Some Remarks on Connected Coalgebras

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SOME REMARKS ON CONNECTED COALGEBRAS

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Dedicated to Freddy Van Oystaeyen, on the occasion of his sixtieth birthday

Abstract. In this paper we introduce the notions of connected, 0-connected and strictly graded coalgebra in the framework of an abelian monoidal category $\mathcal{M}$ and we investigate the relations between these concepts. We recover several results, involving these notions, which are well known in the case when $\mathcal{M}$ is the category of vector spaces over a field K. In particular we characterize when a 0-connected graded bialgebra is a bialgebra of type one.

Introduction

Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums. Given a graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta, \varepsilon)$ in $\mathcal{M}$, we can write $\Delta_{|C_n}$ as the sum of unique components $\Delta_{i,j} : C_{i+j} \to C_i \otimes C_j$ where $i + j = n$. The coalgebra C is defined to be a *strongly* $\mathbb{N}$-*graded coalgebra* (see [AM1, Definition 2.9]) when $\Delta^C_{i,j} : C_{i+j} \to C_i \otimes C_j$ is a monomorphism for every $i, j \in \mathbb{N}$. The associated graded coalgebra

$$gr_C E = C \oplus \frac{C\wedge_E C}{C} \oplus \frac{C\wedge_E C\wedge_E C}{C\wedge_E C} \oplus \cdots ,$$

for a given subcoalgebra C of a coalgebra E in $\mathcal{M}$, is an example of strongly $\mathbb{N}$-graded coalgebra (see [AM2, Theorem 2.10]).

A graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ in a cocomplete monoidal category $\mathcal{M}$ is called 0-*connected* whenever $\varepsilon_C i^C_0 : C_0 \to \mathbf{1}$ is an isomorphism where $i^C_0 : C_0 \to C$ denotes the canonical injection. C is called *strictly graded* whenever it is both strongly $\mathbb{N}$-graded and 0-connected. The associated graded coalgebra $gr_{\mathbf{1}} C$ of a coaugmented coalgebra C in $\mathcal{M}$ is an example of a strictly graded coalgebra (see Theorem 2.10). We also introduce the notion of connected coalgebra in $\mathcal{M}$ (see Definition 2.1).

In Theorem 2.11 we prove the following result. Let $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ be a 0-connected graded coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$. Then

1) $\left((C, \Delta_C, \varepsilon_C), u_C = i^C_{\mathbf{1}}\right)$ is a connected coalgebra where $u_C := i^C_0 \varepsilon_0^{-1} : \mathbf{1} \to C$;
2) $C_0 \wedge_C C_0 = C_0 \oplus P(C)$, where $P(C)$ denotes the primitive part of C.

Moreover, if $\mathcal{M}$ is also complete and satisfies $AB5$, the following assertions are equivalent:

(a) C is a strongly $\mathbb{N}$-graded coalgebra;
(b) $C_1 = P(C)$.

This result is then applied to the following setting. Let H be a braided bialgebra in a cocomplete and complete abelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums and is two-sided exact. Let M be in ${}^H_H\mathcal{M}^H_H$. Let $T = T_H(M)$ be the relative tensor algebra and let $T^c = T^c_H(M)$ be the relative cotensor coalgebra as introduced in [AMS1]. In [AM1], we proved that both T and T^c have a natural structure of graded braided bialgebra and that the natural algebra morphism from T to T^c, which coincides with the canonical injections on H and M, is a graded bialgebra homomorphism. Thus its image is a graded braided

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bialgebra which we denote by $H[M]$ and call, accordingly to [Ni], *the braided bialgebra of type one associated to H and M* (see [AM1, Definition 6.7]).

Let now $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ be a braided graded bialgebra in $(\mathcal{M}, c)$. Assume that B is 0-connected as a coalgebra. Then, by the foregoing, $((B, \Delta_B, \varepsilon_B), u_B)$ is a connected coalgebra. We prove (see Theorem 2.13) that B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if

$$(\oplus_{n\geq 1} B_n)^2 = \oplus_{n\geq 2} B_n \qquad \text{and} \qquad P(B) = B_1.$$

Therefore TOBAs, as introduced in [AG, Definition 3.2.3], are exactly the braided bialgebras of type one in $\mathcal{M} = {}^H_H\mathcal{YD}$ which are 0-connected.

1. Preliminaries and Notations

Notations. Let $[(X, i_X)]$ be a subobject of an object E in an abelian category $\mathcal{M}$, where $i_X = i_X^E : X \hookrightarrow E$ is a monomorphism and $[(X, i_X)]$ is the associated equivalence class. By abuse of language, we will say that (X, i_X) is a subobject of E and we will write $(X, i_X) = (Y, i_Y)$ to mean that $(Y, i_Y) \in [(X, i_X)]$. The same convention applies to cokernels. If (X, i_X) is a subobject of E then we will write $(E/X, p_X) = \mathrm{Coker}(i_X)$, where $p_X = p_X^E : E \to E/X$.
Let $(X_1, i_{X_1}^{Y_1})$ be a subobject of Y_1 and let $(X_2, i_{X_2}^{Y_2})$ be a subobject of Y_2. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms such that $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$. Then there exists a unique morphism, which we denote by $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$, such that $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$:

$$\begin{array}{ccccc}
X_1 & \overset{i_{X_1}^{Y_1}}{\hookrightarrow} & Y_1 & \xrightarrow{p_{X_1}^{Y_1}} & \frac{Y_1}{X_1} \\
{\scriptstyle x}\downarrow & & {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle \frac{y}{x}} \\
X_2 & \overset{i_{X_2}^{Y_2}}{\hookrightarrow} & Y_2 & \xrightarrow{p_{X_2}^{Y_2}} & \frac{Y_2}{X_2}
\end{array}$$

$\delta_{u,v}$ will denote the Kronecker symbol for every $u, v \in \mathbb{N}$.

1.1. **Monoidal Categories.** Recall that (see [Ka, Chap. XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom,* that is the following relation

$$(U \otimes a_{V,W,X}) \circ a_{U,V\otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom,* that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

A *braided monoidal category* $(\mathcal{M}, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is a natural isomorphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for every X, Y, Z in $\mathcal{M}$ satisfying

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \qquad \text{and} \qquad c_{X,Y\otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z).$$

For further details on these topics, we refer to [Ka, Chapter XIII].

It is well known that the Pentagon Axiom completely solves the consistency problem arising out of the possibility of going from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$ in two different ways (see [Mj1, page 420]). This allows the notation $X_1 \otimes \cdots \otimes X_n$ forgetting the brackets for any object obtained from $X_1, \cdots X_n$ using $\otimes$. Also, as a consequence of the coherence theorem, the constraints take care of themselves and can then be omitted in any computation involving morphisms in a monoidal category $\mathcal{M}$.
Thus, for sake of simplicity, from now on, we will omit the associativity constraints.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra A in $\mathcal{M}$ on can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in $\mathcal{M}$, one can define the categories of C-comodules ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$. For more details, the reader is referred to [AMS2].

DEFINITIONS 1.2. Let $\mathcal{M}$ be a monoidal category.
We say that $\mathcal{M}$ is a **coabelian monoidal category** if $\mathcal{M}$ is abelian and both the functors $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and left exact, for any $X \in \mathcal{M}$.

1.3. Let $\mathcal{M}$ be a coabelian monoidal category.
Let (C, i_C^E) and (D, i_D^E) be two subobjects of a coalgebra (E, Δ, ε). Set

$$\Delta_{C,D} := (p_C^E \otimes p_D^E)\Delta : E \to \frac{E}{C} \otimes \frac{E}{D}$$
$$(C \wedge_E D, i_{C\wedge_E D}^E) = \ker(\Delta_{C,D}), \qquad i_{C\wedge_E D}^E : C \wedge_E D \to E$$
$$(\frac{E}{C \wedge_E D}, p_{C\wedge_E D}^E) = \mathrm{Coker}\left(i_{C\wedge_E D}^E\right) = \mathrm{Im}(\Delta_{C,D}), \qquad p_{C\wedge_E D}^E : E \to \frac{E}{C \wedge_E D}$$

Moreover, we have the following exact sequence:

$$0 \longrightarrow C \wedge_E D \overset{i_{C\wedge_E D}^E}{\longrightarrow} E \overset{p_{C\wedge_E D}^E}{\longrightarrow} \frac{E}{C \wedge_E D} \longrightarrow 0. \tag{1}$$

Assume now that (C, i_C^E) and (D, i_D^E) are two subcoalgebras of (E, Δ, ε). Since $\Delta_{C,D} \in {}^E\mathcal{M}^E$, it is straightforward to prove that $C \wedge_E D$ is a coalgebra and that $i_{C\wedge_E D}^E$ is a coalgebra homomorphism.

Let (C, i_C^E) be a subobject of a coalgebra (E, Δ, ε) in a coabelian monoidal category $\mathcal{M}$. We can define (see [AMS2]) the n-th wedge product $\left(C^{\wedge_E n}, i_{C^{\wedge_E n}}^E\right)$ of C in E where $i_{C^{\wedge_E n}}^E : C^{\wedge_E n} \to E$. By definition, we have

$$C^{\wedge_E 0} = 0 \qquad \text{and} \qquad C^{\wedge_E n} = C^{\wedge_E n-1} \wedge_E C, \text{ for every } n \geq 1.$$

One can check that $C^{\wedge_E i} \wedge_E C^{\wedge_E j} = C^{\wedge_E i+j}$ for every $i, j \in \mathbb{N}$.

Assume now that (C, i_C^E) is a subcoalgebra of the coalgebra (E, Δ, ε). Then there is a (unique) coalgebra homomorphism

$$i_{C^{\wedge_E n}}^{C^{\wedge_E n+1}} : C^{\wedge_E n} \to C^{\wedge_E n+1}, \text{ for every } n \in \mathbb{N}.$$

such that $i_{C^{\wedge_E n+1}}^E \circ i_{C^{\wedge_E n}}^{C^{\wedge_E n+1}} = i_{C^{\wedge_E n}}^E$.

1.4. **Graded Objects.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of objects in a cocomplete coabelian monoidal category $\mathcal{M}$ and let

$$X = \bigoplus_{n\in\mathbb{N}} X_n$$

be their coproduct in $\mathcal{M}$. In this case we also say that X is a *graded object of* $\mathcal{M}$ and that the sequence $(X_n)_{n\in\mathbb{N}}$ defines a grading on X. A morphism

$$f : X = \bigoplus_{n\in\mathbb{N}} X_n \to Y = \bigoplus_{n\in\mathbb{N}} Y_n$$

is called a *graded homomorphism* whenever there exists a family of morphisms $(f_n : X_n \to Y_n)_{n\in\mathbb{N}}$ such that $f = \oplus_{n\in\mathbb{N}} f_n$ i.e. such that

$$f \circ i_{X_n}^X = i_{Y_n}^Y \circ f_n, \text{ for every } n \in \mathbb{N}.$$

We fix the following notations. Throughout let

$$p_n^X : X \to X_n \qquad \text{and} \qquad i_n^X : X_n \to X$$

be the canonical projection and injection respectively, for any $n \in \mathbb{N}$.

Given graded objects X, Y in $\mathcal{M}$ we set

$$(X \otimes Y)_n = \oplus_{a+b=n} (X_a \otimes Y_b).$$

Then this defines a grading on $X \otimes Y$ whenever the tensor product commutes with direct sums.

1.5. Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.

Recall that a *graded coalgebra* in $\mathcal{M}$ is a coalgebra (C, Δ, ε) where

$$C = \oplus_{n\in\mathbb{N}} C_n$$

is a graded object of $\mathcal{M}$ such that $\Delta : C \to C \otimes C$ is a graded homomorphism i.e. there exists a family $(\Delta_n)_{n\in\mathbb{N}}$ of morphisms

$$\Delta_n^C = \Delta_n : C_n \to (C \otimes C)_n = \oplus_{a+b=n} (C_a \otimes C_b) \text{ such that } \Delta = \oplus_{n\in\mathbb{N}} \Delta_n.$$

We set

$$\Delta_{a,b}^C = \Delta_{a,b} := \left(C_{a+b} \overset{\Delta_{a+b}}{\to} (C \otimes C)_{a+b} \overset{\omega_{a,b}^{C,C}}{\to} C_a \otimes C_b \right).$$

A homomorphism $f : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D)$ of coalgebras is a graded coalgebra homomorphism if it is a graded homomorphism too.

Definition 1.6. [AM1, Definition 2.9] Let $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta, \varepsilon)$ be a graded coalgebra in $\mathcal{M}$. In analogy with the group graded case (see [NT]), we say that C is a *strongly* $\mathbb{N}$*-graded coalgebra* whenever

$$\Delta_{i,j}^C : C_{i+j} \to C_i \otimes C_j \text{ is a monomorphism for every } i, j \in \mathbb{N},$$

where $\Delta_{i,j}^C$ is the morphism defined in Definition 1.5.

2. Connected coalgebras

Definitions 2.1. Let $\mathcal{M}$ be a coabelian monoidal category. A *coaugmented coalgebra* $((C, \Delta, \varepsilon), u)$ in $\mathcal{M}$ consists of a coalgebra (C, Δ, ε) endowed with a coalgebra homomorphism $u : \mathbf{1} \to C$ called *coaugmentation* of C. Note that u is a monomorphism as $\varepsilon u = \mathrm{Id}_{\mathbf{1}}$. Given a coaugmented coalgebra $((C, \Delta, \varepsilon), u)$ define

$$\alpha_C := (C \otimes u_C) \circ r_C^{-1} + (u_C \otimes C) \circ l_C^{-1} - \Delta_C : C \to C \otimes C,$$
$$(P(C), i_{P(C)}) = \ker(\alpha_C).$$

$(P(C), i_{P(C)})$ is called the *primitive part* of the coaugmented coalgebra C.

A *connected coalgebra* in $\mathcal{M}$ is a coaugmented coalgebra $((C, \Delta, \varepsilon), u)$ in $\mathcal{M}$ such that

$$\varinjlim (\mathbf{1}^{\wedge_C^n})_{n\in\mathbb{N}} = C.$$

Remark 2.2. Let $\mathcal{M}$ be the category of vector spaces over a field K and let $((C, \Delta, \varepsilon), u)$ be a connected coalgebra in $\mathcal{M}$ accordingly to the previous definition. Then $C_{(0)} := Corad(C) \subseteq \mathrm{Im}(u)$ (see e.g. [AMS1, Lemma 5.2]) and hence $C_{(0)} = \mathrm{Im}(u)$ so that C is connected in the usual sense. On the other hand, since $C = \varinjlim (C_{(0)}^{\wedge_C^n})_{n\in\mathbb{N}}$ it is clear that an ordinary connected coalgebra C is also a connected coalgebra in $\mathcal{M}$.

Remark 2.3. Let C be a connected coalgebra in the monoidal category of vector spaces over a field K. Then, the cotensor coalgebra $T^c = T_C^c(M)$ is strongly $\mathbb{N}$-graded and connected for every C-bicomodule M. Nevertheless C needs not to be K, in general.

Question 2.4. Let $\mathcal{M}$ be a cocomplete coabelian monoidal category and let $((C, \Delta, \varepsilon), u)$ be a connected coalgebra in $\mathcal{M}$. Let M be a C-bicomodule in $\mathcal{M}$. Is it true that the cotensor coalgebra $T_C^c(M)$ is a connected coalgebra?

Lemma 2.5. *Let (C, u_C) be a coaugmented coalgebra and let $f : C \to D$ be a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Then (D, u_D) is a coaugmented coalgebra where $u_D = f \circ u_C$. Moreover*

$$\alpha_D \circ f = (f \otimes f) \circ \alpha_C.$$

Proof. Clearly (D, u_D) is a coaugmented coalgebra. Moreover, we have

$$\begin{aligned}
\alpha_D \circ f &= \left[(D \otimes u_D) \circ r_D^{-1} + (u_D \otimes D) \circ l_D^{-1} - \Delta_D\right] \circ f \\
&= (D \otimes f \circ u_C) \circ r_D^{-1} \circ f + (f \circ u_C \otimes D) \circ l_D^{-1} \circ f - \Delta_D \circ f \\
&= (D \otimes f \circ u_C) \circ (f \otimes \mathbf{1}) \circ r_C^{-1} + (f \circ u_C \otimes D) \circ (\mathbf{1} \otimes f) \circ l_C^{-1} - (f \otimes f) \circ \Delta_C \\
&= (f \otimes f) \circ \left[(C \otimes u_C) \circ r_C^{-1} + (u_C \otimes C) \circ l_C^{-1} - \Delta_C\right] = (f \otimes f) \circ \alpha_C.
\end{aligned}$$

□

LEMMA 2.6. *Let $i_F^E : F \to E$ and $i_G^E : G \to E$ be monomorphisms which are coalgebra homomorphisms in a coabelian monoidal category $\mathcal{M}$. Then*

$$\left(\frac{F \wedge_E G}{G}, {}^F\rho_{\frac{F \wedge_E G}{G}} : \frac{F \wedge_E G}{G} \to F \otimes \frac{F \wedge_E G}{G}\right)$$

is a left F-comodule where ${}^F\rho_{\frac{F \wedge_E G}{G}}$ is uniquely defined by

$${}^E\rho_{\frac{F \wedge_E G}{G}} = \left(i_F^E \otimes \frac{F \wedge_E G}{G}\right) \circ {}^F\rho_{\frac{F \wedge_E G}{G}}.$$

Furthermore the following diagram

$$\begin{array}{ccc}
F \wedge_E G & \xrightarrow{\Delta_{F \wedge_E G}} & (F \wedge_E G) \otimes (F \wedge_E G) \\
\Big\downarrow p_G^{F \wedge_E G} & & \\
\frac{F \wedge_E G}{G} & & \Big\downarrow (F \wedge_E G) \otimes p_G^{F \wedge_E G} \\
\Big\downarrow {}^F\rho_{\frac{F \wedge_E G}{G}} & & \\
F \otimes \frac{F \wedge_E G}{G} & \xrightarrow{i_F^{F \wedge_E G} \otimes \frac{F \wedge_E G}{G}} & (F \wedge_E G) \otimes \frac{F \wedge_E G}{G}
\end{array}$$

is commutative and

$${}^{\mathbf{1}}\rho_{\frac{\mathbf{1} \wedge_E G}{G}} = l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}}$$

whenever $F = \mathbf{1}$.

Proof. The first part of the statement follows by [Ar, Lemma 2.14].

Let us prove the commutativity of the diagram. We have

$$\begin{aligned}
&\left(i_{F \wedge_E G}^E \otimes \frac{F \wedge_E G}{G}\right) \circ \left(i_F^{F \wedge_E G} \otimes \frac{F \wedge_E G}{G}\right) \circ {}^F\rho_{\frac{F \wedge_E G}{G}} \circ p_G^{F \wedge_E G} \\
= &\left(i_F^E \otimes \frac{F \wedge_E G}{G}\right) \circ {}^F\rho_{\frac{F \wedge_E G}{G}} \circ p_G^{F \wedge_E G} = {}^E\rho_{\frac{F \wedge_E G}{G}} \circ p_G^{F \wedge_E G} \\
= &\left(E \otimes p_G^{F \wedge_E G}\right) \circ {}^E\rho_{F \wedge_E G} = \left(i_{F \wedge_E G}^E \otimes p_G^{F \wedge_E G}\right) \circ \Delta_{F \wedge_E G}.
\end{aligned}$$

Since the tensor product is left exact, then $i_{F \wedge_E G}^E \otimes \frac{F \wedge_E G}{G}$ is a monomorphism so that we obtain the commutativity of the diagram. Finally, since

$$l^{-1}{}_{\frac{F \wedge_E G}{G}} = \left(\varepsilon_F \otimes \frac{F \wedge_E G}{G}\right) \circ {}^F\rho_{\frac{F \wedge_E G}{G}} \qquad \text{and} \qquad \varepsilon_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}},$$

when $F = \mathbf{1}$ we obtain the last equality in the statement. □

LEMMA 2.7. *Let $\left(E, u_E = i_{\mathbf{1}}^E\right)$ be a coaugmented coalgebra in a coabelian monoidal category $\mathcal{M}$. Then*

$$\left(\mathbf{1}^{\wedge_E^n}, u_{\mathbf{1}^{\wedge_E^n}} = i_{\mathbf{1}}^{\mathbf{1}^{\wedge_E^n}}\right)$$

is a coaugmented coalgebra for every $n \in \mathbb{N}$. Furthermore, for every $n \in \mathbb{N}$, there exists a unique morphism $\tau_n : \mathbf{1}^{\wedge_E^{n+1}} \to \mathbf{1}^{\wedge_E^n} \otimes \mathbf{1}^{\wedge_E^n}$ such that the following diagram

$$\begin{array}{ccc} & & \mathbf{1}^{\wedge_E^{n+1}} \\ & \overset{\tau_n}{\swarrow} & \downarrow \alpha_{\mathbf{1}^{\wedge_E^{n+1}}} \\ \mathbf{1}^{\wedge_E^n} \otimes \mathbf{1}^{\wedge_E^n} & \xrightarrow[i_{\mathbf{1}^{\wedge_E^n}}^{\mathbf{1}^{\wedge_E^{n+1}}} \otimes i_{\mathbf{1}^{\wedge_E^n}}^{\mathbf{1}^{\wedge_E^{n+1}}}]{} & \mathbf{1}^{\wedge_E^{n+1}} \otimes \mathbf{1}^{\wedge_E^{n+1}} \end{array}$$

is commutative.

Proof. Set $\mathbf{1}^n := \mathbf{1}^{\wedge_E^n}$, for every $n \in \mathbb{N}$.
Since $(\mathbf{1}, u_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}})$ is a coaugmented coalgebra and $i_{\mathbf{1}^1}^{\mathbf{1}^n}$ is a coalgebra homomorphism, in view of Lemma 2.5, it is clear that $\left(\mathbf{1}^{\wedge_E^n}, u_{\mathbf{1}^{\wedge_E^n}} = i_{\mathbf{1}^1}^{\mathbf{1}^n}\right)$ is also a coaugmented coalgebra.
Consider the following exact sequence

$$0 \longrightarrow \mathbf{1}^n \xrightarrow{i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} \mathbf{1}^{n+1} \xrightarrow{p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \longrightarrow 0 \tag{2}$$

were $p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}$ denotes the canonical projection. By applying the functor $\mathbf{1}^{n+1} \otimes (-)$ we get

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbf{1}^{n+1} \otimes \mathbf{1}^n & \xrightarrow{\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} & \mathbf{1}^{n+1} \otimes \mathbf{1}^{n+1} & \xrightarrow{\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} & \mathbf{1}^{n+1} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \\ & & \overset{\beta_n}{\nwarrow} & \uparrow \alpha_{\mathbf{1}^n} & & \\ & & & \mathbf{1}^{n+1} & & \end{array}$$

By Lemma 2.6, we have

$$\left(i_{\mathbf{1}}^{\mathbf{1} \wedge_E \mathbf{1}^n} \otimes \frac{\mathbf{1} \wedge_E \mathbf{1}^n}{\mathbf{1}^n}\right) \circ {}^{\mathbf{1}}\rho_{\frac{\mathbf{1} \wedge_E \mathbf{1}^n}{\mathbf{1}^n}} \circ p_{\mathbf{1}^n}^{\mathbf{1} \wedge_E \mathbf{1}^n} = \left[(\mathbf{1} \wedge_E \mathbf{1}^n) \otimes p_{\mathbf{1}^n}^{\mathbf{1} \wedge_E \mathbf{1}^n}\right] \circ \Delta_{\mathbf{1} \wedge_E \mathbf{1}^n}$$

and $l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} = {}^{\mathbf{1}}\rho_{\frac{\mathbf{1} \wedge_E G}{G}}$ so that

$$\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\right) \circ l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} \circ p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} = \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}}. \tag{3}$$

We compute

$$\begin{aligned} & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \alpha_{\mathbf{1}^{n+1}} \\ = \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\begin{array}{c} \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} i_{\mathbf{1}^1}^{\mathbf{1}^n}\right) \circ r_{\mathbf{1}^{n+1}}^{-1} + \left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} + \\ - \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}} \end{array} \right] \\ = \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} - \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}}\right] \\ \overset{(3)}{=} \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} - \left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\right) \circ l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} \circ p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right] = 0, \end{aligned}$$

where the last equality follows by naturality of the unit constraint.

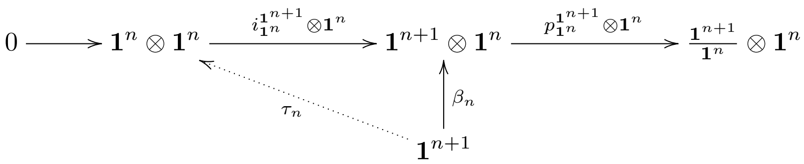
Since $i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}$ is a monomorphism, we obtain

$$\left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \alpha_{\mathbf{1}^{n+1}} = 0 \tag{4}$$

so that, as the above sequence is exact, by the universal property of kernels, there exists a unique morphism $\beta_n : \mathbf{1}^{n+1} \to \mathbf{1}^{n+1} \otimes \mathbf{1}^n$ such that

$$\left(\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \beta_n = \alpha_{\mathbf{1}^{n+1}}. \tag{5}$$

By applying the functor $(-) \otimes \mathbf{1}^n$ to (2), we get



We have

$$\left(\frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^n\right) \circ \beta_n = \left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^{n+1}\right) \circ \left(\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \beta_n$$

$$\overset{(5)}{=} \left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^{n+1}\right) \circ \alpha_{\mathbf{1}^{n+1}} = 0$$

where the last equality can be proved similarly to (4). Since $\frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}$ is a monomorphism we get $\left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^n\right) \circ \beta_n = 0$ so that, as the previous sequence is exact, by the universal property of kernels there exists a unique morphism $\tau_n : \mathbf{1}^{n+1} \to \mathbf{1}^n \otimes \mathbf{1}^n$ such that $\left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^n\right) \circ \tau_n = \beta_n$. Finally we have

$$\left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \tau_n = \left(\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} \otimes \mathbf{1}^n\right) \circ \tau_n = \left(\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \beta_n = \alpha_{\mathbf{1}^{n+1}}.$$

□

Theorem 2.8. *Let $\left((E, \Delta_E, \varepsilon_E), u_E = i_{\mathbf{1}}^E\right)$ be a coaugmented coalgebra in a coabelian monoidal category $\mathcal{M}$. Then $\varepsilon_E \circ i_{P(E)} = 0$ and*

$$\left(\mathbf{1} \wedge_{\mathbf{E}} \mathbf{1} = \mathbf{1}^{\wedge_E^2}, i_{\mathbf{1}^{\wedge_E^2}}^E\right) = \left(\mathbf{1} \oplus P(E), \nabla\left(u_E, i_{P(E)}\right)\right),$$

where $\nabla\left(u_E, i_{P(E)}\right) : \mathbf{1} \oplus P(E) \to E$ denotes the codiagonal morphism associated to u_E and $i_{P(E)}$.

Proof. Set $P = P(E)$. Since (E, u_E) is a coaugmented coalgebra, we apply Lemma 2.5 to the coalgebra homomorphism $\varepsilon_E : E \to \mathbf{1}$. Thus $\left(\mathbf{1}, u_{\mathbf{1}} = \varepsilon_E i_{\mathbf{1}}^E = \mathrm{Id}_{\mathbf{1}}\right)$ is a coaugmented coalgebra and $\alpha_{\mathbf{1}} \circ \varepsilon_E = (\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E$. We have

$$\alpha_{\mathbf{1}} \circ \varepsilon_E \circ i_P = (\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E \circ i_P = 0. \tag{6}$$

By definition, we have

$$\alpha_{\mathbf{1}} = (\mathbf{1} \otimes u_{\mathbf{1}}) \circ r_{\mathbf{1}}^{-1} + (u_{\mathbf{1}} \otimes \mathbf{1}) \circ l_{\mathbf{1}}^{-1} - \Delta_{\mathbf{1}} = r_{\mathbf{1}}^{-1} + l_{\mathbf{1}}^{-1} - l_{\mathbf{1}}^{-1} = r_{\mathbf{1}}^{-1}. \tag{7}$$

Since $\alpha_{\mathbf{1}}$ is an isomorphism, in view of (6), we obtain $\varepsilon_E \circ i_P = 0$. Consider the following exact sequence

$$0 \to \mathbf{1} \xrightarrow{i_{\mathbf{1}}^E} E \xrightarrow{p_{\mathbf{1}}^E} \frac{E}{\mathbf{1}} \to 0.$$

Since $\varepsilon_E \circ i_{\mathbf{1}}^E = \mathrm{Id}_{\mathbf{1}}$, there exists a unique morphism $a : \frac{E}{\mathbf{1}} \to E$ such that $\mathrm{Id}_E = i_{\mathbf{1}}^E \varepsilon_E + a p_{\mathbf{1}}^E$. Clearly the following sequence

$$0 \to \frac{E}{\mathbf{1}} \xrightarrow{a} E \xrightarrow{\varepsilon_E} \mathbf{1} \to 0.$$

is exact. From $\varepsilon_E \circ i_P = 0$, we get that there exists a unique morphism $i'_P : P \to \frac{E}{\mathbf{1}}$ such that $a \circ i'_P = i_P$. Thus

$$\nabla\left(i_{\mathbf{1}}^E, a\right) \circ \left(\mathrm{Id}_{\mathbf{1}} \oplus i'_P\right) = \nabla\left(i_{\mathbf{1}}^E \circ \mathrm{Id}_{\mathbf{1}}, a \circ i'_P\right) = \nabla\left(i_{\mathbf{1}}^E, i_P\right).$$

where $\nabla\left(i_{\mathbf{1}}^E, a\right) : \mathbf{1} \oplus \frac{E}{\mathbf{1}} \to E$ is the codiagonal morphism associated to $i_{\mathbf{1}}^E$ and a. Since $\nabla\left(i_{\mathbf{1}}^E, a\right)$ is an isomorphism and $\mathrm{Id}_{\mathbf{1}} \oplus i'_P$ is a monomorphism, we get that $\nabla\left(i_{\mathbf{1}}^E, i_P\right)$ is a monomorphism. Let us prove that

$$\alpha_E \circ \left(i_{\mathbf{1}^{\wedge_E^2}}^E - i_{\mathbf{1}}^E \circ \varepsilon_E \circ i_{\mathbf{1}^{\wedge_E^2}}^E\right) = 0.$$

By Lemma 2.5 and Lemma 2.7, we have

$$\alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}^{\wedge^2_E}} \otimes i^E_{\mathbf{1}^{\wedge^2_E}}\right) \circ \alpha_{\mathbf{1}^2} = \left(i^E_{\mathbf{1}^{\wedge^2_E}} \otimes i^E_{\mathbf{1}^{\wedge^2_E}}\right) \circ \left(i^{\mathbf{1}^{\wedge^2_E}}_{\mathbf{1}} \otimes i^{\mathbf{1}^{\wedge^2_E}}_{\mathbf{1}}\right) \circ \tau_1 = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1$$

for a suitable $\tau_1 : \mathbf{1}^{\wedge^2_E} \to \mathbf{1} \otimes \mathbf{1}$. Then, by Lemma 2.5, we have

$$(\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E = \alpha_{\mathbf{1}} \circ \varepsilon_E \overset{(7)}{=} r^{-1}_{\mathbf{1}} \circ \varepsilon_E. \tag{8}$$

so that

$$\tau_1 = (\varepsilon_E \otimes \varepsilon_E) \circ \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1 = (\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} \overset{(8)}{=} r^{-1}_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Then

$$\alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1 = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ r^{-1}_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

On the other, by Lemma 2.5, hand we have

$$\alpha_E \circ i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}}\varepsilon_E \otimes i^E_{\mathbf{1}}\varepsilon_E\right) \circ \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} \overset{(8)}{=} \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ r^{-1}_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Hence $\alpha_E \circ \left(i^E_{\mathbf{1}^{\wedge^2_E}} - i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}\right) = 0$ so that, there exists a unique morphism $b : \mathbf{1}^{\wedge^2_E} \to P$ such that $i_P \circ b = i^E_{\mathbf{1}^{\wedge^2_E}} - i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}$. Let

$$\Delta\left(\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) : \mathbf{1}^{\wedge^2_E} \to \mathbf{1} \oplus P$$

be the diagonal morphism of $\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}} : \mathbf{1}^{\wedge^2_E} \to \mathbf{1}$ and $b : \mathbf{1}^{\wedge^2_E} \to P$. We have

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} + i_P \circ b = i^E_{\mathbf{1}^{\wedge^2_E}}$$

so that

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = i^E_{\mathbf{1}^{\wedge^2_E}}. \tag{9}$$

We have

$$\begin{aligned} & \left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ \nabla\left(i^E_{\mathbf{1}}, i_P\right) \\ = \ & \nabla\left[\left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ i^E_{\mathbf{1}}, \left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ i_P\right] \\ = \ & \nabla\left\{\left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \Delta_E, \left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \left[\left(E \otimes i^E_{\mathbf{1}}\right) \circ r^{-1}_E + \left(i^E_{\mathbf{1}} \otimes E\right) \circ l^{-1}_E\right]\right\} = 0 \end{aligned}$$

so that there exists a unique morphism $\Gamma\left(i^E_{\mathbf{1}}, i_P\right) : \mathbf{1} \oplus P \to \mathbf{1}^{\wedge^2_E}$ such that

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) = i^E_{\mathbf{1}^{\wedge^2_E}} \circ \Gamma\left(i^E_{\mathbf{1}}, i_P\right).$$

Since $\nabla\left(i^E_{\mathbf{1}}, i_P\right)$ is a monomorphism, so is $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$. On the other hand, we get

$$i^E_{\mathbf{1}^{\wedge^2_E}} \circ \Gamma\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) \overset{(9)}{=} i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Since $i^E_{\mathbf{1}^{\wedge^2_E}}$ is a monomorphism, we get $\Gamma\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = \mathrm{Id}_{\mathbf{1}^{\wedge^2_E}}$ and hence $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$ is also an epimorphism. Thus $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$ and $\Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right)$ are mutual inverses. □

Definition 2.9. A graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ in a cocomplete monoidal category $\mathcal{M}$ is called 0-*connected* whenever $\varepsilon^C_0 = \varepsilon_C i^C_0 : C_0 \to \mathbf{1}$ is an isomorphism

A graded coalgebra C in a cocomplete monoidal category $\mathcal{M}$ is called *strictly graded* whenever

1) C is 0-connected;
2) C is a strongly $\mathbb{N}$-graded coalgebra.

Next theorem provides our main example of a strictly graded coalgebra.

THEOREM 2.10. *Let $\mathcal{M}$ be a cocomplete coabelian monoidal category such that the tensor product commutes with direct sums. Let $((C,\Delta,\varepsilon),u_C)$ be a coaugmented coalgebra in $\mathcal{M}$.*

Then the associated graded coalgebra

$$gr_{\mathbf{1}}C=\mathbf{1}\oplus\frac{\mathbf{1}\wedge_C\mathbf{1}}{\mathbf{1}}\oplus\frac{\mathbf{1}\wedge_C\mathbf{1}\wedge_C\mathbf{1}}{\mathbf{1}\wedge_C\mathbf{1}}\oplus\cdots$$

is a strictly graded coalgebra.

Proof. By [AM2, Theorem 2.10], we have that $\left(gr_{\mathbf{1}}C,\Delta_{gr_{\mathbf{1}}C},\varepsilon_{gr_{\mathbf{1}}C}=\varepsilon_C\circ u_C\circ p_0^{gr_{\mathbf{1}}C}\right)$ is a strongly $\mathbb{N}$-graded coalgebra. Since u_C is a coalgebra homomorphism, we get

$$\varepsilon_{gr_{\mathbf{1}}C}=\varepsilon\circ u_C\circ p_0^{gr_{\mathbf{1}}C}=p_0^{gr_{\mathbf{1}}C}.$$

It is now clear that $\varepsilon_0^{gr_{\mathbf{1}}C}:=\varepsilon_{gr_{\mathbf{1}}C}\circ i_0^{gr_{\mathbf{1}}C}=\mathrm{Id}_{\mathbf{1}}$ so that $gr_{\mathbf{1}}C$ also 0-connected and hence it is a strictly graded coalgebra. □

THEOREM 2.11. *Let $(C=\oplus_{n\in\mathbb{N}}C_n,\Delta_C,\varepsilon_C)$ be a 0-connected graded coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ (e.g. C is strictly graded). Then*

1) $\left((C,\Delta_C,\varepsilon_C),u_C=i_{\mathbf{1}}^C\right)$ is a connected coalgebra where $u_C:=i_0^C\varepsilon_0^{-1}:\mathbf{1}\to C$.

2) $\left(C_0^{\wedge_C^2},i^C_{C_0^{\wedge_C^2}}\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right)$, where $\nabla\left(i_0^C,i_{P(C)}\right):C_0\oplus P(C)\to C$ denotes the diagonal morphism associated to i_0^C and $i_{P(C)}$.

Moreover if $\mathcal{M}$ is also complete and satisfies $AB5$, then the following assertions are equivalent:

(a) C is a strongly $\mathbb{N}$-graded coalgebra.

(b) $\left(C_1,i_1^C\right)=\left(P(C),i_{P(C)}\right)$.

In particular, when (b) holds, C is a strictly graded coalgebra.

Proof. 1) By Proposition [AM1, Proposition 2.5], $\left(C_0,\Delta_0=\Delta_{0,0},\varepsilon_0=\varepsilon i_0^C\right)$ is a coalgebra in $\mathcal{M}$ and i_0^C is a coalgebra homomorphism. Hence ε_0 and i_0^C are both coalgebra homomorphisms so that $\delta:=i_0^C\varepsilon_0^{-1}$ is a coalgebra homomorphism and hence $((C,\Delta_C,\varepsilon_C),\delta)$ is a coaugmented coalgebra. By [AMS1, Proposition 3.3], we have $C=\varinjlim(C_0^{\wedge_C^t})_{t\in\mathbb{N}}$. Since ε_0 is a coalgebra isomorphism, we conclude that $\varinjlim(\mathbf{1}^{\wedge_C^n})_{n\in\mathbb{N}}=C$ i.e. that $((C,\Delta_C,\varepsilon_C),\delta)$ is a connected coalgebra.

2) It follows by 1) and in view of Theorem 2.8.

Now, assume that $\mathcal{M}$ is also complete and satisfies $AB5$ and let us prove that (a) and (b) are equivalent. By 2), we have

$$\left(C_0^{\wedge_C^2},\delta_2\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right).$$

Then, by [AM1, Theorem 2.22], (a) is equivalent to

$$\left(C_0\oplus C_1,\nabla\left(i_0^C,i_1^C\right)\right)=\left(C_0^{\wedge_C^2},\delta_2\right)$$

and hence to

$$\left(C_0\oplus C_1,\nabla\left(i_0^C,i_1^C\right)\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right). \tag{10}$$

$(b)\Rightarrow(10)$ It is trivial.

$(10)\Rightarrow(b)$ By hypothesis there exists an isomorphism $\Lambda:C_0\oplus P(C)\to C_0\oplus C_1$ such that $\nabla\left(i_0^C,i_{P(C)}^C\right)=\nabla\left(i_0^C,i_1^C\right)\circ\Lambda$.

Let $\pi_{C_a}^{C_0\oplus C_1}:C_0\oplus C_1\to C_a$ be the canonical projection for $a=0,1$. We have

$$\varepsilon_C\circ\nabla\left(i_0^C,i_1^C\right)=\nabla\left(\varepsilon_C\circ i_0^C,\varepsilon_C\circ i_1^C\right)=\nabla\left(\varepsilon_0,0_{Hom(C_1,\mathbf{1})}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus C_1}$$

and by Theorem 2.8, we have

$$\varepsilon_C\circ\nabla\left(i_0^C,i_{P(C)}\right)=\nabla\left(\varepsilon_C\circ i_0^C,\varepsilon_C\circ i_{P(C)}^C\right)=\nabla\left(\varepsilon_0,0_{Hom(P(C),\mathbf{1})}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus P(C)}.$$

Hence, by definition of Λ we get

$$\varepsilon_0\circ\pi_{C_0}^{C_0\oplus C_1}\circ\Lambda=\varepsilon_C\circ\nabla\left(i_0^C,i_1^C\right)\circ\Lambda=\varepsilon_C\circ\nabla\left(i_0^C,i_{P(C)}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus P(C)}.$$

Since ε_0 is an isomorphism, we get that

$$\pi_{C_0}^{C_0\oplus C_1}\circ\Lambda=\pi_{C_0}^{C_0\oplus P(C)}. \tag{11}$$

Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P(C) & \xrightarrow{i_{P(C)}^{C_0\oplus P(C)}} & C_0\oplus P(C) & \xrightarrow{\pi_{C_0}^{C_0\oplus P(C)}} & C_0 & \longrightarrow & 0\\
 & & \downarrow b & & \Lambda\downarrow & & \downarrow \mathrm{Id}_{C_0} & & \\
0 & \longrightarrow & C_1 & \xrightarrow{i_{C_1}^{C_0\oplus C_1}} & C_0\oplus C_1 & \xrightarrow{\pi_{C_0}^{C_0\oplus C_1}} & C_0 & \longrightarrow & 0
\end{array}$$

where the rows are exact and the right square commutes. Hence there is a unique morphism $b:P(C)\to C_1$ such that the left square commutes too. Clearly b is an isomorphism. Moreover

$$i_1^C\circ b=\nabla\left(i_0^C,i_1^C\right)\circ i_{C_1}^{C_0\oplus C_1}\circ b=\nabla\left(i_0^C,i_1^C\right)\circ\Lambda\circ i_{P(C)}^{C_0\oplus P(C)}=\nabla\left(i_0^C,i_{P(C)}^C\right)\circ i_{P(C)}^{C_0\oplus P(C)}=i_{P(C)}^C$$

so that $\left(C_1,i_1^C\right)=\left(P\left(C\right),i_{P(C)}^C\right)$. □

Remark 2.12. Let $(C=\oplus_{n\in\mathbb{N}}C_n,\Delta_C,\varepsilon_C)$ be a graded coalgebra in a cocomplete and complete coabelian monoidal category $\mathcal{M}$ satisfying $AB5$. In view of Theorem 2.11, C is strictly graded if and only if it is 0-connected and

$$\left(C_1,i_1^C\right)=\left(P\left(C\right),i_{P(C)}\right).$$

Note that, when $\mathcal{M}$ is the category of vector spaces over a field K, our definition agrees with Sweedler's one in [Sw, page 232].

Theorem 2.13. *Let $(B,m_B,u_B,\Delta_B,\varepsilon_B)$ be a braided graded bialgebra in a cocomplete and complete braided monoidal category $(\mathcal{M},c)$ such that $\mathcal{M}$ is abelian satisfying $AB5$. Assume that the tensor products are additive, commute with direct sums and are (two-sided) exact. Assume that B is 0-connected as a coalgebra.*
Then $((B,\Delta_B,\varepsilon_B),u_B)$ is a connected coalgebra.
Moreover B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if

$$\left(\oplus_{n\geq 1}B_n\right)^2=\oplus_{n\geq 2}B_n\qquad\textit{and}\qquad P\left(B\right)=B_1.$$

Proof. By [AM1, Theorem 6.10] and Theorem 2.11, $((B,\Delta_B,\varepsilon_B),\delta)$ is a connected coalgebra where $\delta:=i_0^B\varepsilon_0^{-1}:\mathbf{1}\to B$. Moreover B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if $\left(\oplus_{n\geq 1}B_n\right)^2=\oplus_{n\geq 2}B_n$ and $P\left(B\right)=B_1$.

Let us prove that $\delta=u_B$.

By [AM1, Propositions 2.5 and 3.4], $\varepsilon_B=\varepsilon_0\circ p_0^B$ and $u_B=i_0^B\circ u_0$ where $u_0=p_0^B\circ u_B$. Thus

$$\mathrm{Id}_{\mathbf{1}}=\varepsilon_B\circ u_B=\varepsilon_0\circ p_0^B\circ i_0^B\circ u_0=\varepsilon_0\circ u_0$$

and hence $u_0=\varepsilon_0^{-1}$. Then we get $u_B=i_0^B\circ u_0=i_0^B\circ\varepsilon_0^{-1}$. □

Remark 2.14. Recall that a *TOBA* (also called braided Hopf algebra of type one) [AG, Definition 3.2.3] in the category ${}_H^H\mathcal{YD}$ of Yetter Drinfeld modules over an ordinary K-Hopf algebra H is a graded bialgebra $T=\oplus_{n\in\mathbb{N}}T_n$ in this category such that

$$T_0\simeq K,\qquad\left(\oplus_{n\geq 1}T_n\right)^2=\oplus_{n\geq 2}T_n,\qquad\text{and}\qquad P\left(T\right)=T_1.$$

Therefore TOBAs are exactly the bialgebras described in Theorem 2.13 in the case when $\mathcal{M}={}_H^H\mathcal{YD}$.

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