"Dig where you stand" 2

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The proposals of the School of Peano on the rational teaching of Geometry

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Abstract

Between 1890 and 1940, the School of Peano formulated many proposals for the renewal of the teaching of mathematics, showing a sensitivity regarding the pedagogical tasks and the syllabuses. In this context we intend to analyze the reflections of some members of this team about the introduction of the results of their research activity on the foundations of geometry into education, illustrate the ways used to train teachers in this field, and examine the debates that the 'rational teaching' of geometry ignited. An examination of the textbooks by G. Ingrami and A. Pensa will make explicit the epistemological assumptions, elaborated by Pieri, Peano and Padoa, on the possibility of presenting maths in classroom as an hypothetical-deductive abstract system and the use of the logical symbolism.

Introduction

Between the late 19th century and the 1920s, many of the best Italian mathematicians were particularly aware of logic-foundational issues, sometimes spurred by their own scientific activity, at other times by interests of a philosophical or methodological nature. It is in this context that we may consider the reflections of members of the School of Giuseppe Peano[1] on the so-called *rational teaching* of Geometry, aimed at renovating its traditional treatment from elementary school to University.

First of all it is necessary to determine what is being alluded to with the expression *rational teaching*. In a general sense, it refers to a didactic praxis which substantially uses the output of the foundational research. More precisely, it is articulated along two branches: *hypothetical-deductive* and *axiomatic-deductive* teaching. The two trends, although coinciding in many aspects, greatly differ in relation to the nature of mathematical objects. In fact, according to the hypothetical-deductive approach, geometrical concepts must be viewed and presented in classroom *à la* Hilbert, i.e. as mere names attributed to classes of abstract objects, and without any reference to their physical substratum. On the contrary, according to the axiomatic-deductive method, opportunely revitalized by some positivist educationalists such as Carlo Cattaneo, Roberto Ardigò, Aristide Gabelli and Andrea Angiulli, the fundamental geometric ideas should always

[1] The characteristics of the School of Peano (its members, the role of Peano as a *Maestro*, the choice of research themes, ...) are problems currently debated in historiography. Cf. (Luciano & Roero, 2012; Roero, 2010).

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maintain a concrete connotation and they must be transposed in teaching activities through adequate interpretations and 'metaphors', derived from the real world.

Once this distinction is made, we can say that the majority of Peano's collaborators dedicated themselves to examining how to adapt the advanced research on the foundations of mathematics to textbooks for elementary, middle and secondary schools and how to insert it into teacher training, showing only occasionally an interest in the psychological aspects linked to these themes. Strangely enough, the strong theoretical engagement on the part of the School of Peano in the field of hypothetic-deductive teaching did not correspond to an analogous commitment to school publishing. Indeed, almost none of the authors of manuals of rational geometry published in Italy belonged, strictly speaking, to the School of Peano, even though they maintained varying levels of relationship with some of its members.

The institutional context

Around 1890, large sectors of the Italian mathematical community agreed on the advisability of illustrating in Universities and *Scuole di Magistero* (schools for teacher education, see Roero 1999, Furinghetti & Giacardi, 2012) the results of research carried out in the foundations of mathematics from the scientific, critical and pedagogical point of view. The main objective was to train up teachers for middle schools who were aware of the importance of these studies in order to deal with elementary mathematics in the classroom in a more clear and precise way. There soon arose, however, a divergence of opinions regarding both the expediency of establishing formal courses in this discipline, and on the contents and outlines that should characterize them. On one side there was the School of Peano, whose members maintained that logic and foundations ought to be strictly related to each other, and that the presentation of such studies necessarily involved the use of ideographic symbolism. On the opposing side was the School of Algebraic Geometry, whose members believed that the principles of mathematics ought to be addressed in connection with the so-called *elementary mathematics from an advanced standpoint*, and taking into account the physiological and cognitive aspects, in the wake of Felix Klein and Henri Poincaré's influences.

In spite of the emergence of signs of hostility, in 1897 Luigi Certo presented for the first time a proposal to create chairs in Foundations of Mathematics in the second biennium of Universities, and to teach the simplest and most essential notions of logic (conceived as an introduction to the mathematics) in secondary schools, technical institutes and normal schools.[2] In particular, he invited the colleagues to take as a starting point the volumes by Wilhelm Killing *Einführung in die Grundlagen der Geometrie* (Paderborn, Schöningh, 1893, 1898) and the works by Hermann Grassmann, Hermann Helmholtz, Bernhard Riemann, Adrien-Marie Legendre, Jules Hoüel, Felix Klein, Moritz Pasch and Giuseppe Peano. This idea was then presented and received with enthusiasm at the first national congress of the Mathesis Association, held in Turin, in the "citadel where these methods were championed so pertinaciously and valiantly".[3]

In the months that followed, within the local sessions of the Mathesis, perplexities emerged about Certo's proposal concerning mathematical logic, although his suggestion of establishing courses in foundations enjoyed a general consensus (Certo 1899, pp. 114-116). In spite of the insistence of the Peanian followers, the project to introduce a formal teaching of logic was firmly rejected during the next congress of the Mathesis, held in Livorno. On the contrary, Giulio Pittarelli, who had taken over from Certo as a speaker, continued to underline the expediency of devoting some of the lectures in the *Scuole di Magistero* to questions pertaining the primitives ideas and postulates of the science, taking Federigo Enriques' *Questioni riguardanti la Geometria Elementare* (Bologna, Zanichelli, 1900) as a model, see (Pittarelli 1902, p. 163)

Meanwhile, the debate was growing on the idea of introducing courses in *Metodologia Matematica* (*Mathematical Methodology*) or *Matematiche Elementari da un Punto di Vista Superiore* (*Elementary Mathematics from an Advanced Standpoint*), which were expected to include elements of logic and considerations on the foundations of mathematics, along with didactic issues and the history of mathematics.[4] Regarding this question, the majority of Italian mathematicians, whether of the Peanian *entourage* or not, recommended tackling questions such as the deductive method, the partial arbitrariness in the choice of primitive concepts and postulates of a theory, the various axiomatic systems for the different branches of geometry, etc.

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[2] Estratto del verbale dell'adunanza tenuta in Palermo nel giorno 27 febbraio 1897, fra i soci di "Mathesis", professori Certo, Pepoli e Rozzolino. *Boll. Mathesis* (1897-98), *2*, p. 9.

[3] (Certo, 1899, p. 116): "nella cittadella dove quei metodi sono così pertinacemente e con tanto valore propugnati".

[4] Among the dozens of interventions on the introduction of foundational research in the teachers training we mention: (Pincherle, 1903, p. 47, p. 49; Loria, 1906; Pittarelli, 1908, p. 35, p. 36, pp. 38-39; Loria & Padoa, 1909; Padoa, 1909, pp. 110-111; Pincherle, 1911, p. 11, p. 13; Fano, 1922, pp. 107-110).

Furthermore, a suitable education on these themes was judged to be important also for the training of teachers in kindergartens and elementary schools. In this case, the cultural background to be imparted had to be especially calibrated according to the exigencies of this kind of educators, charged with the difficult task of teaching young pupils (3-10 years old) the first concepts of mathematics, the very same concepts whose treatment naturally implied many delicate foundational questions.[5] In this context, authors such as Rodolfo Bettazzi, concentrated on naive mathematical education identifying a typical Achilles' heel of pre-school teaching of geometry in the linguistic aspects, wherein the foundational research itself had focused attention, see (Bettazzi 1939, pp. 72-77).

In the absence of courses specifically dedicated to the principles of geometry, there is no shortage of mathematicians, like Corrado Segre, who hinted at these themes in their university lectures of Higher Geometry and even scholars, like Mario Pieri, who entirely restructured his courses in Projective Geometry after his studies on foundations.[6] Furthermore, it can't be forget the long lasting experience of Giuseppe Veronese at the University and *Scuola di Magistero* of Padua, beginning at the end of the century and producing the well known volume *Fondamenti di Geometria a più dimensioni e a più specie di unità rettilinee, esposti in forma elementare* (Padova, Tip. del Seminario, 1891).[7]

In the 1920s, the institution of *Matematiche complementari* chairs provided the ideal context for the presentation of critical research, crystallizing a situation of substantial differences from place to place. Looking at that period we can see an authentic proliferation of courses entitled *Fondamenti di Geometria*, held by Luigi Berzolari, Enea Bortolotti, Gaspare Mignosi, Giuseppe Marletta and Alfredo Perna in the Universities of Pavia, Bologna, Palermo, Catania, and Rome.[8] Logic ideography combined with the axiomatic treatment of arithmetic and geometry were integral parts of the lectures of *Matematiche complementari* given in Turin by Peano and in

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[5] Cf. (Bettazzi & Burali-Forti, 1899; Conti, 1912, pp. 121-123, pp. 125-126, pp. 137-138; Bisson-Minio, 1910, p. 98).

[6] Cf. (Segre, C. (1916-17). Vedute superiori sulla Geometria elementare, Quad. 30, pp. 7-27. In Giacardi (Ed.) 2002 and Pieri M. (1902-03). *Lezioni di Geometria Superiore dettate dal Prof. Mario Pieri nella R. Università di Catania nell'a.a. 1902-03.* Catania: Bianca, lithographic print, Dipartimento di Matematica dell'Università di Pavia, Archive L. Brusotti). On Pieri's teaching at the Universities of Turin and Parma cf. also Marchisotto, 2010, pp. 340-342, pp. 350-353.

[7] Veronese held courses specifically entitled *Fondamenti* or *Principi di Geometria* in 1910-11, 1912-14, 1916-17 cf. *L'Enseignement Mathématique*, *12*, 1910, p. 343; *14*, 1912, p. 334; *15*, 1913, p. 357; *18*, 1916, p. 362.

[8] *L'Enseignement Mathématique*, *25*, 1926, p. 131; *26*, 1927, p. 148, p. 149; *29*, 1930, p. 168; *31*, 1932, p. 128, p. 129.

Milan by Ugo Cassina,[9] and were often chosen as the subject of degree dissertations (*tesi di laurea*) by their students.[10] Also some algebraic geometers approached these questions, dedicating time to non-Euclidean and non-Archimedean geometries, the concepts of elementary geometry from the point of view of groups of transformation, according to the *Erlangen Program* etc. Open to international influence, they repeatedly discussed the problems related to the transposition of foundational research into education with Felix Klein and David Hilbert, on the occasion of their visits to Göttingen and during trips by their German colleagues to Italy.[11]

The lectures and seminars on logic and foundations

The teaching carried out in Universities and *Scuole di magistero* was flanked by a great number of initiatives that fell halfway between propedeutic education and high-level popularization: the series of lectures and seminars on logic and foundations given by Cesare Burali-Forti, Alessandro Padoa, Giovanni Vacca and Michele Cipolla in Italy and abroad, which were widely distributed in lithographic or printed form, see (Luciano 2009 and 2010). In this context, the principal figure was Padoa, who gave series of seminars in Brussels (1898), Pavia (1899), Rome (1900), Geneva (1910) and Genoa (1932-36), which were imprinted by a single approach, outlined in the paper *Logica matematica e matematica elementare* (Padoa, 1902) not by chance defined as the "manifesto of Italian logicians".[12] Convinced that "only a few misunderstandings stand today between Mathematical Logic and the eminent place that it seems to merit among the most important manifestations of human thought",[13] his intention was to combine the teaching of ideographic formalism with that

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[9] *L'Enseignement Mathématique*, *27*, 1928, p. 153; *28*, 1929, p. 321; *29*, 1930, p. 168, p. 169; *30*, 1931, p. 151; *31*, 1932, p. 129.

[10] For example the Peano's student Cesarina Boccalatte presented a degree thesis entitled *La geometria basata sulle idee di punto e angolo retto*, where she illustrated an axiomatic system for elementary geometry based on the primitive ideas of point and right angle, taking her cue from Pasch, Pieri and Peano's works. The dissertation was published in *Atti della R. Accademia delle Scienze di Torino*, *64*, 1928-29, pp. 47-55.

[11] These relationships with Hilbert and Klein, (closer than those maintained by the Peano School), were also behind the different importance given to the presentation of Hilbert's *Grundlagen der Geometrie* in Castelnuovo and Enriques' courses in Rome and Bologna, and in the lectures given by Segre in Turin. Cf. (Giacardi, 2003; Gario, 2006, pp. 254-258; Giacardi, 2012; Luciano & Roero, 2012).

[12] G. Vailati to G. Vacca, 19 February 1902, c. 1r-v, *Peano-Vacca Archive*, Turin.

[13] (Padoa, 1902, p. 186): "soltanto alcuni equivoci contrastino oggi ancora alla Logica matematica il posto eminente che sembra spettarle fra le più importanti manifestazioni dell'umano pensiero".

of foundational themes. So, for example, in the first section of his Belgian lectures, Padoa introduced the meaning and syntax of the principal Peanian symbols and, in the second part, commented on a *selecta* of essays on the foundations of geometry by Peano and Pieri, see (Padoa, 1898, pp. 78-79).

In spite of the strong legacy of Peano's views, these series of lectures also displayed elements of originality: indeed, until 1910, research in and teaching of logic and the foundations proceeded on parallel tracks, and the best of the results were anticipated or re-examined in such didactic contexts. For example, in a series of *Conferenze su l'Algebra e la Geometria quali teorie deduttive*, given in Rome in 1900, Padoa developed the problem of absolute independence of a system of primitive propositions in an axiomatic theory of arithmetic and geometry. Just a few months later, the same topic would have been the subject of his celebrated lectures at the International Congresses of Philosophy and Mathematicians held in Paris.[14]

The warm reception by the public of these teaching initiatives confirmed the faith of the Peano School in the rapid establishment of their scientific style. However, the commitment of these scholars was not sufficient to clear the hurdle constituted by the hostility of many colleagues with regard to foundational research, which was denied the stamp of originality. This seems evident, if we take into consideration the various bureaucratic difficulties that Burali-Forti or Padoa had to face, in order to attain a habilitation (*libera docenza*) in Logic or Foundations of Mathematics, see (Borga, Fenaroli & Garibaldi 2010, pp. 283-285).

In addition to university and para-university teaching, the members of the School of Peano were engaged in putting together a set of editorial initiatives aimed at the communication of their studies. Noteworthy among these is the *Rivista di Matematica* (*RdM*) directed by Peano in the years 1891-1906. The editorial policy of the journal, conceived as essentially didactic, gradually placed the accent on the ideographical address to the point that the *Rivista* ended up as a journal of mathematical symbolism, with the majority of articles being compiled in Peanian language. It was in relation to this journal that the School of Peano began to devise the idea of inserting the output of scientific activity in logic and foundations into middle teaching and textbooks. On the other hand, it was precisely in the terrain of these questions that the *Rivista* offered to

[14] (Padoa, 1900, pp. 18-20). The interchange between scientific and didactic activity was continuous so much so that it is precisely at the origins also of the numerous works by Pieri and Padoa on Euclidean and neutral geometry that followed one another in the years 1898-1900. Cf. (Luciano, 2012, pp. 49-52).

consolidate a collaboration between university professors and teachers in upper and lower level secondary schools. The foundations of mathematics, in fact, had to be initially studied for purely scientific purposes, but in order to be transmitted into educational practice, they had to be investigated by teachers themselves, being the only figures who could get a feedback of the reactions of students to this new methodology of teaching. In the words of Pieri:

> Reconciling the needs of Schools with the ideality of the deductive method is such an undertaking that, if it ever comes to be, it will only be thanks to the work and fatigue of many.[15]

The epistemological frame of the rational teaching of geometry

The overall initiatives outlined so far led to the maturation of a composite framework of why, where and how far it was reasonable and useful to promote the rational teaching of geometry. More precisely, the reflections related to the transposition of foundations touched on 1) the criteria for choosing primitive concepts and propositions; 2) the varying degrees of concern to avoid implied admissions; 3) the advisability of mentioning the problems of independence, completeness and coherence of axiomatic systems; 4) the different ways of schematising language, between the opposite poles of everyday expressions and the symbolism of mathematical logic and 5) the importance given to the adherence to physical or psychological reality *vs.* the logical-deductive structure.

Faced by these problems, we can say - without trivializing excessively the positions of the single authors - that a general consensus was shared within the Peano School as far as many aspects were concerned,[16] first of all a propos the ascertainment that the first aim of teaching is to develop and promote "the practice of reasoning with exactness; that is, the sure knowledge of the logical relationship between principle and consequence: in short, the art or capacity of correctly arguing and concluding".[17] Equally shared by Bettazzi, Burali-Forti, Padoa, Peano, Pieri and Vailati was a lucid examination of the defects in the way geometry was presented in

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[15] (Pieri, 1899, p. 181): "conciliare i bisogni della Scuola con le idealità del metodo deduttivo è tale un'impresa, da non poter maturare, se mai, che per opera di molti e a fatica".

[16] Cf. for example (Bettazzi, 1891; Peano, 1894, pp. 51-55; Burali-Forti, 1898; Bettazzi, 1899; Pieri, 1898-99; Pieri, 1901; Vailati, 1907a; Vailati, 1907b; Pieri, 1908; Peano, 1910; Padoa, 1910).

[17] (Pieri, 1908, p. 447): "la pratica del ragionare con esattezza, vale a dire la cognizione sicura dei rapporti logici di principio e conseguenza: insomma l'arte o la facoltà di rettamente argomentare e concludere".

classrooms.[18] The solution to these problems, according to the School of Peano, was to be looked for in clever exploitation of foundational studies in oral teaching and textbooks. This practical role rather justified the interest in this type of scientific activity and provided proof of its usefulness. At the basis of hypothetical-deductive teaching of geometry there was thus a set of contents that the Peanians unanimously identified in these terms: the model was provided by the Peano system based on the primitive ideas of point and segment and on seventeen postulates, see (Peano, 1894). However, the knowledge of other systems – for example the *point-motion* and *point-distance* systems of Pieri – was recommended both in oral teaching and as a theoretical basis for the editing of textbooks. Finally, the greatest attention was given, by the members of the Peano's School, in describing how to apply the critical studies to the selection of logical and mathematical vocabularies for the textbooks:

> The logical introduction of our scientific treatise on Elementary Mathematics ought to be formed of the lists of the logical symbols and of the logical propositions that will be used therein. These two lists, provided respectively by the Logical Ideography and by Mathematical Logic, complete each other by turns: one is the vocabulary, the other the grammar […]. The list of the undefined mathematical symbols and of the unproven mathematical propositions compose the first chapter, the premise of the entire deductive theory.[19]

While the collaborators of the School of Peano were alike in sharing these scientific and educational opinions on the rational teaching of geometry, the divergences – or at least – differences in opinions about other issues were certainly not lacking.

First of all, there was debate concerning the *concrete* ways in which the foundational contents should be transmitted. Indeed, all of Peano's collaborators agreed on the fact that an axiomatic approach constituted the only way to bridge the gap between pre-university teaching - which *could* be intuitive - and university teaching - which *must* instead be formal

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[18] (Pieri, 1901, p. 377): "Les conséquences n'y découlent pas toujours des prémisses par la Logique pure: les arguments d'*évidence* (ou, comme on dit à présent, d'*intuition*) se dissimulent derrière les syllogismes les mieux ajustés, ou même sont invoqués ouvertement. Les notions primitives y sont plus nombreuses qu'il n'est besoin; etc.".

[19] (Padoa, 1902, pp. 194-195): "Quindi, l'*Introduzione logica* del nostro trattato scientifico di Matematica elementare dovrebbe esser formata dagli elenchi dei *simboli logici* e delle *proposizioni logiche* di cui sarà fatto uso. Questi due elenchi, forniti rispettivamente dall'*Ideografia logica* e dalla *Logica matematica*, si completano a vicenda: l'uno è il *vocabolario*, l'altro è la *grammatica* […]. Gli elenchi dei simboli matematici non definiti e delle proposizioni non dimostrate formano il Primo capitolo, la premessa dell'intera teoria deduttiva".

and hypothetical-deductive. However, the majority of these scholars usually limited themselves to observing that 'many parts of their works' could be used to advantage, even if it not be suitable for adoption in all details in teaching.

The use of symbols constituted the first object of discussion. According to Peano, in fact, one of the features of rational teaching was singled out with the purpose of presenting to the students the symbols and algorithms of logic that best rendered common procedures of proof rigorous, without debasing the ideography to mere tachigraphy. In this regard the translation of Euclid's *Elements* into ideographical language took on a special significance. With it the Peano School entered into the debates in Italy over the use of Euclid in secondary school teaching.

Strongly persuaded that it was above all "in the field of teaching that logic can demonstrate its brilliant simplicity"[20], as early as 1898 Peano maintained that it was possible to impart an axiomatic and symbolic teaching since middle schools, through the use of textbooks modeled on the *Formulaire*. Strangely enough, even though recommended as a guide to the compilation of textbooks, the *Formulaire* did not include the majority of results on the foundations of geometry, as opposed to the situation regarding the foundations of arithmetic.

At their turn, many of his collaborators frequently underlined their recourse to ideography, recognising in it a valuable tool "in virtue of the intellectual skills which [it is] capable of teaching and promoting, and also for certain of its evocative capacities, which often point the way to observations and investigations that would otherwise go unseen"[21].

This notwithstanding, scholars like Peano or Pieri were perfectly aware of the risks connected to a completely formal treatment of mathematics, and they warned the teachers against an indiscriminate introduction of symbols in classrooms.[22]

The most delicate element of the rational teaching was, however, that related to the different opinions about the epistemological status of geometrical objects.[23] Among the figures who participated most actively in

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[20] (Peano, 1919, p. 960): "è nel campo dell'insegnamento che la Logica manifesta al meglio la sua fulgida semplicità".

[21] (Pieri, 1898-99, p. 177): "in virtù degli abiti intellettuali, che i metodi e le dottrine di questa scienza si manifestan capaci di educare e promuovere, ed anche per certa loro facoltà suggestiva, che guida spesso ad osservazioni e ricerche non curate altrimenti".

[22] Cf. (G. Peano to E. Catalan, 25 January 1892, in Jongmans 1981, pp. 307-308 and Pieri 1903, p. 293).

[23] Also the on-going research in mathematics education deal with the problem of "the contrast between the abstract nature of mathematical objects, which are usually seen as having no perceptual existence, and their representations, which are tangible and upon

these discussions were Pieri who, in effect, is one of the mathematicians of the School of Peano that more warmly defended the *hypothetical-deductive* teaching of geometry.[24] His positions were the fruit of several works of a scientific nature – namely, his famous publications on the foundations of projective and Euclidean geometries – and were consolidated in relation to his teaching experience in the technical schools in Livorno and Pisa.

Pieri's starting point, taken from Edmond Goblot and Filippo Masci's papers,[25] was the natural evolution of *all* sciences, *in primis* of geometry, towards the structure of a hypothetical-deductive system, corresponding to the evolution of teaching in a rational-abstract direction.

More precisely, for Pieri, the propositions at the base of geometry correlate primitive concepts that are not associated with any meaning. As a consequence, the notions of point and space are not unequivocally determined *a priori*; rather, they are characterised by formal conditions, freely imposed, and are resolved in the set of all the interpretations that they satisfy, subject only to the constraint of consistency. Maintaining that the postulates of geometry are nothing other than Euclidean forms of the intuitive concept of space, means remaining tied to a rigid and subjective representation. Consequently, Pieri's ideal was that of presenting Euclidean geometry in classrooms as an abstract speculative system, a doctrine or a science "of all that is capable of being figured or represented".[26] In this type of teaching, geometrical objects must be introduced as pure creations of the spirit, the postulates as simple acts of our will, i.e. "artefacts of the mind and truths by definition"[27], arbitrary in so far as their ordering is determined - according to the pragmatist philosophical view - by the deductive end that the author sets for himself. As a consequence, concluded Pieri, a mind educated in general ideas and supported by a discreet faculty for abstraction "becomes capable of perceiving not only the abstract logical meaning, but also the nexus of the various propositions and their deductive outcomes"[28], appealing only to the properties that the axioms confer on primitive notions and referring only to the definitions of the various objects.

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which subjects' activities can develop in a concrete way" (Arzarello, Bosch, Gascón, & Sabena, 2008, p. 179).

[24] An other member of the Peanian team who partially agreed with Pieri on these questions was Padoa. Cf. (Padoa, 1902, pp. 194-200; Ferrera, Furinghetti, & Ortica, 2010, pp. 387-404).

[25] Cf. (Goblot, 1898 ; Masci, 1885).

[26] (Pieri, 1908, p. 447): "scienza di tutto ciò che è figurabile ovvero rappresentabile".

[27] (Pieri, 1901, p. 373): "scelte dello spirito o verità di definizione".

[28] (Pieri, 1908, p. 447): "diventa infatti capace di percepire sia il senso logico astratto sia il nesso delle varie proposizioni e le loro veci deduttive".

It is easy to understand that this conviction was a significant point of rift from other mathematicians such as Peano, or Cassina who, in accordance with the dictates of positivistic pedagogy of that time (from the *concrete* to the *abstract)*, held the physical interpretation of geometrical objects and the practical or experimental nature of the postulates to be indispensable.[29]

Pieri himself, however, shared Enriques's interest towards the psychological foundations of geometry,[30] and admitted that the selection of primitive ideas should not only be the most appropriate from the point of view of logic, but also from that of perception, and confessed that there was no reasonable way, in schools, to avoid presenting geometry *also* as a "mathematical physics of the extended bodies":[31] a guise that history, educational traditions and the results of cognitive research necessarily conferred on it. Attentive to the criticisms that emerged during the debates on rigour and intuition, Pieri could then recommend to teachers that a mote of dust, or the hole made by the point of a needle in a sheet of paper, could be usefully exploited to provide an image of a point. And again, systems of rigid sticks arranged in simple structures, or threads attached to a frame, could lend themselves to experimental verifications of axioms.

By the way, resorting to concrete representations of fundamental objects did not mean remaining silent regarding the hypothetical-deductive connotation of geometry, and in fact Pieri himself provided some suggestions for setting up school activities, describing how the first mathematics lecture in a secondary school might begin.[32]

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[29] Cf. (Peano, 1894, p. 54, p. 75; Cassina, 1961, pp. 197-201, p. 203).

[30] Pieri knew the scheme for a positive gnoseology worked out by Enriques in the Bolognese period, and significantly opened one of his papers referring to the conclusions of the search carried out by Enriques in the wake of Herbert Spencer, George Romanes, George H. Lewes, Wilhelm Wundt and Victor Henri, saying (Pieri, 1908, p. 345): "All in all, it would seem that the primordial constructive elements which are most evidently involved in the creation of tactile-muscular space, are not notions of line and plane, but rather of distance and thus of circles and spheres"; "Tutto sommato, parrebbe che gli elementi costruttivi primordiali, che più spicciamente intervengono a formare lo spazio tattile-muscolare, non siano le nozioni della retta e del piano, ma sì della distanza e quindi dei cerchi e delle sfere".

[31] (Pieri, 1901, p. 377): "fisica matematica dell'estensione".

[32] (Pieri, 1908, p. 447): "The first time the teacher ought to address his disciples thus: Allow me the truth of these primitive propositions; and I will lead you step by step by means of successive deductions, to having to recognize the truth of all the other geometric propositions. The axioms are like the seeds of all geometric truths: but the fruits of these do not grow from the seeds if they are not fertilized by reason. In this way can be grounded, for example, Geometry and Algebra; in brief, in what consists the deductive process, which informs all of pure mathematics"; "La prima volta il Maestro così parli ai

The textbooks of Ingrami and Pensa

An examination of some textbooks that attempted the transposition of results of logical-foundational research into teaching according to the suggestions of the School of Peano makes it possible to see how the appeals for collaboration between the world of research and that of schools, launched by the *RdM*, translated into concrete terms.[33]

The first example is constituted by the *Elementi di Geometria per le scuole secondarie superiori* (Bologna, Tip. Cenerelli, 1899) published by Giuseppe Ingrami, who did not attend himself Peano's lectures, but all the same decided to apply the rational approach, after having autonomously studied Peano and Veronese's works. In this textbook the rigorous deductive treatment began with three primitive concepts (point, segment and a congruence relation between two segments) and constituted the first application of the reflections of Pasch, Peano and Pieri in a teaching context. "Fruit of long meditations and a careful and intelligent study of the most recent works on the foundations of geometry",[34] is particularly the generation of geometrical entities. Taking the point and segment as primitive concepts, the line is defined as the set of infinite points formed by a segment and its extension; the plane and space are generated by projecting the point towards the perimeter of a triangle and a tetrahedron respectively, by means of radii whose origins are inside this triangle and this tetrahedron, etc.. Ingrami's textbook received an excellent review in the *RdM*, where Pieri concluded:

> While (didactically speaking) Elementary Geometry does not for now show signs of that degree of hypothetical and frankly deductive science that we so much admire in Arithmetic, nevertheless this work of Prof. Ingrami, in which some aims speculatively hoped for by a few, and relatively recently, have begun to be put into practice in concrete

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discepoli: Concedetemi la verità di codeste proposizioni primitive; ed io vi conduco man mano, per via di successive deduzioni, a dover riconoscere la verità di tutte le altre proposizioni geometriche. Gli assiomi son come il seme di tutte le verità geometriche: ma i germi di queste non si svolgon da quelli, se non sian fecondati dal raziocinio. A questo modo s'istituisce, p. es., la Geometria e l'Aritmetica; in ciò consiste sommariamente il processo deduttivo, che informa tutta quanta la Matematica pura".

[33] Many other Italian textbooks of Geometry used the output of the research on the foundations, but they were published by authors such as F. Enriques and U. Amaldi, outside the Peano's School or according different kinds of approach. Cf. (Natucci, 1967; Mammana, 2000, pp. 240-251; Giacardi, 2004, pp. CVIII-CXIX; Giacardi, 2006, pp. 592-594; Menghini & Cannizzaro, 2006).

[34] (Palatini, 1900, p. 85): "frutto di lunghe meditazioni e di uno studio accurato e intelligente dei più recenti lavori sui fondamenti della geometria".

and practical ways, is already an excellent sign, and a sure presage of new and increasingly greater steps forward in that direction.[35]

Another book that is emblematic of the 'modern wave of logic' in the teaching of geometry was Angelo Pensa's *Elementi di Geometria ad uso delle Scuole secondarie inferiori* (Torino, Petrini-Gallizio, 1912). This work, published on the recommendation of Burali-Forti by a former student of Peano in the courses of Infinitesimal calculus at the Turin University, assumes as its scientific underpinnings the paper by Pieri *La Geometria elementare istituita sulle nozioni di punto e sfera*. In place of the concept of sphere, however, is the more common and intuitive idea of distance, rather, the material one of a cord stretched between two points, or that given by the compass. The treatment is distinguished by its simplicity and at no point is it dry or boring, thanks to the fact that "the concern for the logic of the whole resides solely in the mind of the author"[36]. Physical interpretations are provided for all of the primitive ideas, renouncing those pseudo-definitions that mathematical logic had unmasked as vicious circles and inserting in the text an evocative set of illustrations. Moreover, making the suggestions of Pieri and Peano his own, Pensa courageously suppressed the majority of proofs, and substituted them with experimental justifications, obtained through superposition. All else aside, according to Pieri, Burali-Forti and Peano, this is the best form of transposition of Pieri's system, since its complete justification is not possible even in upper-level secondary school.

The debates on rational geometry

The concept of teaching maintained by the Peano School gave rise to a long series of debates. Generally, it might be said that the detractors thought that an excessive use of the axiomatic method, matched with the new logical language, would mask the natural paths of geometric reasoning, lead to mechanicalness in learning, in addition to obscuring the connections to the applications of geometry to the physical and natural sciences. In essence, it was feared that rational teaching of geometry would be suitable only for students who were exceptionally brilliant or exceptionally mediocre. In some sense, these criticisms are similar to those

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[35] (Pieri, 1899, p. 182): "se (didatticamente parlando) la Geometria elementare non accenna per ora a quel grado di scienza ipotetica e schiettamente deduttiva, che tanto ammiriamo nell'Aritmetica, non di meno quest'opera del prof. Ingrami, dove alcuni propositi vagheggiati speculativamente da pochi, e da non molto tempo, cominciano ad attuarsi in forma concreta e pratica, è già un ottimo pegno, e un affidamento sicuro di nuovi e sempre maggiori progressi su quella via".

[36] (Moglia, 1912, p. 195): "la preoccupazione logica dell'insieme risiede soltanto nella mente dell'autore".

aimed in the 1960s at textbooks compiled with a view to the so-called Modern Mathematics and imprinted with the concepts of the Bourbakis.

Furthermore, a serious problem was posed by the fact that the majority of textbooks of rational geometry required constant and complex epistemic-cognitive mediation, which could not be left up to the good intentions of the teachers, but necessitated adequate teacher training. Testimony gathered in the experiments regarding the use in class of this type of textbooks effectively show a wide range of different behaviours.

There were teachers who considered these works as a basis for preparing their lectures, but offering the relevant contents to the students only after adaptations and simplifications made by themselves without any specific technical competence.[37] Others, instead, adopted them in a traditional way, although they misrepresented, in some sense, Peano's advices. This was the case of Michele De Franchis who, in his textbook *Geometria elementare ad uso dei Licei e dei Ginnasi superiori* (Milano, Sandron, 1909) presented geometry as a rational science founded on experimental bases, following an exclusively axiomatic treatment, based on the primitive objects of point and segment, motion being defined as an affinity. He appreciated the foundational approach and inserted, for the first time in a geometry textbook, a short accounts of mathematical logic. However, De Franchis rejected the ideography, holding it to be an educational impediment and, by doing so, he risked rendering logic almost as an artificial and useless appendage to his manual, even more so because the symbols explained in the first chapter were then never used in the rest of the book.

Despite a decent editorial success, the textbooks of rational geometry aroused huge criticism - even in the ranks of the Peano School - from the

[37] Cf. (Nannei, 1904, p. 24): "I was once told, for example, that a colleague in an Italian technical school explained arithmetic and geometry using Peano's logical symbols. I don't dispute the method, although, for all that I am an admirer of the professor of the University of Turin and his work [...] I doubt that future merchants, salesmen, or even future students who would come out of that school, would profit much from it. But I was also told that the teacher did not use a textbook. And I feel a shudder of compassion when I think of how the students at home must have cursed that poor teacher, when they couldn't make head or tails of all those symbols!"; "Mi fu detto una volta, per es., che un collega di una scuola tecnica italiana spiegava l'aritmetica e la geometria usando i simboli logici del Peano. Io non discuto il metodo, benché, per quanto ammiratore del professore dell'università torinese e dell'opera sua [...] dubito che i futuri commercianti, futuri commessi, o anche futuri studenti d'Istituto che saranno usciti da quella scuola, abbiano potuto trarne molto profitto. Ma mi si disse anche che quel professore non adoprava libro di testo. E io sento un brivido di compassione a pensare alle benedizioni che avran mandato da casa gli alunni a quel povero insegnante, quando non riuscivano a raccapezzarsi fra tutti quei simboli!".

moment of the publication of the *Elementi di Geometria ad uso dei Licei e degli istituti tecnici (1° biennio)* by Giuseppe Veronese and Paolo Gazzaniga (Padova, Drucker, 1897).[38] Even more problematic was the reception of Pensa's textbook, which despite eleven successive reprints, was strongly opposed by the author's colleagues in Turin and by inspectors for the Ministry, who went so far as to prohibit its adoption, see (T. Boggio to G. Peano, 1 July 1912, in Roero 2010, p. 120). The climax of these disputes arrived in 1913 when Paolo Ricaldone stated, in a lecture to the Piedmont session of the Mathesis Association: "Some books are informed by the principles of pure logic. Even while admitting that mathematical logic has in certain cases a beneficial effect, the lecturer is contrary to adopting in middle schools books informed by them."[39]

This type of vicissitude, along with a series of quite heated debates due to a complex web of scientific reasoning and academic quarrels, led to a contraposition and eventually to a rupture between the Schools of Peano and Segre. The different ways of considering the research activity strongly conditioned the approach of the two Schools towards educational problems. What carried weight was not so much the component 'foundations', as much as the divergent conceptions of logic, see (Enriques 1906, pp. 69-78; 1921, p. 8; 1938, p. 188; 1942, pp. 65-67).

The methodological instruction supplying the Italian *curricula* well reflects the tensions within the Italian scientific world as regards the advisability of embodying in the mathematics at the pre-university level the 'philosophical' reflexions on the foundations, see (Vita 1986, pp. 16-17, pp. 22-27, p. 29, pp. 40-43, p. 49, pp. 64-65, pp. 70-72, p. 75, pp. 78-79, p. 80).

On the other hand, this kind of debate was not confined to within the Italian public. It's just the case to remember, in this respect, that in his report of the inquiry on rigour and intuition in secondary teaching commissioned by the ICMI, Henri Fehr stated that not a single country had adopted in a systematic way the entirely logical teaching method of

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[38] Cf. (Padoa, 1899, pp. 3-22; F. Klein to M. Pieri, 31 March 1897, M. Pieri to F. Klein, 9 Avril 1897, in Luciano & Roero, 2012, pp. 188-190).

[39] *Bollettino della Mathesis*, *5*, 1913, p. 49: "Alcuni libri sono informati ai principii della logica pura. Pur ammettendo che la logica matematica abbia in certi casi un'azione benefica, il conferenziere, prof. Ricaldone, è contrario ad adottare nelle scuole medie libri ad essa informati". As regards this debate, which involved Peano, Pieri, Burali-Forti, Sebastiano Catania, Alpinolo Natucci, Giacomo Bellacchi, Francesco Gerbaldi, Francesco Giudice, Enrico Nannei, Michele Cipolla, Giuseppe Marletta, Giuseppe Sforza, Guido Castelnuovo and Gaetano Scorza, cf. (Mammana & Tazzioli, 2001, pp. 223-232; Luciano, 2009).

Peano, David Hilbert, and George B. Halsted, which had instead been tried out only by isolated teachers, see (Fehr, 1911, pp. 462-464).

The hypothetical-deductive teaching after 1910

After 1910, the School of Peano made a serious self-criticism of the proposals regarding the rational teaching of mathematics. An overall re-assessment became indispensable following the outcome of debates on rigor and intuition, and the ascertainment of the difficulties faced by the teachers who had experienced the hypothetical-deductive approach. It was even further necessary in the light of some extra-mathematical circumstances, such as the clash in the Turin Faculty of Sciences between Peano and Segre in March of 1910, see (Luciano & Roero 2008, pp. 65-68, pp. 135-143). However, the pedagogical context too had changed in the meanwhile, and Masci's reflections were now recognized as outdated in some popular volumes on the teaching of mathematics, such as those by Carlo Leoni (1915, pp. 178-221) and Jacob William Albert Young (1924, pp. 236-253, pp. 345-349). In this period indeed, and still more in the Twenties and Thirties, we can observe an authentic flourishing of criticism concerning the use of logic-foundational studies in education, and a sharp defense of intuition, also on the part of scholars such as Vincenzo Cavallaro (1928, pp. 80-81) and Alpinolo Natucci (1928, p. 269), who had been fervent supporters of the Peanian trend.

The School of Peano continued to play an active part in teacher training, through the establishment of the *Conferenze Matematiche Torinesi* and, to confirm the cultural influence which this team exerted, we point out the ample space given to the logic and foundations of geometry, in accordance with Peano and his collaborators' views, in a series of books dedicated to prospective teachers, such as the *Questioni riguardanti le Matematiche Elementari* and the *Enciclopedia delle Matematiche Elementari*.[40]

On the other hand, with reference to publishing, textbooks of geometry which properly adopted the rational approach disappeared almost completely, with a few exceptions such as the manuals by Giuseppe Marletta (1911), Piero Benedetti and Carlo Rosati (1924), and the appendix to Francesco Severi's *Elementi di Geometria* (1926, pp. 175-184).

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[40] Cf. Enriques, F. (Ed.) (1924-1927). *Questioni riguardanti le matematiche elementari*. Bologna: Zanichelli, I, p. 46, pp. 87-89, p. 114, p. 212; Berzolari, L., Gigli, D., & Vivanti, G. (Eds.) (1929-1949). *Enciclopedia delle Matematiche Elementari*. Milano: Hoepli, 1_1, pp. 1-79, 2_1, pp. 3-118; 3_2, pp. 800-802, pp. 892-893, pp. 900-902, pp. 924-927, pp. 954-958, p. 968, pp. 977-1014.

Irrespective of the results (more or less successful) of the efforts of the School of Peano in the field of education, there remains the fact that the contribution of these scholars was by no means insignificant in the evolution of Italian teaching. In this sense, the question of the transposition of the foundations can provide an effective key for illustrating the practices of shared creation, socialisation and transmission of mathematical knowledge typical of the illustrious Schools of Peano and Segre.

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