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ON THE COMBINATORIAL RANK OF A GRADED BRAIDED BIALGEBRA

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Abstract. Let B be a graded braided bialgebra. Let $S(B)$ denote the algebra obtained dividing out B by the two sided ideal generated by homogeneous primitive elements in B of degree at least two. We prove that $S(B)$ is indeed a graded braided bialgebra quotient of B. It is then natural to compute $S(S(B))$, $S(S(S(B)))$ and so on. This process yields a direct system whose direct limit comes out to be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra. Following V. K. Kharchenko, if the direct system is stationary exactly after n steps, we say that B has combinatorial rank n and we write $\kappa(B) = n$. We investigate conditions guaranteeing that $\kappa(B)$ is finite. In particular, we focus on the case when B is the braided tensor algebra $T(V, c)$ associated to a braided vector space (V, c), providing meaningful examples such that $\kappa(T(V, c)) \leq 1$.

1. Introduction

Let K be a fixed ground-field. Let A be a finite dimensional Hopf algebra whose coradical (i.e. the sum of all simple subcoalgebras of A) is a Hopf subalgebra, say H; this happens e.g. if A is pointed i.e. all simple subcoalgebras of A are one-dimensional. It is well known that the graded coalgebra gr$A$, associated to the coradical filtration of A, is a Hopf algebra itself and can be described as a Radford-Majid bosonization $R\#H$ by H of the so called diagram R of A, which is a connected graded braided bialgebra in the braided monoidal category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H. This is the starting point of the so called *lifting method* for the classification of finite dimensional pointed Hopf algebras, due to N. Andruskiewitsch and H.J. Schneider, see e.g. [ASc1]. This example shows how the theory of graded braided bialgebras fits into the classification of finite dimensional Hopf algebras problem. It is remarkable that the notion of braided bialgebra can be defined, following [Ta, Definition 5.1], without the use of braided monoidal categories. In fact a braided bialgebra is a braided vector space (i.e. a vector space endowed with a braiding) which is both an algebra and a coalgebra with structures compatible with the braiding in a natural way. The notion of braided bialgebra admits a graded counterpart which is called a graded braided bialgebra.

The starting point of this paper is the study of graded braided bialgebras whose underlying coalgebra is strongly $\mathbb{N}$-graded. Recall that a graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C^n, \Delta, \varepsilon)$ is *strongly $\mathbb{N}$-graded* whenever the (i, j)-homogeneous component $\Delta^{i,j} : C^{i+j} \to C^i \otimes C^j$ of the comultiplication Δ is a monomorphism for every $i, j \in \mathbb{N}$ (dually the notion of strongly $\mathbb{N}$-graded algebra can be introduced). In Theorem 3.4, using results in [AM], we show that a graded braided bialgebra B is strongly $\mathbb{N}$-graded as a coalgebra if and only if B has no non-zero homogeneous primitive elements of degree at least two. This can be stated equivalently as follows. Denote by $S(B)$ the algebra obtained dividing out B by the two sided ideal generated by the space $E(B)$ spanned by the homogeneous primitive elements in B of degree at least two. Then B is strongly $\mathbb{N}$-graded as a coalgebra if and only if the canonical projection $\pi_S : B \to S(B)$ is an isomorphism. In Theorem 3.6, we prove that the algebra $S(B)$ is indeed a graded braided bialgebra such that π_S is a graded braided bialgebra homomorphism. It is then natural to compute $S^{[2]}(B) := S(S(B))$, $S^{[3]}(B) := S(S(S(B)))$ and so on. This process yields a direct system whose direct limit $S^{[\infty]}(B)$ is a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra (Theorem 3.14) and that coincides with the image of

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a canonical graded braided bialgebra homomorphism $\psi_B : B \to T^c_{B^0}\left(B^1\right)$, the latter being the cotensor coalgebra (Theorem 3.16). In Corollary 3.17, we prove that $S^{[\infty]}(B)$ coincides also with $B^0\left[B^1\right]$, the braided bialgebra of Type one associated to B^0 and B^1 (see 3.15), whenever B is, in addition, strongly $\mathbb{N}$-graded as an algebra.

Adopting the terminology of V. K. Kharchenko in [Kh2, Definition 5.4], we say that B *has combinatorial rank* n, in symbols $\kappa(B) = n$, if the direct system above is stationary exactly after n steps. It stems from our results that combinatorial rank is a measure of how far B is to be strongly $\mathbb{N}$-graded as a coalgebra.

We then deal with the existence of an upper bound for the combinatorial rank $\kappa(B)$. In Proposition 4.2, we investigate how combinatorial rank behaves with respect to surjective graded braided bialgebra homomorphisms. In Theorem 4.4 we show that, if the set of non-zero homogeneous components of $S^{[\infty]}(B)$ is finite, then $\kappa(B)$ is finite. On the other hand, in Corollary 4.5, we prove that, when B is strongly $\mathbb{N}$-graded as an algebra and $B^0\left[B^1\right]$, as a graded braided bialgebra, divides out $T_{B^0}\left(B^1\right)$ by relations in degree not greater than $N \in \mathbb{N}$, then $\kappa\left(B\right) \leq N - 1$.

Next we focus our attention on case B is the braided tensor algebra $T(V, c)$ associated to a braided vector space (V, c). In particular $\kappa\left(T(V, c)\right)$ is denoted by $\kappa(V, c)$ and it is called the combinatorial rank of (V, c). Moreover $S(T(V, c))$ is simply denoted by $S(V, c)$ and it is called the symmetric algebra of (V, c): we adopt this terminology as $S(V, c)$ reduces to the classical symmetric algebra when c is the canonical flip, see Remark 6.14.

In Section 6, meaningful examples of braided vector spaces of combinatorial rank at most one are given. For instance, we consider braided vector spaces

- of diagonal type whose Nichols algebra is a domain of finite Gelfand-Kirillov dimension (Theorem 6.3);
- two-dimensional and of abelian group type whose symmetric algebra has dimension at most 31 (Proposition 6.6);
- with braiding of Hecke type with regular mark, e.g. the braiding is a symmetry and the characteristic of K is zero (Theorem 6.13);
- whose Nichols algebra is quadratic as an algebra (Proposition 6.16).

In Section 7, we collect some examples of braided vector spaces having combinatorial rank greater than one.

2. Preliminaries

Throughout this paper K will denote a field. All vector spaces will be defined over K and the tensor product over K will be denoted by $\otimes$.

In this section we define the main notions that we will deal with in the paper.

Definition 2.1. Let V be a vector space over a field K. A K-linear map $c = c_V : V \otimes V \to V \otimes V$ is called a **braiding** if it satisfies the quantum Yang-Baxter equation $c_1 c_2 c_1 = c_2 c_1 c_2$ on $V \otimes V \otimes V$, where we set $c_1 := c \otimes V$ and $c_2 := V \otimes c$. The pair (V, c) will be called a **braided vector space**. A morphism of braided vector spaces (V, c_V) and (W, c_W) is a K-linear map $f : V \to W$ such that $c_W(f \otimes f) = (f \otimes f)c_V$.

A general method for producing braided vector spaces is to take an arbitrary braided category $(\mathcal{M}, \otimes, K, a, l, r, c)$, which is a monoidal subcategory of the category of K-vector spaces (here a, l, r denote the associativity, the left and the right unit constraints respectively). Hence any object $V \in \mathcal{M}$ can be regarded as a braided vector space with respect to $c := c_{V,V}$, where $c_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the braiding in $\mathcal{M}$, for all $X, Y \in \mathcal{M}$.

Let $\mathcal{N}$ be either the category of comodules over a coquasitriangular Hopf algebra or the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode. Then the forgetful functor F from $\mathcal{N}$ into the category of K-vector spaces is a strict monoidal functor. Hence $\mathcal{M} = \mathrm{Im}F$ is an example of a category as above.

Definition 2.2 (Baez, [Ba]). A quadruple (A, m, u, c) is called a **braided algebra** if (A, m, u) is an associative unital algebra, (A, c) is a braided vector space and m and u commute with c, that is the following conditions hold:

$$c(m\otimes A)=(A\otimes m)(c\otimes A)(A\otimes c),\qquad c(A\otimes m)=(m\otimes A)\,(A\otimes c)\,(c\otimes A),$$
$$c(u\otimes A)=A\otimes u,\qquad c(A\otimes u)=u\otimes A.$$

A morphism of braided algebras is, by definition, a morphism of ordinary algebras which, in addition, is a morphism of braided vector spaces.

A **braided coalgebra** (C,Δ,ε,c) and a morphism of braided coalgebras are defined analogously.

[Ta, Definition 5.1] A sextuple $(B,m,u,\Delta,\varepsilon,c)$ is a called a **braided bialgebra** if (B,m,u,c) is a braided algebra, (B,Δ,ε,c) is a braided coalgebra and the following relation holds:

$$\Delta_B m=(m\otimes m)(B\otimes c\otimes B)(\Delta\otimes\Delta). \tag{1}$$

Examples of the notions above are algebras, coalgebras and bialgebras in any braided category $(\mathcal{M},\otimes,K,a,l,r,c)$ which is a monoidal subcategory of the category of K-vector spaces.

The notions of algebra, coalgebra and braided bialgebra admit a graded counterpart. For further details we refer to [AMS, 1.8].

Example 2.3. Let (V,c) be a braided vector space. Consider the tensor algebra $T=T(V)$ with multiplication m_T and unit u_T. This is a graded braided algebra with n-th graded component $V^{\otimes n}$. The braiding c_T on T is defined using the braiding of V.

Now $T\otimes T$ becomes itself an algebra with multiplication $m_{T\otimes T}:=(m_T\otimes m_T)\,(T\otimes c_T\otimes T)$. This algebra is denoted by $T\otimes_c T$. The universal property of the tensor algebra yields two algebra homomorphisms $\Delta_T:T\to T\otimes_c T$ and $\varepsilon_T:T\to K$. It is straightforward to check that $(T,m_T,u_T,\Delta_T,\varepsilon_T,c_T)$ is a graded braided bialgebra. Note that Δ_T really depends on c. For example, one has $\Delta_T(z)=z\otimes 1+1\otimes z+(c+\mathrm{Id})(z)$, for all $z\in V\otimes V$.

Definition 2.4. The graded braided bialgebra described in Example 2.3 is called the **braided tensor algebra** and will be denoted by $T(V,c)$.

Definitions 2.5. Let (C,Δ,ε) be a graded coalgebra. Set

$$E_0(C):=0,\qquad E_1(C):=0\qquad\text{and}\qquad E_n(C):=\bigcap_{1\leq i\leq n-1}\ker\left(\Delta^{i,n-i}\right),\ \text{for } n\geq 2$$

and set $E(C):=\bigoplus_{n\in\mathbb{N}}E_n(C)$. Denote by $i^n=i^n_C:C^n\to C$ and $p^n=p^n_C:C\to C^n$ the canonical injection and projection respectively. Following [Ni, 1.2], we set

$$P_1(C):=\ker\left[\Delta-\left(i^0p^0\otimes C\right)\Delta-\left(C\otimes i^0p^0\right)\Delta\right]\subseteq C.$$

Remark 2.6. Note that $P_1(C)$ is just the space $P(C)$ of primitive elements in C whenever C^0 is one dimensional.

The following result relates $E(C)$ and $P_1(C)$.

Lemma 2.7. *Let (C,Δ,ε) be a graded braided bialgebra. Then $P_1(C)=C^1\oplus[\oplus_{n\geq 2}E_n(C)]$. Moreover $C^0\wedge C^0=C^0\oplus C^1\oplus[\oplus_{n\geq2}E_n(C)]$.*

Proof. Set $h:=\left[\mathrm{Id}_{C\otimes C}-i^0p^0\otimes C-C\otimes i^0p^0\right]\circ\Delta:C\to C\otimes C$. For every $a,b\in\mathbb{N},a+b\geq 1$, define a map $h_{a,b}:C^{a+b}\to C^a\otimes C^b$ by setting $h_{a,b}:=0$ if $a=0$ or $b=0$, and $h_{a,b}:=\Delta^{a,b}$, if $a,b\geq 1$. Consider the map $h_n:C^n\to(C\otimes C)^n:=\oplus_{a+b=n}C^a\otimes C^b$ where $h_0:=-\Delta^{0,0}$ and where, for every $n\geq 1$, h_n denotes the diagonal morphism associated to the family $(h_{a,b})_{a+b=n}$. Using that (C,Δ,ε) is a graded coalgebra, it is straightforward to check that $h\circ i^n=i^n_{C\otimes C}\circ h_n$ for all $n\in\mathbb{N}$. This entails that $h:C\to C\otimes C$ is a graded homomorphism so that

$$P_1(C)=\ker(h)=\oplus_{n\in\mathbb{N}}\ker(h_n)=\ker(h_0)\oplus\ker(h_1)\oplus[\oplus_{n\geq2}\ker(h_n)]=C^1\oplus[\oplus_{n\geq2}E_n(C)].$$

Finally, by [Ni, after Proposition 1.2.1], we have that $C^0\wedge C^0=C^0\oplus P_1(C)$. □

2.8. Recall that a coalgebra C is called **connected** if the coradical C_0 of C (i.e the sum of all simple subcoalgebras of C) is one dimensional. In this case there is a unique group-like element $1_C\in C$ such that $C_0=K1_C$. A morphism of connected coalgebras is a coalgebra homomorphisms (clearly it preserves the grouplike element).

By definition, a braided coalgebra (C,c) is **connected** if the underlying coalgebra is connected and, for any $x \in C$, $c(x \otimes 1_C) = 1_C \otimes x$ and $c(1_C \otimes x) = x \otimes 1_C$.

Lemma 2.9. *(cf.* [AMS, Remark 1.12]*) Let (B,c) be a connected braided bialgebra. Let $P = P(B)$ be the space of primitive elements of B. Then $c(P \otimes P) \subseteq P \otimes P$. Let $c_P : P \otimes P \to P \otimes P$ be the restriction of c to $P \otimes P$. Then (P, c_P) is a braided vector space.*

3. The construction of $S(B)$ and its iteration

In this section we associate to any graded braided bialgebra B a new algebra $S(B)$. We will show how this definition can help to characterize graded braided bialgebras which are strongly $\mathbb{N}$-graded as coalgebras. Then we will show that $S(B)$ carries a graded braided bialgebra structure itself. Iterating the process we will obtain another graded braided bialgebra, denoted by $S^{[\infty]}(B)$, and we will investigate its properties.

Definition 3.1. For a graded braided bialgebra B, we set

$$S(B) = \frac{B}{(E(B))}, \tag{2}$$

where $E(B)$ is as in Definition 2.5. Denote by $\pi_S := \pi_S^B : B \to S(B)$ the canonical projection.

3.1. Strongly $\mathbb{N}$-graded (co)algebras and $S(B)$.

In this subsection we introduce and investigate the notion of combinatorial rank of a graded braided algebra an coalgebra. In particular we deal with graded braided bialgebras B which are strongly $\mathbb{N}$-graded as coalgebras.

Definition 3.2. [AM, Definition 3.5] Let (A, m, u) be a graded algebra in $\mathcal{M}$. In analogy with the group graded case, we say that A is a *strongly* $\mathbb{N}$*-graded algebra* whenever $m^{i,j} : A^i \otimes A^j \to A^{i+j}$ is an epimorphism for every $i, j \in \mathbb{N}$.

[AM, Definition 2.9] (see also [MS, Lemma 2.3]) Let (C, Δ, ε) be a graded coalgebra in $\mathcal{M}$. We say that C is a *strongly* $\mathbb{N}$*-graded coalgebra* whenever $\Delta^{i,j} : C^{i+j} \to C^i \otimes C^j$ is a monomorphism for every $i, j \in \mathbb{N}$.

3.3. Recall that, given a coalgebra C and a C-bicomodule M, one can consider the cotensor coalgebra $(T^c := T^c_C(M), \Delta_{T^c}, \varepsilon_{T^c})$. It is defined as follows. As a vector space $T^c = C \oplus M \oplus M^{\square_C 2} \oplus M^{\square_C 3} \oplus \cdots$, where $\square_C$ denotes the cotensor product over C. Furthermore, $\Delta^{i,j}_{T^c}$ is given by the comodule structure map for $i = 0$ or $j = 0$; when $i, j > 0$, it is induced by the map $m_1 \otimes \cdots \otimes m_n \mapsto (m_1 \otimes \cdots \otimes m_i) \otimes (m_{i+1} \otimes \cdots \otimes m_n)$. The counit is zero on elements of positive homogeneous degree. The cotensor coalgebra fulfils a suitable universal property that resemble the universal property of the tensor algebra (see [Ni, Proposition 1.4.2]). Note that T^c is a strongly $\mathbb{N}$-graded coalgebra.

Theorem 3.4. *Let B be a graded braided bialgebra. The following assertions are equivalent.*

1. *The canonical projection $\pi_S : B \to S(B)$ is an isomorphism.*
2. *$E_n(B) = 0$ for every $n \geq 2$.*
3. *B is strongly $\mathbb{N}$-graded as a coalgebra.*
4. *the (n,1)-homogeneous component $\Delta^{n,1}$ of the comultiplication of B is injective for all $n \in \mathbb{N}$.*
5. *The canonical map $\psi : B \to T^c_{B^0}(B^1)$ is injective.*
6. *$B^0 \oplus B^1 \oplus \cdots \oplus B^n = \underbrace{B^0 \wedge_B \cdots \wedge_B B^0}_{n+1}$, for every $n \in \mathbb{N}$.*
7. *$B^0 \oplus B^1 = B^0 \wedge_B B^0$.*

Proof. The equivalences (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follows by applying [AM, Theorem 2.22] to the monoidal category of vector spaces. (1) $\Leftrightarrow$ (2) It is trivial. (2) $\Leftrightarrow$ (7) By Lemma 2.7, we have $B^0 \wedge_B B^0 = B^0 \oplus B^1 \oplus [\oplus_{n \geq 2} E_n(B)]$. $\square$

Note that the previous result is closely related to [MS, Lemma 2.3].

3.2. $S(B)$ **is a graded braided bialgebras.** In this subsection we investigate the properties of the algebra $S(B)$ associated to any graded braided bialgebra B (see Definition 3.1).

REMARK 3.5. For B a graded braided bialgebra and for all $s,t\in\mathbb{N}$, one has

$$c\left(B^{s}\otimes E_{t}\left(B\right)\right)\subseteq E_{t}\left(B\right)\otimes B^{s}\qquad\text{and}\qquad c\left(E_{t}\left(B\right)\otimes B^{s}\right)\subseteq B^{s}\otimes E_{t}\left(B\right). \tag{3}$$

Let as prove, for instance, the second relation. For all $1\leq i\leq t-1$, we have

$$\left(B^{s}\otimes\Delta^{i,t-i}\right)c^{t,s}\left(E_{t}\left(B\right)\otimes B^{s}\right)=\left(c^{i,s}\otimes B^{t-i}\right)\left(B^{i}\otimes c^{t-i,s}\right)\left(\Delta^{i,t-i}\otimes B^{s}\right)\left(E_{t}\left(B\right)\otimes B^{s}\right)=0.$$

Then

$$c^{t,s}\left(E_{t}\left(B\right)\otimes B^{s}\right)\subseteq\bigcap_{1\leq i\leq t-1}\left[B^{s}\otimes\ker\left(\Delta^{i,t-i}\right)\right]=B^{s}\otimes\left[\bigcap_{1\leq i\leq t-1}\ker\left(\Delta^{i,t-i}\right)\right]=B^{s}\otimes E_{t}\left(B\right).$$

THEOREM 3.6. *For any graded braided bialgebra B, $S\left(B\right)$ has a unique graded braided bialgebra structure such that the canonical projection $\pi_{S}:B\rightarrow S\left(B\right)$ is a graded braided bialgebra homomorphism.*

Proof. Denote by m,u,Δ,ε,c the structure maps of B. Since $I:=\left(E\left(B\right)\right)$ is generated by homogeneous elements, it is a graded ideal of B with graded component $I_{n}=I\cap B^{n}$. Hence $S:=S\left(B\right)$ has a unique graded algebra structure such that $\pi_{S}:B\rightarrow S\left(B\right)$ is a graded algebra homomorphism. Let us check it is also a graded coalgebra.

Now, for every $z\in E_{t}\left(B\right),x\in B^{u},y\in B^{v},s\in B^{w}$, one can check that $c^{u+t+v,w}\left(xzy\otimes s\right)\in B^{w}\otimes I_{u+t+v}$ so that

$$c\left(B^{u}\otimes I_{t}\right)\subseteq I_{t}\otimes B^{u}\qquad\text{and}\qquad c\left(I_{t}\otimes B^{u}\right)\subseteq B^{u}\otimes I_{t}. \tag{4}$$

Using (1) and (4), it is straightforward to prove that $\Delta^{a,b}\left(I_{a+b}\right)\subseteq B^{a}\otimes I_{b}+I_{a}\otimes B^{b}$, for every $a,b\in\mathbb{N}$. In the other hand, since B is a graded coalgebra and $E_{t}\left(B\right)=0$ if $n=0,1$, we have $\varepsilon\left(I\right)=0$ so that I is a graded coideal and hence $S:=S\left(B\right)$ carries a unique graded coalgebra structure such that π_{S} is a coalgebra homomorphism. Let us prove that c factors trough a braiding c_{S} of S. By the foregoing we get

$$c^{a,b}\left(\ker\left(\pi_{S}^{a}\otimes\pi_{S}^{b}\right)\right)=c^{a,b}\left(B^{a}\otimes I_{b}+I_{a}\otimes B^{b}\right)\overset{(4)}{\subseteq}I_{b}\otimes B^{a}+B^{b}\otimes I_{a}=\ker\left(\pi_{S}^{b}\otimes\pi_{S}^{a}\right)$$

so that, since $\pi_{S}\otimes\pi_{S}$ is surjective, there exists a unique K-linear map $c_{S}^{a,b}:S^{a}\otimes S^{b}\rightarrow S^{b}\otimes S^{a}$ such that

$$c_{S}^{a,b}\left(\pi_{S}^{b}\otimes\pi_{S}^{a}\right)=\left(\pi_{S}^{a}\otimes\pi_{S}^{b}\right)c^{a,b}. \tag{5}$$

This relation can be used to prove that (S,c_{S}) is a graded braided bialgebra with structure induced by π_{S}. $\square$

REMARK 3.7. Let $f:B\rightarrow B'$ be a graded braided bialgebra homomorphism. Then one has $\Delta_{B'}^{a,b}\circ f_{a+b}=\left(f_{a}\otimes f_{b}\right)\circ\Delta_{B}^{a,b}$, for every $a,b\in\mathbb{N}$. This entails that $f_{t}\left[E_{t}\left(B\right)\right]\subseteq E_{t}\left(B'\right)$ for every $t\geq2$.

PROPOSITION 3.8. *For any graded braided bialgebra homomorphism $f:B\rightarrow B'$, there is a unique algebra homomorphism*

$$S\left(f\right):S\left(B\right)\rightarrow S\left(B'\right)$$

such that $S\left(f\right)\circ\pi_{S}=\pi_{S'}\circ f$. Moreover $S\left(f\right)$ is a graded braided bialgebra homomorphism.

Proof. Let $n\geq2$. Set $I:=\left(E_{n}\left(B\right)\right)$ and $I':=\left(E_{n}\left(B'\right)\right)$. Denote by I_{n} and I_{n}' the n-th graded components of these ideals. By Remark 3.7, $f_{n}\left[E_{n}\left(B\right)\right]\subseteq E_{n}\left(B'\right).$ Since f is a graded algebra homomorphism, we get $f\left(I_{n}\right)\subseteq I_{n}'$ so that f is a morphism of graded ideals. Hence the first part of the statement follows. Furthermore $S\left(f\right)$ is a morphism of graded algebras with graded component $S\left(f^{n}\right):S^{n}\rightarrow\left(S'\right)^{n}$ uniquely defined by $S\left(f^{n}\right)\circ\pi_{S}^{n}=\pi_{S'}^{n}\circ f^{n}$. Since π_{S} is an epimorphism, one gets that $S\left(f\right)$ is a coalgebra homomorphism whence a graded coalgebra homomorphism. $\square$

DEFINITION 3.9. Let $f : B \to B'$ be a graded braided bialgebra homomorphism. In view of Remark 3.7, there exists a unique map $E_t(f) : E_t(B) \to E_t(B')$ such that $j_t^{B'} \circ E_t(f) = f_t \circ j_t^B$, for every $t \geq 2$, where $j_t^B : E_t(B) \to B^t$ and $j_t^{B'} : E_t(B') \to (B')^t$ are the canonical inclusions. We set $E(f) := \oplus_{n \in \mathbb{N}} E_t(f)$.

3.3. **The graded braided bialgebra $S^{[\infty]}(B)$.** In this subsection we introduce and investigate the graded braided bialgebra $S^{[\infty]}(B)$ associated to a graded braided bialgebra B.

DEFINITION 3.10. Let B be a graded braided bialgebra. Define recursively $S^{[n]}(B)$ by setting

$$S^{[0]}(B) := B \qquad \text{and} \qquad S^{[n]}(B) := S(S^{[n-1]}(B)), \text{ for } n \geq 1.$$

This yields a direct system $((S^{[i]}(B))_{i \in \mathbb{N}}, (\pi_i^j)_{i,j \in \mathbb{N}})$

$$B = S^{[0]}(B) \xrightarrow{\pi_0^1} S^{[1]}(B) \xrightarrow{\pi_1^2} S^{[3]}(B) \to \cdots$$

where π_i^j is defined in an obvious way for every $j \geq i$. Denote by

$$(S^{[\infty]}(B), \pi_i^\infty) = \varinjlim S^{[i]}(B)$$

the direct limit of this direct system.

The following technical results are needed to prove Theorem 3.14.

LEMMA 3.11. *Let $(B, m, u, \Delta, \varepsilon, c)$ be a graded braided bialgebra. Assume there exists $n \in \mathbb{N}$ such that $\Delta^{t,n-t} : B^n \to B^t \otimes B^{n-t}$ is injective for every $0 \leq t \leq n$. Then*

$$\ker\left(\Delta^{n,1}\right) = \ker\left(\Delta^{n-1,2}\right) = \cdots = \ker\left(\Delta^{1,n}\right).$$

Furthermore, for every $a, b \geq 1$ such that $a + b = n + 1$, we have $E_{n+1}(B) = \ker\left(\Delta^{a,b}\right)$.

Proof. For every $a, b \geq 1$ such that $a + b = n + 1$, we have

$$\ker\left(\Delta^{a,b}\right) = \ker\left[\left(\Delta^{a-1,1} \otimes B^b\right)\Delta^{a,b}\right] = \ker\left[\left(B^{a-1} \otimes \Delta^{1,b}\right)\Delta^{a-1,b+1}\right] = \ker\left(\Delta^{a-1,b+1}\right).$$

The last assertion follows by definition of $E_{n+1}(B)$. □

PROPOSITION 3.12. *Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a graded braided bialgebra. Let $n \in \mathbb{N}$ and let $S_n := S^{[n]}(B)$. For every $t \in \mathbb{N}$, set $I_t^{S_n} := (E(S_n)) \cap [S_n]^t$. The following assertions hold.*

i) $I_t^{S_n} = \{0\}$ *for every* $0 \leq t \leq n + 1$.

ii) $I_{n+2}^{S_n} = \ker\left(\Delta_{S_n}^{a,b}\right)$, *for every* $a, b \geq 1$ *such that* $a + b = n + 2$.

iii) $\Delta_{S_n}^{a,b}$ *is injective for every* $a, b \geq 1$ *such that* $0 \leq a + b \leq n + 1$.

Proof. Fix $n \in \mathbb{N}$ and let us prove that *i)* and *ii)* follow by *iii)*. Set $A := S_n = S^{[n]}(B)$.

iii) $\Rightarrow$ *i)* We have $I_0^A := (E(A)) \cap A^0 = \{0\}$ and $I_1^A := (E(A)) \cap A^1 = \{0\}$. From this and iii), since, $\Delta_A^{t,1}\left(I_{t+1}^A\right) \subseteq A^t \otimes I_1^A + I_t^A \otimes A^1$, for every $t \in \mathbb{N}$, one has that i) follows by induction on t.

iii) $\Rightarrow$ *ii)* Let $a, b \geq 1$ be such that $a + b = n + 2$. Using i), we obtain $\Delta_A^{a,b}\left(I_{n+2}^A\right) \subseteq A^a \otimes I_b^A + I_a^A \otimes A^b = \{0\}$ so that $I_{n+2}^A \subseteq \ker\left(\Delta_A^{a,b}\right)$. By iii) and Lemma 3.11, we have $E_{n+2}(A) = \ker\left(\Delta_A^{a,b}\right)$ so that $\ker\left(\Delta_A^{a,b}\right) \subseteq (E_{n+2}(A)) \cap A^{n+2} \subseteq (E(A)) \cap A^{n+2} = I_{n+2}^A$. Hence ii) holds.

Let us check that iii) holds, by induction on $n \in \mathbb{N}$. For $n = 0$, there is nothing to prove. Let $n \geq 1$ and assume that iii) is true for $n - 1$. Let us prove it for n. Let $R := S^{[n-1]}(B)$. Then, by inductive hypothesis and the first part, we have

i′) $I_t^R = \{0\}$ for every $0 \leq t \leq n$.

ii′) $I_{n+1}^R = \ker\left(\Delta_A^{a,b}\right)$, for every $a, b \geq 1$ such that $a + b = n + 1$.

iii′) $\Delta_R^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq n$.

For every $i \in \mathbb{N}$, let $\pi_A^i : R^i \to A^i := R^i / I_i^R$ be the canonical projection. Recall that $\Delta_A^{a,b}$ is uniquely defined by $\Delta_A^{a,b} \circ \pi_A^{a+b} = \left(\pi_A^a \otimes \pi_A^b\right) \circ \Delta_R^{a,b}$. By i'), π_A^i is an isomorphism for every $0 \leq i \leq n$. Using this fact and hypothesis iii'), from the last displayed equality, we deduce that $\Delta_A^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a+b \leq n$. To conclude it remains to check that $\Delta_A^{a,b}$ is injective for $a, b \geq 1$ such that $a+b = n+1$. By ii'), we have that

$$A^{n+1} = \frac{R^{n+1}}{I_{n+1}^R} = \frac{R^{n+1}}{\ker\left(\Delta_R^{a,b}\right)}$$

so that $\Delta_R^{a,b}$ factors to a unique map $\overline{\Delta_R^{a,b}} : A^{n+1} \to R^a \otimes R^b$ such that $\overline{\Delta_R^{a,b}} \circ \pi_A^{n+1} = \Delta_R^{a,b}$. Furthermore $\overline{\Delta_R^{a,b}}$ is injective. We have $\Delta_A^{a,b} \circ \pi_A^{n+1} = \left(\pi_A^a \otimes \pi_A^b\right) \circ \Delta_R^{a,b} = \left(\pi_A^a \otimes \pi_A^b\right) \circ \overline{\Delta_R^{a,b}} \circ \pi_A^{n+1}$ and hence $\Delta_A^{a,b} = \left(\pi_A^a \otimes \pi_A^b\right) \circ \overline{\Delta_R^{a,b}}$. Since π_A^a and π_A^b are isomorphisms and $\overline{\Delta_R^{a,b}}$ is injective, it is clear that also $\Delta_A^{a,b}$ is injective. □

LEMMA 3.13. *Let K be a field and let $((B_i)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ be a direct system of K-vector spaces, where, for $i \leq j$, $\xi_i^j : B_i \to B_j$. Assume that each B_i is endowed with a graded braided bialgebra structure $(B_i, m_{B_i}, u_{B_i}, \Delta_{B_i}, \varepsilon_{B_i}, c_{B_i})$ such that ξ_i^j is a graded braided bialgebra homomorphism, for every $i, j \in \mathbb{N}$. Then $\varinjlim B_i$ carries a natural graded braided bialgebra structure that makes it the direct limit of $((B_i)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of graded braided bialgebras. Furthermore the n-th graded component of $\varinjlim B_i$ is $\varinjlim B_i^n$, where B_i^n denotes the n-th graded component of B_i.*

THEOREM 3.14. *Let B be a graded braided bialgebra. Then $S^{[\infty]}(B)$ is a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra. Furthermore, the n-th graded component $S^{[\infty]}(B)^n$ of $S^{[\infty]}(B)$ is given by B^0, for $n = 0$, and by $S^{[n-1]}(B)^n$, for $n \geq 1$. Therefore $S^{[\infty]}(B) = B^0 \oplus S^{[0]}(B)^1 \oplus S^{[1]}(B)^2 \oplus \cdots$.*

Proof. Set $S := S^{[\infty]}(B)$ and set $S_i := S^{[i]}(B)$ for each $i \in \mathbb{N}$.

By Lemma 3.13, S carries a natural graded braided bialgebra structure that makes it the direct limit of $((S_i)_{i\in\mathbb{N}}, (\pi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of graded braided bialgebras. Furthermore the n-th graded component of $S^{[\infty]}(B)$ is $S^n := \varinjlim (S_i)^n$, where $(S_i)^n$ denotes the n-th graded component of S_i. Let us prove that S is strongly $\mathbb{N}$-graded as a coalgebra. By construction, each morphism $\Delta_S^{a,b} : S^{a+b} \to S^a \otimes S^b$ is uniquely determined by $\Delta_S^{a,b} \circ (\pi_i^\infty)^{a+b} = \left((\pi_i^\infty)^a \otimes (\pi_i^\infty)^b\right) \circ \Delta_{S_i}^{a,b}$, for every $i \in \mathbb{N}$, where $(\pi_i^\infty)^n : (S_i)^n \to S^n$ is the structural morphism of $S^n = \varinjlim (S_i)^n$. By Proposition 3.12, $\Delta_{S_i}^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a+b \leq i+1$. In particular for $i = a+b-1$ we get the equality $\Delta_S^{a,b} \circ \left(\pi_{a+b-1}^\infty\right)^{a+b} = \left((\pi_{a+b-1}^\infty)^a \otimes (\pi_{a+b-1}^\infty)^b\right) \circ \Delta_{S_{a+b-1}}^{a,b}$ where $\Delta_{S_{a+b-1}}^{a,b}$ is injective. Note also that the system

$$(S_0)^t \overset{(\pi_0^1)^t}{\to} (S_1)^t \overset{(\pi_1^2)^t}{\to} (S_2)^t \to \cdots \to (S_{t-1})^t \overset{(\pi_{t-1}^t)^t}{\to} (S_t)^t \underset{\sim}{\overset{(\pi_t^{t+1})^t}{\to}} (S_{t+1})^t \underset{\sim}{\overset{(\pi_{t+1}^{t+2})^t}{\to}} \cdots$$

is stationary as, in view of Proposition 3.12, the projection $\left(\pi_n^{n+1}\right)^t : (S_n)^t \to (S_{n+1})^t = (S_n)^t / I_t^{S_n}$ is an isomorphism for every $0 \leq t \leq n+1$. This entails that $\left(\pi_{a+b-1}^\infty\right)^t : (S_{a+b-1})^t \to S^t$ is an isomorphism for $t \in \{a, b, a+b-1\}$. Therefore $\Delta_S^{a,b} = ((\pi_{a+b}^\infty)^a \otimes (\pi_{a+b}^\infty)^b) \circ \Delta_{S_{a+b}}^{a,b} \circ [(\pi_{a+b}^\infty)^{a+b}]^{-1}$ is injective for every $a, b \geq 1$ and hence for every $a, b \in \mathbb{N}$. Since S is a graded braided bialgebra, this means it is strongly $\mathbb{N}$-graded as a coalgebra.

Let us prove the last part of the statement. For every $t \in \mathbb{N}$, the morphism $\left(\pi_t^{t+1}\right)^0 : (S_t)^0 \to (S_{t+1})^0$ is an isomorphism so that we can identify S^0 with $(S_0)^0 = B^0$. The morphism $(\pi_t^\infty)^{t+1} : (S_t)^{t+1} \to S^{t+1}$ is an isomorphism so that we can identify S^{t+1} with $(S_t)^{t+1}$ for every $t \in \mathbb{N}$. In particular we have $S^1 = (S_0)^1 = B^1$. □

3.15. Let $(B, m, u, \Delta, \varepsilon, c)$ be a graded braided bialgebra. Then $\left(B^0, m^{0,0}, u^0, \Delta^{0,0}, \varepsilon^0, c^{0,0}\right)$ is a braided bialgebra. Denote by ${}_{B^0}^{B^0}\mathfrak{M}_{B^0}^{B^0}$ the category of Hopf bimodules over B^0. It is straightforward to check that $\left(B^1, m^{0,1}, m^{1,0}, \Delta^{0,1}, \Delta^{1,0}\right)$ is an object in ${}_{B^0}^{B^0}\mathfrak{M}_{B^0}^{B^0}$. Thus it makes sense to consider the tensor algebra $T_{B^0}\left(B^1\right)$ and the cotensor coalgebra $T_{B^0}^c\left(B^1\right)$. Both of them carry a graded braided bialgebra structure arising from their respective universal properties. Moreover by the universal property of the tensor algebra, there is a morphism

$$F : T_{B^0}\left(B^1\right) \to T_{B^0}^c\left(B^1\right)$$

of graded algebras which is the identity on B^0 and B^1 respectively. F is in fact a graded braided algebra homomorphism. Thus the image of F is a graded braided bialgebra which is called **the braided bialgebra of Type one associated to** B^0 **and** B^1 and denoted by $B^0\left[B^1\right]$. In case when c is the canonical flip map, this definition and notation goes back to [Ni]. For further details, the reader is referred also to [AM, Lemma 6.1 and Theorem 6.8]. The proofs there are performed inside braided monoidal categories, but can be easily adapted to a non categorical framework. This essentially works because of the definition of graded braided bialgebra which includes the compatibility of c with graded components of B.

THEOREM 3.16. *Let B be a graded braided bialgebra. Then, there is a unique coalgebra homomorphism $\psi_B : B \to T_{B^0}^c\left(B^1\right) = T^c$ such that $p_0^{T^c}\circ\psi_B = p_0^B$ and $p_1^{T^c}\circ\psi_B = p_1^B$, where $p_i^{T^c} : T^c \to (T^c)^i$ and $p_i^B : B \to B^i$ are the canonical projections. The map ψ_B is indeed a graded braided bialgebra homomorphism and as graded braided bialgebras we have $\left(S^{[\infty]}(B), \pi_0^\infty\right) \cong Im(\psi_B)$.*

Proof. Set $S := S^{[\infty]}(B)$. The first part concerning ψ_B follows by the universal property of the cotensor coalgebra. By Theorem 3.4, the canonical map $\psi_S : S \to T_{S^0}^c\left(S^1\right) = T_{B^0}^c\left(B^1\right)$ is injective. For $t = 1, 2$ we have $p_t^{T_{S^0}^c(S^1)}\circ\psi_S\circ\pi_0^\infty = p_t^S\circ\pi_0^\infty = (\pi_0^\infty)^t\circ p_t^B = (\pi_0^\infty)^t\circ p_t^{T_{B^0}^c(B^1)}\circ\psi_B = p_t^{T_{S^0}^c(S^1)}\circ\psi_B$. Since π_0^∞ is a graded braided bialgebra homomorphism, the universal property of cotensor coalgebra entails that $\psi_S\circ\pi_0^\infty = \psi_B$. Since ψ_S is injective and π_0^∞ surjective we get that $S^{[\infty]}(B) = \mathrm{Im}(\psi_B)$. □

COROLLARY 3.17. *Let B be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. Then $S^{[\infty]}(B)$ is isomorphic as graded braided bialgebras to $B^0\left[B^1\right]$.*

Proof. Set $S := S^{[\infty]}(B)$. By Theorem 3.14, $S^{[\infty]}(B)$ is a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra so that, by Theorem 3.4, the canonical map $\psi_S : S \to T_{S^0}^c\left(S^1\right) = T_{B^0}^c\left(B^1\right)$ is injective. As a quotient of B, which is strongly $\mathbb{N}$-graded as an algebra, by [AM, Proposition 3.6], S is strongly $\mathbb{N}$-graded as an algebra too. By [AM, Theorem 3.11], this means that the canonical map $\varphi_S : T_{S^0}\left(S^1\right) \to S$ is surjective. Now, the composition $\psi_S\circ\varphi_S : T_{S^0}\left(S^1\right) \to T_{S^0}^c\left(S^1\right)$ is the unique graded braided bialgebra homomorphism that restricted to S^0 and S^1 gives the respective inclusion (compare with [AM, Theorem 6.8]). Hence, its image is, by definition, $S^0\left[S^1\right]$ that is the braided bialgebra of Type one associated to S^0 and S^1. We conclude by observing that $S^0 = B^0$ and $S^1 = B^1$. □

REMARK 3.18. When B is the braided tensor algebra $T(V, c)$ of a braided vector space (V, c), Corollary 3.17 should be compared with [Kh2, page 553]. It is practically [Mas, Proposition 3.2], even for pre-Nichols algebras.

4. Combiantorial rank $\kappa(B)$

Next aim is to introduce the notion of combinatorial rank for a graded braided bialgebra and show that under suitable assumptions it is finite.

DEFINITION 4.1. (cf. [Kh2, Definition 5.4]) We say that a graded braided bialgebra B has **combinatorial rank** n if the system of Definition 3.10 is stationary and

$$n = \min\{t \in \mathbb{N} \mid \pi_t^{t+1} : S^{[t]}(B) \to S^{[t+1]}(B) \text{ is an isomorphism}\}.$$

We will write $\kappa(B) = n$. If if the system is not stationary, we will write $\kappa(B) = \infty$.

We now investigate the behaviour of combinatorial ranks under graded braided bialgebra homomorphisms.

Proposition 4.2. *Let $f : B \to B'$ be a graded braided bialgebra homomorphism. If f is surjective and f^0 and f^1 are injective, then $\kappa(B') \leq \kappa(B)$.*

Proof. Without loss of generality we can assume $\kappa(B) = N < \infty$. By Proposition 3.8, for each $n \in \mathbb{N}$, there is a map $S^{[n]}(f) : S^{[n]}(B) \to S^{[n]}(B')$ such that $(\pi_n^{n+1})' \circ S^{[n]}(f) = S^{[n+1]}(f) \circ \pi_n^{n+1}$. Since f is surjective, by induction, it is clear that $S^{[n]}(f)$ is surjective for each $n \in \mathbb{N}$. Now, by [Sw, Proposition 11.1.1], the coradical of $S^{[N+1]}(B)$ is contained in $S^{[N+1]}(B)^0$. By (1) $\Rightarrow$ (7) of Theorem 3.4, the second term of the coradical filtration of $S^{[N+1]}(B)$ is included in $S^{[N+1]}(B)^0 \oplus S^{[N+1]}(B)^1 = B^0 \oplus B^1$. On the other hand, the restriction of $S^{[N+1]}(f)$ to $B^0 \oplus B^1$ coincide with the restriction of $(\pi_0^{N+1})' \circ f$ to $B^0 \oplus B^1$ which is injective as f^0 and f^1 are injective. Therefore, by the Heyneman-Radford theorem [Mo, Theorem 5.3.1], we get that $S^{[N+1]}(f)$ is injective whence bijective. Since π_N^{N+1} is bijective too, we get that $(\pi_n^{N+1})' \circ S^{[N]}(f)$ is bijective. Thus $S^{[N]}(f)$ is injective whence bijective so that $(\pi_n^{N+1})'$ is bijective. This means $\kappa(B') \leq N$. $\square$

Remark 4.3. Let $N \in \mathbb{N}$ and let B be a graded braided bialgebra. It stems from the definition that $\kappa(B) \leq N$ if and only if $\pi_N^{\infty} : S^{[N]}(B) \to S^{[\infty]}(B)$ is an isomorphism if and only if $S^{[N]}(B)$ is strongly $\mathbb{N}$-graded as a coalgebra.

The following results give conditions sufficient for B to have finite combinatorial rank.

Theorem 4.4. *Let B be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. The following assertions are equivalent for $N \geq 1$.*

(1) $S^{[\infty]}(B)^N = 0$.
(2) $S^{[\infty]}(B)^n = 0$, *for every* $n \geq N$.
(3) $S^{[N-1]}(B)^N = 0$.
(4) $S^{[N-1]}(B)^n = 0$, *for every* $n \geq N$.

If one of these conditions if fulfilled, then $\kappa(B) \leq N - 1$.

Proof. Since B is strongly $\mathbb{N}$-graded as an algebra, by [AM, Proposition 3.6], each graded algebra quotient of B is strongly $\mathbb{N}$-graded too. This entails that both $S^{[\infty]}(B)$ and $S^{[N]}(B)$ are strongly $\mathbb{N}$-graded as an algebras. Clearly, if A is a graded algebra which is strongly $\mathbb{N}$-graded then $A^n = A^{n-N} \cdot A^N$, for every $n \geq N$. Hence we deduce that $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$.

By Theorem 3.14, $S^{[\infty]}(B)^N = S^{[N-1]}(B)^N$ so that $(1) \Leftrightarrow (3)$.

Now assume (4) and let us prove that $\kappa(B) \leq N$. By Remark 4.3, it is enough to prove that $S_{N-1} := S^{[N-1]}(B)$ is strongly $\mathbb{N}$-graded as a coalgebra. By Proposition 3.12, $\Delta_{S_{N-1}}^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq N$. On the other hand $(S_{N-1})^n = 0$, for every $n \geq N$, so that $\Delta_{S_{N-1}}^{a,b} : \{0\} = (S_{N-1})^{a+b} \to (S_{N-1})^a \otimes (S_{N-1})^b$ is injective for every for every $a, b \geq 1$ such that $a + b \geq N$. We have so proved that S_{N-1} is strongly $\mathbb{N}$-graded as a coalgebra. $\square$

Corollary 4.5. *Let B be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. Assume that $B^0 \left[B^1\right]$ as a graded braided bialgebra divides out $T_{B^0}\left(B^1\right)$ by relations in degree not greater than N. Then $\kappa(B) \leq N - 1$.*

Proof. Set $S := S^{[\infty]}(B)$ and set $S_i := S^{[i]}(B)$ for each $i \in \mathbb{N}$. Since B is strongly $\mathbb{N}$-graded as an algebra, by Corollary 3.17, we have $B^0 \left[B^1\right] = S$. By Proposition 3.12, $\Delta_{S_{N-1}}^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq N$. Now, by definition we have

$$S_N = S(S_{N-1}) = \frac{S_{N-1}}{(E(S_{N-1}))} = \frac{S_{N-1}}{(\oplus_{n \in \mathbb{N}} E_n(S_{N-1}))} = \frac{S_{N-1}}{(\oplus_{n \geq N+1} E_n(S_{N-1}))}.$$

Since $S^n = \varinjlim S_i^n$ and S is defined by relations in degree not greater than N, from $E_n(S_{N-1}) \subseteq (S_{N-1})^n$ we deduce that $\oplus_{n \geq N+1} E_n(S_{N-1}) = 0$. Hence $S_{N-1} \cong S_N$, so that S_{N-1} is strongly $\mathbb{N}$-graded as a coalgebra. By Remark 4.3, $\mathrm{sdeg}(B) \leq N$. $\square$

Denote by $\mathfrak{h}(B) := \sum_{n\in\mathbb{N}} \dim_K(B^n)X^n \in K[[X]]$ the Hilbert-Poincaré sery of a graded algebra B.

Corollary 4.6. *Let B be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. If $B^0\left[B^1\right]$ is finite dimensional, then $\kappa(B) \leq \deg\left(\mathfrak{h}\left(B^0\left[B^1\right]\right)\right) - 1 \leq \dim_K B^0\left[B^1\right] - 1$.*

Proof. Since B is strongly $\mathbb{N}$-graded as an algebra, by Corollary 3.17, we have $B^0\left[B^1\right] = S^{[\infty]}(B)$. Since $B^0\left[B^1\right]$ is finite dimensional, then $N := \deg\left(\mathfrak{h}\left(B^0\left[B^1\right]\right)\right)$ is finite so that $S^{[\infty]}(B)^n = 0$, for every $n \geq N$. Either by Theorem 4.4 or by Corollary 4.5, we conclude. □

5. Passing to the case $B = T(V,c)$

Definition 5.1. Let $(B, m, u, \Delta, \varepsilon, c)$ be a graded braided bialgebra. For every $a, b, n \in \mathbb{N}$, we set

$$\Gamma^B_{a,b} := m^{a,b}\Delta^{a,b} \qquad \text{and} \qquad \Gamma^B_n := \begin{cases} \mathrm{Id}_{B^0} & \text{if } n = 0 \\ m^{n-1,1}\left(\Gamma^B_{n-1} \otimes B^1\right)\Delta^{n-1,1} & \text{if } n \geq 1 \end{cases}. \tag{6}$$

Remark 5.2. Let (V,c) be a braided vector space and let $T = T(V,c)$ be the associated braided tensor algebra. Then $m^{a,b}_T$ is just the juxtaposition. This entails that $\Gamma^T_{a,b} = \mathrm{S}_{a,b}$ and $\Gamma^T_n = \mathrm{S}_n$ where $\mathrm{S}_{a,b}$ and S_n are the morphisms defined in [Sch, page 2815].

Although many other results could be obtained for a general graded braided bialgebra B, we focus our attention on $B = T(V,c)$, the braided tensor algebra of a braided vector space (V,c).

In particular $\kappa(T(V,c))$ is denoted by $\kappa(V,c)$ and it is called the **combinatorial rank of** (V,c). Moreover $S(T(V,c))$ is simply denoted by $S(V,c)$ and it is called the **symmetric algebra of** (V,c). We will denote by $\pi_S : T(V,c) \to S(V,c)$ the canonical projection. Note that since π_S is surjective, by [Mo, Corollary 5.3.5], $S(V,c)$ is a connected coalgebra. The symmetric algebra admits an obvious universal property.

We will simply write $E_n(V,c)$ and $E(V,c)$ instead of $E_n(T(V,c))$ and $E(T(V,c))$ respectively (see Definition 2.5). Given a morphism of braided vector spaces $f : (V, c_V) \to (W, c_W)$, we will also write $E_n(f)$ and $E(f)$ instead of $E_n(T(f))$ and $E(T(f))$ respectively (see Definition 3.9).

5.3. Let (V,c) be a braided vector space and let $T = T(V,c)$. By the universal property of the tensor algebra there is a unique algebra homomorphism

$$\Gamma^T : T(V,c) \to T^c(V,c)$$

such that $\Gamma^T_{|V} = \mathrm{Id}_V$, where $T^c(V,c)$ denotes the quantum shuffle algebra (see 3.15). This is a morphism of graded braided bialgebras. By [Sch, page 2815] (see also [AG1, page 25-26]), it is clear that the n-th graded component of Γ^T is the map $\Gamma^T_n : V^{\otimes n} \to V^{\otimes n}$ in the sense of Definition 5.1 (see Remark 5.2). The bialgebra of Type one generated by V over K, also known as **Nichols algebra**, is by definition

$$\mathcal{B}(V,c) = \mathrm{Im}\left(\Gamma^T\right) \cong \frac{T(V,c)}{\ker\left(\Gamma^T\right)}.$$

Remark 5.4. Let (V,c) be a braided vector space whose Nicholas algebra $\mathcal{B}(V,c)$ is finite dimensional. Since the braided tensor algebra $T := T_K(V,c)$ is strongly $\mathbb{N}$-graded as an algebra and $\mathcal{B}(V,c)$ identifies with $T^0\left[T^1\right]$, then Corollary 4.6 applies. This shows how the investigation of braided vector spaces with finite combinatorial rank fits into the classification of finite dimensional Hopf algebras problem. In fact, one of the steps of the lifting method by Andruskiewitsch and Schneider (see e.g. [ASc1]) is the classification of finite dimensional Nichols algebras.

We would like to point out that, at the best of our knowledge, the following questions, raised in [Kh2, page 555] and [KA, end of the introduction]) respectively, still remain open: does there exist a graded braided bialgebra with infinite combinatorial rank? does the combinatorial rank of a finitely generated graded braided bialgebra remain finite? Also it is unknown if the combinatorial rank of a finite dimensional vector space is finite.

6. Examples for $\kappa(V, c) \leq 1$

In this section meaningful examples of braided vector space of combinatorial rank at most one are given.

Theorem 6.1. *Let $L \subseteq E(V,c)$ and let $R = T(V,c)/(L)$. If R inherits from $T(V,c)$ a braided bialgebra structure such that the space $P(R)$ of primitive elements of R identifies with the image of V in R, then $S(V,c) = R$ and $\kappa(V,c) \leq 1$.*

Proof. Set $T := T(V,c)$ and $S := S(V,c)$. Denote by $\tau : R \to S$, $\pi_S : T \to S$ and $\pi_R : T \to R$ the canonical projections. Since $\tau\pi_R = \pi_S$ and both π_R and π_S are coalgebra homomorphism, then τ is a coalgebra homomorphism too. By [Mo, Lemma 5.3.3], we obtain that τ is injective as its restriction to $P(R)$ is injective (note that the map $i_S : V \to S$ is injective). Hence τ is an isomorphism so that $S = R$. Therefore $P(S)$ identifies with V whence S is strongly $\mathbb{N}$-graded as a coalgebra which means $\kappa(V,c) \leq 1$. $\square$

Definition 6.2. Recall that a braided vector space (V,c) is of **diagonal type** is there is a basis $x_1, \ldots, x_n$ of V over K and a $n \times n$ matrix $(q_{i,j})$, $q_{i,j} \in K \backslash \{0\}$, such that $c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, 1 \leq i,j \leq n$.

Theorem 6.3. *Let (V,c) be a braided vector space of diagonal type such that $\mathcal{B}(V,c)$ is a domain and its Gelfand-Kirillov dimension is finite. Then $\kappa(V,c) \leq 1$.*

Proof. By hypothesis there is a basis $x_1, \ldots, x_n$ of V over K and a matrix $(q_{i,j})$, $q_{i,j} \in K \backslash \{0\}$, such that $c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, 1 \leq i,j \leq n$. By [AAn, Corollary 3.11] (which was obtained from results of Lusztig and Rosso), $R := \mathcal{B}(V,c)$ as a graded braided bialgebra divides out $T(V,c)$ by the ideal generated by set L whose elements are $z_{i,j} := (ad_c x_i)^{m_{i,j}+1}(x_j)$, $1 \leq i,j \leq n$, where $i \neq j$ and $m_{i,j}$ is a suitable non-negative integer. Since $z_{i,j}$ is primitive in $T(V,c)$ (see e.g. [Kh1, Theorem 6.1], where $z_{i,j}$ is denoted by $W(x_j, x_i)$ in formulae (18) and (19), or see [ASc2, Lemma A.1]) and of degree at least two, we get $L \subseteq E(V,c)$. By Theorem 6.1, $\kappa(V,c) \leq 1$. $\square$

Example 6.4. Assume K is algebraically closed of characteristic 0. Let (V,c) be a n-dimensional braided vector space of diagonal type. Keeping the notations of Definition 6.2, assume that $q_{i,i} \neq 1$ are primitive N_i-th roots of unity ($N_i > 1$) and $q_{i,j}q_{j,i} = 1$ for $i \neq j$. By means of the quantum binomial formula (see [ASc3, Lemma 3.6]) one gets that $x_i \otimes x_j - q_{i,j} x_j \otimes x_i \in E_2(V,c)$ for $i \neq j$ and $x_i^{\otimes N_i} \in E_{N_i}(V,c)$. Consider the **quantum linear space** $R = K < x_1, \ldots, x_n \mid x_1^{N_1} = 0; \ldots; x_n^{N_n} = 0; x_i x_j = q_{i,j} x_j x_i >$. By Theorem 6.1, $S(V,c) = R$ and $\kappa(V,c) \leq 1$. Since $S(V,c) \neq T(V,c)$ it is clear that $\kappa(V,c) = 1$.

Definition 6.5. [ASc3, Definition 1.6] Recall that a braided vector space (V,c) is of **abelian group type** if V has a basis $x_1, \ldots, x_n$ such that $c(x_i \otimes x_j) = \gamma_i(x_j) \otimes x_i$ for some $\gamma_1, \ldots, \gamma_n \in GL(V)$ and the subgroup $G(V,c)$ of $GL(V)$ generated by $\gamma_1, \ldots, \gamma_n$ is abelian.

Proposition 6.6. *Let (V,c) be a two dimensional braided vector space of abelian group type. Suppose that $\dim_K S(V,c) < 32$. Then $\kappa(V,c) \leq 1$.*

Proof. We have two graded braided bialgebra homomorphisms $\pi_S : T(V,c) \twoheadrightarrow S(V,c)$ and $\pi_1^\infty : S(V,c) \twoheadrightarrow \mathcal{B}(V,c)$. By a famous Takeuchi's result (see [Mo, Lemma 5.2.10]), these bialgebras are indeed Hopf algebras whence π_S and π_1^∞ are in fact graded braided Hopf algebra homomorphisms. It is well known that V can be regarded as an object in the monoidal category $\mathcal{M}$ of Yetter-Drinfeld modules over $G(V,c)$. Furthermore c coincides with the braiding of V in the category. Clearly $T(V,c)$ and $\mathcal{B}(V,c)$ are Hopf algebras in $\mathcal{M}$ and the same is true for $S(V,c)$ as $E(V,c)$ is an object in $\mathcal{M}$. Furthermore π_S and π_1^∞ are morphism in $\mathcal{M}$. Since $\mathcal{B}(V,c)$ is a quotient of $S(V,c)$, one has $\dim_K B(V,c) \leq \dim_K S(V,c) < 32$. By [Gr, Lemma 6.2], we have that π_1^∞ is an isomorphism i.e. $\kappa(V,c) \leq 1$. $\square$

6.1. **Braidings of Hecke type.** In order to prove the main results concerning braidings of Hecke type, we first investigate some general properties of graded braided bialgebras.

PROPOSITION 6.7. *Let $(B,m,u,\Delta,\varepsilon,c)$ be a connected graded braided bialgebra. Assume that $m^{1,1}$ is an isomorphism and $c^{1,1}$ has minimal polynomial $f(X) \in K[X]$ where $f(-1)=0$ i.e. $f(X)=(X+1)h(X)$ for some $h \in K[X]$. Then $\kappa(B)=\kappa(S(B))+1$ and $m_{S(B)}^{1,1}h(c_{S(B)}^{1,1})=0$.*

Proof. Note that, if $\kappa(B)=0$, then $\Delta^{1,1}$ is injective and hence $c^{1,1}+\mathrm{Id}_{B^2}=\Delta^{1,1}m^{1,1}$ is injective too. In this case $f\left(c^{1,1}\right)=0$ implies $h\left(c^{1,1}\right)=0$ so that f is not the minimal polynomial for $c^{1,1}$, a contradiction. We have so proved that $\kappa(B)\neq 0$. Thus $\kappa(B)=\kappa(S(B))+1$.

By (5), we have $c_S^{1,1}\left(\pi_S^1\otimes\pi_S^1\right)=\left(\pi_S^1\otimes\pi_S^1\right)c^{1,1}$. Since π_S^1 is bijective, we infer that $c_S^{1,1}$ has minimal polynomial f. Thus $0=f(c_S^{1,1})=(c_S^{1,1}+\mathrm{Id}_{S^2})h(c_S^{1,1})=\Delta_S^{1,1}m_S^{1,1}h(c_S^{1,1})$. By Proposition 3.12, $\Delta_S^{1,1}$ is injective whence $m_S^{1,1}h(c_S^{1,1})=0$. □

DEFINITION 6.8. For every $n\geq 1$, we set

$$(n)_X := \frac{X^n-1}{X-1} = 1+X+\cdots+X^{n-1}, \quad (n)_X! := (1)_X\cdot(2)_X\cdot\cdots\cdot(n)_X\,.$$

Following [AMS, Definition 2.13], we will say that an element $q\in K$ is **regular** whenever $(n)_q\neq 0$, for all $n\geq 2$. Note that $q\neq 1$ is regular if and only if q is not a root of unity, while 1 is regular if and only if $\mathrm{char}(K)=0$.

PROPOSITION 6.9. *Let $(B,m,u,\Delta,\varepsilon,c)$ be a graded braided bialgebra. Assume that B is strongly $\mathbb{N}$-graded as an algebra, B^0 is one dimensional and $m^{1,1}\left(c^{1,1}-q\mathrm{Id}_{B^2}\right)=0$ for some regular $q\in K$ (e.g. $c^{1,1}$ has minimal polynomial of degree 1). Then the canonical projection $\pi_S : B\to S(B)$ is an isomorphism i.e. $\kappa(B)=0$.*

Proof. By [AMS, Theorem 2.15], B is strongly $\mathbb{N}$-graded as a coalgebra. By Theorem 3.4, the canonical projection $\pi_S : B\to S(B)$ is an isomorphism i.e. $\kappa(B)=0$. □

PROPOSITION 6.10. *Let (B,c) be a graded braided bialgebra. Assume that B is strongly $\mathbb{N}$-graded as an algebra, B^0 is one dimensional, $m^{1,1}$ is an isomorphism and $c^{1,1}$ has minimal polynomial $f(X)=(X+1)(X-q)$, for some regular $q\in K$. Then $\kappa(B)=1$.*

Proof. By Proposition 6.7, we have $\kappa(B)=\kappa(S(B))+1$ and $m_{S(B)}^{1,1}\left(c_{S(B)}^{1,1}-q\right)=0$. Therefore, by Proposition 6.9, we get that $\kappa(S(B))=0$ whence $\kappa(B)=1$. □

Part i) in the following result has already been concerned in [ASc3, Example 3.5].

THEOREM 6.11. *Let $V\neq\{0\}$ be a K-vector space, let $q\in K\setminus\{0\}$ and set $c=q\mathrm{Id}_{V\otimes V}$.*
Then (V,c) is a braided vector space and
i) *$\kappa(V,c)=0$, if q is regular.*
ii) *$\kappa(V,c)=1$, otherwise. In this case q is a primitive N-th root of unity for some $N\geq 2$ and $S(V,c)=T(V,c)/\left(V^N\right)$.*

Proof. Clearly $c : V\otimes V\to V\otimes V$ is a braiding on V. Proposition 6.9 implies i). It remains to check ii). Set $T:=T(V,c)$ and $S:=S(V,c)$. Adapting the ideas in the proof of (3) ⇒ (1) in [AMS, Theorem 2.15], it is straightforward to check that $\Gamma_{n,1}^T=(n+1)_q\,\mathrm{Id}_{V^{\otimes n+1}}$, for every $n\geq 1$. Now, since the multiplication in T is the juxtaposition, we have that $\Gamma_n^T=\left(\Gamma_{n-1}^T\otimes V\right)\Gamma_{n-1,1}^T$ so that, inductively, we obtain $\Gamma_n^T=(n)_q!\mathrm{Id}_{V^{\otimes n}}$, for every $n\geq 2$. Since q is not regular, there is a minimal $N\geq 2$ such that $\Gamma_N^T=0$. Clearly q is a primitive N-th root of unity and

$$\ker\left(\Gamma_n^T\right)=\begin{cases} 0, & \text{for every } 0\leq n\leq N-1,\\ V^n, & \text{for every } N\leq n,\end{cases}$$

whence $\mathcal{B}(V,c)=T(V,c)/\left(V^N\right)$. Now, for every $a,b\geq 1, a+b=N$, we have $0=\Gamma_N^T=\Gamma_{a+b}^T\overset{(11)}{=}\left(\Gamma_a^T\otimes\Gamma_b^T\right)\Gamma_{a,b}^T$. Since $a,b\leq N-1$, then Γ_a^T and Γ_b^T are injective so that $\Delta_T^{a,b}=\Gamma_{a,b}^T=0$. Thus $E_N(V,c)=V^N$ and hence $\left(V^N\right)=\left(E_N(V,c)\right)\subseteq\left(E(V,c)\right)\subseteq\ker\left(\Gamma^T\right)=\left(V^N\right)$. Therefore

$(E(V,c)) = \ker\left(\Gamma^T\right)$ and hence $S(V,c) = \mathcal{B}(V,c)$. Since $\mathcal{B}(V,c) \neq T(V,c)$, we conclude that $\kappa(V,c) = 1$. □

Definition 6.12. (see e.g. [ASc3, Definition 3.3] or [AbA, Definition 3.1.1]) We say that a braiding c on a vector space V is of **Hecke type (or a Hecke symmetry) with mark** q if c is a root in $\mathrm{End}(V \otimes V)$ of the polynomial $(X+1)(X-q)$ for some $q \in K\setminus\{0\}$.

The following result explains how braided vector spaces with braiding of Hecke type fits into our study of braided vector spaces of combinatorial rank at most 1.

Theorem 6.13. *Let (V,c) be braided vector space such that c is a braiding of Hecke type with regular mark q. Then* $\mathrm{sdeg}(V,c) \leq 1$.

Proof. By hypothesis, c has minimal polynomial a divisor f of the polynomial $(X+1)(X-q)$.

Now, the braided tensor algebra $T := T(V,c)$ is strongly $\mathbb{N}$-graded as an algebra and $m_T^{1,1}$ is an isomorphism. Moreover $c_T^{1,1} = c$. Thus, if $f = (X+1)(X-q)$, by Proposition 6.10, we have $\kappa(V,c) := \kappa(T(V,c)) = 1$. On the other hand, by Theorem 6.11, if $f = X+1$, we get $\kappa(V,c) = 1$ while, if $f = X-q$, we obtain $\kappa(V,c) = 0$. □

Remark 6.14. By [ASc3, Proposition 3.4], another way to prove Theorem 6.13 is to apply Proposition 6.16. Moreover, again by [ASc3, Proposition 3.4], one has $S(V,c) = T(V,c)/(E_2(V,c))$. Note that any vector space V is a braided vector space with braiding the flip map $\tau : V \otimes V \to V \otimes V, \tau(v \otimes w) = w \otimes v$. Therefore $E_2(V,\tau) = \mathrm{Ker}(\tau + \mathrm{Id}_{V\otimes V})$. If $\mathrm{char}(K) \neq 2$, this equals $\mathrm{Im}(\tau - \mathrm{Id}_{V\otimes V})$ whence $S(V,\tau) = T(V,\tau)/(w \otimes v - v \otimes w \mid v, w \in V)$, the classical symmetric algebra.

6.2. **Quadratic algebras.** This subsection is devoted to the study of quadratic algebras, see [Man, page 19].

Corollary 6.15. *Let B be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. Then $\kappa(B) \leq 1$ whenever $B^0\left[B^1\right]$ is a quadratic algebra.*

Proof. By definition, $B^0 = K$ and the ideal of relations among elements of B^1 is generated by the subspace of all quadratic relations $R(B) \subseteq B^1 \otimes B^1$. By Corollary 4.5, we get $\kappa(B) \leq 1$. □

Proposition 6.16. *Let (V,c) be braided vector space. The following assertions are equivalent: (i)* $\mathrm{S}(V,c)$ *is a quadratic algebra and $\kappa(V,c) \leq 1$; (ii) $\mathcal{B}(V,c)$ is a quadratic algebra.*

Proof. First observe that $\kappa(V,c) \leq 1$ if and only if $S(V,c) = \mathcal{B}(V,c)$. Therefore we have only to prove that (ii) implies $\kappa(V,c) \leq 1$. This follows by applying Corollary 6.15 to the case $B := T(V,c)$ once observed that $B^0\left[B^1\right] = \mathcal{B}(V,c)$. □

Remark 6.17. Examples of braided vector spaces (V,c) such that $\mathcal{B}(V,c)$ is a quadratic algebra can be found e.g. in [MS, AG2, ASt]. Another example is given by braided vector spaces of Hecke-type with regular mark, see Remark 6.14.

Example 6.18. Let K be a field of characteristic 0 and let $m \in K$ be such that $m^2 \neq 0, 1$ is regular. Consider the vector space V, appeared in [Gu2, page 325], with basis $\{e_0, e_1, e_2\}$ and braiding given by $c(e_i \otimes e_j) = e_j \otimes e_i$ for $i = 0$ or $j = 0$, $c(e_i \otimes e_i) = m^2 e_i \otimes e_i$ for $i = 1, 2$, $c(e_2 \otimes e_1) = m e_1 \otimes e_2 + \left(m^2 - 1\right) e_2 \otimes e_1$ and $c(e_1 \otimes e_2) = m e_2 \otimes e_1$. One can check that $E_2(V,c)$ is generated over K by the elements $e_1 \otimes e_0 - e_0 \otimes e_1, e_2 \otimes e_0 - e_0 \otimes e_2$ and $e_2 \otimes e_1 - m e_1 \otimes e_2$. Now $R := T(V,c)/(E_2(V,c))$ is a braided bialgebra quotient of $T(V,c)$. Thus, by [Mo, Corollary 5.3.5], R is connected as $T(V,c)$ is connected. Moreover R has basis $\{e_0^{n_0} e_1^{n_1} e_2^{n_2} | n_0, n_1, n_2 \in \mathbb{N}\}$ and, using regularity of m^2, one gets (by the same argument as in the proof of [ASc1, Lemma 3.3]) that the braided vector space of primitive elements in R identifies with (V,c). By Theorem 6.1, $R = S(V,c)$ and $\kappa(V,c) \leq 1$. By Proposition 6.16, $\mathcal{B}(V,c)$ is a quadratic algebra. Moreover $\mathcal{B}(V,c) = S(V,c) = R$.

7. Examples for $\kappa(V,c)\geq 2$

In this section we collect some examples of braided vector spaces with combinatorial rank greater than one.

Lemma 7.1. *Let (V,c) be a braided vector space. Then $I=(E(V,c))$ is a graded ideal of $T(V,c)$ with graded component $I_n := I\cap V^{\otimes n}$. Moreover $I_n = I_{n-1}\otimes V + V\otimes I_{n-1} + E_n(V,c)$, for all $n\in\mathbb{N}$.*

Proof. The first part follows by the proof of Theorem 3.6, the latter one by induction on $n\in\mathbb{N}$. □

Example 7.2. From [AG1, Example 3.3.22], we quote the following example. Take $H = K\mathbb{D}_4$, where $\mathbb{D}_4 = \left\{\sigma,\rho \mid o(\sigma)=2, o(\rho)=4, \rho\sigma=\sigma\rho^{-1}\right\}$. Consider the module V in ${}_H^H\mathcal{YD}$ with basis $\{z_0,z_1,z_2,z_3\}$, with the structure given by $\delta(z_i)=\rho^i\sigma\oplus z_i$, $\rho^i\rhd z_i = z_{i+2j}$ and $\sigma\rhd z_i = -z_{-i}$ where we take subindexes of the z_i to be on $\mathbb{Z}/4\mathbb{Z}$. The braiding is then given by $c(z_i\otimes z_j) = -z_{2i-j}\otimes z_i$. Let $T:=T(V,c)$, let $a:=z_1z_2+z_0z_1$, $b:=z_1z_0+z_2z_1$ and let Z be the set consisting of the following elements:

$$z_0^2,\quad z_1^2,\quad z_2^2,\quad z_3^2,\quad z_0z_2+z_2z_0,\quad z_1z_3+z_3z_1, \tag{7}$$

$$z_0z_1+z_1z_2+z_2z_3+z_3z_0,\quad z_0z_3+z_1z_0+z_2z_1+z_3z_2 \tag{8}$$

$$a^2,\quad b^2,\quad ab+ba. \tag{9}$$

Then the Nichols algebra $\mathcal{B}(V,c)$ comes out to be $T/(Z)$. One can check that:

- $E_2(V,c)$ is generated over K by the elements in (7) and (8);
- $E_3(V,c)$ is generated over K by the elements $z_1^2z_3-z_3z_1^2$, $z_1z_3^2-z_3^2z_1$, $z_2^2z_0-z_0z_2^2$, $z_2z_0^2-z_0^2z_2$, $z_1(z_0z_2+z_2z_0)-(z_0z_2+z_2z_0)z_1$, $z_3(z_0z_2+z_2z_0)-(z_0z_2+z_2z_0)z_3$, $z_2(z_1z_3+z_3z_1)-(z_1z_3+z_3z_1)z_2\in E_2(V,c)\otimes V+V\otimes E_2(V,c)$;
- $E_4(V,c)=\{0\}$.

Thus, by Lemma 7.1, we obtain $(E(V,c))\cap V^{\otimes 4}\subseteq E_2(V,c)\otimes V\otimes V+V\otimes E_2(V,c)\otimes V+V\otimes V\otimes E_2(V,c)$. Since $a^2 = z_1z_2z_1z_2+z_1z_2z_0z_1+z_0z_1^2z_2+z_0z_1z_0z_1$ is not an element in the ideal generated by $E_2(V,c)$, then $\kappa(V,c)=\kappa(T)>1$. In fact the elements in (9) belongs to $E_4(S(V,c))$ so that we obtain $\kappa(V,c)=2$.

Example 7.3. At the end of [Kh2], an example of a two dimensional braided vector space (V,c) of combinatorial rank 2 is given. The braiding c is of diagonal type of the form $c(x_i\otimes x_j) = q_{i,j}x_j\otimes x_i, 1\leq i,j\leq 2$, where x_1,x_2 is a basis of V over K, $q_{1,2}=1\neq -1$ and $q_{i,j}=-1$ for all $(i,j)\neq(1,2)$.

Example 7.4. In [KA], one can find the combinatorial rank of a braided vector space with diagonal braiding of Cartan type A_n. That is, if $c(x_i\otimes x_j)=q_{i,j}x_j\otimes x_i$ with

$$q_{i,i}=q_{jj}=q,\qquad q_{i,i+1}q_{i+1,i}=q^{-1},\qquad q_{i,j}q_{j,i}=1, i-j>1$$

where q is a primitive t-th root of unity, $t>2$, then the combinatorial rank of the related braided vector space equals $\lceil 1+log_2(n)\rceil$. If q is not a root of unity, then the combinatorial rank is one.

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Appendix A. Some properties of the map Γ_n^B

In Definition 5.1, we introduce the map Γ_n^B, for a given braided bialgebra B. In 5.3, we used this map to describe the so called Nichols algebra. We show here some further properties of Γ_n^B that have been used to prove Theorem 6.11.

Lemma A.1. *Let $\pi : B\to E$ be a graded braided bialgebra homomorphism. Then*

$$\Gamma_{a,b}^E\circ\pi^{a+b}=\pi^{a+b}\circ\Gamma_{a,b}^B,\ \text{for every } a,b\in\mathbb{N}, a+b\geq 1. \tag{10}$$

where, for every $n\in\mathbb{N}$, $\pi^n : B^n\to E^n$ denotes the n-th graded component of π.

Proof. We have $\Gamma^E_{a,b}\pi^{a+b} = m^{a,b}_E \Delta^{a,b}_E \pi^{a+b} = m^{a,b}_E \left(\pi^a \otimes \pi^b\right) \Delta^{a,b}_B = \pi^{a+b} m^{a,b}_B \Delta^{a,b}_B = \pi^{a+b}\Gamma^B_{a,b}$. □

LEMMA A.2. *Let $(B, m, u, \Delta, \varepsilon, c)$ be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. Then*

$$\Gamma^B_{a+b} = m^{a,b}\left(\Gamma^B_a \otimes \Gamma^B_b\right)\Delta^{a,b}, \text{ for every } a, b \in \mathbb{N}. \tag{11}$$

Proof. By [AM, Theorem 3.11], there is a unique algebra homomorphism $\varphi : T := T_{B^0}\left(B^1\right) \to B$ that restricted to B^0 and B^1 gives the canonical inclusions. Furthermore each graded component of φ is an epimorphism as B is strongly $\mathbb{N}$-graded as an algebra. By [Sch, (1)], we have $m^{a,b}_T\left(\Gamma^T_a \otimes \Gamma^T_b\right)\Delta^{a,b}_T = \Gamma^T_{a+b}$. Using this equality and (10), we have $m^{a,b}\left(\Gamma^B_a \otimes \Gamma^B_b\right)\Delta^{a,b}\varphi^{a+b} = m^{a,b}\left(\Gamma^B_a\varphi^a \otimes \Gamma^B_b\varphi^b\right)\Delta^{a,b}_T = m^{a,b}\left(\varphi^a\Gamma^T_a \otimes \varphi^b\Gamma^T_b\right)\Delta^{a,b}_T = \varphi^{a+b}m^{a,b}_T\left(\Gamma^T_a \otimes \Gamma^T_b\right)\Delta^{a,b}_T = \varphi^{a+b}\Gamma^T_{a+b} = \Gamma^B_{a+b}\varphi^{a+b}$. Since φ^{a+b} is an epimorphism, we obtain (11). □

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