_C \otimes x) = x \otimes 1_C$.

Remark 2.7. Let $C = \bigoplus_{n\in\mathbb{N}} C^n$ be a graded braided coalgebra. By [Sw, Proposition 11.1.1], if $(C_n)_{n\in\mathbb{N}}$ is the coradical filtration, then $C_n \subseteq \bigoplus_{0\le m\le n} C^m$. Therefore, C is connected if C^0 is a one dimensional vector space.

Definitions 2.8. For a braided vector space (V,c), set

$$E_0(V,c) := 0, \qquad E_1(V,c) := 0 \qquad \text{and} \qquad E_n(V,c) := \bigcap_{1\le i\le n-1} \mathrm{Ker}\left(\Delta_T^{i,n-i}\right), \text{ for } n \ge 2$$

and set $E(V,c) := \bigoplus_{n\in\mathbb{N}} E_n(V,c)$.

Remark 2.9. It is easy checked (see e.g. [Ar, Lemma 2.7 and notations in Section 5]) that the space of primitive elements in $T(V,c)$ is given by $P(T(V,c)) = V \oplus [\oplus_{n\ge2} E_n(V,c)]$. Moreover $K \wedge K = K \oplus V \oplus [\oplus_{n\ge2} E_n(V,c)]$.

In the connected case, the following result follows from [Kh5, page 4]: in fact P is a categorical subspace of B.

Lemma 2.10. *Let (B,c) be a braided bialgebra. Let $P = P(B)$ be the space of primitive elements of B. Then $c(P \otimes B) \subseteq B \otimes P$, $c(B \otimes P) \subseteq P \otimes B$ and $c(P \otimes P) \subseteq P \otimes P$. Let $c_P : P \otimes P \to P \otimes P$ be the restriction of c to $P \otimes P$. Then (P, c_P) is a braided vector space.*

Proof. Denote by Δ the comultiplication of B and by $i : P \to B$ the canonical inclusion. Let $\alpha : B \to B \otimes B, \alpha(b) = b \otimes 1_B + 1_B \otimes b$. Then we have the equalizer

$$0 \longrightarrow P \xrightarrow{\ i\ } B \underset{\Delta}{\overset{\alpha}{\rightrightarrows}} B$$

Consider the following diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & P \otimes B & \xrightarrow{i\otimes B} & B \otimes B & \underset{\Delta\otimes B}{\overset{\alpha\otimes B}{\rightrightarrows}} & B \otimes B \otimes B \\
& \vdots\, c_{P,B} & & \downarrow c & & \downarrow (c\otimes B)(B\otimes c) \\
0 \longrightarrow & B \otimes P & \xrightarrow{B\otimes i} & B \otimes B & \underset{B\otimes\Delta}{\overset{B\otimes\alpha}{\rightrightarrows}} & B \otimes B \otimes B
\end{array}$$

The right square commutes by (6) and

$$\begin{aligned}
& (c \otimes B)(B \otimes c)(\alpha \otimes B)(x \otimes y) \\
= \; & (c \otimes B)(B \otimes c)(x \otimes 1_B \otimes y + 1_B \otimes x \otimes y) \\
= \; & (c \otimes B)(x \otimes y \otimes 1_B + 1_B \otimes c(x \otimes y)) \\
= \; & c(x \otimes y) \otimes 1_B + (c \otimes B)\left[1_B \otimes \sum y^i \otimes x^i\right] \\
= \; & \sum y^i \otimes x^i \otimes 1_B + \sum y^i \otimes 1_B \otimes x^i \\
= \; & (B \otimes \alpha)\left(\sum y^i \otimes x^i\right) \\
= \; & (B \otimes \alpha)\, c(x \otimes y).
\end{aligned}$$

where $c(x\otimes y):=\sum y^{i}\otimes x^{i}$. Then, by universal property of equalizers there is a unique map $c_{P,B}:P\otimes B\rightarrow B\otimes P$ such that $c(i\otimes B)=(B\otimes i)\,c_{P,B}$. In other words $c(P\otimes B)\subseteq B\otimes P$. Similarly one gets that $c(B\otimes P)\subseteq P\otimes B$. Then

$$c(P\otimes P)\subseteq c(P\otimes B)\cap c(B\otimes P)\subseteq[B\otimes P]\cap[P\otimes B]=P\otimes P.$$

□

Definition 2.11. With same assumptions and notations as in Lemma 2.10, (P,c_{P}) will be called the **infinitesimal part of** B while c_{P} will be called the **infinitesimal braiding of** B.

3. Universal enveloping algebra of a braided vector space

In this section we introduce and investigate the notion of universal enveloping algebra of a braided vector space endowed with a bracket.

3.1. Let (V,c) be a braided vector space and set $T:=T(V,c)$. It is easy checked that

$$c_{T}\left(V^{\otimes u}\otimes E_{t}\left(V,c\right)\right)\subseteq E_{t}\left(V,c\right)\otimes V^{\otimes u}\qquad\text{and}\qquad c_{T}\left(E_{t}\left(V,c\right)\otimes V^{\otimes u}\right)\subseteq V^{\otimes u}\otimes E_{t}\left(V,c\right)$$

hold for each $u\in\mathbb{N}$ (see [Ar, Remark 3.5]). Hence there exists a unique morphism

$$c_{E_{t}(V,c),V^{\otimes u}}:E_{t}\left(V,c\right)\otimes V^{\otimes u}\rightarrow V^{\otimes u}\otimes E_{t}\left(V,c\right)$$

such that

$$\left(V^{\otimes u}\otimes j^{t}\right)\circ c_{E_{t}(V,c),V^{\otimes u}}=c_{T}^{t,u}\circ\left(j^{t}\otimes V^{\otimes u}\right),\tag{9}$$

where $j^{t}:E_{t}\left(V,c\right)\rightarrow V^{\otimes t}$ denotes the canonical injection. Similarly one gets $c_{V^{\otimes u},E_{t}(V,c)}$. Define now

$$\begin{aligned}c_{E(V,c),V^{\otimes u}}&:=\bigoplus_{t\in\mathbb{N}}c_{E_{t}(V,c),V^{\otimes u}}:E\left(V,c\right)\otimes V^{\otimes u}\rightarrow V^{\otimes u}\otimes E\left(V,c\right),\\ c_{V^{\otimes u},E(V,c)}&:=\bigoplus_{t\in\mathbb{N}}c_{V^{\otimes u},E_{t}(V,c)}:V^{\otimes u}\otimes E\left(V,c\right)\rightarrow E\left(V,c\right)\otimes V^{\otimes u}.\end{aligned}$$

Let $f:(V,c_{V})\rightarrow(W,c_{W})$ be a morphism of braided vector spaces. The universal property of the braided tensor algebra yields, in an obvious way, a graded braided bialgebra homomorphism $T(f):(V,c_{V})\rightarrow(W,c_{W})$. It is easy to check that, for all $t\geq2$, $T(f)$ restricts to a map

$$E_{t}(f):E_{t}(V,c_{V})\rightarrow E_{t}(W,c_{W}).$$

We set $E\left(f\right):=\bigoplus_{n\in\mathbb{N}}E_{t}\left(f\right)$.

Definition 3.2. A **braided bracket** or simply a **bracket** on a braided vector space (V,c) is a K-linear map $b:E\left(V,c\right)\rightarrow V$ such that

$$c\left(b\otimes V\right)=\left(V\otimes b\right)c_{E(V,c),V}\qquad c\left(V\otimes b\right)=\left(b\otimes V\right)c_{V,E(V,c)}.\tag{10}$$

The restriction of b to $E_{t}\left(V,c\right)$ will be denoted by $b^{t}:E_{t}\left(V,c\right)\rightarrow V$. A **morphism of brackets** $f:(V,c_{V},b_{V})\rightarrow(W,c_{W},b_{W})$ is a morphism of braided vector spaces $f:(V,c_{V})\rightarrow(W,c_{W})$ such that

$$f\circ b_{V}^{t}=b_{W}^{t}\circ E_{t}\left(f\right).\tag{11}$$

Lemma 3.3. *Let (V,c) be a braided vector space endowed with a bracket $b:E\left(V,c\right)\rightarrow V$. Then, for every $t\in\mathbb{N}$, the following equalities hold*

$$c_{T}^{1,t}\left(b\otimes V^{\otimes t}\right)=\left(V^{\otimes t}\otimes b\right)c_{E(V,c),V^{\otimes t}},\qquad c_{T}^{t,1}\left(V^{\otimes t}\otimes b\right)=\left(b\otimes V^{\otimes t}\right)c_{V^{\otimes t},E(V,c)}.\tag{12}$$

Proof. Set $T:=T\left(V,c\right)$. We proceed by induction on $t\geq1$. For $t=1$ there is nothing to prove as $c_{T}^{1,1}=c$. Assume that the formulas are true for $t-1$. Then, by construction of c_{T}, we have

$$\begin{aligned}c_{T}^{1,t}\left(b\otimes V^{\otimes t}\right)&=\left(V^{\otimes t-1}\otimes c_{T}^{1,1}\right)\left(c_{T}^{1,t-1}\otimes V\right)\left(b\otimes V^{\otimes t-1}\otimes V\right)\\&=\left(V^{\otimes t-1}\otimes c_{T}^{1,1}\right)\left(V^{\otimes t-1}\otimes b\otimes V\right)\left(c_{E(V,c),V^{\otimes t-1}}\otimes V\right)\end{aligned}$$

$$= \left(V^{\otimes t-1} \otimes V \otimes b\right) \left(V^{\otimes t-1} \otimes c_{E(V,c),V}\right) \left(c_{E(V,c),V^{\otimes t-1}} \otimes V\right)$$
$$= \left(V^{\otimes t} \otimes b\right) c_{E(V,c),V^{\otimes t}}$$

and similarly we get the second equality. □

REMARK 3.4. Note that (12) is related to [Ta, Lemma 1.7, page 304]. Nevertheless, in our case, the braided vector spaces involved need not to be finite dimensional.

DEFINITION 3.5. If b is a bracket for a braided vector space (V, c), we define **the universal enveloping algebra of** (V, c, b) to be

$$U(V, c, b) := \frac{T(V, c)}{((\mathrm{Id} - b)[E(V, c)])}.$$

We will denote by $i_U : V \to U(V, c, b)$ the restriction to V of the canonical projection $\pi_U : T(V, c) \to U(V, c, b)$. Note that 0 is always a bracket for (V, c) so that it makes sense to consider

$$S(V, c) := U(V, c, 0) = \frac{T(V, c)}{(E(V, c))}.$$

This algebra will be called **the symmetric algebra of** (V, c) and it coincides with the one introduced in [Ar, Section 5]. We will denote by $\pi_S : T(V, c) \to S(V, c)$ its canonical projection.

THEOREM 3.6. *Let (V, c) be a braided vector space endowed with a bracket $b : E(V, c) \to V$. Then $U(V, c, b)$ has a unique braided bialgebra structure such that the canonical projection $\pi_U : T(V, c) \to U(V, c, b)$ is a braided bialgebra homomorphism.*

Proof. Set $T := T(V,c)$ and set $I = ((\mathrm{Id} - b)[E(V, c)])$. Let us check that I is a coideal i.e. that $\Delta_T(I) \subseteq I \otimes T + T \otimes I$ and $\varepsilon_T(I) = 0$. First, for every $z \in E_t(V, c), x \in V^{\otimes u}, y \in V^{\otimes v}, s \in V^{\otimes w}$, using the compatibility of c_T and Δ_T with m_T and using (12) one can prove that $c_T(x(z - b(z))y \otimes s) \in V^{\otimes w} \otimes I$ so that

$$c_T\left(I \otimes V^{\otimes w}\right) \subseteq V^{\otimes w} \otimes I \qquad \text{and} \qquad c_T\left(V^{\otimes w} \otimes I\right) \subseteq I \otimes V^{\otimes w} \tag{13}$$

where the second equality can be obtained similarly. Let $z \in E_t(B)$. By Remark 2.9, one has

$$u = z - b(z) \in E_t(V, c) + V \subseteq V \oplus [\oplus_{n\geq 2} E_n(V, c)] = P(T)$$

so that $\Delta_T(u) = 1 \otimes u + u \otimes 1 \in T \otimes I + I \otimes T$. Let $x \in V^{\otimes c}$. Using (8), the last relation and (13) we get that $\Delta(xu) \in T \otimes I + I \otimes T$. Using (8), the last relation and (13), for every $y \in V^{\otimes d}$ we obtain $\Delta(xuy) \in T \otimes I + I \otimes T$.

In conclusion $\Delta(I) \subseteq I \otimes T + T \otimes I$. Therefore I is coideal and hence $U = U(V, c, b)$ carries a unique coalgebra structure such that the canonical projection $\pi_U : T \to U$ is a coalgebra homomorphism. In fact there are unique morphisms $\Delta_U : T/I \to T/I \otimes T/I$ and $\varepsilon_U : T/I \to K$ such that $\Delta_U \pi_U = (\pi_U \otimes \pi_U)\Delta$ and $\varepsilon_U \pi_U = \varepsilon$. Using that π_U is an epimorphism and that T is a coalgebra one gets that $(U, \Delta_U, \varepsilon_U)$ is a coalgebra too. Let us prove that c_T factors through a braiding c_U of U that makes U a braided bialgebra. We get

$$c_T(\mathrm{Ker}(\pi_U \otimes \pi_U)) = c_T(I \otimes T + T \otimes I) \overset{(13)}{\subseteq} I \otimes T + T \otimes I = \mathrm{Ker}(\pi_U \otimes \pi_U)$$

so that, since $\pi_U \otimes \pi_U$ is surjective, there exists a unique K-linear map $c_U : U \otimes U \to U \otimes U$ such that $c_U(\pi_U \otimes \pi_U) = (\pi_U \otimes \pi_U)c_T$. This relation can be used to prove that $(U, m_U, u_U, \Delta_U, \varepsilon_U, c_U)$ is a braided bialgebra with structure induced by π_U. Here $m_U : U \otimes U \to U$ and $u_U : K \to U$ denote the multiplication and the unit of U respectively. □

REMARK 3.7. Since both π_U and π_S are an epimorphisms, then $U(V, c, b)$ and $S(V, c)$ are connected coalgebras (cf. [Mo, Corollary 5.3.5]).

4. Braided Lie algebras

Definition 4.1. We say that (V,c,b) is a **braided Lie algebra** whenever

- (V,c) is a braided vector space;
- $b:E(V,c)\to V$ is a bracket on (V,c);
- the canonical map $i_U:V\to U(V,c,b)$ is injective i.e. $V\cap \mathrm{Ker}(\pi_U)=\mathrm{Ker}(i_U)=0$.

Injectivity of i_U plays here the role of an implicit Jacobi identity. On the other hand, antisymmetry of the bracket is encoded in the choice of the domain of b. This becomes clearer when c is of Hecke type as in Theorem 6.3. There it is shown that b can be replaced by the c-antisymmetric map $[-]_b:V\otimes V\to V$ defined by setting $[z]_b=b(c-q\mathrm{Id}_{V\otimes V})(z)$, for every $z\in V\otimes V$.

A first example of braided Lie algebra is given in the following proposition.

Proposition 4.2. *Let (V,c) is a braided vector space. Then $(V,c,0)$ is a braided Lie algebra.*

Proof. We already observed that 0 is always a bracket for (V,c) whence it remains to prove that i_U is injective. Set $A:=\frac{T(V,c)}{(V\otimes V)}$ where $(V\otimes V)$ is the two sided ideal of $T(V,c)$ generated by $V\otimes V$. Denote by $\pi:T(V,c)\to A$ and $\pi_S:T(V,c)\to S(V,c)$ the canonical projections. Since $E(V,c)\subseteq(V\otimes V)$, then $\pi:T(V,c)\to A$ quotients to an algebra homomorphism $\overline{\pi}:S(V,c)=U(V,c,0)\to A$ such that $\overline{\pi}\circ\pi_S=\pi$. Hence $\overline{\pi}\circ i_U=\overline{\pi}\circ\pi_S\circ i_T=\pi\circ i_T$ where $i_T:V\to T(V,c)$ is the canonical injection. Note that as a vector space $A\simeq K\oplus V$ and through this isomorphism, $\pi\circ i_T$ identifies with the canonical injection of V in $K\oplus V$ whence i_U is injective too. □

We now give some conditions guaranteeing that a braided Lie algebra has trivial bracket.

Proposition 4.3. *Let (V,c,b) be a braided Lie algebra. Let $p:T(V,c)\to\mathcal{B}(V,c)$ be the canonical projection. Suppose the ideal $\mathrm{Ker}(p)$ is generated by a subset of $\mathrm{Ker}(\pi_U)$. Then b is the trivial bracket of (V,c) and $U(V,c,b)=\mathcal{B}(V,c)$.*

Proof. Since the ideal $\mathrm{Ker}(p)$ is generated by a subset of $\mathrm{Ker}(\pi_U)$, there exists a unique algebra homomorphism $\phi:\mathcal{B}(V,c)\to U(V,c,b)$ such that $\phi\circ p=\pi_U$. We have $i_Ub(E(V,c))=\pi_Ub(E(V,c))=\pi_U(E(V,c))=\phi p(E(V,c))=0$. Since i_U is injective we get that $b(E(V,c))=0$ i.e. $b=0$. Now, using surjectivity of p one deduce that ϕ is indeed a braided bialgebra homomorphism. Moreover ϕ is surjective as π_U is. Note that $\phi_{|V}=(\phi\circ p)_{|V}=(\pi_U)_{|V}=i_U$ which is injective. Since $P(\mathcal{B}(V,c))\cong V$, by [Mo, Lemma 5.3.3], ϕ is injective whence bijective. □

Corollary 4.4. *Let (V,c,b) be a braided Lie algebra. Let $p:T(V,c)\to\mathcal{B}(V,c)$ be the canonical projection. Suppose the ideal $\mathrm{Ker}(p)$ is generated by a subset of $\mathrm{Ker}(b)$. Then b is the trivial bracket of (V,c) and $U(V,c,b)=\mathcal{B}(V,c)$.*

Proof. Since the ideal $\mathrm{Ker}(p)$ is generated by a set $W\subset\mathrm{Ker}(b)$, we have $\pi_U(W)=\pi_Ub(W)=0$. Hence $W\subset\mathrm{Ker}(\pi_U)$. We conclude by applying Proposition 4.3. □

Next aim is to associate some braided Lie algebras to any braided bialgebra.

Proposition 4.5. *Let (A,c) be a braided algebra and denote by m its multiplication. Then (A,c,b) is a braided Lie algebra, where $b(z):=m^{t-1}(z)$, for every $z\in E_t(A,c)$.*

Proof. Let us check that b is a bracket on the braided vector space (A,c). For every $z\in E_t(A,c)$ and $x\in A$, we have $c(b^t\otimes A)(z\otimes x)=c\left(m^{t-1}\otimes A\right)(z\otimes x)=\left(A\otimes m^{t-1}\right)c_T^{t,1}(z\otimes x)=(A\otimes b^t)\,c_{E_t(A,c),A}(z\otimes x)$ so that the left-hand side of (10) is proved. Similarly one proves also the right-hand side. By the universal property of $T(A,c)$ (see [AMS1, Theorem 1.17]), Id_A can be lifted to a unique braided algebra homomorphism $\varphi:T(A,c)\to A$.

For every $z\in E_t(A,c)$, we have $\varphi b(z)=b(z)=m^{t-1}(z)=\varphi(z)$ so that φ quotients to a morphism $\overline{\varphi}:U(A,c,b)\to A$. If we denote by $i_U:P\to U(A,c,b)$ the canonical map, we get that $\overline{\varphi}\circ i_U=\mathrm{Id}_A$. In particular i_U is injective whence (A,c,b) is a braided Lie algebra. □

Lemma 4.6. *Let (V,c_V,b_V) a braided Lie algebra and let $h:(W,c_W)\to(V,c_V)$ be an injective morphism of braided vector spaces. If there exists a K-linear map $b_W:E(W,c_W)\to W$ such that $h\circ b_W=b_V\circ E(h)$, then (W,c_W,b_W) is a braided Lie algebra too.*

Proof. We have

$$\begin{aligned}(h\otimes h)\,c_W\,(b_W\otimes W) &= c_V\,(h\otimes h)\,(b_W\otimes W)=c_V\,(b_V\otimes V)\,(E(h)\otimes h)\\ &\overset{(10)}{=} (V\otimes b_V)\,c_{E(V,c_V),V}\,(E(h)\otimes h)=(V\otimes b_V)\,(h\otimes E(h))\,c_{E(W,c_W),W}\\ &= (h\otimes h)\,(W\otimes b_W)\,c_{E(W,c_W),W}.\end{aligned}$$

Since h is injective, we deduce that $c_W\,(b_W\otimes W)=(W\otimes b_W)\,c_{E(W,c_W),W}$ so that the left-hand side of (10) is proved for (W,c_W). Similarly one proves also the right-hand side. If we denote by $i_U^W:W\to U(W,c_W,b_W)$ and $i_U^V:V\to U(V,c_V,b_V)$ the canonical maps, by the universal property of the universal enveloping algebra, there is a unique algebra homomorphism $U(h):U(W,c_W)\to U(V,c_{V,V})$ such that $U(h)\circ i_U^W=i_U^V\circ h$. Since both i_U^V and h are injective, we deduce that i_U^W is injective too. Therefore (W,c_W,b_W) is a braided Lie algebra. □

LEMMA 4.7. *Let A be a braided bialgebra and let (P,c_P) be the infinitesimal part of A. If m is the multiplication of A, then $m^{t-1}(E_t(P,c_P))\subseteq P$ for every $t\geq 2$, so we can define $b_P:E(P,c_P)\to P$ by $b_P(z):=m^{t-1}(z)$, for every $z\in E_t(P,c_P)$. Moreover (P,c_P,b_P) is a braided Lie algebra.*

Proof. Set $T=T(P,c_P)$. By Remark 2.9, we have $E_t(P,c_P)\subseteq P(T)$ for every $t\geq 2$. By the universal property of the braided tensor algebra, the canonical inclusion $i:P\to A$ lifts to a braided bialgebra homomorphism $\varphi:T\to A$. Then $m^{t-1}(E_t(P,c_P))=\varphi(E_t(P,c_P))\subseteq\varphi(P(T))\subseteq P(A)=P$, for every $t\geq 2$. Let b_P be defined as in the statement. Then $i\circ b_P=b\circ E(i)$, where b is the map of Proposition 4.5. By the same proposition and Lemma 4.6 we have that (P,c_P,b_P) is a braided Lie algebra. □

DEFINITION 4.8. With the same assumptions and notations of Lemma 4.7, (P,c_P,b_P) will be called the **infinitesimal braided Lie algebra of** A.

THEOREM 4.9 (The universal property of the universal enveloping algebra). *Let A be a braided bialgebra and let (P,c_P,b_P) be its infinitesimal braided Lie algebra. Then, every morphism of braided brackets $f:(V,c,b)\to(P,c_P,b_P)$ can be lifted to a morphism of braided bialgebras $\overline{f}:U(V,c,b)\to A$.*

Proof. By the universal property of $T(V,c)$, f can be lifted to a unique braided bialgebra homomorphism $f':T(V,c)\to A$. We have

$$f'b(z)=fb^t(z)\overset{(11)}{=}b_P^tE_t(f)(z)=m_A^{t-1}f^{\otimes t}(z)=f'(z)$$

for every $z\in E_t(V,c)$. Therefore f' quotients to a morphism $\overline{f}:U(V,c,b)\to A$. □

4.10. Let (V,c) be a braided vector space and let $b:E(V,c)\to V$ be a c-bracket. Set $T:=T(V,c)$ and $U:=U(V,c,b)$. By construction, the projection $\pi_U:T\to U$ is a morphism of braided bialgebras. Mimicking [AMS2, 4.11], set $T_{(n)}:=\oplus_{0\leq t\leq n}V^{\otimes t}$ and

$$U_n':=\pi_U\left(T_{(n)}\right).$$

Then $(U_n')_{n\in\mathbb{N}}$ is both an algebra and a coalgebra filtration on U which will be called the *standard filtration on* U. Note that this filtration is not the coradical filtration $(U_n)_{n\in\mathbb{N}}$ in general. Still one has $U_n'=\pi_U\left(T_{(n)}\right)\subseteq\pi_U(T_n)\subseteq U_n$ where T_n and U_n denote the n-th terms of the coradical filtrations of T and U respectively. Denote by

$$\mathrm{gr}'(U):=\oplus_{n\in\mathbb{N}}\frac{U_n'}{U_{n-1}'}.$$

the graded coalgebra associated to the standard filtration (see [Sw, page 228]).

If $b=0$, then $S(V,c)=U(V,c,0)$ is a graded bialgebra $S(V,c)=\oplus_{n\in\mathbb{N}}S^n(V,c)$. The standard filtration on $S(V,c)$ is the filtration associated to this grading.

LEMMA 4.11. *Let (V,c) be a braided vector space and let $b:E(V,c)\to V$ be a bracket. Consider the map $\theta_1:V\to U_1'/U_0':v\mapsto\pi_U(v)+U_0'$. Then $\mathrm{Ker}(\theta_1)=V\cap\mathrm{Ker}(\pi_U)$.*

Proof. Set $T := T(V,c)$. Let $x \in \mathrm{Ker}(\theta_1)$. Then $\pi_U(x) \in U'_0 = \pi_U(K)$ so that there exists $k \in K$ such that $\pi_U(x) = \pi_U(k)$. From this we get $k = \varepsilon_U \pi_U(k) = \varepsilon_U \pi_U(x) = \varepsilon_T(x) = 0$ where the last equality holds as $V \subseteq P(T)$. Thus $\pi_U(x) = 0$ and hence $x \in V \cap \mathrm{Ker}(\pi_U)$. The other inclusion is trivial. □

PROPOSITION 4.12. *Let (V,c) be a braided vector space and let $b : E(V,c) \to V$ be a bracket. Then, the following assertions are equivalent.*

(i) *(V,c,b) is a braided Lie algebra.*
(ii) $V \cap \mathrm{Ker}(\pi_U) = 0$.
(iii) *The map θ_1 of Lemma 4.11 is injective.*

Proof. It follows by Definition 4.1 and Lemma 4.11. □

The following result is inspired to [AMS2, Proposition 4.19].

PROPOSITION 4.13. *Let (V,c) be a braided vector space and let $b : E(V,c) \to V$ be a bracket. Then $\mathrm{gr}'(U(V,c,b))$ is a graded braided bialgebra and there is a canonical morphism of graded braided bialgebras $\theta : S(V,c) \to \mathrm{gr}'(U(V,c,b))$ which is surjective and lifts the map $\theta_1 : V \to U'_1/U'_0$ of Lemma 4.11.*

Proof. Set $T := T(V,c)$, $U := U(V,c,b)$, $G := \mathrm{gr}'(U(V,c,b))$, $G^n := U'_n/U'_{n-1}$ and let $p_n : U'_n \to G^n$ be the canonical projection, for every $n \in \mathbb{N}$. We have $c_U(U'_a \otimes U'_b) = c_U(\pi_U \otimes \pi_U)\left(T_{(a)} \otimes T_{(b)}\right)$ $= (\pi_U \otimes \pi_U) c_T \left(T_{(a)} \otimes T_{(b)}\right) \subseteq (\pi_U \otimes \pi_U)\left(T_{(b)} \otimes T_{(a)}\right) = U'_b \otimes U'_a$. Hence c_U induces a braiding $c_G : G \otimes G \to G \otimes G$. Now, we have already observed that G carries a coalgebra structure $(G, \Delta_G, \varepsilon_G)$. Since $(U'_n)_{n\in\mathbb{N}}$ is also an algebra filtration on U, the multiplication of U induces, for every $a,b \in \mathbb{N}$, maps $m_G^{a,b} : G^a \otimes G^b \to G^{a+b}$ and the unit of U induces a map $u_G^0 : K \to G^0$. There is a unique map $m_G : G \otimes G \to G$ such that $m_G\left(i_G^a \otimes i_G^b\right) = i_G^{a+b} m_G^{a,b}$, for every $a,b \in \mathbb{N}$, where $i_G^t : G^t \to G$ is the canonical map. Moreover $\left(G, m_G, u_G = i_G^0 u_G^0\right)$ is a graded algebra (see e.g. [AM, Proposition 3.4]). It is straightforward to prove that $(G, m_G, u_G, \Delta_G, \varepsilon_G, c_G)$ is indeed a graded braided bialgebra.

Set $P := P(G)$. Since G is a connected graded coalgebra, it is clear that $\mathrm{Im}(\theta_1) \subseteq G^1 \subseteq P$. Moreover $\theta_1 : (V,c) \to (P, c_P)$ is a morphism of braided vector spaces as, for every $u,v \in V$, we have

$$\begin{aligned} c_G(\theta_1 \otimes \theta_1)(u \otimes v) &= c_G\left[(\pi_U(u) + U'_0) \otimes (\pi_U(v) + U'_0)\right] \\ &= (p_1 \otimes p_1) c_U(\pi_U(u) \otimes \pi_U(v)) \\ &= (p_1 \otimes p_1) c_U(\pi_U \otimes \pi_U)(u \otimes v) \\ &= (p_1 \otimes p_1)(\pi_U \otimes \pi_U) c_T(u \otimes v) = (\theta_1 \otimes \theta_1) c(u \otimes v). \end{aligned}$$

For every $t \geq 2$ and $z \in E_t(V,c)$, we have

$$\begin{aligned} & m_G^{t-1} E_t(\theta_1)(z) = m_G^{t-1} \theta_1^{\otimes t}(z) = p_t m_U^{t-1} \pi_U^{\otimes t}(z) = p_t \pi_U m_T^{t-1}(z) \\ = \; & p_t \pi_U(z) = \pi_U(z) + U'_{t-1} \overset{\text{def. } U}{=} \pi_U b^t(z) + U'_{t-1} = 0 \end{aligned}$$

so that $m_G^{t-1} \circ E_t(\theta_1) = 0$. By Theorem 4.9, there is a canonical morphism of graded braided bialgebras $\theta : S(V,c) \to G$ lifting θ_1. By construction θ lifts the canonical map $T(V,c) \to G$ which is surjective as θ_1 is surjective and G is generated as an K-algebra by G^1. Thus θ is surjective too. □

DEFINITION 4.14. Let (V,c) be a braided vector space and let $b : E(V,c) \to V$ be a bracket. Following [BG, Definition, page 316], we will say that $U(V,c,b)$ is **Poincaré-Birkhoff-Witt (PBW) type** whenever the projection $\theta : S(V,c) \to \mathrm{gr}'(U(V,c,b))$ of Proposition 4.13 is an isomorphism (compare with [Hu, page 92] for justifying this terminology).

5. The class $\mathcal{S}$

This section mainly concerns the characterization of braided vector spaces in the class $\mathcal{S}$.

5.1. Let (V,c) be a braided vector space and let $T = T(V,c)$. By the universal property of the tensor algebra there is a unique algebra homomorphism

$$\Gamma : T(V,c) \to T^c(V,c)$$

such that $\Gamma_{|V} = \mathrm{Id}_V$, where $T^c(V,c)$ denotes the quantum shuffle algebra. This is a morphism of graded braided bialgebras. The bialgebra of type one generated by V over K, also called **Nichols algebra**, is by definition

$$\mathcal{B}(V,c) = \mathrm{Im}(\Gamma) \simeq \frac{T(V,c)}{\mathrm{Ker}(\Gamma)}.$$

Definition 5.2. Denote by $\mathcal{S}$ the class of those braided vector spaces (V,c) such that $P(S) = \mathrm{Im}(i_S)$, where $S := S(V,c)$ and $i_S : V \to S$ denotes the canonical map.

Remark 5.3. Let (V,c) be braided vector space. Then $(V,c) \in \mathcal{S}$ if and only if $\kappa(V,c) \leq 1$ in the sense of [Ar, Section 5] (cf. [Ar, Lemma 2.7 and Theorem 3.4]). This amounts to say that $S(V,c)$ is isomorphic to $\mathcal{B}(V,c)$ as a graded braided bialgebra i.e. that $\mathrm{Ker}(\Gamma) = (E(V,c))$.

We point out that the class $\mathcal{S}$ is so large to enclose both the class of all braided vector spaces of diagonal type (see Definition 9.1) whose Nichols algebra is a domain of finite Gelfand-Kirillov dimension and also the class of all two dimensional braided vector spaces of abelian group type whose symmetric algebra has dimension at most 31. We note that in either one of these classes there are braided vector spaces (V,c) whose Nichols algebra is not quadratic and whose braiding has minimal polynomial of degree greater than two. These and other results on the class $\mathcal{S}$ can be found in [Ar].

Theorem 5.4 (PBW type Theorem). *Let $(V,c) \in \mathcal{S}$ and let $b : E(V,c) \to V$ be a bracket. Then, the following assertions are equivalent.*

(i) *(V,c,b) is a braided Lie algebra.*

(ii) *$U(V,c,b)$ is of PBW type.*

Proof. Set $S := S(V,c)$. First observe that the canonical map $i_S : V \to S$ is injective (cf. Proposition 4.2).

$(i) \Rightarrow (ii)$ Since $(V,c) \in \mathcal{S}$, we have that $P(S) = \mathrm{Im}(i_S)$. Therefore, since, by Proposition 4.12, the map $\theta_1 : V \to U_1'/U_0' : v \mapsto \pi_U(v) + U_0'$ is injective and since $\theta_1 = \theta \circ i_S$, we get that the restriction of θ to $P(S)$ is injective. Therefore, by [Mo, Lemma 5.3.3], we obtain that θ is injective whence bijective.

$(ii) \Rightarrow (i)$ $\theta_1 = \theta \circ i_S$. □

Corollary 5.5. *Let (V,c,b) be a braided Lie algebra such that $(V,c) \in \mathcal{S}$. Then $i_U : V \to U(V,c,b)$ induces an isomorphism between V and $P(U(V,c,b))$.*

Proof. Set $U := U(V,c,b)$. By hypothesis i_U is injective so that we just have to prove that $P(U) = \mathrm{Im}(i_U)$. Let $z \in P(U)$. Since $z \in U$, there is $n \in \mathbb{N}$ such that $z \in U_n' \backslash U_{n-1}'$. Let $\overline{z} := z + U_{n-1}' \in U_n'/U_{n-1}'$. By definition of the bialgebra structure of $G := \mathrm{gr}'(U(V,c,b))$ one has that $\overline{z} \in P(G)$. By Theorem 5.4, U is of PBW type so that we have $P(G) = \theta(P(S)) = \theta i_S(V) = \theta_1(V) = U_1'/U_0'$. Thus $\overline{z} \in \left(U_n'/U_{n-1}'\right) \cap (U_1'/U_0')$. Since $\overline{z} \neq 0$, we get $n = 1$. Hence $z \in U_n' = U_1' = \pi_U\left(T_{(1)}\right) = \pi_U(K \oplus V)$. Therefore, there are $k \in K$ and $v \in V$ such that $z = \pi_U(k) + \pi_U(v)$ so that $k = \varepsilon_U \pi_U(k) = \varepsilon_U(z) - \varepsilon_U \pi_U(v) = \varepsilon_U(z) - \varepsilon_T(v) = 0$ where the last equality holds as $z \in P(U)$ and $v \in P(T)$. Then $z = \pi_U(k) + \pi_U(v) = \pi_U(v) = i_U(v)$. We have so proved that $P(U) \subseteq \mathrm{Im}(i_U)$. The other inclusion is trivial. □

Definition 5.6. A braided bialgebra A is called **primitively generated** if it is generated as a K-algebra by its space $P(A)$ of primitive elements.

The following is one of the main results of this section.

Theorem 5.7. *Let A be a primitively generated braided bialgebra and let (P, c_P, b_P) be its infinitesimal braided Lie algebra. If $(P, c_P) \in \mathcal{S}$, then A is isomorphic to $U(P, c_P, b_P)$ as a braided bialgebra.*

Proof. Set $U := U(P, c_P, b_P)$. By Theorem 4.9, the identity map of P can be lifted to a morphism of braided bialgebras $\alpha : U \to A$. Since P generates A as a K-algebra, this morphism is surjective. On the other hand, by Corollary 5.5, $i_U : P \to U$ induces an isomorphism between P and $P(U)$. Hence the restriction of α on $P(U)$ is injective. By [Mo, Lemma 5.3.3] α is injective. □

Proposition 5.8. *Let B be a primitively generated braided bialgebra. Then the underlying braided coalgebra is connected.*

Proof. Let (P, c_P) be the infinitesimal part of B. By the universal property of the braided tensor algebra $T = T(P, c_P)$ (see [AMS1, Theorem 1.17]), there exists a unique braided bialgebra homomorphism $\varphi : T \to B$ giving the inclusion when restricted to P. Since B is primitively generated, the map φ is surjective. Since the coalgebra T is connected whence pointed, by [Mo, Corollary 5.3.5] we have $\mathrm{Corad}(B) = \varphi(\mathrm{Corad}(T)) = \varphi(K) = K$. □

6. Braidings of Hecke type

In this section we investigate primitively generated braided bialgebras with infinitesimal braiding oh Hecke type.

Definition 6.1. Let (V, c) be a braided vector space. We say that c is a braiding of **Hecke type with mark** q if c is a root in $\mathrm{End}(V \otimes V)$ of the polynomial $(X+1)(X-q)$ for some $q \in K \setminus \{0\}$.

An element $q \in K$ will be called **regular** if it satisfies one of the following conditions:

- q is not a root of unity;
- $q = 1$ and $\mathrm{char}(K) = 0$.

For further details we refer to [ASc3, Definition 3.3] or [AA, Definition 3.1.1].

Theorem 6.2. *Let A be a primitively generated braided bialgebra and let (P, c_P, b_P) be its infinitesimal braided Lie algebra. If c_P is a braiding of Hecke type with regular mark q, then A is isomorphic to $U(P, c_P, b_P)$ as a braided bialgebra.*

Proof. By Remark 5.3 and [Ar, Theorem 6.13], we have that $(P, c_P) \in \mathcal{S}$ whence Theorem 5.7 applies. □

Using the results in [AMS1, AMS2, Ar], we will now give a simple description of the universal enveloping algebra of a braided Lie algebra such that the braiding is of Hecke type with regular mark. We point out that this description does not depend on the bracket in the non-symmetric case whenever $\mathrm{char}(K) \neq 2$.

Theorem 6.3. *Let (V, c, b) be a braided Lie algebra such that c is of Hecke type with regular mark q. Then*

$$U(V, c, b) = \frac{T(V, c)}{(c(z) - qz - [z]_b \mid z \in V \otimes V)}$$

where $[z]_b = b(c(z) - qz)$, for every $z \in V \otimes V$. Moreover if $q \neq 1$ and $\mathrm{char}(K) \neq 2$, then $[-]_b = 0$.

Proof. We set $T := T(V, c)$. Let us prove the following equality

$$\mathrm{Im}(c - q\mathrm{Id}_{V \otimes V}) = E_2(V, c). \tag{14}$$

Since $(c + \mathrm{Id}_{V\otimes V})(c - q\mathrm{Id}_{V\otimes V}) = 0$, it is clear that $\mathrm{Im}(c - q\mathrm{Id}_{V\otimes V}) \subseteq \mathrm{Ker}(c + \mathrm{Id}_{V\otimes V})$. Let $z \in \mathrm{Ker}(c + \mathrm{Id}_{V\otimes V})$. Then $(c - q\mathrm{Id}_{V\otimes V})(z) = c(z) - qz = -(1+q)z = -(2)_q z$. Since q is regular, we have $(2)_q \neq 0$, so that $z = -(2)_q^{-1}(c - q\mathrm{Id}_{V\otimes V})(z) \in \mathrm{Im}(c - q\mathrm{Id}_{V\otimes V})$. Hence $\mathrm{Im}(c - q\mathrm{Id}_{V\otimes V}) = \mathrm{Ker}(c + \mathrm{Id}_{V\otimes V}) = \mathrm{Ker}\left(\Delta_T^{1,1}\right) = E_2(V, c)$. We have so proved (14). By (14), we have that $(c - q\mathrm{Id}_{V\otimes V})(z)$ lies in the domain of b for every $z \in V \otimes V$. Hence we can define a map $[-]_b : V \otimes V \to V$ by setting $[z]_b = b(c - q\mathrm{Id}_{V\otimes V})(z)$, for every $z \in V \otimes V$.

Let us check that $[-]_b$ is a c-bracket in the sense of [AMS2, Definition 4.1] i.e. that $c([-]_b \otimes V) = (V \otimes [-]_b) c_1 c_2$ and $c(V \otimes [-]_b) = ([-]_b \otimes V) c_2 c_1$. By (10), b is a bracket on (V,c) if $c(b \otimes V) = (V \otimes b) c_{E(V,c),V}$ and $c(V \otimes b) = (b \otimes V) c_{V,E(V,c)}$. By (14), the previous formulas rereads as follows

$$\begin{aligned} c(b \otimes V)[(c - q\mathrm{Id}_{V\otimes V}) \otimes V](z) &= (V \otimes b) c_1 c_2 [(c - q\mathrm{Id}_{V\otimes V}) \otimes V](z) \quad \text{and} \\ c(V \otimes b)[V \otimes (c - q\mathrm{Id}_{V\otimes V})](z) &= (b \otimes V) c_2 c_1 [V \otimes (c - q\mathrm{Id}_{V\otimes V})](z), \end{aligned}$$

for every $z \in V^{\otimes 3}$. Thus $[-]_b$ is a c-bracket.

Set $U_H(V,c,[-]_b) := T(V,c) / (c(z) - qz - [z]_b \mid z \in V \otimes V)$. This is the enveloping algebra as defined in [AMS1, Definition 2.1]. Let us prove that $U_H(V,c,[-]_b) = U(V,c,b)$.

By definition of $[-]_b$ and (14), we have

$$(c(z) - qz - [z]_b \mid z \in V \otimes V) = ((\mathrm{Id}_{V\otimes V} - b)[E_2(V,c)]).$$

Therefore there is a canonical projection $\gamma : U_H(V,c,[-]_b) \to U(V,c,b)$. Since $i_U : V \to U(V,c,b)$ is injective and $\gamma \circ i_{U_H} = i_U$, it is clear that the canonical map $i_{U_H} : V \to U_H(V,c,[-]_b)$ is injective too. By [AMS2, Theorem 4.20 and Corollary 4.23], the injectivity of i_{U_H} imply that $U_H(V,c,[-]_b)$ is strictly graded as a coalgebra. Therefore $P(U_H(V,c,[-]_b))$ identifies with V so that the restriction of the bialgebra homomorphism γ to the space of primitive elements of $U_H(V,c,[-]_b)$ is injective. By [Mo, Lemma 5.3.3] this entails that γ itself is injective and hence bijective. We have so proved that $U_H(V,c,[-]_b) = U(V,c,b)$.

Assume now $q \neq 1$ and $\mathrm{char}(K) \neq 2$. Then, by [AMS1, Theorem 4.3], since q is regular, the map $[-]_b$ is necessarily zero. □

Remark 6.4. Note that, when $q = 1$, then regularity of q means $\mathrm{char}(K) = 0$. Moreover $(V,c,[-]_b)$ becomes a Lie algebra in the sense of [Gu1] and $U_H(V,c,[-]_b)$ is the corresponding enveloping algebra. In this case $[-]_b$ is not necessarily trivial in general.

7. Quadratic algebras

Definition 7.1. Recall that a **quadratic algebra** [Man, page 19] is an associative graded K-algebra $A = \oplus_{n\in\mathbb{N}} A^n$ such that:

1) $A^0 = K$;
2) A is generated as a K-algebra by A^1;
3) the ideal of relations among elements of A^1 is generated by the subspace of all quadratic relations $R(A) \subseteq A^1 \otimes A^1$.

Equivalently A is a graded K-algebra such that the natural map $\pi : T(A^1) \to A$ from the tensor algebra generated by A^1 is surjective and $\mathrm{Ker}(\pi)$ is generated as a two sided ideal in $T(A^1)$ by $\mathrm{Ker}(\pi) \cap [A^1 \otimes A^1]$.

Theorem 7.2. *Let A be a braided bialgebra such that the graded coalgebra* gr$A$ *associated to the coradical filtration of A is a quadratic algebra with respect to its natural braided bialgebra structure. Let (P, c_P, b_P) be the infinitesimal braided Lie algebra of A. Then $\mathcal{B}(P, c_P)$ is a quadratic algebra and A is isomorphic to $U(P, c_P, b_P)$ as a braided bialgebra.*

Proof. Since $B := \mathrm{gr}A$ is quadratic, then A is connected. Therefore B is indeed a braided bialgebra. Denote by $(A_n)_{n\in\mathbb{N}}$ the coradical filtration of A. Since B is a quadratic algebra, then B is generated as a K-algebra by $B^1 = A_1/A_0$ so that B is strongly $\mathbb{N}$-graded as an algebra. Since, by construction, it is also strongly $\mathbb{N}$-graded as a coalgebra, we get that $B \simeq B_0[B_1] \simeq \mathcal{B}(P, c_P)$. Hence $\mathcal{B}(P, c_P)$ is a quadratic algebra. By [Ar, Proposition 6.16] and Remark 5.3, we have that $(P, c_P) \in \mathcal{S}$. On the other hand B is generated as a K-algebra by $B^1 = A_1/A_0$ implies that P generates A as a K-algebra. Therefore, by Theorem 5.7, A is isomorphic to $U(P, c_P, b_P)$ as a braided bialgebra. □

Remark 7.3. Examples of braided vector spaces (V,c) such that $\mathcal{B}(V,c)$ is a quadratic algebra can be found e.g. in [MS, AG]. By [ASc3, Proposition 3.4], another example is given by braided vector spaces of Hecke-type with regular mark.

We now give and example of braided vector spaces (V,c) admitting a non-trivial bracket and such that $\mathcal{B}(V,c)$ is a quadratic algebra. For this examples, using results in [Ar, ASt], we will determine all primitively generated braided bialgebras whose infinitesimal part is (V,c).

Example 7.4. Let K be a field of characteristic 0 and let $m \in K$ be such that $m^2 \neq 0,1$ is regular. Consider the vector space V, appeared in [Gu2, page 325], with basis $\{e_0, e_1, e_2\}$ and braiding given by $c(e_i \otimes e_j) = e_j \otimes e_i$ for $i = 0$ or $j = 0$, $c(e_i \otimes e_i) = m^2 e_i \otimes e_i$ for $i = 1,2$, $c(e_2 \otimes e_1) = me_1 \otimes e_2 + (m^2 - 1) e_2 \otimes e_1$ and $c(e_1 \otimes e_2) = me_2 \otimes e_1$. One can check that $E_2(V,c)$ is generated over K by the elements $u_{0,1} := e_1 \otimes e_0 - e_0 \otimes e_1, u_{0,2} := e_2 \otimes e_0 - e_0 \otimes e_2$ and $u_{1,2} := e_2 \otimes e_1 - me_1 \otimes e_2$. Moreover, by [Ar, Example 6.18] and Remark 5.3, we have that $(V,c) \in \mathcal{S}$ and

$$\mathcal{B}(V,c) = S(V,c) = \frac{T(V,c)}{(E_2(V,c))} = \frac{T(V,c)}{(e_1 \otimes e_0 - e_0 \otimes e_1, e_2 \otimes e_0 - e_0 \otimes e_2, e_2 \otimes e_1 - me_1 \otimes e_2)}.$$

Let $b : E(V,c) \to V$ be a bracket on (V,c) such that (V,c,b) is a braided Lie algebra. By [ASt, Theorem 3.11], we have that

$$U(V,c,b) = \frac{T(V,c)}{((\mathrm{Id} - b)[E_2(V,c)])}.$$

Set $b(u_{a,b}) = \sum_{0 \leq i \leq 2} b_{a,b,i} e_i$. By (10), for all $0 \leq i \leq 2$, we have that

$$c(b \otimes V)(u_{a,b} \otimes e_1) = (V \otimes b) c_{V^{\otimes 2},V}(u_{a,b} \otimes e_1). \tag{15}$$

Computing (15) in cases $(a,b) = (0,1), (0,2)$ and $(1,2)$ and using that $m^2 \neq 0,1$ and regularity of m^2, it is straightforward to get $b(u_{0,1}) = b_{0,1,1} e_1, b(u_{0,2}) = b_{0,1,1} e_2$ and $b(u_{1,2}) = 0$ whence

$$U(V,c,b) = \frac{T(V,c)}{(e_1 \otimes e_0 - e_0 \otimes e_1 - b_{0,1,1} e_1, e_2 \otimes e_0 - e_0 \otimes e_2 - b_{0,1,1} e_2, e_2 \otimes e_1 - me_1 \otimes e_2)}.$$

If $b_{0,1,1} \neq 0$, via the change of basis $e_0' := e_0 / b_{0,1,1}$, $e_1' := e_1$ and $e_2' := e_2$ we can assume $b_{0,1,1} = 1$. By Theorem 5.7,

$$\mathcal{B}(V,c) \qquad \text{and} \qquad A := \frac{T(V,c)}{(e_1 \otimes e_0 - e_0 \otimes e_1 - e_1, e_2 \otimes e_0 - e_0 \otimes e_2 - e_2, e_2 \otimes e_1 - me_1 \otimes e_2)}$$

are the only candidate primitively generated braided bialgebras with infinitesimal part (V,c) (note that A already appeared in [Gu2, page 325]). Let us prove that A has the required property.

One easily checks that A is a braided bialgebra quotient of $T(V,c)$. Thus, by [Mo, Corollary 5.3.5], A is connected as $T(V,c)$ is connected. Let (P, c_P, b_P) be the infinitesimal braided Lie algebra of A. Then A has basis $\{e_0^{n_0} e_1^{n_1} e_2^{n_2} | n_0, n_1, n_2 \in \mathbb{N}\}$ and, using regularity of q, one gets (by the same argument as in the proof of [ASc1, Lemma 3.3]) that (P, c_P) identifies with (V,c).

8. Pareigis-Lie algebras

In this section we investigate the relation between our notions of braided Lie algebra and universal enveloping algebra and the ones introduced by Pareigis in [Pa]. Although these constructions were performed for braided vector spaces which are in addition objects inside the category of Yetter-Drinfeld modules, we will not take care of this extra structure.

8.1. Let $\mathcal{B}_n$ be the Artin braid group with generators $\tau_i, i \in \{1, \ldots, n-1\}$ and relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \qquad \text{and} \qquad \tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i-j| \geq 2.$$

Let $\nu_n : \mathcal{B}_n \to \mathcal{S}_n : \tau \mapsto \widetilde{\tau}$ be the canonical quotient homomorphism from the braid group onto the symmetric group and set $s_i := \widetilde{\tau_i}$ for every $i \in \{1, \ldots, n-1\}$.

Let $\sigma \in \mathcal{S}_n$. The length $l(\sigma)$ of σ is the smallest number m such that there is a decomposition $\sigma = s_{i_1} s_{i_2} \cdots s_{i_m}$. Such a decomposition will be called reduced. A permutation σ may have several reduced decompositions, but if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_m}$ is one of them, then the element $\tau_{i_1} \tau_{i_2} \cdots \tau_{i_m}$ is uniquely determined in $\mathcal{B}_n$.

It is well-known that the ν_n has a canonical section $\iota_n : \mathcal{S}_n \to \mathcal{B}_n$. By definition,

$$\iota_n(\sigma) := \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m},$$

where $\sigma = s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced decomposition of σ. Note that ι_n is just a map, but

$$\iota_n(\sigma\tau) = \iota_n(\sigma)\iota_n(\tau),$$

for any $\sigma, \tau \in \mathcal{S}_n$ such that $l(\sigma\tau) = l(\sigma) + l(\tau)$.

Recall that $\sigma \in \mathcal{S}_{p+q}$ is a (p,q)-shuffle if $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. The set of (p,q)-shuffles will be denoted by $(p \mid q)$.

8.2. Let (V,c) be a braided vector space. For $n \in \mathbb{N}^*$, there is a canonical linear representation $\rho_n : \mathcal{B}_n \to \mathrm{Aut}_K(V^{\otimes n})$ such that $\rho_n(\tau_i) = c_i$, where

$$c_i = V^{\otimes(i-1)} \otimes c \otimes V^{\otimes(n-i-1)}.$$

The induced action will be denoted by

$$\rhd : \mathcal{B}_n \times V^{\otimes n} \to V^{\otimes n}.$$

On generators we have $\tau_i \rhd x = c_i(x)$ for each $x \in V^{\otimes n}$. Note that, since ι_n is not a group homomorphism, this does not restrict in a natural way to an action of $\mathcal{S}_n$ on $V^{\otimes n}$.

Let $n \geq 2$ and let $\zeta \in K \setminus \{0\}$. Following [Pa, Definition 2.3], we set

$$V^{\otimes n}(\zeta) := \left\{ x \in V^{\otimes n} \mid \left(\varphi^{-1}\tau_i^2\varphi\right) \rhd x = \zeta^2 x, \text{ for every } \varphi \in \mathcal{B}_n, i \in \{1,\ldots,n-1\} \right\}.$$

There is an action $\blacktriangleright : \mathcal{S}_n \times V^{\otimes n}(\zeta) \to V^{\otimes n}(\zeta)$ defined on generators by

$$s_i \blacktriangleright x = \zeta^{-1}\tau_i \rhd x. \tag{16}$$

Remark 8.3. Let (V,c) be a braided vector space and set $T := T(V,c)$. By [AMS2, Corollary 1.22], we have

$$\Delta_T^{i,n-i}(x) = \sum_{\sigma \in (i|n-i)} \iota_n(\sigma^{-1}) \rhd x, \text{ for all } 0 \leq i \leq n \text{ and } x \in V^{\otimes n}. \tag{17}$$

Lemma 8.4. *Let (V,c) be a braided vector space. For every $n \geq 2$ consider the map*

$$\Pi_\zeta^n : V^{\otimes n}(\zeta) \to V^{\otimes n}(\zeta), \Pi_\zeta^n(x) = \sum_{\sigma \in \mathcal{S}_n} \sigma \blacktriangleright x$$

for $\zeta \in K \setminus \{0\}$. If ζ is a primitive n-th root of unity then $\Pi_\zeta^n(x) \in E_n(V,c)$ for each $x \in V^{\otimes n}(\zeta)$.

Proof. We set $T := T(V,c)$. Note that, for every $y \in V^{\otimes n}(\zeta)$ we have

$$\sigma^{-1} \blacktriangleright y \overset{(16)}{=} \zeta^{-l(\sigma^{-1})}\iota_n(\sigma^{-1}) \rhd y = \zeta^{-l(\sigma)}\iota_n(\sigma^{-1}) \rhd y$$

as $l(\sigma^{-1}) = l(\sigma)$ for every $\sigma \in \mathcal{S}_n$. Therefore, if $x \in V^{\otimes n}(\zeta)$ we get

$$\begin{aligned}
\Delta_T^{i,n-i}\left[\Pi_\zeta^n(x)\right] &\overset{(17)}{=} \sum_{\sigma \in (i|n-i)} \iota_n(\sigma^{-1}) \rhd \Pi_\zeta^n(x) = \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in \mathcal{S}_n} \iota_n(\sigma^{-1}) \rhd (\gamma \blacktriangleright x) \\
&= \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in \mathcal{S}_n} \zeta^{l(\sigma)}\sigma^{-1} \blacktriangleright (\gamma \blacktriangleright x) = \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in \mathcal{S}_n} \zeta^{l(\sigma)}\left(\sigma^{-1}\gamma\right) \blacktriangleright x \\
&= \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in \mathcal{S}_n} \zeta^{l(\sigma)}\gamma \blacktriangleright x = \left(\sum_{\sigma \in (i|n-i)} \zeta^{l(\sigma)}\right)\Pi_\zeta^n(x) = \binom{n}{i}_\zeta \Pi_\zeta^n(x).
\end{aligned}$$

where the last equality holds in view of [AMS2, 1.29]. If ζ is a primitive n-th root of unity, one has that $\binom{n}{i}_\zeta = 0$ unless $i = 0$ or $i = n$. We have so proved that $\Pi_\zeta^n(x) \in E_n(V,c)$ for each $x \in V^{\otimes n}(\zeta)$. We point out that the calculations above are inspired by part of the proof of [Pa, Theorem 5.3]. □

Definition 8.5. (cf. [Pa, Definition 4.1]) We will call a **Pareigis-Lie algebra** any braided vector space (V,c) together with a K-linear map

$$[-] : \bigoplus_{n \in \mathbb{N}, \zeta \in \mathbb{P}_n} V^{\otimes n}(\zeta) \to V,$$

uniquely defined by its restriction $[-]_\zeta^n : V^{\otimes n}(\zeta) \to V$ to $V^{\otimes n}(\zeta)$, such that the following identities hold:

$$[\sigma \blacktriangleright x]_\zeta^n = [x]_\zeta^n\,, \text{ for every } \sigma \in \mathcal{S}_n, x \in V^{\otimes n}(\zeta)\,, \tag{18}$$

$$\sum_{i=1}^{n+1} \left\{ [-]_{-1}^2 \circ \left(V \otimes [-]_\zeta^n \right) \right\} ((1,\ldots,i) \blacktriangleright x) = 0, \text{ for every } x \in V^{\otimes n+1}(\zeta)\,, \tag{19}$$

$$\left\{ [-]_{-1}^2 \circ \left(V \otimes [-]_\zeta^n \right) \right\}(x) = \sum_{i=1}^{n} \left\{ [-]_\zeta^n \circ \left(V^{\otimes i-1} \otimes [-]_{-1}^2 \otimes V^{\otimes n-i} \right) \right\} ((\tau_{i-1}\cdots\tau_1) \rhd x)\,, \tag{20}$$
$$\text{for every } x \in V^{\otimes n+1}(-1,\zeta)$$

where

$$V^{\otimes n+1}(-1,\zeta) := \left\{ x \in V \otimes V^{\otimes n}(\zeta) \mid (1 \otimes \varphi)^{-1} \blacktriangleright \left(\tau_1^2 \rhd ((1\otimes\varphi) \blacktriangleright x) \right) = x, \forall \varphi \in \mathcal{S}_n \right\}.$$

Given a Pareigis-Lie algebra $(V,c,[-])$ one can define (see [Pa]) **the universal enveloping algebra**

$$U_P(V,c,[-]) := \frac{T(V,c)}{\left(\Pi_\zeta^n(z) - [z]_\zeta^n \mid n \in \mathbb{N}, \zeta \in \mathbb{P}_n, z \in V^{\otimes n}(\zeta) \right)}.$$

Remark 8.6. V. K. Kharchenko drew our attention to [Kh4, Corollary 7.5] where it is proven there exist at least $(n-2)!$ possibilities to define symmetric Pareigis-Lie algebras in the case when the underline braided vector space is an object in the category of Yetter-Drinfeld modules over an abelian group algebra.

Next result associates a Pareigis-Lie algebra to any braided Lie algebra.

Theorem 8.7. *Let (V,c,b) be a braided Lie algebra. Then $(V,c,[-])$ is a Pareigis-Lie algebra where $[x]_\zeta^n := b_n \Pi_\zeta^n(x)$ for every $x \in V^{\otimes n}(\zeta)$. Moreover there is a canonical braided bialgebra projection $U_P(V,c,[-]) \to U(V,c,b)$ which is the identity map whenever*

$$\sum_{\zeta \in \mathbb{P}_n} \mathrm{Im}\left(\Pi_\zeta^n\right) = E_n(V,c)\,, \text{ for every } n \in \mathbb{N} \tag{21}$$

is fulfilled.

Proof. We keep many of the notations used in [Pa] and apply some properties proved therein. First, observe that, by Lemma 8.4, $\Pi_\zeta^n(x)$ lies in the domain of b_n, for every $x \in V^{\otimes n}(\zeta)$ whence $[-]_\zeta^n$ is well defined. For every $\sigma \in \mathcal{S}_n, x \in V^{\otimes n}(\zeta)$, we have

$$\begin{aligned} [\sigma \blacktriangleright x]_\zeta^n &= b_n \Pi_\zeta^n(\sigma \blacktriangleright x) = b_n \left[\sum_{\gamma \in \mathcal{S}_n} \gamma \blacktriangleright (\sigma \blacktriangleright x) \right] = b_n \left[\sum_{\gamma \in \mathcal{S}_n} (\gamma\sigma) \blacktriangleright x \right] \\ &= b_n \left[\sum_{\gamma \in \mathcal{S}_n} \gamma \blacktriangleright x \right] = b_n \Pi_\zeta^n(x) = [x]_\zeta^n \end{aligned}$$

so that (18) holds. In order to obtain (19) we adapt the proof of [Pa, Theorem 3.4] as follows. For every $x \in V^{\otimes n+1}(\zeta)$ and $\zeta \in \mathbb{P}_n$, we have

$$\sum_{i=1}^{n+1} \left\{ [-]_{-1}^2 \circ \left(V \otimes [-]_\zeta^n \right) \right\} ((1,\ldots,i) \blacktriangleright x) = b_2 \left(\sum_{i=1}^{n+1} (\mathrm{Id}_{V^{\otimes 2}} - c) \left\{ \left(V \otimes [-]_\zeta^n \right) ((1,\ldots,i) \blacktriangleright x) \right\} \right).$$

Let us focus on the argument of b_2 that we denote by a. Note that $a \in E_2(V,c)$ and observe that

$$c\left(V \otimes [-]_\zeta^n \right)(z) \overset{(10)}{=} (b_n \otimes V)\, c_{V,E_n(V,c)} \left(V \otimes \Pi_\zeta^n \right)(z) = \left([-]_\zeta^n \otimes V \right) \{ (\tau_n \cdots \tau_1) \rhd z \}$$

so that

$$c\left(V \otimes [-]_\zeta^n \right)(z) = \left([-]_\zeta^n \otimes V \right) \{ (\tau_n \cdots \tau_1) \rhd z \}\,, \text{ for every } z \in V \otimes V^{\otimes n}(\zeta)\,. \tag{22}$$

Note also that $V^{\otimes n+1}(\zeta) \subseteq V \otimes V^{\otimes n}(\zeta)$ (see [Pa, Proof of Proposition 3.1]). We get

$$\begin{aligned} a &= \sum_{i=1}^{n+1} \left(V \otimes [-]_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) - \sum_{i=1}^{n+1} c\left(V \otimes [-]_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) \\ &\overset{(22)}{=} \sum_{i=1}^{n+1} \left(V \otimes [-]_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) - \sum_{i=1}^{n+1} \left([-]_{\zeta}^{n} \otimes V\right)\{(\tau_n \cdots \tau_1) \rhd ((1,\ldots,i) \blacktriangleright x)\} \\ &= \sum_{i=1}^{n+1} \left(V \otimes [-]_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) - \sum_{i=1}^{n+1} \left([-]_{\zeta}^{n} \otimes V\right)\{(n+1,\ldots,1) \blacktriangleright (1,\ldots,i) \blacktriangleright x\} \\ &= \sum_{i=1}^{n+1} \left(V \otimes [-]_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) - \sum_{i=1}^{n+1} \left([-]_{\zeta}^{n} \otimes V\right)\{(n+1,\ldots,i) \blacktriangleright x\} \\ &= \sum_{i=1}^{n+1} \left(V \otimes b_n \Pi_{\zeta}^{n}\right)((1,\ldots,i) \blacktriangleright x) - \sum_{i=1}^{n+1} \left(b_n \Pi_{\zeta}^{n} \otimes V\right)\{(n+1,\ldots,i) \blacktriangleright x\} \\ &= \begin{pmatrix} (V \otimes b_n)\left(\sum_{i=1}^{n+1} \sum_{\sigma \in \mathcal{S}_n} \{(1 \otimes \sigma) \blacktriangleright (1,\ldots,i) \blacktriangleright x\}\right) + \\ -(b_n \otimes V)\left(\sum_{i=1}^{n+1} \sum_{\sigma \in \mathcal{S}_n} \{(\sigma \otimes 1) \blacktriangleright (n+1,\ldots,i) \blacktriangleright x\}\right) \end{pmatrix} \\ &= (V \otimes b_n)\left(\sum_{\sigma \in \mathcal{S}_{n+1}} \sigma \blacktriangleright x\right) - (b_n \otimes V)\left(\sum_{\sigma \in \mathcal{S}_{n+1}} \sigma \blacktriangleright x\right) \\ &= (V \otimes b_n)\left(\Pi_{\zeta}^{n+1}(x)\right) - (b_n \otimes V)\left(\Pi_{\zeta}^{n+1}(x)\right) \end{aligned}$$

Thus

$$\sum_{i=1}^{n+1} \left\{[-]_{-1}^{2} \circ \left(V \otimes [-]_{\zeta}^{n}\right)\right\}((1,\ldots,i) \blacktriangleright x) = b_2\left\{(V \otimes b_n)\left(\Pi_{\zeta}^{n+1}(x)\right) - (b_n \otimes V)\left(\Pi_{\zeta}^{n+1}(x)\right)\right\}.$$

Note that it must be true that $\Pi_{\zeta}^{n+1}(x) \in (V \otimes E_n(V)) \cap (E_n(V) \otimes V)$.

In order to prove (19), it suffices to check that

$$b_2\left\{(V \otimes b_n)\left(\Pi_{\zeta}^{n+1}(x)\right) - (b_n \otimes V)\left(\Pi_{\zeta}^{n+1}(x)\right)\right\} = 0.$$

Let $F := (\mathrm{Id} - b)[E(V,c)]$. Since $a \in E_2(V,c)$ and $\Pi_{\zeta}^{n+1}(x) \in (V \otimes E_n(V)) \cap (E_n(V) \otimes V)$, we get that

$$\begin{aligned} u \quad &:= b_2\left\{(V \otimes b_n)\left(\Pi_{\zeta}^{n+1}(x)\right) - (b_n \otimes V)\left(\Pi_{\zeta}^{n+1}(x)\right)\right\} \\ &= (b_2 - \mathrm{Id}_{V^{\otimes 2}})(a) + \\ &\quad + (V \otimes (b_n - \mathrm{Id}_{V^{\otimes n}}))\left(\Pi_{\zeta}^{n+1}(x)\right) + ((\mathrm{Id}_{V^{\otimes 2}} - b_n) \otimes V)\left(\Pi_{\zeta}^{n+1}(x)\right) \\ &\in F + V \otimes F + F \otimes V \end{aligned}$$

so that $u \in V \cap (F) = V \cap \mathrm{Ker}(\pi_U)$ which is zero by Definition 4.1. Hence (19) is proved.

Let us check that (20) is true. We will adapt the proof of [Pa, Theorem 3.5]. We get

$$\begin{aligned} &\sum_{i=1}^{n} \left\{[-]_{\zeta}^{n} \circ \left(V^{\otimes i-1} \otimes [-]_{-1}^{2} \otimes V^{\otimes n-i}\right)\right\}((\tau_{i-1} \cdots \tau_1) \rhd x) \\ = \;& b_n\left(\sum_{i=1}^{n} \Pi_{\zeta}^{n}\left(V^{\otimes i-1} \otimes [-]_{-1}^{2} \otimes V^{\otimes n-i}\right)((\tau_{i-1} \cdots \tau_1) \rhd x)\right) \\ = \;& b_n\left(\sum_{i=1}^{n} \Pi_{\zeta}^{n}\left(V^{\otimes i-1} \otimes b_2(\mathrm{Id}_{V^{\otimes 2}} - c) \otimes V^{\otimes n-i}\right)((\tau_{i-1} \cdots \tau_1) \rhd x)\right) \end{aligned}$$

$$
\begin{array}{ll}
= & b_n \left(\begin{array}{c} \sum_{i=1}^n \Pi_\zeta^n \left(V^{\otimes i-1} \otimes b_2 \otimes V^{\otimes n-i}\right) \left(\left(\tau_{i-1}\cdots\tau_1\right) \rhd x\right) + \\ -\sum_{j=1}^n \Pi_\zeta^n \left(V^{\otimes j-1} \otimes b_2 \otimes V^{\otimes n-j}\right)\left(V^{\otimes j-1} \otimes c \otimes V^{\otimes n-j}\right)\left(\left(\tau_{j-1}\cdots\tau_1\right) \rhd x\right) \end{array} \right) \\
= & b_n \left(\begin{array}{c} \sum_{i=1}^n \sum_{\sigma\in\mathcal{S}_n} \sigma \blacktriangleright \left(V^{\otimes i-1} \otimes b_2 \otimes V^{\otimes n-i}\right) \left(\left(\tau_{i-1}\cdots\tau_1\right) \rhd x\right) + \\ -\sum_{j=1}^n \sum_{\sigma\in\mathcal{S}_n} \sigma \blacktriangleright \left(V^{\otimes j-1} \otimes b_2 \otimes V^{\otimes n-j}\right)\left(\left(\tau_j\tau_{j-1}\cdots\tau_1\right) \rhd x\right) \end{array} \right) \\
= & b_n \left(\begin{array}{c} \sum_{i=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \varphi \rhd \left(V^{\otimes i-1} \otimes b_2 \otimes V^{\otimes n-i}\right) \left\{\left(\tau_{i-1}\cdots\tau_1\right) \rhd x\right\} + \\ -\sum_{j=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \varphi \rhd \left(V^{\otimes j-1} \otimes b_2 \otimes V^{\otimes n-i}\right)\left\{\left(\tau_j\tau_{j-1}\cdots\tau_1\right) \rhd x\right\} \end{array} \right) \\
\overset{(*)}{=} & b_n \left(\begin{array}{c} \sum_{i=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(V^{\otimes \widetilde{\varphi}(i)-1} \otimes b_2 \otimes V^{\otimes n-\widetilde{\varphi}(i)}\right) \left\{\left(\varphi_{(i)}\tau_{i-1}\cdots\tau_1\right) \rhd x\right\} + \\ -\sum_{j=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(V^{\otimes \widetilde{\varphi}(j)-1} \otimes b_2 \otimes V^{\otimes n-\widetilde{\varphi}(j)}\right)\left\{\left(\varphi_{(j)}\tau_j\tau_{j-1}\cdots\tau_1\right) \rhd x\right\} \end{array} \right) \\
\overset{(**)}{=} & b_n \left(\begin{array}{c} \sum_{i=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(V^{\otimes \widetilde{\varphi}(i)-1} \otimes b_2 \otimes V^{\otimes n-\widetilde{\varphi}(i)}\right) \left\{\left(\tau_{\widetilde{\varphi}(i)-1}\cdots\tau_1\left(1\otimes\varphi\right)\right) \rhd x\right\} + \\ -\sum_{j=1}^n \sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(V^{\otimes \widetilde{\varphi}(j)-1} \otimes b_2 \otimes V^{\otimes n-\widetilde{\varphi}(j)}\right)\left\{\left(\tau_{\widetilde{\varphi}(j)}\tau_{\widetilde{\varphi}(j)-1}\cdots\tau_1\left(1\otimes\varphi\right)\right) \rhd x\right\} \end{array} \right) \\
= & b_n \left(\begin{array}{c} \left(b_2 \otimes V^{\otimes n-1}\right)\left(\sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(1\otimes\varphi\right) \rhd x\right) + \\ -\left(V^{\otimes n-1}\otimes b_2\right)\left(\sum_{\varphi\in \mathrm{Im}(\iota_n)} \zeta^{-l(\widetilde{\varphi})} \left(\left(\tau_n\tau_{n-1}\cdots\tau_1\right)\left(1\otimes\varphi\right)\right) \rhd x\right) \end{array} \right) \\
= & b_n \left(\begin{array}{c} \left(b_2 \otimes V^{\otimes n-1}\right)\left(\sum_{\sigma\in\mathcal{S}_n} \left(1\otimes\sigma\right) \blacktriangleright x\right) + \\ -\left(V^{\otimes n-1}\otimes b_2\right)\left(\sum_{\sigma\in\mathcal{S}_n} \left(\tau_n\tau_{n-1}\cdots\tau_1\right) \rhd \left(\left(1\otimes\sigma\right) \blacktriangleright x\right)\right) \end{array} \right) \\
= & b_n \left(\left(b_2 \otimes V^{\otimes n-1}\right)\left(V\otimes \Pi_\zeta^n\right)(x) - \left(V^{\otimes n-1}\otimes b_2\right) c_{V,V^{\otimes n}}\left(V\otimes\Pi_\zeta^n\right)(x)\right) \\
= & b_n \left(\left(b_2 \otimes V^{\otimes n-1}\right) - \left(V^{\otimes n-1}\otimes b_2\right) c_{V,V^{\otimes n}}\right)\left(V\otimes\Pi_\zeta^n\right)(x),
\end{array}
$$

where in (*) we used that, for every $x' \in V^{\otimes i-1} \otimes E_2(V,c) \otimes V^{\otimes n-i}$,

$$
\varphi \rhd \left(V^{\otimes i-1} \otimes b_2 \otimes V^{\otimes n-i}\right)(x') = \left(V^{\otimes \widetilde{\varphi}(i)-1} \otimes b_2 \otimes V^{\otimes n-\widetilde{\varphi}(i)}\right)\varphi_{(i)}(x'),
$$

where $\varphi_{(i)} \in \mathcal{B}_{n+1}$ is a suitable braid depending on φ and that was defined in [Pa, Appendix]; in (**) we applied [Pa, Proposition 8.1].

On the other hand, we have

$$
\begin{array}{ll}
 & [-]_{-1}^2 \left(V \otimes [-]_\zeta^n\right)(x) \\
= & b_2\left(\mathrm{Id}_{V^{\otimes 2}} - c\right)\left(V \otimes [-]_\zeta^n\right)(x) \\
= & b_2\left\{\left(V \otimes [-]_\zeta^n\right)(x) - c\left(V \otimes [-]_\zeta^n\right)(x)\right\} \\
= & b_2\left\{\left(V\otimes b_n\Pi_\zeta^n\right)(x) - c\left(V\otimes b_n\Pi_\zeta^n\right)(x)\right\} \\
\overset{(10)}{=} & b_2\left\{\left(V\otimes b_n\right)\left(V\otimes\Pi_\zeta^n\right)(x) - \left(b_n\otimes V\right)c_{V,E_n(V,c)}\left(V\otimes\Pi_\zeta^n\right)(x)\right\} \\
= & b_2\left\{\left(V\otimes b_n\right) - \left(b_n\otimes V\right)c_{V,E_n(V,c)}\right\}\left(V\otimes\Pi_\zeta^n\right)(x).
\end{array}
$$

Let $w := \left(V\otimes\Pi_\zeta^n\right)(x)$. Hence it remains to check that

$$
b_n\left(\left(b_2 \otimes V^{\otimes n-1}\right) - \left(V^{\otimes n-1}\otimes b_2\right)c_{V,V^{\otimes n}}\right)(w) = b_2\left\{\left(V\otimes b_n\right) - \left(b_n\otimes V\right)c_{V,E_n(V,c)}\right\}(w).
$$

We have

$$
\begin{array}{rcl}
s & := & \left(\begin{array}{c} b_n\left(\left(b_2 \otimes V^{\otimes n-1}\right) - \left(V^{\otimes n-1}\otimes b_2\right)c_{V,V^{\otimes n}}\right)(w) + \\ -b_2\left\{\left(V\otimes b_n\right) - \left(b_n\otimes V\right)c_{V,E_n(V,c)}\right\}(w) \end{array} \right) \\
& = & \left(\begin{array}{c} \left(b_n - \mathrm{Id}\right)\left(\left(b_2 \otimes V^{\otimes n-1}\right) - \left(V^{\otimes n-1}\otimes b_2\right)c_{V,V^{\otimes n}}\right)(w) + \\ +\left(\left(\left(b_2-\mathrm{Id}\right)\otimes V^{\otimes n-1}\right) - \left(V^{\otimes n-1}\otimes\left(b_2-\mathrm{Id}\right)\right)c_{V,V^{\otimes n}}\right)(w) + \\ \left(\mathrm{Id}-b_2\right)\left\{\left(V\otimes b_n\right) - \left(b_n\otimes V\right)c_{V,E_n(V,c)}\right\}(w) + \\ -\left\{\left(V\otimes\left(b_n-\mathrm{Id}\right)\right) - \left(\left(b_n-\mathrm{Id}\right)\otimes V\right)c_{V,E_n(V,c)}\right\}(w) \end{array} \right).
\end{array}
$$

so that $s \in V \cap (F) = V \cap \mathrm{Ker}\left(\pi_U\right)$ which is zero by Definition 4.1. Hence (20) is proved.

Thus $(V, c, [-])$ is a Pareigis-Lie algebra. By the universal property of its universal enveloping algebra (see [Pa, Section 6]), there is a unique braided bialgebra homomorphism $U_P(V, c, [-]) \to U(V, c, b)$ that lifts the map $i_U : V \to U(V, c, b)$. Assume that (21) is fulfilled. Then

$$\begin{aligned} U(V,c,b) &= \frac{T(V,c)}{((\mathrm{Id}-b)\{E(V,c)\})} = \frac{T(V,c)}{\left((\mathrm{Id}-b_n)\left\{\sum_{\zeta\in\mathbb{P}_n}\mathrm{Im}\left(\Pi^n_\zeta\right)\right\} \mid n\in\mathbb{N}\right)} \\ &= \frac{T(V,c)}{\left((\mathrm{Id}-b_n)\left\{\Pi^n_\zeta(z)\right\} \mid n\in\mathbb{N}, \zeta\in\mathbb{P}_n, z\in V^{\otimes n}(\zeta)\right)} \\ &= \frac{T(V,c)}{\left(\Pi^n_\zeta(z)-[z]^n_\zeta \mid n\in\mathbb{N}, \zeta\in\mathbb{P}_n, z\in V^{\otimes n}(\zeta)\right)} = U_P(V,c,[-]). \end{aligned}$$

□

Remark 8.8. Note that, by [Mo, Lemma 5.3.3], the projection $U_P(V, c, [-]) \to U(V, c, b)$ in Theorem 8.7 becomes the identity whenever $P(U_P(V, c, [-]))$ identifies with V through the canonical map $i_{U_P} : V \to U_P(V, c, [-])$. In view of Corollary 5.5, the converse is true if $(V, c) \in \mathcal{S}$.

Theorem 8.7 leads to the following problem.

Problem 8.9. To determine under which conditions (21) is true for a given braided vector space (V, c). Note that $\mathrm{Im}\left(\Pi^n_\zeta\right)$ is a vector subspace of $V^{\otimes n}(\zeta)$ so that a necessary condition is $E_n(V, c) \subseteq \sum_{\zeta\in\mathbb{P}_n} V^{\otimes n}(\zeta)$.

Remark 8.10. Assume $\mathrm{char}(K) \neq 2$ and let (V, c) be a braided vector space. Let us show (21) holds for $n = 2$ i.e. that $\mathrm{Im}\left(\Pi^2_{-1}\right) = E_2(V, c)$. By definition, $E_2(V, c) = \left\{x \in V^{\otimes 2} \mid c(x) = -x\right\}$ and

$$\begin{aligned} V^{\otimes 2}(-1) &= \left\{x\in V^{\otimes 2} \mid \left(\varphi^{-1}\tau_1^2\varphi\right) \rhd x = x, \text{ for every } \varphi\in\mathcal{B}_2\right\} \\ &= \left\{x\in V^{\otimes 2} \mid \tau_1^2 \rhd x = x\right\} = \left\{x\in V^{\otimes 2} \mid c^2(x) = x\right\}. \end{aligned}$$

For each $x \in V^{\otimes 2}(-1)$, we have $\Pi^2_{-1}(x) = \sum_{\sigma\in\mathcal{S}_2} \sigma \blacktriangleright x = x - \tau_1 \rhd x = x - c(x)$. Let $\gamma : E_2(V, c) \to V^{\otimes 2}(-1)$ be defined by $\gamma(x) = x/2$. Then, for every $x \in E_2(V, c)$, we have $\Pi^2_{-1}\gamma(x) = \frac{1}{2}(x - c(x)) = x$, where the last equality holds as $x \in E_2(V, c)$. Hence $\mathrm{Im}\left(\Pi^2_{-1}\right) = E_2(V, c)$.

Note that Π^2_{-1} is not injective in general. For, if $x \in V^{\otimes 2}(-1)$ we have that $\Pi^2_{-1}(x) \in V^{\otimes 2}(-1)$ so that it makes sense to compute $\Pi^2_{-1}\Pi^2_{-1}(x) = x - c(x) - c(x) + c^2(x) = 2\Pi^2(x)$. Thus injectivity of Π^2_{-1} implies $\Pi^2_{-1}(x) = 2x$ i.e. $c(x) = -x$ for every $x \in V^{\otimes 2}(-1)$ which is not true in general (for example choose V to be a two dimensional vector space and c to be the canonical flip map defined by $c(v \otimes w) = w \otimes v$ for every $v, w \in V$).

In view of Theorem 8.7, we can now recover two meaningful examples of Pareigis-Lie algebra.

Corollary 8.11. *(cf.* [Pa, Corollaries 4.2 and 5.4]*) Let (A, m, u, c) be a braided algebra. Then $(A, c, [-])$ is a Pareigis-Lie algebra where $[x]^n_\zeta := m^{n-1}\Pi^n_\zeta(x)$, for every $x \in V^{\otimes n}(\zeta)$. Moreover, if A is a connected braided bialgebra, then the space $P(A)$ of primitive elements of A forms a Pareigis-Lie algebra too.*

Proof. By Proposition 4.5 (A, c, b) is a braided Lie algebra, where $b(z) := m^{t-1}(z)$, for every $z \in E_t(A, c)$. By Lemma 4.7 also $P(A)$ carries a braided Lie algebra structure which is induced by that of A. We conclude by applying Theorem 8.7. □

9. Braided vector spaces of diagonal type

In these section, we collect general results on braided vector spaces of diagonal type. In Theorem 9.5, we will prove that a braided vector space of diagonal type endowed with a bracket must fulfil suitable identities. This result can be used to establish conditions on the bracket for a given braided vector space.

For all $n \in \mathbb{N}$, we will use the following notation

$$\underline{n} := \{1, \ldots, n\}.$$

DEFINITION 9.1. We will say that a braided vector space (V, c) **is of diagonal type with basis** $x_1, \ldots, x_n$ **and matrix** $(q_{i,j})_{i,j \in \underline{n}}$ whenever $x_1, \ldots, x_n$ form a basis of V such that for all $1 \leq i, j \leq n$

$$c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i$$

for some $q_{i,j} \in K$. For all $\mathbf{m} = (m_1, \ldots, m_t) \in \underline{n}^t$, we set

$$x_{\mathbf{m}} := x_{m_1} \otimes x_{m_2} \otimes \cdots \otimes x_{m_t} \in V^{\otimes t}.$$

Note that $\{x_{\mathbf{m}} \mid \mathbf{m} \in \underline{n}^t\}$ is a basis of the vector space $V^{\otimes t}$. Furthermore, for all $\sigma \in \mathcal{S}_t$ we set

$$\sigma(\mathbf{m}) := \left(m_{\sigma(1)}, \ldots, m_{\sigma(t)}\right).$$

For $\mathbf{m}, \mathbf{n} \in \underline{n}^t$ we write $\mathbf{m} \sim \mathbf{n}$ if there exists $\sigma \in \mathcal{S}_t$ such that $\sigma(\mathbf{m}) = \mathbf{n}$. The equivalence class of $\mathbf{m}$ will be denoted by

$$\mathcal{S}_t \mathbf{m} := \{\sigma(\mathbf{m}) \mid \sigma \in \mathcal{S}_t\}.$$

Let $P(T)$ denote the space of primitive elements in the tensor algebra $T := T(V, c)$. A **homogeneous quantum operation for** T (cf. [Kh1, Definition 2.2]) is an element $u \in P(T) \cap V^{\otimes t}$ of the form

$$u = \sum_{\mathbf{m} \in \mathcal{S}_t \mathbf{n}} u_{\mathbf{m}} x_{\mathbf{m}}$$

for some $\mathbf{n} \in \underline{n}^t$ and $u_{\mathbf{m}} \in K$.

The following result is probably well known (e.g. it seems to be implicitly understood in [Kh1]), but we could not find a precise reference for it.

THEOREM 9.2. *Let (V, c) be a braided vector space of diagonal type. Then the vector space $P(T)$ of primitive elements in the tensor algebra $T := T(V, c)$ is generated by homogeneous quantum operations for T.*

Proof. First one has $P(T) = \oplus_{t \in \mathbb{N}} P(V^{\otimes t})$ where $P(V^{\otimes t}) = P(T) \cap V^{\otimes t}$. Note that $P\left(V^{\otimes 0}\right) = P(K) = 0$ and $P\left(V^{\otimes 1}\right) = P(V) = V$. It remains to prove that each $P(V^{\otimes t})$ is generated by homogeneous quantum operations. Let $u \in P(V^{\otimes t})$. Let $x_1, \ldots, x_n$ be a basis for V. Then $\{x_{\mathbf{m}} \mid \mathbf{m} \in \underline{n}^t\}$ is a basis of $V^{\otimes t}$ so that u can be written in the form $u = \sum_{\mathbf{m} \in \underline{n}^t} u_{\mathbf{m}} x_{\mathbf{m}}$ for unique elements $u_{\mathbf{m}} \in K$. Since $P(V^{\otimes t}) = \bigcap_{0<i<t} \operatorname{Ker}\left(\Delta_T^{i,t-i}\right)$, we get that $0 = \Delta_T^{i,t-i}(u) = \sum_{\mathbf{m} \in \underline{n}^t} u_{\mathbf{m}} \Delta_T^{i,t-i}(x_{\mathbf{m}})$, for all $0 < i < n$. Since the action $\triangleright$ permutes the terms in the tensor product $x_{\mathbf{m}} = x_{m_1} \otimes x_{m_2} \otimes \cdots \otimes x_{m_t}$, using (17), it is clear that $\Delta_T^{i,t-i}(x_{\mathbf{m}})$ and $\Delta_T^{i,t-i}(x_{\mathbf{n}})$ are linearly independent unless $\mathbf{m} \sim \mathbf{n}$. Therefore, if $\mathbf{m}^1, \ldots, \mathbf{m}^s$ is a set of equivalence class representatives for the equivalence relation $\sim$, we get that

$$0 = \sum_{\mathbf{m} \in \underline{n}^t} u_{\mathbf{m}} \Delta_T^{i,t-i}(x_{\mathbf{m}}) = \sum_{1 \leq a \leq s} \left(\sum_{\mathbf{m} \in \mathcal{S}_t \mathbf{m}^a} u_{\mathbf{m}} \Delta_T^{i,t-i}(x_{\mathbf{m}}) \right)$$

implies $0 = \sum_{\mathbf{m} \in \mathcal{S}_t \mathbf{m}^a} u_{\mathbf{m}} \Delta_T^{i,t-i}(x_{\mathbf{m}})$, for all $1 \leq a \leq s$ and $0 < i < n$.

Hence $u_a := \sum_{\mathbf{m} \in \mathcal{S}_t \mathbf{m}^a} u_{\mathbf{m}} x_{\mathbf{m}} \in P(V^{\otimes t})$, for all $1 \leq a \leq s$. Then u_a is a homogeneous quantum operation for T and $u = \sum_{1 \leq a \leq s} u_a$. □

REMARK 9.3. Theorem 9.2 is not true for non-diagonal braided vector spaces in general. For example let V be the braided vector space with basis x_1, x_2 and braiding defined by

$$\begin{aligned} c(x_1 \otimes x_1) &= x_1 \otimes x_1, \qquad c(x_2 \otimes x_2) = x_2 \otimes x_2, \\ c(x_2 \otimes x_1) &= x_1 \otimes x_1 + x_1 \otimes x_2, \qquad c(x_1 \otimes x_2) = -x_1 \otimes x_1 + x_2 \otimes x_1. \end{aligned}$$

Then $x_1 x_1 + x_1 x_2 - x_2 x_1$ is the unique non-zero primitive element in $V^{\otimes 2}$ and it is not a homogeneous quantum operation. The braided vector space (V, c) has been investigated in [ASt].

LEMMA 9.4. *Let (V,c) be a braided vector space of diagonal type with basis $x_1,\ldots,x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. Let $u=\sum_{\mathbf{m}\in\mathcal{S}_t\mathbf{n}}u_{\mathbf{m}}x_{\mathbf{m}}$ be a homogeneous quantum operation for $T(V,c)$ for some $\mathbf{n}=(n_1,\ldots,n_t)\in\underline{n}^t$ and set $d_i:=\#\{j\mid n_j=i\}$ for all $i\in\{1\ldots n\}$. Then*

$$c_{V,V^{\otimes t}}(x_i\otimes u)=u\otimes\prod_{1\le l\le n}q_{i,l}^{d_l}x_i \qquad \text{and} \qquad c_{V^{\otimes t},V}(u\otimes x_i)=\prod_{1\le l\le n}q_{l,i}^{d_l}x_i\otimes u. \tag{23}$$

Proof. It follows from the equalities $c_{V,V^{\otimes t}}(x_i\otimes x_{\mathbf{m}})=x_{\mathbf{m}}\otimes\prod_{1\le l\le n}q_{i,l}^{d_l}x_i$ and $c_{V^{\otimes t},V}(x_{\mathbf{m}}\otimes x_i)=\prod_{1\le l\le n}q_{l,i}^{d_l}x_i\otimes x_{\mathbf{m}}$ which hold for all $\mathbf{m}\in\mathcal{S}_t\mathbf{n}$. □

THEOREM 9.5. *Let (V,c) be a braided vector space of diagonal type with basis $x_1,\ldots,x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. Let $u=\sum_{\mathbf{m}\in\mathcal{S}_t\mathbf{n}}u_{\mathbf{m}}x_{\mathbf{m}}$ be a homogeneous quantum operation for $T(V,c)$ for some $\mathbf{n}=(n_1,\ldots,n_t)\in\underline{n}^t$ and set $d_i:=\#\{j\mid n_j=i\}$ for all $i\in\{1\ldots n\}$. Suppose (V,c) has a bracket b, and write $b(u)=\sum_{1\le i\le n}b_ix_i$ where $b_i\in K$, for all $i\in\{1,\ldots,n\}$. Then*

$$b_i\left(q_{i,j}-\prod_{1\le l\le n}q_{l,j}^{d_l}\right)=0,\ \text{for all } 1\le i,j\le n, \tag{24}$$

$$b_j\left(q_{i,j}-\prod_{1\le l\le n}q_{i,l}^{d_l}\right)=0,\ \text{for all } 1\le i,j\le n. \tag{25}$$

Proof. By (10), for $1\le j\le n$ we have that $c(b\otimes V)(u\otimes x_j)=(V\otimes b)c_{V^{\otimes t},V}(u\otimes x_j)$. The first term is $c(b\otimes V)(u\otimes x_j)=\sum_{1\le i\le n}b_ic(x_i\otimes x_j)=\sum_{1\le i\le n}b_iq_{i,j}x_j\otimes x_i$. The second term is

$$(V\otimes b)c_{V^{\otimes t},V}(u\otimes x_j)\overset{(23)}{=}\prod_{1\le l\le n}q_{l,j}^{d_l}x_j\otimes b(u)=\sum_{1\le i\le n}b_i\prod_{1\le l\le n}q_{l,j}^{d_l}x_j\otimes x_i.$$

Therefore we get (24).

Similarly, by (10), for $1\le i\le n$ we have that $c(V\otimes b)(x_i\otimes u)=(b\otimes V)c_{V,V^{\otimes t}}(x_i\otimes u)$. The first term is $c(V\otimes b)(x_i\otimes u)=\sum_{1\le j\le n}b_jc(x_i\otimes x_j)=\sum_{1\le j\le n}b_jq_{i,j}x_j\otimes x_i$. The second term is

$$(b\otimes V)c_{V,V^{\otimes t}}(x_i\otimes u)\overset{(23)}{=}\prod_{1\le l\le n}q_{i,l}^{d_l}b(u)\otimes x_i=\sum_{1\le j\le n}b_j\prod_{1\le l\le n}q_{i,l}^{d_l}x_j\otimes x_i.$$

Therefore we get (25). □

COROLLARY 9.6. *Let (V,c) be a braided vector space of diagonal type with basis $x_1,\ldots,x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. Let $u=\sum_{\mathbf{m}\in\mathcal{S}_t\mathbf{n}}u_{\mathbf{m}}x_{\mathbf{m}}$ be a homogeneous quantum operation for $T(V,c)$ for some $\mathbf{n}=(n_1,\ldots,n_t)\in\underline{n}^t$ and set $d_i:=\#\{j\mid n_j=i\}$, for all $i\in\{1\ldots n\}$. Suppose (V,c) has a bracket b. If $b(u)\neq0$, there is $a\in\underline{n}$ such that the following relations hold:*

$$q_{a,j}=\prod_{1\le l\le n}q_{l,j}^{d_l},\ \text{for all } 1\le j\le n, \tag{26}$$

$$q_{i,a}=\prod_{1\le l\le n}q_{i,l}^{d_l},\ \text{for all } 1\le i\le n. \tag{27}$$

Proof. Let $b(u)=\sum_{1\le i\le n}b_ix_i$. From $b(u)\neq0$, we get there is $a\in\underline{n}$ such that $b_a\neq0$. The conclusion follows by Theorem 9.5. □

COROLLARY 9.7. *Let (V,c) be a braided vector space of diagonal type with basis $x_1,\ldots,x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. Suppose (V,c) has a bracket b which is not the restriction of the canonical projection $T(V,c)\to V$. Then there are $a\in\underline{n}$ and $d_1,\ldots,d_n\in\mathbb{N}$ with $d_1+\cdots+d_n>1$ such that (26) and (27) hold.*

Proof. Since b is not the restriction of the canonical projection $T(V,c)\to V$, by Theorem 9.2, there exists $t>1$ and a homogeneous quantum operation $u=\sum_{\mathbf{m}\in\mathcal{S}_t\mathbf{n}}u_{\mathbf{m}}x_{\mathbf{m}}$ for $T(V,c)$ for some $\mathbf{n}=(n_1,\ldots,n_t)\in\underline{n}^t$ such that $b(u)\neq0$. Set $d_i:=\#\{j\mid n_j=i\}$, for all $i\in\{1\ldots n\}$. Then $d_1+\cdots+d_n=t>1$. The conclusion follows by Corollary 9.6. □

We now give an example of a braided vector space (V,c) of diagonal type admitting a non-trivial bracket. Furthermore we determine all primitively generated braided bialgebras with infinitesimal part (V,c).

Example 9.8. Let K be a fixed field with $\mathrm{char}(K) = 0$. In [ASt, Theorem 4.5], the following example of a two dimensional braided vector space (V,c) is given. (V,c) is of diagonal type with basis x_1, x_2 and matrix $(q_{i,j})_{i,j\in\{1,2\}}$, where $q_{1,1} = \gamma \in K$ and $q_{i,j} = 1$ for all $(i,j) \neq (1,1)$. Assume γ is not a root of unity. Then, by [ASt, Theorem 4.7], one has that

$$\mathcal{B}(V,c) = \frac{T(V,c)}{(x_2x_1 - x_1x_2)} = K[x_1,x_2].$$

Thus, as an algebra, $\mathcal{B}(V,c)$ is the polynomial ring in 2 variables with coefficients in K. Note that the endomorphism $c : V \otimes V \to V \otimes V$ has minimal polynomial $(X-\gamma)\left(X^2-1\right)$.

Let then b be a bracket on (V,c) such that (V,c,b) is a braided Lie algebra. Since $\mathcal{B}(V,c)$ is quadratic as an algebra, by [Ar, Proposition 6.16] and Remark 5.3, the braided vector space (V,c) is in $\mathcal{S}$. By [ASt, Theorem 3.11]

$$U(V,c,\beta) = \frac{T(V,c)}{([\mathrm{Id}-b][E_2(V,c)])}$$

where $E_2(V,c)$ is the space of primitive elements in $V\otimes V$. Set $u := x_2\otimes x_1 - x_1\otimes x_2$. It is easy to check that $E_2(V,c) = Ku$. Thus

$$U(V,c,b) = \frac{T(V,c)}{(x_2x_1 - x_1x_2 - b(u))}.$$

If $b(u) = 0$, then by Corollary 4.4, b is the trivial bracket of (V,c).

Assume now $b(u) \neq 0$. Note that this does not contradict Corollary 9.6 as we can choose $a = 1$. Set $b(u) = b_1x_1 + b_2x_2$. Note that u is a homogeneous quantum operation for $T(V,c)$ for $\mathbf{n} = (n_1,n_2) = (1,2)$. Then $d_i := \#\{j \mid n_j = i\} = 1$ for all $i = 1,2$. By Theorem 9.5, we get $0 = b_2\left(q_{2,1} - \prod_{1\le l\le 2} q_{l,1}^{d_l}\right) = b_2(1 - q_{1,1}q_{2,1}) = (1-\gamma)b_2$. Since γ is not a root of unity, we get $b_2 = 0$ so that $b(u) = b_1x_1$ and $b_1 \neq 0$. Via the change of basis $(y_1,y_2) := (x_1, x_2/b_1)$, we can assume $b_1 = 1$ obtaining

$$U(V,c,b) = \frac{T(V,c)}{(x_2x_1 - x_1x_2 - x_1)}$$

In conclusion, by Theorem 5.7,

$$\mathcal{B}(V,c) = K[x_1,x_2] \qquad \text{and} \qquad A := \frac{T(V,c)}{(x_2x_1 - x_1x_2 - x_1)}$$

are the unique candidate primitively generated braided bialgebras with infinitesimal part (V,c). It remains to prove that A has the required property. If we apply a new change of basis $(y_1,y_2) := (x_1, x_2/1+\gamma)$, we get that that $A = T(V,c)/((1+\gamma)(y_2y_1 - y_1y_2) - y_1)$, which is the seventh case in [ASt, Table 1]. Thus, by [ASt, Corollary 3.9], the space of primitive elements in this connected braided bialgebra identify with V as required.

In the last part of this section we prove that the infinitesimal part of a quantum linear space and the braided vector spaces with braiding of Drinfeld-Jimbo type admit only trivial brackets. We will also determine all primitively generated braided bialgebras having one of these braided vector spaces as infinitesimal part.

Example 9.9. Assume K is algebraically closed of characteristic 0. Let (V,c) be a braided vector space of diagonal type with basis $x_1,\ldots,x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. For $i,j\in\underline{n}$, assume that $q_{i,i}\neq 1$ are primitive N_ith roots of unity $(N_i > 1)$ and $q_{i,j}q_{j,i} = 1$ for $i \neq j$.

Set $T := T(V,c)$. By means of the quantum binomial formula (see [ASc3, Lemma 3.6]) one gets that $x_i\otimes x_j - q_{i,j}x_j\otimes x_i \in E(V,c)$ for $i\neq j$ and $x_i^{\otimes N_i} \in E(V,c)$. It is known that $\mathcal{B}(V,c)$ is the so called **quantum linear space**

$$\mathcal{B}(V,c)=\frac{T(V,c)}{\left(x_1^{N_1};\qquad \ldots \qquad x_n^{N_n};\qquad x_ix_j-q_{i,j}x_jx_i,\, i>j\right)}.$$

and that $(V,c)\in\mathcal{S}$ (see [Ar, Example 6.4] and Remark 5.3). Note that $\mathcal{B}(V,c)$ has dimension $N_1\cdot\cdots\cdot N_n$.

Let b be a bracket on (V,c). Set $b\left(x_t^{\otimes N_t}\right)=\sum_{1\leq i\leq n}b_{t,i}x_i$. Note that $x_t^{\otimes N_t}$ is a homogeneous quantum operation for T for $\mathbf{n}=(n_1,\ldots,n_{N_t})=(t,\ldots,t)$. Then $d_l:=\#\{j\mid n_j=l\}=N_t\delta_{l,t}$ for all $l\in\{1,\ldots,n\}$. By Theorem 9.5, we get

$$0=b_{t,i}\left(q_{i,j}-\prod_{1\leq l\leq n}q_{l,j}^{d_l}\right)=b_{t,i}\left(q_{i,j}-q_{t,j}^{N_t}\right)\quad,\text{ for all }1\leq i,j\leq n,$$

$$0=b_{t,j}\left(q_{i,j}-\prod_{1\leq l\leq n}q_{i,l}^{d_l}\right)=b_{t,j}\left(q_{i,j}-q_{i,t}^{N_t}\right)\quad,\text{ for all }1\leq i,j\leq n.$$

Suppose there exists $s\in\{1,\ldots,N_t\}$ such that $b_{t,s}\neq 0$. Then

$$q_{s,i}=q_{t,i}^{N_t},\qquad\text{and}\qquad q_{i,s}=q_{i,t}^{N_t},\qquad\text{for all }1\leq i\leq n.$$

Then $q_{s,s}=q_{t,s}^{N_t}=(q_{t,t}^{N_t})^{N_t}=1$ a contradiction. Therefore $b\left(x_t^{\otimes N_t}\right)=0$ for all $t\in\{1,\ldots,N_t\}$. For all $a\neq b$ we set $u_{a,b}:=x_ax_b-q_{a,b}x_bx_a$ and $b(u_{a,b})=\sum_{1\leq i\leq n}b_{a,b,i}x_i$. Note that $u_{a,b}$ is a homogeneous quantum operation for T for $\mathbf{n}=(n_1,n_2)=(a,b)$. Then $d_l:=\#\{j\mid n_j=l\}=\delta_{l,a}+\delta_{l,b}$ for all $l\in\{1,\ldots,n\}$. By Theorem 9.5, we get

$$0=b_{a,b,i}\left(q_{i,j}-\prod_{1\leq l\leq n}q_{l,j}^{d_l}\right)=b_{a,b,i}\left(q_{i,j}-q_{a,j}q_{b,j}\right),\text{ for all }1\leq i,j\leq n,$$

$$0=b_{a,b,j}\left(q_{i,j}-\prod_{1\leq l\leq n}q_{i,l}^{d_l}\right)=b_{a,b,j}\left(q_{i,j}-q_{i,a}q_{i,b}\right),\text{ for all }1\leq i,j\leq n.$$

Suppose there exists $s\in\{1,\ldots,n\}$ such that $b_{a,b,s}\neq 0$. Then

$$q_{s,i}=q_{a,i}q_{b,i},\qquad\text{and}\qquad q_{i,s}=q_{i,a}q_{i,b}\qquad\text{for all }1\leq i\leq n.\tag{28}$$

From the first equality, if $s\in\{a,b\}$, we get $q_{a,j}=1$ or $q_{b,j}=1$ for all $1\leq j\leq n$ whence $q_{a,a}=1$ or $q_{b,b}=1$ a contradiction. Thus $s\notin\{a,b\}$. We have

$$1=q_{s,a}q_{a,s}\overset{(28)}{=}q_{a,a}q_{b,a}q_{a,a}q_{a,b}=q_{a,a}^2,$$

$$1=q_{s,b}q_{b,s}\overset{(28)}{=}q_{a,b}q_{b,b}q_{b,a}q_{b,b}=q_{b,b}^2.$$

Since $q_{a,a}\neq 1\neq q_{b,b}$, we get $q_{a,a}=-1$ and $q_{b,b}=-1$. Hence

$$q_{s,s}\overset{(28)}{=}q_{a,s}q_{b,s}\overset{(28)}{=}q_{a,a}q_{a,b}q_{b,a}q_{b,b}=q_{a,a}q_{b,b}=1$$

a contradiction. Therefore $b_{a,b,s}=0$ for all $s\in\{1,\ldots,n\}$ whence $b(u_{a,b})=0$. By Corollary 4.4, b is the trivial bracket of (V,c) and $U(V,c,b)=\mathcal{B}(V,c)$.

We are now able to determine all primitively generated braided bialgebras A with infinitesimal part (V,c). Let A be such a braided bialgebra. By Theorem 5.7, A is isomorphic to the enveloping algebra $U(P,c_P,b_P)$ associated to its infinitesimal braided Lie algebra (P,c_P,b_P). Since $(P,c_P)=(V,c)$, by the foregoing, we must have $b_P=0$. In conclusion $A\cong U(P,c_P,b_P)=S(P,c_P)=\mathcal{B}(V,c)$ so that $\mathcal{B}(V,c)$ is the unique primitively generated braided bialgebra with infinitesimal part (V,c).

DEFINITION 9.10. [ASc4, Definition 1.1] Assume char$K = 0$. Let (V, c) be a braided vector space of diagonal type with basis $x_1, \ldots, x_n$ and matrix $(q_{i,j})_{i,j\in\underline{n}}$. To such a braiding one associates a graph as follows. The vertices of the graph are the elements $\{1, \ldots, n\}$, and there is an edge between two vertices i and j if and only if they are distinct and $q_{i,j}q_{j,i} \neq 1$. Denote by $\mathbb{X}$ the set of connected components of the graph.

The braiding c is called of **Cartan type** if $q_{i,i} \neq 1$ for all $1 \leq i \leq n$, and there are integers $\alpha_{i,j}$ such that

- $\alpha_{i,i} = 2$, for $1 \leq i \leq n$,
- $0 \leq -\alpha_{i,j} < \mathrm{ord}\,(q_{i,i})$ (which could be infinite), for $1 \leq i \neq j \leq n$,
- $q_{i,j}q_{j,i} = q_{i,i}^{\alpha_{i,j}}$, for all $1 \leq i, j \leq n$.

Since $\alpha_{i,j} = 0$ implies $\alpha_{j,i} = 0$ for all $1 \leq i \neq j \leq n$, then $(\alpha_{i,j})$ is a generalized Cartan matrix.

A braiding c of Cartan type is called of **Drinfeld-Jimbo type** (**DJ-type** for shortness) if, in addition, there exist positive integers $\delta_1, \ldots, \delta_n$ such that

- $\delta_i\alpha_{i,j} = \delta_j\alpha_{j,i}$, for all $1 \leq i, j \leq n$,
- for all $I \in \mathbb{X}$, there exists $q_I \in K$ not a root of unity, such that $q_{i,j} = q_I^{\delta_i\alpha_{i,j}}$ for all $i \in I$,$1 \leq j \leq n$.

Note that, since we are in characteristic zero, in view of the latter condition, for $1 \leq i, j \leq n$ one has $q_{i,j}$ is a root of unity if and only if $q_{i,j} = 1$. In fact if $n \neq 0$ is such that $q_{i,j}^n = 1$, then $1 = q_{i,j}^n = q_I^{\delta_i\alpha_{i,j}n}$. Since $q_I \in K$ is not a root of unity, we get $\delta_i\alpha_{i,j}n = 0$ so that $\delta_i\alpha_{i,j} = 0$ whence $q_{i,j} = q_I^{\delta_i\alpha_{i,j}} = 1$.

If $i \neq j$ are not connected in the graph, then $1 = q_{i,j}q_{j,i} = q_{i,i}^{\alpha_{i,j}}$. By the foregoing, $q_{ii} \neq 1$ implies $\alpha_{i,j} = 0$ whence $q_{ij} = q_I^{\delta_i\alpha_{i,j}} = 1$.

EXAMPLE 9.11. Let (V, c) be a braided vector space with braiding of DJ-type as above. Set $T := T(V, c)$ and recall that the braided adjoint representation is the linear map $\mathrm{ad}_c : T \to \mathrm{End}T$ given by $(\mathrm{ad}_c u)(v) = u \otimes v - c_T(u \otimes v)$ for all $u, v \in T$. Denote by $(\mathrm{ad}_c u)^t$ the t-th power of $\mathrm{ad}_c(u)$ in End$T$. By [ASc4, Theorem 2.9], one has

$$\mathcal{B}(V, c) = \frac{T(V, c)}{\left((\mathrm{ad}_c x_i)^{1-\alpha_{i,j}}(x_j), \quad 1 \leq i \neq j \leq n\right)}.$$

For all $1 \leq a \neq b \leq n$, set $u_{a,b} := (\mathrm{ad}_c x_a)^{1-\alpha_{a,b}}(x_b)$. One has $u_{a,b} \in E(V, c)$ (see e.g. [Kh1, Theorem 6.1], where $u_{a,b}$ is denoted by $W(x_b, x_a)$ in formulae (18) and (19), or see [ASc2, Lemma A.1]). By [Ar, Theorem 6.1], and Remark 5.3, we have that $(V, c) \in \mathcal{S}$.

Suppose (V, c) has a bracket b and set $b(u_{a,b}) = \sum_{1\leq i\leq n} b_{a,b,i}x_i$. Now, $u_{a,b}$ is a homogeneous quantum operation for $\mathbf{n} = (n_1, \ldots, n_{2-\alpha_{a,b}}) = (a, \ldots, a, b)$. Then $d_l := \#\{j \mid n_j = l\} = (1 - \alpha_{a,b})\delta_{l,a} + \delta_{l,b}$ for all $l \in \{1, \ldots, N_t\}$. By Theorem 9.5, we get

$$\begin{aligned} 0 = b_{a,b,i}\left(q_{i,j} - \prod_{1\leq l\leq n} q_{l,j}^{d_l}\right) &= b_{a,b,i}\left(q_{i,j} - q_{a,j}^{1-\alpha_{a,b}}q_{b,j}\right), \text{ for all } 1 \leq i, j \leq n, \\ 0 = b_{a,b,j}\left(q_{i,j} - \prod_{1\leq l\leq n} q_{i,l}^{d_l}\right) &= b_{a,b,j}\left(q_{i,j} - q_{i,a}^{1-\alpha_{a,b}}q_{i,b}\right), \text{ for all } 1 \leq i, j \leq n. \end{aligned}$$

Suppose there exists $s \in \{1, \ldots, n\}$ such that $b_{a,b,s} \neq 0$. Then

$$q_{s,i} = q_{a,i}^{1-\alpha_{a,b}}q_{b,i}, \qquad \text{and} \qquad q_{i,s} = q_{i,a}^{1-\alpha_{a,b}}q_{i,b} \qquad \text{for all } 1 \leq i \leq n. \tag{29}$$

We have

$$\begin{aligned} q_{s,s}^2 &= q_{s,s}q_{s,s} \overset{(29)}{=} q_{a,s}^{1-\alpha_{a,b}}q_{b,s}q_{s,a}^{1-\alpha_{a,b}}q_{s,b} = (q_{s,a}q_{a,s})^{1-\alpha_{a,b}}\,q_{s,b}q_{b,s} \\ &= \left(q_{s,s}^{\alpha_{s,a}}\right)^{1-\alpha_{a,b}}q_{s,s}^{\alpha_{s,b}} = q_{s,s}^{\alpha_{s,a}(1-\alpha_{a,b})+\alpha_{s,b}}. \end{aligned}$$

Since $q_{s,s}$ is not a root of unity, we get $2 = \alpha_{s,a}(1 - \alpha_{a,b}) + \alpha_{s,b}$. Suppose $s \notin \{a, b\}$. Then $\alpha_{s,a}, \alpha_{a,b}, \alpha_{s,b} \leq 0$ so that $\alpha_{s,a}(1 - \alpha_{a,b}) + \alpha_{s,b} \leq 0$, a contradiction. Hence $s \in \{a, b\}$.

If $s = a$, we get $2 = \alpha_{s,a}(1 - \alpha_{a,b}) + \alpha_{s,b} = \alpha_{a,a}(1 - \alpha_{a,b}) + \alpha_{a,b} = 2 - \alpha_{a,b}$ so that $\alpha_{a,b} = 0$ whence

$$q_{a,b} = q_{s,b} \overset{(29)}{=} q_{a,b}^{1-\alpha_{a,b}} q_{b,b} = q_{a,b} q_{b,b}.$$

Since $q_{a,b}q_{b,a} = a_{a,a}^{\alpha_{a,b}} = 1$, we get that $q_{a,b}$ is invertible so that $q_{b,b} = 1$, a contradiction. Thus $s = b$. We have $2 = \alpha_{s,a}(1 - \alpha_{a,b}) + \alpha_{s,b} = \alpha_{b,a}(1 - \alpha_{a,b}) + \alpha_{b,b} = \alpha_{b,a}(1 - \alpha_{a,b}) + 2$ whence $\alpha_{b,a} = 0$ as $a \neq b \Rightarrow \alpha_{a,b} \leq 0$. Since $\alpha_{b,a} = 0$ implies $\alpha_{a,b} = 0$, as in case $s = a$, we arrive to a contradiction. In conclusion $b(u_{a,b}) = 0$ for all $1 \leq a \neq b \leq n$. By Corollary 4.4, b is the trivial bracket of (V, c) and $U(V, c, b) \cong \mathcal{B}(V, c)$. In view of Theorem 5.7, $\mathcal{B}(V, c)$ is then the unique primitively generated braided bialgebra A whose infinitesimal part is (V, c).

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