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Pliska Stud. Math. Bulgar. **21** (2012), 97–112

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WINDOWED-WIGNER REPRESENTATIONS, INTERFERENCES AND OPERATORS

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In honor of the 65th Birthday of Prof. Petar Popivanov

Abstract. "Windowed-Wigner" representations, denoted by Wig_ψ and Wig^*_ψ, were introduced in [2] in connection with uncertainty principles and interferences problems. In this paper we present a more precise analysis of their behavior obtaining an estimate of the L^2-norm of interferences of couples of "model" signals. We further define a suitable functional framework for the associated operators and show that they form a class of pseudo-differential operators which define a natural "path" between the multiplication, Weyl and Fourier multipliers operators.

1. Introduction. Given a signal, i.e. a function $f(x)$ of the time variable x (or more generally $x \in \mathbb{R}^d$), its energy distributions with respect to time and to frequency are classically represented by $|f(x)|^2$ and $|\hat{f}(\omega)|^2$ respectively. Here f is supposed to be in some space of Fourier transformable functions. On the other hand a function, or a distribution, $Qf(x,\omega)$ defined on the time-frequency plane $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$ which can be, in some sense, interpreted as the energy distribution of f with respect to both time and frequency is called "time-frequency distribution" or "representation". Some of the most natural requirements which connect $Qf(x,\omega)$ with $|f(x)|^2$ and $|\widehat{f}(\omega)|^2$, and allow Q to be considered a time-frequency representation, are the following (see e.g. [4]):

2010 *Mathematics Subject Classification:* 42B10, 47A07, 35S05.
Key words: Wigner type representations, pseudodifferential operators, interference.

– *Positivity*: $Q(f)(x,\omega) \geq 0$ for all x, ω;

– *No spreading effect*: $\operatorname{supp} f \subseteq I$ for an interval $I \subseteq \mathbb{R}^d$ implies $\Pi_x \operatorname{supp} Q(f) \subseteq I$ (Π_x orthogonal projection $\mathbb{R}^d_x \times \mathbb{R}^d_\omega \to \mathbb{R}^d_x$) and, analogously, $\operatorname{supp} \hat{f} \subseteq J$ implies $\Pi_\omega \operatorname{supp} Q(f) \subseteq J$;

– *Marginal distributions condition*: $\int_{\mathbb{R}^d} Q(f)(x,\omega)dx = |\hat{f}(\omega)|^2$ and $\int_{\mathbb{R}^d} Q(f)(x,\omega)d\omega = |f(x)|^2$.

The uncertainty principle for time-frequency analysis makes however these requirements incompatible (see [6], [7], [14]) and this justifies the development of a wide literature about time-frequency distributions which approximate these requirements in some sense. For an extensive discussion on these issues as well as a for a review of the most used representations see e.g. [5], [7], [8], [9], [10].

A wide class of quadratic time-frequency representations is the so-called *Cohen Class.*

A generic representation in the Cohen class is of the form

$$Q(f,g) = \sigma * \operatorname{Wig}(f,g), \tag{1}$$

where $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is the "kernel" and Wig is the classical Wigner transform

$$\operatorname{Wig}(f,g)(x,\omega) = \int e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)}\, dt,$$

$f, g \in \mathcal{S}(\mathbb{R}^d)$ (other functional settings for Q can be considered as well, by choosing f, g, σ in such a way that (1) makes sense). Of course the Wigner itself belongs to the Cohen class, for $\sigma = \delta$, and it plays actually a central role in the development of time-frequency analysis.

For the Wigner transform both the support properties in time and in frequency are satisfied but, as a counterpart, some problems concerning the interferences arise. Indeed it shows an interference, or "ghost frequency", in the "middle" of any couple of "true" frequencies in the time-frequency plane. Many attempts have been made in order to find representations with better behavior to this respect. The Cohen class itself is a way to filter the Wigner transform, and for some choices of the kernel σ interferences can be considerably reduced (see [1], [3], [5]). In the lines of these works, in [2], two possible modifications of the Wigner distribution were considered in dependence on a *window function* (or distribution) $\psi(t)$. The two *windowed-Wigner* representations are defined as

$$\mathrm{Wig}_\psi(f,g)(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i t\omega}\psi(t) f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\,dt \tag{2}$$

and

$$\mathrm{Wig}^*_\psi(f,g)(x,\omega) := \int_{\mathbb{R}^d} e^{2\pi i t x}\psi(t) \hat{f}\left(\omega+\frac{t}{2}\right)\overline{\hat{g}\left(\omega-\frac{t}{2}\right)}\,dt, \tag{3}$$

with $f, g, \psi \in \mathcal{S}(\mathbb{R}^d)$ (see [2] for the definition in a distributional framework).

It is clear that ψ acts as a *window* in time for Wig_ψ, whereas the classical formula

$$\mathrm{Wig}(f,g)(x,\omega) = \mathrm{Wig}(\hat{f},\hat{g})(\omega,-x)$$

shows that it acts as a window in frequency for Wig^*_ψ.

Actually the following relation between the two windowed-Wigner representations holds

$$\mathrm{Wig}^*_\psi(f,g)(x,\omega) = \mathrm{Wig}_\psi(\hat{f},\hat{g})(\omega,-x).$$

In [2] we proved that Wig_ψ and Wig^*_ψ define subclasses of the Cohen class “joining” in suitable sense the Wigner representation with the two basic one-variable representations $|f(x)|^2$ and $|\hat{f}(\omega)|^2$. We studied there marginal properties, boundedness properties and various forms of uncertainty principles. For what interferences are concerned, we restricted however our study essentially to the behavior of Wig_ψ, proving that no interferences appear for classes of signals whose support satisfies suitable conditions.

The purpose of this paper is to study in more detail the problem of interferences (section 2) and, secondly, the type of operators which turn out to be naturally associated with the two windowed-Wigner transforms (section 3).

We present in section 2 a detailed study of signals with two frequencies in two different time intervals, which represent a model of the behavior of more general signals. We give in particular a precise L^2-estimate of the interferences, showing that they go to zero in suitable situations.

In section 3 we obtain explicit expressions for the pseudo-differential operators associated with Wig_ψ and Wig^*_ψ. We show that they turn out to be a sort of ”windowed” Weyl operators which approach the multipliers and Fourier multiplier operators respectively as the window function approaches the Dirac distribution. As "middle point" of this "path" of operators we find again the Weyl transform. Finally we state for these operators natural boundeness and Hilbert-Schmidt properties in the L^p functional setting.

2. Reduction of interference for Wig^*_ψ. We want now to give some properties concerning reduction of interference for the representations Wig_ψ and Wig^*_ψ. We have already pointed out, cf. [1], [2], that for particular choices of ψ the form Wig_ψ does not show ghosts at all. In particular, consider a signal f with two frequencies ω_0 and ω_1 in two disjoint time intervals, say $[k, k+\alpha]$ and $[h, h+\beta]$, with $k+\alpha < h$. We can write f in the form

$$f = f_1 + f_2 \tag{4}$$

with $f_1(t) = e^{2\pi i t\omega_0}\chi_{[k,k+\alpha]}(t)$ and $f_2(t) = e^{2\pi i t\omega_1}\chi_{[h,h+\beta]}(t)$, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$. Considering the classical Wigner transform, since $\mathrm{Wig}(f,g) = \overline{\mathrm{Wig}(g,f)}$ we have

$$\mathrm{Wig}(f,f) = \mathrm{Wig}(f_1,f_1) + 2\Re\,\mathrm{Wig}(f_1,f_2) + \mathrm{Wig}(f_2,f_2). \tag{5}$$

The interference ("ghost") showed by the Wigner transform in the middle of the two frequencies is represented by the cross term $2\Re\,\mathrm{Wig}(f_1,f_2)$. We want now to consider $\mathrm{Wig}_\psi(f,f)$, where $\psi = \chi_{[-R,R]}$. We have the following result.

Proposition 1. *Suppose that*

$$h - k \geq \max\{2\alpha, \beta - \alpha\}. \tag{6}$$

Then there exists $R > 0$ such that:

(i)

$$\chi_{[-R,R]}(t) f_j(x+t/2)\overline{f_j(x-t/2)} = f_j(x+t/2)\overline{f_j(x-t/2)}, \quad j = 1,2,$$

for every $t, x \in \mathbb{R}$ and for $j = 1,2$;

(ii)

$$\chi_{[-R,R]}(t) f_1(x+t/2)\overline{f_2(x-t/2)} \equiv 0.$$

The previous Proposition is proved in [2], where a geometrical interpretation is also given. By simple computations we have that for every $f, g \in L^2$

$$\mathrm{Wig}_\psi(g,f) = \overline{\mathrm{Wig}_{\overline{\psi}}(f,g)},$$

where $\tilde{\psi}(t) := \psi(-t)$. In particular, setting $\psi_R = \chi_{[-R,R]}$ we have

$$\mathrm{Wig}_{\psi_R}(g,f) = \overline{\mathrm{Wig}_{\psi_R}(f,g)}.$$

Then, for f as in (4), if (6) is satisfied and we choose R as in Proposition 1 we have

$$\begin{aligned} \mathrm{Wig}_{\psi_R}(f,f) &= \mathrm{Wig}_{\psi_R}(f_1,f_1) + 2\Re\,\mathrm{Wig}_{\psi_R}(f_1,f_2) + \mathrm{Wig}_{\psi_R}(f_2,f_2) \\ &= \mathrm{Wig}(f_1,f_1) + \mathrm{Wig}(f_2,f_2). \end{aligned} \tag{7}$$

Comparing (7) with (5) we observe that the effect of Wig_ψ on this class of signals is to delete the cross terms, and in fact the graphical representation of $\mathrm{Wig}_\psi(f,f)$ shows no ghost frequencies.

Let us consider now the representation Wig^*_ψ. Since

$$\mathrm{Wig}^*_\psi(f,g)(x,\omega) = \mathrm{Wig}_\psi(\hat{f},\hat{g})(\omega,-x)$$

we could apply Proposition 1 on the Fourier transform side, and conclude that for a signal f whose Fourier transform has support contained in two disjoint intervals $[k,k+\alpha]$ and $[h,h+\beta]$, if condition (6) is satisfied then we can choose $\psi_R = \chi_{[-R,R]}$ with a suitable R such that $\mathrm{Wig}^*_{\psi_R}$ shows no interferences at the Fourier transform level. On the other hand, the requirement that the Fourier transform $\hat{f}$ of the signal f is compactly supported means that the signal itself cannot have compact support in time, and then it is not a "true" signal. Observe that condition (6) means that the silence between the two components f_1 and f_2 of the signal has to be sufficiently large (larger than their existence time). We want to consider a similar situation for frequencies. We then analyze the representation Wig^*_ψ applied to a (compactly supported) signal containing two different frequencies, that we shall fix sufficiently far away from one another. Let us define

$$f_\sigma(t) = e^{2\pi i\sigma t}\chi_{[a,b]}(t),$$

for a fixed $\sigma \in \mathbb{R}$ and $0 < a < b$. We have the following result.

Theorem 1. *Fix $\psi_R(t) = \chi_{[-R,R]}(t)$, $R > 0$.*

(i) *Let f and g be two L^2 functions; we then have*

$$\|\mathrm{Wig}^*_{\psi_R}(f,g)\|_{L^2} \to \|\mathrm{Wig}(f,g)\|_{L^2} \tag{8}$$

as $R \to +\infty$. Furthermore $\|\mathrm{Wig}^*_{\psi_R}(f_\sigma,f_\sigma)\|_{L^2}$ and $\|\mathrm{Wig}(f_\sigma,f_\sigma)\|_{L^2}$ do not depend on σ.

(ii) *Consider two frequencies* $\omega_0, \omega_1 \in \mathbb{R}$, *i.e.* $f_{\omega_0}(t) = e^{2\pi i\omega_0 t}\chi_{[a,b]}(t)$ *and* $f_{\omega_1}(t) = e^{2\pi i\omega_1 t}\chi_{[a,b]}(t)$. *For fixed* R,

$$\|\operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|_{L^2} \to 0$$

as $|\omega_0 - \omega_1| \to +\infty$. *Moreover, for* $|\omega_0 - \omega_1| > R$ *we have*

$$\|\operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|_{L^2} \leq \frac{\sqrt{4R(b-a)}}{\pi\sqrt{(\omega_0-\omega_1)^2 - R^2}} \tag{9}$$

Remark. Let us consider a signal f of the form

$$f = f_{\omega_0} + f_{\omega_1}.$$

As in (7) we have

$$\operatorname{Wig}^*_{\psi_R}(f,f) = \operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_0}) + 2\Re\operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1}) + \operatorname{Wig}^*_{\psi_R}(f_{\omega_1}, f_{\omega_1});$$

then in view of Theorem 1, (i), we can choose R sufficiently large in such a way that the energy (represented by the L^2 norm) of the auto terms $\operatorname{Wig}^*_{\psi_R}(f_{\omega_j}, f_{\omega_j})$, $j = 1, 2$ becomes as near as we want to $\|\operatorname{Wig}(f_{\omega_j}, f_{\omega_j})\|_{L^2}$ (that represents the energy of the corresponding components f_{ω_j} since the Wigner transform satisfies conservation of energy). Observe that this R does not depend on ω_j, $j = 1, 2$. Then if the frequencies ω_0 and ω_1 are sufficiently far away from one another, from Theorem 1, (ii), the energy of the cross term $2\Re\operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})$ (that represents the interference) can be made arbitrarily small.

Proof of Theorem 1. (i) For $f, g \in L^2(\mathbb{R})$ we have that

$$\chi_{[-R,R]}(t)\hat{f}(\omega + t/2)\overline{\hat{g}(\omega - t/2)} \in L^2(\mathbb{R}^2);$$

then, since

$$\operatorname{Wig}^*_{\psi_R}(f,g) = \mathcal{F}^{-1}_{t\to x}\left(\chi_{[-R,R]}(t)\hat{f}(\omega + t/2)\overline{\hat{g}(\omega - t/2)}\right),$$

we have that

$$\begin{aligned}\|\operatorname{Wig}^*_{\psi_R}(f,g)\|^2_{L^2} &= \|\chi_{[-R,R]}(t)\hat{f}(\omega + t/2)\overline{\hat{g}(\omega - t/2)}\|^2_{L^2} \\ &= \int \chi_{[-R,R]}(t)\left|\hat{f}(\omega + t/2)\overline{\hat{g}(\omega - t/2)}\right|^2 dt\,d\omega.\end{aligned} \tag{10}$$

Since $\chi_{[-R,R]}(t)\left|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\right|^2$ tends to $\left|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\right|^2$ almost everywhere and

$$\chi_{[-R,R]}(t)\left|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\right|^2 \leq \left|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\right|^2 \in L^1(\mathbb{R}^2)$$

for every $R>0$, from the Dominated Convergence Theorem we have that

$$\|\operatorname{Wig}^*_{\psi_R}(f,g)\|^2_{L^2} \longrightarrow \|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\|^2_{L^2}. \tag{11}$$

Now,

$$\begin{aligned}\|\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\|_{L^2} &= \|\mathcal{F}^{-1}_{t\to x}\left(\hat{f}(\omega+t/2)\overline{\hat{g}(\omega-t/2)}\right)\|_{L^2}\\ &= \|\operatorname{Wig}(\hat{g},\hat{f})(\omega,-x)\|_{L^2} = \|\operatorname{Wig}(f,g)\|_{L^2},\end{aligned} \tag{12}$$

and so from (11) and (12) we have the thesis. Now we want to prove the independence of the norms $\|\operatorname{Wig}^*_{\psi_R}(f_\sigma,f_\sigma)\|_{L^2}$ and $\|\operatorname{Wig}(f_\sigma,f_\sigma)\|_{L^2}$ of σ. We indicate translation and modulation by τ_a and M_b, respectively, i.e., for real parameters a and b and an L^2 function f we set $\tau_a f(x)=f(x-a)$ and $M_b f(x)=e^{2\pi ibx}f(x)$. Then we can write $f_\sigma(s)=M_\sigma\chi_{[a,b]}(s)$. From (10) and the standard properties of the Fourier transform we have that

$$\begin{aligned}\|\operatorname{Wig}^*_{\psi_R}(f_\sigma,f_\sigma)\|^2_{L^2} &= \int \chi_{[-R,R]}(t)\left|(\tau_\sigma\hat{\chi}_{[a,b]})(\omega+t/2)\overline{(\tau_\sigma\hat{\chi}_{[a,b]})(\omega-t/2)}\right|^2 dt\,d\omega\\ &= \int \chi_{[-R,R]}(t)\left|\hat{\chi}_{[a,b]}(\omega-\sigma+t/2)\overline{\hat{\chi}_{[a,b]}(\omega-\sigma-t/2)}\right|^2 dt\,d\omega\\ &= \int \chi_{[-R,R]}(t)\left|\hat{\chi}_{[a,b]}(\omega+t/2)\overline{\hat{\chi}_{[a,b]}(\omega-t/2)}\right|^2 dt\,d\omega,\end{aligned}$$

and so $\|\operatorname{Wig}^*_{\psi_R}(f_\sigma,f_\sigma)\|_{L^2}$ does not depend on σ. The same conclusion can be proved in a similar way for $\|\operatorname{Wig}(f_\sigma,f_\sigma)\|_{L^2}$.
Observe that (8) is true also for functions $f,g\in L^2(\mathbb{R}^d)$, $d\geq 1$, taking a cut-off function of the form $\psi=\psi_R=\chi_{[-R,R]^d}$.

(ii) From (10) we have that

$$\begin{aligned}\|\operatorname{Wig}^*_{\psi_R}(f_{\omega_0},f_{\omega_1})\|^2_{L^2} &= \|\psi_R(t)\widehat{f_{\omega_0}}(\omega+t/2)\overline{\widehat{f_{\omega_1}}(\omega-t/2)}\|^2_{L^2}\\ &= \int_{-R}^{R}\int_{-\infty}^{+\infty}\left|\widehat{f_{\omega_0}}(\omega+t/2)\overline{\widehat{f_{\omega_1}}(\omega-t/2)}\right|^2 d\omega\,dt.\end{aligned}$$

So, by the change of variables $\omega + t/2 = \xi$ and using the standard properties of the Fourier transform we get

$$
\begin{aligned}
\| \operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|^2_{L^2} &= \int_{-R}^{R}\int_{-\infty}^{+\infty} \left| \widehat{f_{\omega_0}}(\xi) \left(\overline{\tau_t \widehat{f_{\omega_1}}}\right)(\xi)\right|^2 d\xi\, dt \\
&= \int_{-R}^{R}\int_{-\infty}^{+\infty} \left| \widehat{f_{\omega_0}}(\xi)\overline{\widehat{M_t f_{\omega_1}}}(\xi)\right|^2 d\xi\, dt \\
&= \int_{-R}^{R} \left\| \mathcal{F}_{s\to\xi}\left(f_{\omega_0} * (\overline{\widetilde{M_t f_{\omega_1}}})\right)\right\|^2_{L^2(\mathbb{R}_\xi)} dt \qquad (13)\\
&= \int_{-R}^{R}\int_{-\infty}^{+\infty} \left| \left(f_{\omega_0} * \overline{\widetilde{M_t f_{\omega_1}}}\right)(s)\right|^2 ds\, dt,
\end{aligned}
$$

where we have indicated $\tilde{g}(x) = g(-x)$. Now, we compute

$$\left(f_{\omega_0} * \overline{\widetilde{M_t f_{\omega_1}}}\right)(s) = e^{2\pi i\omega_0 s}\int e^{-2\pi i y(\omega_0-\omega_1-t)}\chi_{[a,b]}(-y)\chi_{[a,b]}(s-y)\, dy.$$

Observe that

$$\chi_{[a,b]}(-y)\chi_{[a,b]}(s-y) = \begin{cases} 0 & \text{if } s \notin [a-b, b-a] \\ \chi_{[-b,s-a]}(y) & \text{if } s \in [a-b, 0] \\ \chi_{[s-b,-a]}(y) & \text{if } s \in [0, b-a] \end{cases}$$

We then have that for $s \notin [a-b, b-a]$, $\left(f_{\omega_0} * \overline{\widetilde{M_t f_{\omega_1}}}\right)(s) \equiv 0$. Concerning the other cases, we recall that for every $\alpha < \beta$ we have

$$\int_{\alpha}^{\beta} e^{-2\pi i y z}\, dy = e^{-\pi i(\alpha+\beta)z}(\beta-\alpha)\operatorname{sinc}(\pi(\beta-\alpha)z),$$

where $\operatorname{sinc} x$ is the continuous extension on $\mathbb{R}$ of the function $\frac{\sin x}{x}$. Then, for $s \in [a-b, 0]$ we get

$$
\begin{aligned}
&\left(f_{\omega_0} * \overline{\widetilde{M_t f_{\omega_1}}}\right)(s) = e^{2\pi i\omega_0 s}\int_{-b}^{s-a} e^{-2\pi i y(\omega_0-\omega_1-t)}\, dy \\
&\quad = e^{2\pi i\omega_0 s} e^{-\pi i(s-a-b)(\omega_0-\omega_1-t)}(s-a+b)\operatorname{sinc}(\pi(s-a+b)(\omega_0-\omega_1-t)).
\end{aligned}
$$

Reasoning in an analogous way for $s \in [0, b-a]$ and using that $\operatorname{sinc}(x)$ is an even function we have

$$\left(f_{\omega_0} * \widetilde{\overline{M_t f_{\omega_1}}}\right)(s) = e^{2\pi i \omega_0 s} e^{-\pi i (s-a-b)(\omega_0-\omega_1-t)} \varphi(s) \operatorname{sinc}(\pi\varphi(s)(t-\omega_0+\omega_1)),$$

where

$$\varphi(s) = \begin{cases} 0 & \text{if } s \notin [a-b, b-a] \\ s-a+b & \text{if } s \in [a-b, 0] \\ -s-a+b & \text{if } s \in [0, b-a] \end{cases}$$

We then have from (13) that

$$\begin{aligned} \| \operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|^2_{L^2} &= \int_{-R}^{R} \int_{-\infty}^{+\infty} |\varphi(s) \operatorname{sinc}(\pi\varphi(s)(t-\omega_0+\omega_1))|^2 \, ds \, dt \\ &= \int_{-R-(\omega_0-\omega_1)}^{R-(\omega_0-\omega_1)} \int_{a-b}^{b-a} |\varphi(s) \operatorname{sinc}(\pi\varphi(s)y)|^2 \, ds \, dy, \end{aligned} \tag{14}$$

since the function $\varphi(s)$ is compactly supported. Observe that $|\varphi(s)\operatorname{sinc}(\pi\varphi(s)y)|^2 \in L^1(\mathbb{R}^2)$, and given an arbitrary compact set $K \subset \mathbb{R}^2$ the set $[a-b, b-a] \times [-R-(\omega_0-\omega_1), R-(\omega_0-\omega_1)]$ does not intersect K for $|\omega_0-\omega_1|$ sufficiently large. Then we conclude that

$$\| \operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|^2_{L^2} \to 0$$

as $|\omega_0 - \omega_1| \to +\infty$.
The estimate (9) follows from (14). We have in fact that for $|\omega_0 - \omega_1| > R$ the L^2 norm of $\operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})$ can be estimated as follows:

$$\begin{aligned} \| \operatorname{Wig}^*_{\psi_R}(f_{\omega_0}, f_{\omega_1})\|_{L^2} &\le \left(\int_{-R-(\omega_0-\omega_1)}^{R-(\omega_0-\omega_1)} \int_{a-b}^{b-a} \frac{1}{\pi^2 y^2} \, ds \, dy \right)^{1/2} \\ &= \frac{\sqrt{4R(b-a)}}{\pi\sqrt{(\omega_0-\omega_1)^2 - R^2}}. \end{aligned}$$

□

3. Operators associated with the windowed-Wigner distributions Wig_ψ and Wig_ψ^*. The Weyl calculus and time-frequency representations are deeply connected, see for instance [11], [12], [13], [15] for general references. In particular we recall here from [3] a result concerning the existence of a bijection between pseudo-differential operators and bounded conjugate linear forms.

Proposition 2. *Let E, E_1, E_2 be three Banach spaces and suppose E_2 to be reflexive.*

(i) *Let $\varphi : E_2 \times E_1 \to E$ be a bounded skew-linear map. Then there exists a unique linear and bounded map $a \in E^* \to T_a \in B(E_1, E_2^*)$ such that for every $v \in E_2$*

$$(T_a u, v) = (a, \varphi_{v,u}), \qquad \forall u \in E_1. \tag{15}$$

(ii) *Suppose now that the map $a \in E^* \to T_a \in B(E_1, E_2^*)$ is continuous and linear; then* (15) *defines a skew-linear bounded map $\varphi : E_2 \times E_1 \to E$.*

Observe that the dual spaces considered are taken as conjugate-linear functional spaces, therefore the notation $(\cdot, \cdot)$ extends the scalar product of L^2.

According to this Proposition, we want to focalize on the operators associated to the windowed-Wigner distributions Wig_ψ and Wig_ψ^*. More precisely, writing T_ψ^a and U_ψ^a for the operators associated to Wig_ψ and Wig_ψ^* respectively, we have the following result.

Proposition 3. *For every $f, g \in L^2(\mathbb{R}^d)$ and $a \in \mathcal{S}(\mathbb{R}^d)$ we have:*

$$(T_\psi^a f, g) = (a, \mathrm{Wig}_\psi(g, f)), \tag{16}$$

with

$$T_\psi^a f(u) = \int_{\mathbb{R}^{2d}} a\left(\frac{u+v}{2}, \omega\right) \overline{\psi(u-v)} e^{2\pi i (u-v)\omega} f(v) dv d\omega. \tag{17}$$

Concerning the Wig_ψ^* *we have that*

$$(U_\psi^a f, g) = (a, \mathrm{Wig}_\psi^*(g, f)),$$

with

$$U_\psi^a f(m) = \int_{\mathbb{R}^{3d}} a(x,y) \overline{\widehat{\psi\left(x - \frac{s+m}{2}\right)}} e^{2\pi i y (m-s)} f(s) ds dy dx.$$

P r o o f. We first consider the case of the Wig_ψ. To get the expression of the operators associated with the Wig_ψ we start by writing explicitly the inner product $(a, \mathrm{Wig}_\psi(g,f))$ then, using suitable changes of variables, we obtain

$$\int_{\mathbb{R}^{2d}} a(x,\omega)\left(\overline{\int e^{-2\pi i\omega t}\psi(t)g\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}dt}\right)dxd\omega =$$

$$2^d\int_{\mathbb{R}^{2d}} a(x,\omega)\left(\overline{\int_{\mathbb{R}^d} e^{-2\pi i\omega(2u-2x)}\psi(2u-2x)g(u)\overline{f(2x-u)}du}\right)dxd\omega =$$

$$2^d\int_{\mathbb{R}^{3d}} a(x,\omega)\overline{\psi(2u-2x)}e^{2\pi i(2u-2x)\omega}\overline{g(u)}f(2x-u)dudxd\omega =$$

$$\int_{\mathbb{R}^{3d}} a\left(\frac{u+v}{2},\omega\right)\overline{\psi(u-v)}e^{2\pi i(u-v)\omega}f(v)\overline{g(u)}dudvd\omega.$$

Then

$$T^a_\psi f(u) = \int_{\mathbb{R}^{2d}} a\left(\frac{u+v}{2},\omega\right)\overline{\psi(u-v)}e^{2\pi i(u-v)\omega}f(v)dvd\omega,$$

is the operator we were looking for.

Consider now the case of the Wig^*_ψ. As before we start by writing explicitely the scalar product $(a, \mathrm{Wig}^*_\psi(\hat{g},\hat{f}))$:

$$\int_{\mathbb{R}^{2d}} a(x,\omega)\left(\overline{\int e^{2\pi ixt}\psi(t)\hat{g}\left(\omega+\frac{t}{2}\right)\overline{\hat{f}\left(\omega-\frac{t}{2}\right)}dt}\right)dxd\omega =$$

$$2^d\int_{\mathbb{R}^{2d}} a(x,\omega)\left(\overline{\int_{\mathbb{R}^d} e^{2\pi ix(2u-2\omega)}\psi(2u-2\omega)\hat{g}(u)\overline{\hat{f}(2\omega-u)}du}\right)dxd\omega =$$

$$2^d\int_{\mathbb{R}^{3d}} a(x,\omega)\overline{\psi(2u-2\omega)}e^{-2\pi ix(2u-2\omega)}\overline{\hat{g}(u)}\hat{f}(2\omega-u)dudxd\omega =$$

$$\int_{\mathbb{R}^{3d}} a\left(x,\frac{u+v}{2}\right)\overline{\psi(u-v)}e^{-2\pi ix(u-v)}\hat{f}(v)\overline{\hat{g}(u)}dudvdx.$$

Now, in order to highlight the inner product between the operator and the function g, it is necessary to write out the expressions of $\hat{f}$ and $\hat{g}$; then

$$\int_{\mathbb{R}^{5d}} a\left(x, \frac{u+v}{2}\right) \overline{\psi(u-v)} e^{-2\pi i x(u-v)} e^{-2\pi i v s} e^{2\pi i u m} f(s) \overline{g(m)} dm ds du dv dx =$$

$$\int_{\mathbb{R}^{5d}} a(x,y) \overline{\psi(z)} e^{-2\pi i x z} e^{-2\pi i \left(y - \frac{z}{2}\right) s} e^{2\pi i \left(y + \frac{z}{2}\right) m} f(s) \overline{g(m)} dm ds dy dz dx =$$

$$\int_{\mathbb{R}^{5d}} a(x,y) \overline{\widehat{\psi}\left(x - \frac{s+m}{2}\right)} e^{2\pi i y(m-s)} f(s) \overline{g(m)} dm ds dy dx,$$

Finally we have that

$$U_\psi^a f(m) = \int_{\mathbb{R}^{3d}} a(x,y) \overline{\widehat{\psi}\left(x - \frac{s+m}{2}\right)} e^{2\pi i y(m-s)} f(s) ds dy dx,$$

is the operator associated with the Wig_ψ^*. □

Observe that it is possible to rewrite the operator associated with the Wig_ψ^* in order to point out the correspondance with the Weyl operators. More precisely, given $a \in \mathcal{S}(\mathbb{R}^{2d})$, we consider $b(x,\omega) = a(-\omega, x)$. Then

$$(a, \mathrm{Wig}_\psi^*(g,f)) = (b(\omega,-x), \mathrm{Wig}_{\check{\psi}}(\hat{g},\hat{f})(\omega,-x)) =$$

$$\int_{\mathbb{R}^{2d}} b(\omega,-x) \left(\overline{\int e^{2\pi i x t} \psi(t) \hat{g}\left(\omega + \frac{t}{2}\right) \overline{\hat{f}\left(\omega - \frac{t}{2}\right)} dt} \right) dx d\omega =$$

$$2^d \int_{\mathbb{R}^{3d}} b(\omega,-x) \overline{\psi(2u-2\omega)} e^{-2\pi i (2u-2\omega)x} \overline{\hat{g}(u)} \hat{f}(2\omega - u) du dx d\omega =$$

$$\int_{\mathbb{R}^{3d}} b\left(\frac{u+v}{2}, -x\right) \overline{\psi(u-v)} e^{2\pi i (u-v)(-x)} \hat{f}(v) \overline{\hat{g}(u)} du dv dx,$$

and, writing explicitly $\hat{g}(u)$, we have that the associated operator is given by the expression

$$U_\psi^a f(u) = \int_{\mathbb{R}^{3d}} b\left(\frac{u+v}{2}, x\right) e^{2\pi i (u-v)x} e^{2\pi i s u} \overline{\psi(u-v)} \hat{f}(v) dv dx ds, \tag{18}$$

where $b(x,\omega) = a(-\omega, x)$.

Remark. Obviously, if $\psi \equiv 1$ then:

$$f \in \mathcal{S}(\mathbb{R}^d) \to T^a f(u) = \int_{\mathbb{R}^{2d}} a\left(\frac{u+v}{2}, \omega\right) e^{2\pi i(u-v)\omega} f(v) dv d\omega \in \mathcal{S}(\mathbb{R}^d),$$

which is the Weyl pseudo-differential operator.

Remark. Take $\psi = \delta$. Concerning the Wig_ψ we have that the associated operator

$$T^a_\psi f(u) = \int a(v, \omega) f(v) d\omega = A(v) f(v)$$

is the multiplication operator with $A(v) = \int a(v, \omega) d\omega$. Concerning the Wig^*_ψ, from (18) choosing $\psi = \delta$, it follows

$$U^a_\psi f(s) = \int_{\mathbb{R}^{2d}} b(u, x) e^{2\pi i s u} \hat{f}(u) dx du = \int_{\mathbb{R}^d} B(u) e^{2\pi i s u} \hat{f}(u) du,$$

which is the Fourier multiplier operator associated to the Wig^*_ψ with $B(u) = \int b(u, x) dx$.

Remark. From [2] we know that

$$\mathrm{Wig}^{(*)}_\psi : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}),$$

are continuous maps. Then, from Proposition 2, we have that

$$a \in L^2(\mathbb{R}^{2d}) \to T^a_\psi \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)) \tag{19}$$

and

$$a \in L^2(\mathbb{R}^{2d}) \to U^a_\psi \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)) \tag{20}$$

are continuous maps. Moreover, since the map $\mathrm{Wig}_\psi : L^p(\mathbb{R}^d) \times L^{p'}(\mathbb{R}^d) \to L^\infty(\mathbb{R}^{2d})$ is well defined as bounded map (for more details see Proposition 2 in [2]), and since $L^1(\mathbb{R}^{2d}) \subset \left(L^\infty(\mathbb{R}^{2d})\right)^*$, it follows also that $a \in L^1(\mathbb{R}^{2d}) \to T^a_\psi \in B(L^p(\mathbb{R}^d), L^p(\mathbb{R}^d))$ is continuous for $1 < p < \infty$.

In the L^2 case we can prove that the corresponding operator is Hilbert-Schmidt. Formally we have the following result.

Proposition 4. *Let $a(x,\omega)$ be a function such that*

$$a(x,\omega) = a_1(x)a_2(\omega).$$

(a) *Consider the case of the* Wig_ψ. *Let $a_1 \in L^2(\mathbb{R}^d)$, $a_2 \in L^r(\mathbb{R}^d)$ and $\psi \in L^q(\mathbb{R}^d)$, with $\frac{1}{r} - \frac{1}{q} = \frac{1}{2}$. Then the operator $T_\psi^a \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$, with T_ψ^a defined as in* (17)*, is Hilbert-Schmidt, in particular it is bounded and compact.*

(b) *Consider now the case of the* Wig_ψ^*. *Let $a_2 \in L^2(\mathbb{R}^d)$, $a_1 \in L^r(\mathbb{R}^d)$ and $\psi \in L^q(\mathbb{R}^d)$, with $\frac{1}{r} - \frac{1}{q} = \frac{1}{2}$. Then the operator $U_\psi^a \in B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$, with U_ψ^a defined as in* (18)*, is Hilbert-Schmidt.*

Proof. We prove (a), the second one is similar. Consider the operator T_ψ^a defined in (17) and integrate with respect to ω. Then we have

$$T_\psi^a f(u) = \int_{\mathbb{R}^d} a_1\left(\frac{u+v}{2}\right) \check{a_2}(u-v)\overline{\psi(u-v)} f(v)dv,$$

where $\check{a_2}$ is the inverse Fourier transform of a_2. It follows immediately that the kernel $K(u,v) = a_1\left(\frac{u+v}{2}\right)\check{a_2}(u-v)\overline{\psi(u-v)}$ lies in the space L^2 if $a_1 \in L^2(\mathbb{R}^d)$ and $\check{a_2}\overline{\psi} \in L^2$. But, using the generalized Hölder inequality, we can choose $\check{a_2} \in L^p$ and $\psi \in L^q$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $p \geqslant 2$. Hence we can take $a_2 \in L^r$ with the condition $\frac{1}{r} - \frac{1}{q} = \frac{1}{2}$. Therefore the result is a consequence of the well-known fact that integral operators with Schwartz kernel in L^2 are Hilbert-Schmidt. □

It is possible to generalize this fact to functions $a(x,\omega)$ by using the density of the tensor products of functions in the space $L^{2,r}(\mathbb{R}^{2d})$, for fixed windows $\psi \in L^q(\mathbb{R}^d)$, with the condition $\frac{1}{r} - \frac{1}{q} = \frac{1}{2}$.

We finally observe that Proposition 4 gives a stronger result on the operators T_ψ^a and U_ψ^a in the L^2 case for fixed symbols a, but it does not prove the continuity of the quantizations (19) and (20), which has been deduced from Proposition 2.

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