Trivariate quartic splines on type-6 tetrahedral partitions and their applications

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Abstract

In this paper we develop a 3D quasi-interpolating spline scheme, on a bounded domain, based on trivariate quartic C^2 box splines on type-6 tetrahedral partitions. Then, we describe some applications related both to the reconstruction of gridded volume data and to numerical integration. We also provide some numerical tests.

Keywords: `trivariate spline, quasi-interpolation, type-6 tetrahedral partition, gridded volume data, cubature.`

AMS subject classification: `65D07, 65D32.`

1. Introduction.

This paper deals with a 3D quasi-interpolating (QI) spline scheme, on a bounded domain Ω, based on trivariate quartic C^2 box splines on a type-6 tetrahedral partition $\mathcal{T}_{\mathbf{m}}$. We also provide two significant applications such as the reconstruction of gridded volume data and the numerical evaluation of integrals.

Indeed, on one hand, in many applications a non-discrete model may be helpful to visualize and elaborate volume data, representing a type of density obtained from suitable sensors. This construction is generally quite complex. The problem may be simplified if data are structured on a regular tridimensional grid and applications as medical CT and MRI, seismic phenomena investigation produce structured data. On the other hand, cubature rules are useful tools in several methods to solve integral and differential problems.

The paper is organized as follows. In Section 2 we consider the space $S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$ of C^2 quartic splines on a type-6 tetrahedral partition of a bounded domain and, in such a space, we define a spline QI of near-

best type, i.e. with coefficient functionals obtained by minimizing an upper bound for its infinity norm.

Then, in Section 3.1, we use the near-best QI for the reconstruction of discrete data on volumetric grids, providing some numerical tests and real world applications and, in order to explore the volumetric data, we visualize some isosurfaces.

Finally, in Section 3.2, we define 3D cubature rules and we propose some numerical tests, illustrating their approximation properties.

2. Quasi-interpolating operators in the spline space $S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$.

Let $m_1, m_2, m_3 \geq 5$ be integers, let $\Omega = [0, m_1 h] \times [0, m_2 h] \times [0, m_3 h]$, $h > 0$, be a parallelepiped divided into $m_1 m_2 m_3$ equal cubes and endowed with the type-6 tetrahedral partition $\mathcal{T}_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, m_3)$, where each cube is subdivided into 24 tetrahedra (see Figure 1).
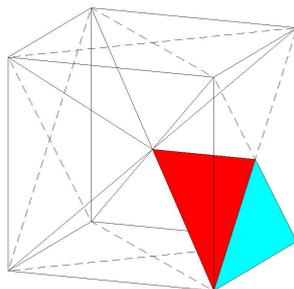

Figure 1. Cube subdivision into 24 tetrahedra with 6 planes.

We consider the seven directional box spline B (see [1] for its definition and [2, Chapter 11], [3] for general results on box splines), whose support is shown in Figure 2, and we consider the scaled translates of B, whose supports overlap with Ω. They are

$$\left\{ B_\alpha(x, y, z) = B\left(\frac{x}{h} - i + 1, \frac{y}{h} - j + 1, \frac{z}{h} - k + 3\right), \ \alpha \in \mathcal{A} \right\},$$

where

$$\mathcal{A} = \left\{ \alpha = (i, j, k), \begin{array}{l} -1 \leq i \leq m_1 + 2, \\ -1 \leq j \leq m_2 + 2, \ \alpha \notin \mathcal{A}' \\ -1 \leq k \leq m_3 + 2; \end{array} \right\},$$

with $\mathcal{A}'$ the set of indices defined by

$$\mathcal{A}' = \left\{ \begin{array}{ll} (i, j, -1), \ (i, j, m_3 + 2), & \text{for} \ -1 \leq i \leq m_1 + 2, \ j = -1, m_2 + 2, \\ (i, j, -1), \ (i, j, m_3 + 2), & \text{for} \ i = -1, m_1 + 2, \ 0 \leq j \leq m_2 + 1, \\ (i, -1, k), \ (i, m_2 + 2, k), & \text{for} \ i = -1, m_1 + 2, \ 0 \leq k \leq m_3 + 1 \end{array} \right\}.$$

We remark that the B_α's supports Ξ_α are centered at the points $\left((i - \frac{1}{2})h,\right.$ $\left.(j - \frac{1}{2})h, \ (k - \frac{1}{2})h\right)$.
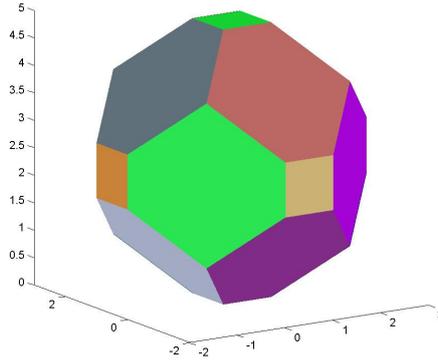

Figure 2. The support of the seven directional box spline B.

Now, we consider the space generated by $\{B_\alpha, \alpha \in \mathcal{A}\}$

$$S_4^2(\Omega, \mathcal{T}_\mathbf{m}) = \left\{ s = \sum_{\alpha \in \mathcal{A}} c_\alpha B_\alpha, \ \ c_\alpha \in \mathbb{R} \right\},$$

and in such a space we consider the near-best quasi-interpolant of the form

$$(1) \qquad\qquad Qf = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(f) B_\alpha.$$

The coefficient functionals λ_α's are linear combinations of function values and they have the following expression

$$(2) \qquad\qquad \lambda_\alpha(f) = \sum_{\beta \in F_\alpha} \sigma_\alpha(\beta) f(M_\beta),$$

where:

- the finite set of points $\left\{ M_\beta, \ \beta \in F_\alpha, \ F_\alpha \subset \mathcal{A}^M \right\}$ lies in some neighbourhood of $\Xi_\alpha \cap \Omega$ and

$$\{ M_\beta = M_{i,j,k} = (s_i, t_j, u_k), \ (i,j,k) \in \mathcal{A}^M \},$$

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with $\mathcal{A}^M = \{(i,j,k), 0 \le i \le m_1 + 1, 0 \le j \le m_2 + 1, 0 \le k \le m_3 + 1\}$ and

$$s_0 = 0, \ s_i = (i - \tfrac{1}{2})h, \ \ 1 \le i \le m_1, \ s_{m_1+1} = m_1 h$$
$$t_0 = 0, \ t_j = (j - \tfrac{1}{2})h, \ \ 1 \le j \le m_2, \ t_{m_2+1} = m_2 h$$
$$u_0 = 0, \ u_k = (k - \tfrac{1}{2})h, \ 1 \le k \le m_3, \ u_{m_3+1} = m_3 h;$$

- the $\sigma_\alpha(\beta)$'s are real numbers, obtained so that $Qf \equiv f$ for all f in $\mathbb{P}_3$, with $\mathbb{P}_3$ the space of trivariate polynomials of total degree at most three, and minimizing an upper bound for the infinity norm of the operator. For this reason Q is a *near-best* QI (see e.g. [4–7]).

For the explicit expression of such coefficient functionals see [8].

Concerning the approximation properties of the above quasi-interpolant, we can deduce the following theorem by means of standard results in approximation theory.

Theorem 1. *Let* $f \in C^4(\Omega)$, *and* $|\gamma| = 0, 1, 2, 3$ *with* $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. *Then there exist constants* $K_\gamma > 0$ *such that*

$$\left\| D^\gamma (f - Qf) \right\|_\infty \le K_\gamma h^{4-|\gamma|} \max_{|\beta|=4} \left\| D^\beta f \right\|_\infty,$$

where $D^\beta = D^{\beta_1 \beta_2 \beta_3} = \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2} \partial z^{\beta_3}}$.

3. Applications.

In the following we show two concrete problems that can be faced with tools based on the approximation scheme introduced in Section 2.

3.1. *Reconstruction of volume data.*

The construction of non-discrete models from given discrete data on volumetric grids is an important problem in many applications, such as scientific visualization, computer graphics and medical imaging, where a precise evaluation, and a high visual quality are the goals of visualization. Indeed, the volume data sets typically represent some kind of density acquired by devices like CT or MRI sensors. Such a type of input data is structured, so that the samples are arranged on a regular three-dimensional grid. In order to process these gridded samples an appropriate non-discrete model is required.

Therefore, we use the quasi-interpolating spline (1) for such a reconstruction, providing some numerical tests and real world applications. For the evaluation of box splines we can refer to [9], where an algorithm based

on the Bernstein-Bézier form of the box spline is proposed. Moreover, we visualize some isosurfaces of (1), generated as a very fine triangular mesh by using the Matlab procedure `isosurface` [10].

Firstly, we approximate the smooth trivariate test function of Franke type

$$f(x, y, z) = \frac{1}{2} e^{-10((x-\frac{1}{4})^2 + (y-\frac{1}{4})^2)} + \frac{3}{4} e^{-16((x-\frac{1}{2})^2 + (y-\frac{1}{4})^2 + (z-\frac{1}{4})^2)}$$
$$+ \frac{1}{2} e^{-10((x-\frac{3}{4})^2 + (y-\frac{1}{8})^2 + (z-\frac{1}{2})^2)} - \frac{1}{4} e^{-20((x-\frac{3}{4})^2 + (y-\frac{3}{4})^2)}$$

on $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$. In Figure 3(a) we report the maximum absolute error

$$E_Q(f) := \max_{(u,v,w) \in G} |f(u, v, w) - Qf(u, v, w)|,$$

for increasing values of $m = m_1 = m_2 = m_3$, $m = 16, 32, 64, 128$, and G a $139 \times 139 \times 139$ uniform three-dimensional grid of points in the domain. We also report an estimate of the approximation order, rf, obtained by the logarithm to base two of the ratio between two consecutive errors. We can notice that theoretical results are confirmed. In Figures 3(b)–(c) a visualization of the approximating spline Qf, for $m = 128$, using the isovalues $\rho = 0.3$ and 0.5, is shown.

m	$E_Q(f)$	rf
16	1.1(−2)	
32	8.0(−4)	3.8
64	5.2(−5)	3.9
128	3.3(−6)	4.0

(a)
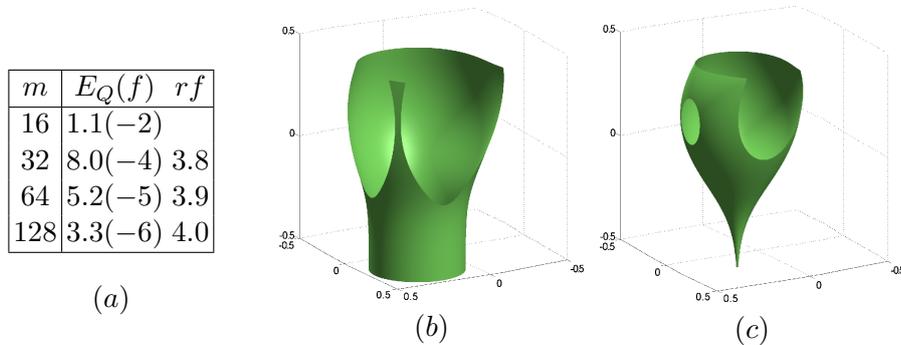
(b)   (c)

Figure 3.   (a) Maximum absolute errors and numerical convergence orders. Isosurfaces of Qf for $m = 128$, with isovalues (b) $\rho = 0.3$, (c) $\rho = 0.5$.

For comparisons with other methods, we refer to [11] and [12], where two C^1 spline QIs of different degree (quadratic and cubic, respectively) on type-6 tetrahedral partitions are presented and to [13], where a near-best QI defined as as blending sum of univariate and bivariate QIs is proposed.

Comparing the results, we can notice that using our C^2 quartic splines, the error decreases faster than using both the quadratic and cubic C^1 piecewise polynomials proposed in [11,12] and the near-best operator given in [13].

In Figure 4 we show ten among the 99 slices of 256×256 pixels. Such a gridded volume data set is obtained from a MR study of head with skull partially removed to reveal brain (courtesy of University of North Carolina). In Figure 5 we show the isosurface of the C^2 quartic spline Qf, resulting from the application of our method in the approximation of such a gridded volume data set. In order to visualize the isosurface, corresponding to the isovalue $\rho = 40$, we evaluate the spline on $N \approx 8,6 \times 10^6$ points.
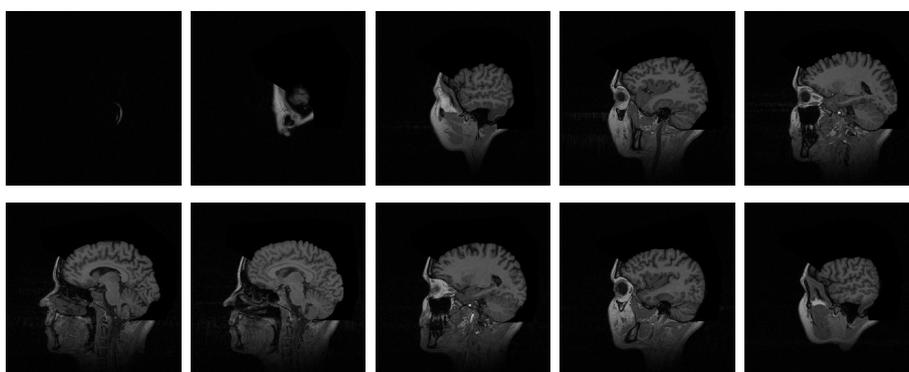

Figure 4.   Ten of $256 \times 256 \times 99$ slices, obtained from a MR study of head with skull partially removed to reveal brain (courtesy of University of North Carolina).
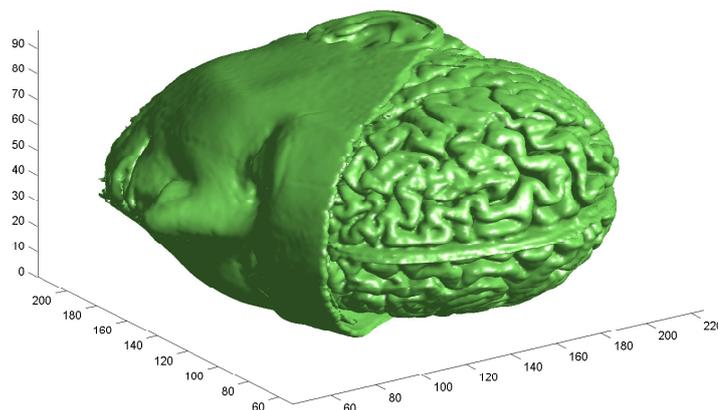

Figure 5.   Isosurface of the C^2 trivariate quartic spline approximating the *MR brain data set* with isovalue $\rho = 40$.

3.2. *Numerical integration.*

For any function $f \in C(\Omega)$, we consider the numerical evaluation of the integral

$$I(f) = \int_{\Omega} f(x, y, z) \, dxdydz$$

by cubature rules

$$I_Q(f) = I(Qf) = \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha}(f) w_{\alpha},$$

where the coefficients $\lambda_{\alpha}(f)$ are given in (2) and the weights

$$w_{\alpha} = \int_{\Xi_{\alpha} \cap \Omega} B_{\alpha}(x, y, z) \, dxdydz,$$

are computed in [14]. The above cubatures have precision degree 3, because Q is exact on $\mathbb{P}_3$. Moreover, from the convergence results of Qf to f, given in Theorem 1, if $f \in C^4(\Omega)$, then

$$E_Q^I(f) = |I(f) - I_Q(f)| = O(h^4).$$

Assuming the standard cube $\Omega = [0, 1]^3$ as integration domain, $m_1 = m_2 = m_3 = m$ and $h = 1/m$, in Table 1 we report the errors $E_Q^I(f)$ and an estimate of the approximation order, considering the test functions

- $f_1(x, y, z) = e^{\left((x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2\right)}$, $I(f_1) = 0.7852115962$,

- $f_2 = \frac{27}{8}\sqrt{1 - |2x - 1|}\sqrt{1 - |2y - 1|}\sqrt{1 - |2z - 1|}$, $I(f_2) = 1$,

for increasing values of m, i.e. $m = 16, 32, 64, 128$. We remark that the function f_1 is a smooth test function, coming from the testing package of Genz [15,16], whereas the function f_2 is only continuous. Concerning the function f_1, the numerical results shown in Table 1 confirm the convergence properties above given.

Table 1. Absolute errors and numerical convergence orders.

m	$E_Q^I(f_1)$	rf_1	$E_Q^I(f_2)$	rf_2
16	2.9(-5)		4.9(-3)	
32	1.9(-6)	3.9	2.4(-3)	1.1
64	1.3(-7)	3.9	9.3(-4)	1.3
128	8.1(-9)	4.0	3.5(-4)	1.4

For comparisons with other methods, we refer to [17,18], where cubature rules for a parallelepiped domain are defined by integrating tensor product

of univariate C^1 quadratic spline QIs and blending sums of C^1 quadratic spline QIs in one and two variables. Such rules are comparable with the formula $I_Q(f)$, based on trivariate quartic spline QIs with higher smoothness C^2, useful, for example, in the numerical treatment of integral equations, where the unknown function can be reconstructed with C^2 smoothness.

We also refer to [14], where cubature rules for 3D integrals based on trivariate C^2 quartic spline QIs are presented. We remark that the points used in the integration formulas there proposed lie also outside the integration domain. Since the function to be integrated may not be defined outside the domain of integration (as in the case of the above test function f_2), here we have considered spline cubature rules that make use of evaluation points inside or on the boundary of the domain.

REFERENCES

1. J. Peters, C^2 surfaces built from zero sets of the 7-direction box spline, in *IMA Conference on the Mathematics of Surfaces* (G. Mullineux, ed.), pp. 463–474, Clarendon Press, 1994.

2. B. D. Bojanov, H. A. Hakopian, and A. A. Sahakian, *Spline functions and multivariate interpolation.* Kluwer, 1993.

3. C. de Boor, K. Höllig, and S. Riemenschneider, *Box splines.* Springer-Verlag, 1993.

4. D. Barrera and M. J. Ibánez-Pérez, Bernstein-Bézier representation and near-minimally normed discrete quasi-interpolation operators, *Applied Numerical Mathematics*, vol. 58, pp. 59–68, 2008.

5. D. Barrera, M. J. Ibánez-Pérez, P. Sablonnière, and D. Sbibih, On near-best discrete quasi-interpolation on a four-directional mesh, *J. Comput. Appl. Math.*, vol. 233, pp. 1470–1477, 2010.

6. S. Remogna, Bivariate C^2 cubic spline quasi-interpolants on uniform Powell-Sabin triangulations of a rectangular domain, *Advances in Computational Mathematics*, vol. 36, pp. 39–65, 2012.

7. S. Remogna, Constructing good coefficient functionals for bivariate C^1 quadratic spline quasi-interpolants, in *Mathematical Methods for Curves and Surfaces* (M. Daehlen, M. Floater, T. Lyche, J. L. Merrien, K. M. rken, and L. L. Schumaker, eds.), pp. 329–346, Springer-Verlag, 2010.

8. C. Dagnino, P. Lamberti, and S. Remogna, Reconstruction of volume data by trivariate quartic C^2 box splines, *Quaderni Scientifici del Dipartimento di Matematica, Università di Torino*, vol. 11, 2012, http://aperto.unito.it/handle/2318/830.

9. M. Kim and J. Peters, Fast and stable evaluation of box-splines via the BB-form, *Numer. Algor.*, vol. 50, no. 4, pp. 381–399, 2009.

10. MATLAB, *Volume Visualization Documentation*. The MathWorks, http://www.mathworks.it/it/help/matlab/volume-visualization.html.

11. G. Nürnberger, C. Rössl, H. P. Seidel, and F. Zeilfelder, Quasi-interpolation by quadratic piecewise polynomials in three variables, *Computer Aided Geometric Design*, vol. 22, pp. 221–249, 2005.

12. T. Sorokina and F. Zeilfelder, Local quasi-interpolation by cubic C^1 splines on type-6 tetrahedral partitions, *IMA J. Numer. Anal.*, vol. 27, no. 1, pp. 74–101, 2007.

13. S. Remogna and P. Sablonnière, On trivariate blending sums of univariate and bivariate quadratic spline quasi-interpolants on bounded domains, *Computer Aided Geometric Design*, vol. 28, pp. 89–101, 2011.

14. C. Dagnino, P. Lamberti, and S. Remogna, Numerical integration based on trivariate C^2 quartic spline quasi-interpolants, submitted, 2013.

15. A. C. Genz, Testing multidimensional integration routines, in *Tools, methods and languages for scientific and engineering computation* (B. Ford, J. Rault, and F. Thomasset, eds.), pp. 81–94, North Holland, 1984.

16. A. C. Genz, A package for testing multiple integration subroutines, in *Numerical integration: recent developments, software and applications* (P. Keast and G. Fairweather, eds.), pp. 337–340, Reidel Publishing, 1987.

17. V. Demichelis and P. Sablonnière, Numerical integration by spline quasi-interpolants in three variables, in *Curve and Surface Fitting: Avignon 2006* (A. Cohen, J. L. Merrien, and L. L. Schumaker, eds.), pp. 131–140, Nashboro Press, 2007.

18. V. Demichelis and P. Sablonnière, Cubature rule associated with a discrete blending sum of quadratic spline quasi-interpolants, *J. Comput. Appl. Math.*, vol. 235, pp. 174–185, 2010.