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UNIVERSITÀ DEGLI STUDI DI TORINO

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**$L^p(\mathbb{R}^n)$ -CONTINUITY OF TRANSLATION INVARIANT
ANISOTROPIC PSEUDODIFFERENTIAL OPERATORS:
A NECESSARY CONDITION**

S. CORIASCO AND M. MURDOCCA

ABSTRACT. We consider certain anisotropic translation invariant pseudodifferential operators, belonging to a class denoted by $\text{op}(\mathcal{A}_{\psi}^{\lambda})$, where λ and $\psi = (\psi_1, \dots, \psi_n)$ are the “order” and “weight” functions, defined on \mathbb{R}^n , for the corresponding space of symbols. We prove that the boundedness of a suitable function $F_p: \mathbb{R}^n \rightarrow [0, +\infty)$, $1 < p < \infty$, associated with λ and ψ , is necessary to let every element of $\text{op}(\mathcal{A}_{\psi}^{\lambda})$ be a $L^p(\mathbb{R}^n)$ -multiplier. Additionally, we show that some results known in the literature can be recovered as special cases of our necessary condition.

1. INTRODUCTION

A translation invariant pseudodifferential operator $\sigma(D)$, or multiplier, is defined by means of a symbol σ which depends only on the covariable $\xi \in \mathbb{R}^n$, that is, as $\widehat{\sigma(D)u} = \sigma \widehat{u}$, where $\widehat{\cdot}$ denotes the Fourier transform. Of course, for such a definition to make sense, σ has to fulfill some suitable additional properties, depending on the domain of definition and the desired properties of $\sigma(D)$. For instance, even just $\sigma \in L^\infty(\mathbb{R}^n)$ is enough to ensure that

$$(1.1) \quad \sigma(D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n): u \mapsto [\sigma(D)u](x) = \frac{1}{(2\pi)^n} \int e^{i\langle \xi, x \rangle} \sigma(\xi) \widehat{u}(\xi) d\xi$$

is a linear continuous map. When the domain of $\sigma(D)$ is a different functional (or distributional) space, more regularity of the symbol is usually needed to achieve continuity. Common choices are the space of temperate distributions $\mathcal{S}'(\mathbb{R}^n)$, for which σ must be smooth and of at most polynomial growth, together with all its derivatives, and $L^p(\mathbb{R}^n)$. In the latter case, the situation is more involved, if one wants to obtain, for any symbol σ belonging to a fixed class, a linear continuous map $\sigma(D): L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, that is, a $L^p(\mathbb{R}^n)$ -multiplier.

The $L^p(\mathbb{R}^n)$ -continuity for pseudodifferential operators is a classical and extensively studied problem, for multipliers as well as for general symbol classes: we mention just a few issues of the vast literature on the subject, which are more strictly related to the situation on which we will be focused. For instance, consider the (global) classes $S_{\rho, \delta}^m(\mathbb{R}^n)$ introduced by L. Hörmander, see [15, 16, 17]:

DEFINITION. Let $m, \rho, \delta \in \mathbb{R}$, and assume $0 \leq \delta \leq \rho \leq 1$. Denote by $S_{\rho, \delta}^m(\mathbb{R}^n)$ the class of functions $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for any $\alpha, \beta \in \mathbb{Z}_+^n$, there exists a constant $C_{\alpha\beta}$ such

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that

$$(1.2) \quad |D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Denote by $A = a(x, D)$ the operator associated with $a(x, \xi)$, given by

$$(1.3) \quad (Au)(x) = [a(x, D)u](x) = \frac{1}{(2\pi)^n} \int e^{i(\xi, x)} a(x, \xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

For the case $p = 2$, we recall the fundamental result proved by A. Calderon and R. Vaillancourt [5]:

THEOREM. *Let $a(x, \xi)$ be a function defined on $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha\beta}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for $\alpha_k, \beta_l = 0, 1, 2, 3, k, l = 1, \dots, n$. Then, the operator (1.3) can be extended to a bounded operator $A: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$.

A version of the Calderon-Vaillancourt Theorem which holds for operators with symbols in the class $S_{\rho, \rho}^0(\mathbb{R}^n)$, $\rho \in [0, 1)$, can be found, e.g., in the book by M. Taylor [34].

The case $1 < p < \infty$, $p \neq 2$, has been investigated by many authors in different situations, see, e.g., R. Beals [3], C. Fefferman [9], L. Hörmander [14], D. S. Kurtz and R. L. Wheeden [20], J. Marcinkiewicz [23], see also [32, 33], G. Mihlin [24, 25], A. Nagel and E. Stein [29]. For the class $S_{1,0}^0(\mathbb{R}^n)$ the following result holds (for a proof see, e.g., the book by M. W. Wong [36] and the references quoted therein):

THEOREM. *Let $\sigma \in S_{1,0}^0(\mathbb{R}^n)$. Then, for $1 < p < \infty$, $\sigma(D)$ can be extended to a bounded operator from $L^p(\mathbb{R}^n)$ to itself.*

A main role in the proof of the previous theorem is played by the following Mihlin-Hörmander Theorem, see the references mentioned above:

THEOREM. *Let $t \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\})$, $k > n/2$. Assume that there exists a positive constant B such that*

$$|(D^\alpha t)(\xi)| \leq B|\xi|^{-|\alpha|}, \quad \xi \neq 0,$$

for any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq k$. Then, for $1 < p < \infty$, there exists a positive constant C , depending only on p and n , such that

$$\|Tu\|_{L^p(\mathbb{R}^n)} \leq CB\|u\|_{L^p(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where T is the pseudo differential operator (1.3) with symbol t .

The investigation of multiplier theorems of Mihlin type is a field of active research: such results can be proved in settings different from the one recalled above, see, e.g., H. Amann [1], M. Girardi and L. Weis [10], T. Hytönen [18] and the references quoted therein.

The definition of pseudodifferential operator has been extended to many other (also non-smooth) symbol classes. For instance, in R. Beals [2], a symbol $a(x, \xi)$ belongs to the class $S_{\Phi, \varphi}^\lambda(\mathbb{R}^n)$, associated with the “order” λ and the “weight functions” Φ, φ , if it satisfies the estimates

$$(1.4) \quad |D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_{\alpha\beta} e^{\lambda(x, \xi)} \Phi(x, \xi)^{-|\alpha|} \varphi(x, \xi)^{-|\beta|}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

with λ, Φ, φ fulfilling suitable hypotheses. Clearly, when $\varphi(x, \xi) = (1 + |\xi|)^{-\delta}$, $\Phi(x, \xi) = (1 + |\xi|)^\rho$ and $\lambda(x, \xi) = m \ln(1 + |\xi|)$, the class $S_{\Phi, \varphi}^\lambda(\mathbb{R}^n)$ coincides with the class $S_{\rho, \delta}^m(\mathbb{R}^n)$ recalled above. Also with the symbols in $S_{\Phi, \varphi}^\lambda(\mathbb{R}^n)$ it is possible to associate the corresponding pseudodifferential operators (1.3), and similar results for the continuity on $L^2(\mathbb{R}^n)$ can be obtained. L. Rodino [30] studied a class of pseudodifferential operators defined by means of amplitudes $c(x, y, \xi)$ rather than symbols, satisfying weighted estimates similar to (1.4), and investigated corresponding conditions for their $L^2(\mathbb{R}^n)$ -boundedness. L. Hörmander [16] has considered an even further generalization of the pseudodifferential calculus on \mathbb{R}^n , see also [17].

To get closer to the results proved in the present paper, we recall the definition of the multiplier class S_ψ considered by R. Beals in [3]:

DEFINITION. Let ψ be a non-decreasing, positive function on \mathbb{R}^n . S_ψ denotes the space of symbols $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that, for $\alpha \in \mathbb{Z}_+^n$, there exists a positive constant C_α , depending only on α , such that

$$|D^\alpha \sigma(\xi)| \leq C_\alpha \psi(|\xi|)^{-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

In that same paper, the following theorem of $L^p(\mathbb{R}^n)$ -boundedness for operators with symbols in S_ψ was proved:

THEOREM. Let $1 < p < \infty$, $p \neq 2$. A necessary and sufficient condition to have that any pseudodifferential operator with symbol $\sigma \in S_\psi$ is a $L^p(\mathbb{R}^n)$ -multiplier is that there exists $\delta > 0$ such that

$$t^{-1} \psi(t) \geq \delta, \quad t > 0.$$

An immediate consequence of the previous theorem is the following

COROLLARY. If $\psi(t) = (1 + t)^\rho$, then any symbol in S_ψ gives rise to a $L^p(\mathbb{R}^n)$ -multiplier if and only if $\rho \geq 1$.

The corollary was well-known: in fact, necessity followed by counterexamples by I.I. Hirschmann [13] and S. Wainger [35], while sufficiency was a consequence of the Marcinkiewicz and Mihlin-Hörmander multiplier theorems. The same observation concerning the sufficiency of the condition can be done for the theorem, where the new aspect was the necessity.

We will deal with an anisotropic generalization of the symbol class S_ψ , denoted by \mathcal{M}_ψ^λ . We remark that “anisotropic structures” often arise in various contexts. In particular, they are a typical feature in the analysis of Carnot-Carathéodory metrics and Carnot groups, as well as of the study of natural operators and functional spaces arising in sub-Riemannian geometry, see, e.g., Der-Chen Chang, I.G. Markina [6], M. Gromov [11], G.L. Leonardi, R. Monti [21], J. Mitchell [26], R. Monti [27], R. Monti, D. Morbidelli [28], and the corresponding reference lists. In particular, classes of anisotropic pseudodifferential operators, Sobolev spaces and Besov spaces have been investigated by many authors, see, e.g., A. Bényi, M. Bownik [4], G. Grubb [12], N. Jacob [19], H.-G. Leopold [22], A. Nagel, E.M. Stein [29], M. Yamazaki [37, 38], and the references quoted therein.

The elements of the symbol class \mathcal{M}_ψ^λ which we consider in this paper, with “order” λ and vector-valued “weight” ψ , are characterized as follows:

1.1. DEFINITION. Let $\psi = (\psi_1, \dots, \psi_n)$, $\psi_j \in \mathcal{C}(\mathbb{R}^n)$ strictly positive, $j = 1, \dots, n$, and $\lambda \in \mathcal{C}(\mathbb{R}^n)$, strictly positive and bounded. We denote by \mathcal{M}_ψ^λ the space of functions $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that, for any $\alpha \in \mathbb{Z}_+^n$ there exists a non-negative constant C_α , satisfying

$$(1.5) \quad |D^\alpha \sigma(\xi)| \leq C_\alpha \lambda(\xi) \psi(\xi)^{-\alpha}, \quad \xi \in \mathbb{R}^n,$$

where $\psi(\xi)^{-\alpha} = \prod_{i=1}^n \psi_i(\xi)^{-\alpha_i}$. We call *symbols* all the functions $\sigma \in \mathcal{M}_\psi^\lambda$.

Clearly, for $\lambda(\xi) = (1 + |\xi|)^m$ and $\psi_j(\xi) = (1 + |\xi|)^\rho$, $j = 1, \dots, n$, $\mathcal{M}_\psi^\lambda = S_{\rho,0}^m(\mathbb{R}^n)$, while for $\lambda(\xi) \equiv 1$ and $\psi_1(\xi) = \dots = \psi_n(\xi) = \psi(|\xi|)$, $\mathcal{M}_\psi^\lambda = S_\psi$. With the standard notation $\text{op}(\mathcal{M}_\psi^\lambda)$ we denote the space of pseudodifferential operators $\sigma(D)$ with symbol $\sigma \in \mathcal{M}_\psi^\lambda$. As usual, we can introduce a family of seminorms p_N on \mathcal{M}_ψ^λ , $N = 1, 2, 3, \dots$, by considering the best constants C_α appearing in (1.5), namely

$$p_N(\sigma) = \sum_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^n} \left\{ \lambda(\xi)^{-1} \psi(\xi)^\alpha |D^\alpha \sigma(\xi)| \right\}, \quad \sigma \in \mathcal{M}_\psi^\lambda.$$

It is immediate to verify that the family of seminorms p_N , $N = 1, 2, \dots$, makes \mathcal{M}_ψ^λ a Fréchet space and that the following results hold:

- 1.2. PROPOSITION. (1) \mathcal{M}_ψ^λ is a closed subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$.
(2) Let $\sigma, \tau \in \mathcal{M}_\psi^\lambda$. Then, $\sigma\tau \in \mathcal{M}_\psi^\lambda$.
(3) Any pseudodifferential operator $\sigma(D) \in \text{op}(\mathcal{M}_\psi^\lambda)$ is a linear continuous map

$$\sigma(D): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n).$$

The present paper is devoted to proving a condition that must be satisfied in order to let any element of $\text{op}(\mathcal{M}_\psi^\lambda)$ be a $L^p(\mathbb{R}^n)$ -multiplier, in the spirit of the paper by R. Beals [2]. Sufficient conditions, in the cases where \mathcal{M}_ψ^λ does not coincide with symbol classes already known in the literature, will be treated in a subsequent paper, where we plan to adapt some of the techniques used by L. Rodino in the paper [30] quoted above, whose reading has partly motivated us to study these topics.

We start by fixing some hypotheses on the “shape” of the “balls” associated with “metric” defined by the functions ψ_i , $i = 1, \dots, n$. We focus on the case $p \neq 2$, since the boundedness of the function λ implies the $L^2(\mathbb{R}^n)$ -continuity of any operator in $\text{op}(\mathcal{M}_\psi^\lambda)$.

1.3. ASSUMPTIONS. Let $1 < p < \infty$, $p \neq 2$. With any $\xi \in \mathbb{R}^n$ associate the set

$$S(\xi) = \{\eta \in \mathbb{R}^n : \psi_j(\eta) \leq \psi_j(\xi), j = 1, \dots, n\}.$$

We assume that there exist suitable positive constants c, C , independent of ξ , such that, for any $\xi \in \mathbb{R}^n$, $|\xi| \geq C$, there exists a n -dimensional interval

$$(1.6) \quad I(\xi) = \{\eta \in \mathbb{R}^n : |\eta_j| \leq l_j(\xi), j = 1, \dots, n\},$$

with $l_j(\xi) \geq c\psi_j(\xi)$, $j = 1, \dots, n$, such that

$$(1.7) \quad I(\xi) \subseteq S(\xi) \quad \text{and} \quad \mu(S(\xi)) \leq C\mu(I(\xi)),$$

where μ is the Lebesgue measure on \mathbb{R}^n .

Let us define the function $F_p(\xi)$ as

$$(1.8) \quad F_p(\xi) = \left(\inf_{\eta \in S(\xi)} \lambda(\eta) \right) \left(\mu(S(\xi)) \prod_{j=1}^n \psi_j(\xi)^{-1} \right)^{\left| \frac{1}{p} - \frac{1}{2} \right|}.$$

Note that, under the Assumptions 1.3, $F_p(\xi)$ is well defined, and assumes non-negative real values for $|\xi| \geq C$. We can now state our main results:

1.4. THEOREM. *Let Assumptions 1.3 be satisfied and let the function $F_p(\xi)$ be unbounded. Then, the map*

$$\begin{aligned} \text{op} : \mathcal{M}_\psi^\lambda &\longrightarrow \mathcal{L}(L^p(\mathbb{R}^n)) \\ \sigma &\longmapsto \sigma(D) \end{aligned}$$

is unbounded. Namely, there exists a sequence of symbols $\{\sigma_k\} \subset \mathcal{M}_\psi^\lambda$ fulfilling (1.5) with constants C_α independent of k , such that

$$\lim_{k \rightarrow \infty} \|\sigma_k(D)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = \infty.$$

Theorem 1.4 is the main step of the argument showing our necessary condition for the $L^p(\mathbb{R}^n)$ -continuity of any operator with symbol in the class \mathcal{M}_ψ^λ , namely, the boundedness of the function $F_p(\xi)$. The proof of the next theorem is easily obtained by contradiction, via a standard application of the Closed Graph Theorem, see Section 2 below:

1.5. THEOREM. *Let Assumptions 1.3 be satisfied and let the function $F_p(\xi)$ be unbounded. Then, there exists a symbol $\sigma \in \mathcal{M}_\psi^\lambda$ such that*

$$\|\sigma(D)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = \infty.$$

Equivalently, under Assumptions 1.3,

$$\text{op}(\mathcal{M}_\psi^\lambda) \subset \mathcal{L}(L^p(\mathbb{R}^n)) \Rightarrow F_p \text{ is bounded.}$$

In the next Section 2 we give the detailed proofs of Theorems 1.4 and 1.5. Some corollaries and remarks are then discussed in Section 3.

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2. PROOF OF THEOREMS 1.4 AND 1.5

For an open subset Ω of \mathbb{R}^n , we denote, as usual, by $\mathcal{C}_0^\infty(\Omega)$ the subspace of all the smooth functions defined on Ω whose support is compact.

2.1. LEMMA. Let $f \in \mathcal{S}(\mathbb{R})$ with $\widehat{f} \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfy $1 = f(0) \geq f(t) \geq 0$ for all $t \in \mathbb{R}$. Then, f can be chosen so that

$$(2.1) \quad \sum_{k \in \mathbb{Z} \setminus \{0\}} f(t-k) \leq \frac{1}{3} \quad \text{if} \quad |t| \leq \frac{1}{2}.$$

Proof. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\chi(\xi) \geq 0$ for any $\xi \in \mathbb{R}$. Defining

$$\lambda(\eta) = \int_{\mathbb{R}} \chi(\xi + \eta) \chi(\xi) d\xi,$$

it is clear that $\lambda \in \mathcal{C}_0^\infty(\mathbb{R})$. Moreover, λ is positive-definite (cfr. Appendix B). Let us now set

$$(2.2) \quad g(\eta) = 2\pi \frac{\lambda(\eta)}{C},$$

where $C = \int_{\mathbb{R}} \lambda(\eta) d\eta$. We then define

$$(2.3) \quad \tilde{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\langle \eta, t \rangle} g(\eta) d\eta = \frac{1}{2\pi} \check{g}(t) = \frac{1}{C} [\widehat{\chi}(t)]^2.$$

By (2.2) and (2.3), and from Theorem B.2, we obtain $\tilde{f} \in \mathcal{S}(\mathbb{R})$, $1 = \tilde{f}(0) \geq \tilde{f}(t) \geq 0$ for any $t \in \mathbb{R}$. Let h be a positive real scalar. We set

$$f(t) = \tilde{f}(ht) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\langle \eta, ht \rangle} g(\eta) d\eta = \frac{1}{2\pi} \check{g}(ht),$$

It follows, obviously, $1 = f(0) \geq f(t) \geq 0$. Since $g \in \mathcal{S}(\mathbb{R})$, $\check{g}(\eta) \leq 2\pi M(1 + \eta^2)^{-1}$ for all $\eta \in \mathbb{R}$, then we find, for $|t| \leq 1/2$,

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} f(t-k) &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\pi} \check{g}(h(t-k)) \leq M \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{1+h^2(t-k)^2} \leq 2M \sum_{k \geq 1} \frac{1}{1+h^2(k-\frac{1}{2})^2} \\ &< \frac{2M}{h^2} \sum_{k \geq 1} \frac{1}{(k-\frac{1}{2})^2} < \frac{2M}{h^2} \left(4 + \sum_{k \geq 1} \frac{1}{k^2} \right) = \frac{A^2}{3h^2}. \end{aligned}$$

The property (2.1) is then fulfilled by choosing $h \geq A$. The proof is complete. \square

Proof of Theorem 1.4. Let $f \in \mathcal{S}(\mathbb{R})$ be chosen as in Lemma 2.1 and pick $r > 0$ such that

$$(2.4) \quad |\tau| \geq \pi r \Rightarrow \widehat{f}(\tau) = 0.$$

Let us also set $L = 4\pi r$ and $f_0(x) = f(x_1) \cdots f(x_n)$. Moreover, by a duality argument, it is not restrictive to assume $1 < p < 2$.

With any given n -tuple of positive integers $N = (N_1, \dots, N_n)$ associate the functions

$$(2.5) \quad \begin{aligned} g_N(x) &= \sum_{\substack{\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_1, \dots, |\gamma_n| \leq N_n}} e^{i\langle L\gamma, x \rangle} f_0(x), \\ h_N(x) &= \sum_{\substack{\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_1, \dots, |\gamma_n| \leq N_n}} e^{i\langle L\gamma, x \rangle} f_0(x - \gamma). \end{aligned}$$

For the functions g_N, h_N defined in (2.5) and $1 < p < 2$ the following estimates hold:

$$(2.6) \quad \begin{aligned} \|g_N\|_{L^p(\mathbb{R}^n)} &\leq C \left(\prod_{j=1}^n N_j \right)^{1-\frac{1}{p}}, \\ \|h_N\|_{L^p(\mathbb{R}^n)} &\geq c \left(\prod_{j=1}^n N_j \right)^{\frac{1}{p}}, \end{aligned}$$

with c, C positive constants independent of N (see, e.g., R. Beals [3], M. Dyachenko [7], M. Dyachenko and S. Tikhonov [8], A. Zygmund [39] and Appendix A).

To prove the unboundedness of $\text{op} : \mathcal{M}_\psi^\lambda \rightarrow \mathcal{L}(L^p(\mathbb{R}^n))$, we will build a sequence $\{\sigma_k\}$, taking values in \mathcal{M}_ψ^λ , such that for any $\alpha \in \mathbb{Z}_+^n$ there exists a constant C_α , independent of k , such that for any $\xi \in \mathbb{R}^n$ we have

$$(2.7) \quad |D^\alpha \sigma_k(\xi)| \leq C_\alpha \lambda(\xi) \psi(\xi)^{-\alpha}.$$

That is, $\{\sigma_k\}$ is a bounded subset of \mathcal{M}_ψ^λ . Moreover, the sequence $\{\sigma_k\}$ is built in such a way that

$$\|\sigma_k(D)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \rightarrow \infty,$$

for $k \rightarrow \infty$.

Let $\{\xi^{(k)}\}$ be a sequence in \mathbb{R}^n satisfying $F_p(\xi^{(k)}) \rightarrow \infty$. The unboundedness hypothesis of F_p and (1.6) imply the existence of a sequence of n -dimensional intervals

$$(2.8) \quad I(\xi^{(k)}) = \left\{ \eta \in \mathbb{R}^n : |\eta_j| \leq l_j(\xi^{(k)}), j = 1, \dots, n \right\},$$

such that

$$(2.9) \quad \lambda_k \left(\prod_{j=1}^n l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1} \right)^{\left(\frac{1}{p} - \frac{1}{2}\right)} \rightarrow \infty,$$

with

$$(2.10) \quad \lambda_k = \inf_{\eta \in S(\xi^{(k)})} \lambda(\eta),$$

and satisfying

$$(2.11) \quad \psi_j(\xi) \leq \psi_j(\xi^{(k)}), \quad j = 1, \dots, n,$$

for any $\xi \in I(\xi^{(k)})$, and

$$(2.12) \quad l_j(\xi^{(k)}) \geq c_1 \psi_j(\xi^{(k)}), \quad j = 1, \dots, n,$$

where c_1 is a suitable positive constant. Let $N_{k,j}$ be the largest positive integer such that

$$(2.13) \quad N_{k,j}(2N_{k,j} + 1) \leq \frac{3}{c_1} l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1}.$$

By (2.12) it follows that $c_1^{-1} l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1} \geq 1$. Then, (2.13) implies $N_{k,j} \geq 1$. Moreover, since

$$(2.14) \quad N_{k,j}(2N_{k,j} + 1) \leq \frac{3}{c_1} l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1} < (N_{k,j} + 1)(2N_{k,j} + 3),$$

dividing (2.14) by $N_{k,j}^2$ we get

$$\frac{3}{c_1 N_{k,j}^2} l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1} < 2 + \frac{5}{N_{k,j}} + \frac{3}{N_{k,j}^2},$$

that is, there exists a constant C such that

$$l_j(\xi^{(k)}) \psi_j(\xi^{(k)})^{-1} \leq C N_{k,j}^2.$$

We then find, in view of (2.9),

$$(2.15) \quad \lambda_k \left(\prod_{j=1}^n N_{k,j} \right)^{\left(\frac{2}{p} - 1\right)} \rightarrow \infty.$$

Let us now choose a cut-off function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(\eta) = \begin{cases} 1 & |\eta_j| \leq L/4, \quad \text{for all } j = 1, \dots, n, \\ 0 & |\eta_j| \geq L/2, \quad \text{for some } j = 1, \dots, n, \end{cases}$$

where $L = 4\pi r > 0$ is the constant present in the expressions (2.5). We start by defining

$$\Phi_k(\eta) = \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} e^{-i\langle \gamma, \eta \rangle} \varphi(\eta - L\gamma),$$

and observing that

$$(2.16) \quad \text{supp } \Phi_k \subset \left\{ \eta \in \mathbb{R}^n : |\eta_j| < L \left(N_{k,j} + \frac{1}{2} \right), j = 1, \dots, n \right\},$$

$$(2.17) \quad \begin{aligned} |D_\eta^\alpha \Phi_k(\eta)| &\leq \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} |D_\eta^\alpha (e^{-i\langle \gamma, \eta \rangle} \varphi(\eta - L\gamma))| \\ &= \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} \sum_{\beta \leq \alpha} |\partial_\eta^\beta e^{-i\langle \gamma, \eta \rangle}| |\partial_\eta^{\alpha-\beta} \varphi(\eta - L\gamma)| \\ &\leq \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} |\gamma|^\alpha \sum_{\beta \leq \alpha} |\partial_\eta^{\alpha-\beta} \varphi(\eta - L\gamma)| \\ &\leq C'_\alpha N_k^\alpha, \end{aligned}$$

with constants C'_α independent of N_k . We then introduce the dilations

$$(2.18) \quad \eta = \tau_k(\xi),$$

where

$$(2.19) \quad \eta_j = \tau_{k,j} \xi_j = \frac{3}{2} L (c_1 \psi_j(\xi^{(k)}) N_{k,j})^{-1} \xi_j, \quad j = 1, \dots, n,$$

and set

$$(2.20) \quad \sigma_k(\xi) = \lambda_k \Phi_k(\tau_k(\xi)).$$

In view of (2.17), (2.18) and (2.20), we have

$$(2.21) \quad |D_\xi^\alpha \sigma_k(\xi)| \leq C''_\alpha \lambda_k \psi(\xi^{(k)})^{-\alpha},$$

and, observing that

$$\text{supp } \sigma_k = \text{supp}(\lambda_k (\Phi_k \circ \tau_k)) = \text{supp}(\Phi_k \circ \tau_k),$$

taking into account (2.16) and (2.19), we find

$$\frac{3}{2} L (c_1 \psi_j(\xi^{(k)}) N_{k,j})^{-1} \xi_j < L \left(N_{k,j} + \frac{1}{2} \right),$$

which implies

$$3(c_1 \psi_j(\xi^{(k)}))^{-1} \xi_j < N_{k,j} (2N_{k,j} + 1).$$

Then, by (2.13), we have $\xi_j < l_j(\xi^{(k)})$, hence

$$\text{supp } \sigma_k \subset I(\xi^{(k)}).$$

In view of (2.8), (2.13), (2.16) and (2.18), the estimates (2.7) then follow by (2.10) and (2.11): we have proved that $\{\sigma_k\} \subset \mathcal{M}_\psi^\lambda$, and that is a bounded set.

We will now show that $\|\sigma_k(D)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \rightarrow \infty$, building a sequence $\{u_k\}$ in $\mathcal{S}(\mathbb{R}^n)$, $u_k \neq 0$, such that

$$\frac{\|\sigma_k(D)u_k\|_{L^p(\mathbb{R}^n)}}{\|u_k\|_{L^p(\mathbb{R}^n)}} \rightarrow \infty.$$

Recalling the definition of g_{N_k} in (2.5), we define u_k as

$$\widehat{u}_k(\xi) = \widehat{g}_{N_k}(\tau_k(\xi)).$$

Now, taking into account (2.4), the definition and properties of f_0 and the fact that $\varphi(\eta) \equiv 1$ for any $\eta \in \text{supp } f_0$, it is immediate to check that

$$\begin{aligned} & \Phi_k(\eta) \widehat{g}_{N_k}(\eta) \\ &= \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} e^{-i\langle \gamma, \eta \rangle} \varphi(\eta - L\gamma) \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} (e^{i\langle L\gamma, x \rangle} f_0(x))^\wedge(\eta) \\ &= \sum_{\substack{\gamma=(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \\ |\gamma_1| \leq N_{k,1}, \dots, |\gamma_n| \leq N_{k,n}}} e^{-i\langle \gamma, \eta \rangle} \widehat{f}_0(\eta - L\gamma) \\ &= \widehat{h}_{N_k}(\eta), \end{aligned}$$

so that we find

$$\sigma_k(\xi) \widehat{u}_k(\xi) = \lambda_k \Phi_k(\tau_k(\xi)) \widehat{g}_{N_k}(\tau_k(\xi)) = \lambda_k \widehat{h}_{N_k}(\tau_k(\xi)).$$

Taking the inverse Fourier transformations and applying the (2.6), we have, for a suitable constant $c > 0$,

$$\begin{aligned} \|\sigma_k(D)\|_{\mathcal{L}(L^p(\mathbb{R}^n))} &= \sup_{v \in L^p(\mathbb{R}^n)} \frac{\|\sigma_k(D)v\|_{L^p(\mathbb{R}^n)}}{\|v\|_{L^p(\mathbb{R}^n)}} \geq \frac{\|\sigma_k(D)u_k\|_{L^p(\mathbb{R}^n)}}{\|u_k\|_{L^p(\mathbb{R}^n)}} \\ &= \lambda_k \frac{\|h_{N_k}\|_{L^p(\mathbb{R}^n)}}{\|g_{N_k}\|_{L^p(\mathbb{R}^n)}} \geq c \lambda_k \left(\prod_{j=1}^n N_{k,j} \right)^{\left(\frac{2}{p}-1\right)}, \end{aligned}$$

so that (2.15) gives the claim. The proof is complete. \square

Proof of Theorem 1.5. If $\text{op}(\mathcal{M}_\psi^\lambda) \in \mathcal{L}(L^p(\mathbb{R}^n))$, it is then easy to check that $\text{op}: \mathcal{M}_\psi^\lambda \rightarrow \mathcal{L}(L^p(\mathbb{R}^n))$ is a linear closed map. Since \mathcal{M}_ψ^λ is a Fréchet space and $\mathcal{L}(L^p(\mathbb{R}^n))$ is a Banach space, the Closed Graph Theorem can be applied, and implies that $\text{op}: \mathcal{M}_\psi^\lambda \rightarrow \mathcal{L}(L^p(\mathbb{R}^n))$ is continuous, that is, bounded. If $F_p(\xi)$ is unbounded, this is a contradiction, by Theorem 1.4. \square

3. COROLLARIES AND REMARKS

The proof of Theorem 1.5 suggests some extensions of the result. For instance, it is clear that, in the hypotheses, we could assume

$$(3.1) \quad \psi_j(\eta) \leq C\psi_j(\xi), \quad j = 1, \dots, n,$$

for any $\eta \in I(\xi)$ and a suitable constant C , independent of ξ , instead of $I(\xi) \subseteq S(\xi)$. Moreover, it is enough to assume that the conditions hold only for a sequence $\{\xi^{(k)}\}$ in \mathbb{R}^n such that $F_p(\xi^{(k)}) \rightarrow \infty$. Further, we could omit the assumption $\mu(S(\xi^{(k)})) \leq C\mu(I(\xi^{(k)}))$ and substitute $F_p(\xi^{(k)}) \rightarrow \infty$ with the following condition:

$$(3.2) \quad \delta_k = \left(\inf_{\eta \in I(\xi^{(k)})} \lambda(\eta) \right) \left(\mu(I(\xi^{(k)})) \prod_{j=1}^n \psi_j(\xi^{(k)})^{-1} \right)^{\left| \frac{1}{p} - \frac{1}{2} \right|} \rightarrow \infty.$$

Let us now assume

$$(3.3) \quad \psi_1(\xi) = \cdots = \psi_n(\xi) = \Psi(|\xi|) \quad \text{e} \quad \lambda(\xi) = \Lambda(|\xi|),$$

with Ψ, Λ continuous and positive functions defined on $[0, \infty)$, Ψ non-decreasing and Λ non-increasing, respectively.

3.1. COROLLARY. *Let ψ, λ be as in (3.3), and let*

$$G_p(t) = \Lambda(t) \left(t \Psi(t)^{-1} \right)^{n \left| \frac{1}{p} - \frac{1}{2} \right|}, \quad t > 0,$$

be unbounded. Then, there exists $\sigma(D)$ in $\text{op}(\mathcal{M}_\psi^\lambda)$ not $L^p(\mathbb{R}^n)$ -bounded.

3.2. REMARK. For instance, when $\Psi(t) = (1+t)^\rho$, $\Lambda(t) = (1+t)^{-m}$, ρ and m non-negative real numbers, we have

$$G_p(t) \sim t^{-m+(1-\rho)n \left| \frac{1}{p} - \frac{1}{2} \right|}, \quad t \rightarrow \infty,$$

which is unbounded if $\rho < 1$ and $m < (1-\rho)n|1/p - 1/2|$. We then reobtain a result proved by C. Fefferman in [9]. For $\Lambda \equiv 1$ we reobtain the result proved by R. Beals in [3].

Proof of Corollary 3.1. Let $\{t_k\}$ be a sequence such that $G_p(t_k) \rightarrow \infty$. Taking into account (3.3), it is possible to build a sequence of n -dimensional cubes $I(t_k)$ whose sidelength l_k is proportional to t_k and such that

$$I(t_k) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq t_k\} \subseteq S(\eta),$$

for $|\eta| = t_k$. Since $G_p(t_k) \rightarrow \infty$ implies $t_k \Psi(t_k)^{-1} \rightarrow \infty$, the condition $l_k \geq c \Psi(t_k)$ is certainly fulfilled. So, in agreement with the observations at the beginning of the section, we only need to check (3.2). In the present case

$$\inf_{\xi \in I(t_k)} \Lambda(|\xi|) \geq \Lambda(t_k),$$

since Λ is non-increasing, and then

$$\delta_k \geq c G_p(t_k).$$

This shows that $\delta_k \rightarrow \infty$ and concludes the proof. \square

Let us now assume the functions ψ e λ to be *slowly varying*, that is

$$(3.4) \quad \begin{aligned} c \leq \psi_j(\xi + \eta) \psi_j(\xi)^{-1} \leq C, \quad j = 1, \dots, n, \\ c \leq \lambda(\xi + \eta) \lambda(\xi)^{-1} \leq C, \end{aligned}$$

for $|\eta_k| \leq c \psi_h(\xi)$, $h = 1, \dots, n$, and fixed constants $c, C > 0$. Moreover, let ψ be decreasing and λ be increasing, respectively, when ‘‘coordinates grow’’, that is

$$(3.5) \quad \begin{aligned} \psi_j(\eta) \leq \psi_j(\xi), \quad j = 1, \dots, n, \\ \lambda(\eta) \geq \lambda(\xi), \end{aligned}$$

for $|\eta_h| < |\xi_h|$, $h = 1, \dots, n$.

3.3. COROLLARY. *Assume that ψ and λ satisfy conditions (3.4) and (3.5), and that the function*

$$F_p^*(\xi) = \lambda(\xi) \left(\prod_{j=1}^n |\xi_j| \psi_j(\xi)^{-1} \right)^{n \left| \frac{1}{p} - \frac{1}{2} \right|},$$

is unbounded. Then, there exists an operator $\sigma(D)$ in $\text{op}(\mathcal{M}_\psi^\lambda)$ which is not $L^p(\mathbb{R}^n)$ -bounded.

3.4. REMARK. Fix a n -tuple of positive integers $L = (L_1, \dots, L_n)$, a n -tuple of real scalars $\rho = (\rho_1, \dots, \rho_n)$, with $0 \leq \rho_j \leq 1$, $j = 1, \dots, n$, and set

$$[\xi]_L = 1 + \sum_{j=1}^n |\xi_j|^{1/L_j}.$$

Consider $\lambda(\xi) = [\xi]_L^{-m}$, $m \geq 0$, and $\psi(\xi) = ([\xi]_L^{\rho_1 L_1}, \dots, [\xi]_L^{\rho_n L_n})$, denoting by $\mathcal{M}_{L,\rho}^{-m}$ the corresponding symbol class, considered by A. Nagel and E. Stein [29]. It is possible to prove that conditions (3.4) and (3.5) are fulfilled, and, evaluating $F_p^*(\xi)$ in the points with coordinates $\xi_j = t^{L_j}$, $j = 1, \dots, n$, $t > 0$, we find that, if

$$m < \left| \frac{1}{p} - \frac{1}{2} \right| \sum_{j=1}^n (1 - \rho_j) L_j,$$

there exists $\sigma(D)$ not $L^p(\mathbb{R}^n)$ -bounded with symbol in $\mathcal{M}_{L,\rho}^{-m}$. Choosing $\lambda \equiv 1$ in Corollary 3.3, we reobtain a result proved by R. Beals in [3].

Proof of Corollary 3.3. Define

$$I(\xi) = \left\{ \eta \in \mathbb{R}^n : |\eta_j| \leq |\xi_j| + \frac{c}{2} \psi_j(\xi), j = 1, \dots, n \right\}.$$

Applying first (3.4) and then (3.5), we see that (3.1) is fulfilled. Taking into account that

$$\inf_{\eta \in I(\xi)} \lambda(\eta) \geq C \lambda(\xi),$$

again in view of (3.4) and (3.5), and of

$$\mu(I(\xi)) > \prod_{j=1}^n |\xi_j|,$$

the statement follows by the observations at the beginning of the section. \square

APPENDIX A. $L^p(\mathbb{R})$ NORMS OF TEMPERED TRIGONOMETRIC POLYNOMIALS

For the convenience of the reader, we give here the proof of the estimates (2.5). They are consequence of the properties of the function f stated in Lemma 2.1, of the following Lemma A.1, see [2], and of the properties of the Dirichlet kernels recalled in Lemma A.2, see, e.g., [7, 8, 39].

A.1. LEMMA. Let $M \in \mathbb{Z}_+$ and define $z_M(t) = \sum_{|j| \leq M} e^{iL_j t} f(t - j)$, $r > 0$, with the function

$f \in \mathcal{S}(\mathbb{R})$ chosen as in Lemma 2.1 and $L = 4\pi r > 0$. Then, $|z_M(t)| \geq \frac{1}{2}$ on the intervals $[k - \delta, k + \delta]$, $\delta = \delta(f) \in \left(0, \frac{1}{2}\right)$, for any $k \in \mathbb{Z}$ such that $|k| \leq M$.

A.2. LEMMA. Let $M \in \mathbb{Z}_+$ and consider the M -th Dirichlet kernel $D_M(t) = \sum_{|j| \leq M} e^{ijt}$. Then, for each $p \in (1, +\infty)$ there exists a suitable positive constant K , depending only on p , such that

$$\|D_M\|_{L^p(0, 2\pi)} \leq KM^{1 - \frac{1}{p}}.$$

A.3. COROLLARY. Let z_M , $M \in \mathbb{Z}_+$, be defined as in Lemma A.1. Then, for any $p \in [1, \infty)$ and a suitable positive constant b , depending only on p and f ,

$$\|z_M\|_{L^p(\mathbb{R})} \geq bM^{\frac{1}{p}}.$$

Proof. Indeed, Lemma A.1 implies

$$\begin{aligned}\|z_M\|_{L^p(\mathbb{R})}^p &= \int_{-\infty}^{+\infty} |z_M(t)|^p dt = \sum_{k \in \mathbb{Z}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} |z_M(t)|^p dt \geq \sum_{|k| \leq M} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} |z_M(t)|^p dt \\ &\geq \sum_{|k| \leq M} \int_{k-\delta}^{k+\delta} |z_M(t)|^p dt \geq \frac{1}{2^p} \sum_{|k| \leq M} 2\delta \\ &\Rightarrow \|z_M\|_{L^p(\mathbb{R})} \geq \left(\delta^{\frac{1}{p}} 2^{\frac{2}{p}-1}\right) M^{\frac{1}{p}},\end{aligned}$$

as claimed. \square

A.4. COROLLARY. *The function $h_N(x)$, $N = (N_1, \dots, N_n)$, defined in (2.5) satisfies the estimate*

$$\|h_N\|_{L^p(\mathbb{R}^n)} \geq c \left(\prod_{j=1}^n N_j \right)^{\frac{1}{p}}, \quad p \in [1, +\infty),$$

with a positive constant c depending only on n , p and f .

Proof. The statement follows immediately from Corollary A.3, observing that, obviously, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$h_N(x) = \prod_{j=1}^n z_{N_j}(x_j).$$

\square

A.5. COROLLARY. *For $p \in (1, +\infty)$, $L = 4\pi r > 0$ and a function $f \in \mathcal{S}(\mathbb{R})$ as in Lemma 2.1, we have*

$$\int_{-\infty}^{+\infty} [f(t)]^p \left| \sum_{|j| \leq M} e^{iLjt} \right|^p dt \leq BM^{p-1},$$

with a suitable positive constant B depending only on L , p and f .

Proof. Rescaling the integration variable by the factor L , recalling that $f(s) = |f(s)| \leq A(1+|s|^2)^{-1}$ for a suitable constant $A > 0$, and denoting by D_M the M -th Dirichlet kernel, we easily obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} [f(t)]^p \left| \sum_{|j| \leq M} e^{iLjt} \right|^p dt &= \frac{1}{L} \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} \left[f\left(\frac{t}{L}\right) \right]^p \left| \sum_{|j| \leq M} e^{ij t} \right|^p dt \\ &= \frac{1}{L} \sum_{k \in \mathbb{Z}} \int_0^{2\pi} \left[f\left(\frac{t+2k\pi}{L}\right) \right]^p |D_M(t)|^p dt \\ &\leq \frac{2}{L} \|D_M\|_{L^p(0,2\pi)}^p \sum_{k \in \mathbb{Z}_+} \left[\frac{A}{1 + \left(\frac{2k\pi}{L}\right)^2} \right]^p,\end{aligned}$$

and the result follows by Lemma A.2 above. \square

A.6. COROLLARY. *The function $g_N(x)$, $N = (N_1, \dots, N_n)$, defined in (2.5) satisfies the estimate*

$$\|g_N\|_{L^p(\mathbb{R}^n)} \leq C \left(\prod_{j=1}^n N_j \right)^{1-\frac{1}{p}}, \quad p \in (1, +\infty),$$

with a positive constant C depending only on n , L , p and f .

Proof. Similarly to Corollary A.4, we observe that

$$\begin{aligned} g_N(x) &= \prod_{j=1}^n \left[f(x_j) \sum_{|\gamma_j| \leq N_j} e^{iL\gamma_j x_j} \right] \\ \Rightarrow \|g_N\|_{L^p(\mathbb{R}^n)} &= \prod_{j=1}^n \int_{-\infty}^{+\infty} |f(x_j)|^p \left| \sum_{|\gamma_j| \leq N_j} e^{iL\gamma_j x_j} \right|^p dx_j, \end{aligned}$$

with $f \in \mathcal{S}(\mathbb{R})$ as in Lemma 2.1. The result then follows immediately by Corollary A.5. \square

Proof of Lemma A.1. Since $f(0) = 1$, the continuity of f implies that there exists $\delta = \delta(f) > 0$ such that $|t| < \delta \Rightarrow f(t) \geq \frac{8}{9}$. Obviously, we can assume $\delta \in \left(0, \frac{1}{2}\right)$. Then, for $t \in [k - \delta, k + \delta]$, $k \in \mathbb{Z}$, $|k| \leq M$, we immediately have

$$\begin{aligned} |z_M(t)| &= \left| \sum_{|j| \leq M} e^{iLjt} f(t-j) \right| = \left| e^{iLkt} f(t-k) + \sum_{|j| \leq M, j \neq k} e^{iLjt} f(t-j) \right| \\ &\geq f(t-k) - \left| \sum_{|j| \leq M, j \neq k} e^{iLjt} f(t-j) \right| \geq f(t-k) - \sum_{|j| \leq M, j \neq k} f(t-j) \\ &\geq f(t-k) - \sum_{j \in \mathbb{Z}, j \neq k} f(t-k-(j-k)) > \frac{1}{2}, \end{aligned}$$

by the choice of δ , since $f(t) \geq 0$ for all $t \in \mathbb{R}$ and (2.1) holds. \square

Proof of Lemma A.2. For $t \in (0, 2\pi)$ we have

$$D_M(t) = \frac{\sin\left[\left(M + \frac{1}{2}\right)t\right]}{\sin\left(\frac{t}{2}\right)},$$

while $D_M(0) = D_M(2\pi) = 2M + 1$. Then, for all $M \geq 1$, $\left|\frac{D_M(t)}{M}\right| \leq 3$ for any $t \in \mathbb{R}$ and

$$\begin{aligned} \|D_M\|_{L^p(0, 2\pi)}^p &= \int_{-\pi}^{\pi} |D_M(t)|^p dt = M^{p-1} \int_{-M\pi}^{M\pi} \left| \frac{\sin\left(s + \frac{s}{2M}\right)}{M \sin\left(\frac{s}{2M}\right)} \right|^p ds \\ &\Rightarrow \frac{\|D_M\|_{L^p(0, 2\pi)}^p}{M^{p-1}} = \int_{-\infty}^{+\infty} \chi_{[-M\pi, M\pi]}(s) \left| \frac{\sin\left(s + \frac{s}{2M}\right)}{M \sin\left(\frac{s}{2M}\right)} \right|^p ds = d_M. \end{aligned}$$

The claimed result follows observing that, by dominated convergence¹, the sequence $\{d_M\}$ admits a finite limit for $M \rightarrow +\infty$, and is then bounded by a positive constant K^p . \square

¹The elementary inequality $|t| \leq \frac{\pi}{2} \Rightarrow |\sin t| \geq \frac{2}{\pi}|t|$ gives $s \in [-M\pi, M\pi] \Rightarrow \left|\sin\left(\frac{s}{2M}\right)\right| \geq \frac{|s|}{M\pi}$. The integrand in the expression of d_M can then be bounded, for all $M \geq 1$, by $\left(\frac{\pi}{|s|}\right)^p$, $p > 1$, for $|s| \geq \varepsilon > 0$, and by a constant for $|s| \leq \varepsilon$.

APPENDIX B. POSITIVE-DEFINITE FUNCTIONS

For the sake of completeness, we recall here a definition and some basic properties of positive-definite functions. For more details, see, e.g., [31].

B.1. DEFINITION. Let f be a complex-valued function defined on \mathbb{R}^n . f is said to be a *positive-definite function* if, for any finite family of vectors $(x_i)_{i=1,\dots,N}$, the matrix

$$(f(x_i - x_j))_{i,j=1,\dots,N}$$

is positive semi-definite, that is, for any finite family of complex scalars $(\rho_i)_{i=1,\dots,N}$, we have

$$(B.1) \quad \sum_{i=1}^N \sum_{j=1}^N f(x_i - x_j) \rho_i \bar{\rho}_j \geq 0.$$

B.2. THEOREM. *Let f be a positive-definite function. Then, f has the following properties:*

- (a) $f(0) \geq 0$,
- (b) $f(-x) = \overline{f(x)}$,
- (c) $|f(x)| \leq f(0)$.

Proof. Let $I = \{1, 2\}$, $x_1 = x$, $x_2 = 0$, $\rho_1 = \lambda \in \mathbb{C}$, $\rho_2 = 1$. Applying (B.1), we find

$$(B.2) \quad f(0) + f(x)\lambda + f(-x)\bar{\lambda} + f(0)|\lambda|^2 \geq 0.$$

Since (B.2) holds for any λ , choosing $\lambda = 0$ we find $f(0) \geq 0$, as claimed.

(B.2) and (a) imply

$$(B.3) \quad f(x)\lambda + f(-x)\bar{\lambda} \in \mathbb{R}.$$

Since $f(x)\lambda + \overline{f(x)\lambda} = 2 \operatorname{Re}(f(x)\lambda) \in \mathbb{R}$, subtracting (B.3), we find $(\overline{f(x)} - f(-x))\bar{\lambda} \in \mathbb{R}$ for any $\lambda \in \mathbb{C}$. Then, choosing $\lambda = i(\overline{f(x)} - f(-x))$ we have

$$(B.4) \quad -i|\overline{f(x)} - f(-x)|^2 \in \mathbb{R},$$

and (B.4) holds if and only if $f(-x) = \overline{f(x)}$, which is property (b).

In view of (b), (B.2) implies also

$$f(0) + \operatorname{Re}(\lambda f(x)) + f(0)|\lambda|^2 \geq 0, \quad \lambda \in \mathbb{C}.$$

If $f(0) = 0$, when $\lambda = -\overline{f(x)}$ we have $-|f(x)|^2 \geq 0 \Rightarrow f(x) = 0$. In the other hand, when $f(0) > 0$, choosing

$$\lambda = -\frac{\overline{f(x)}}{f(0)},$$

we obtain $f(0)^2 \geq |f(x)|^2$. The proof is complete. □

B.3. THEOREM. *Let $f \in L^2(\mathbb{R}^n)$ and g be given by*

$$g(x) = \int_{\mathbb{R}^n} f(x+y) \overline{f(y)} dy.$$

Then, g is a continuous positive-definite function.

Proof. Obviously, $g = f * \tilde{f}$ with $\tilde{f}(x) = \overline{f(-x)}$, which implies the continuity of g on \mathbb{R}^n , by the properties of the convolution. Let $(x_i)_{i=1,\dots,N}$ be a family of vectors and $(\rho_i)_{i=1,\dots,N}$ a family of complex scalars as in Definition B.1. We then have

$$(B.5) \quad \begin{aligned} \sum_{i,j=1}^N g(x_i - x_j) \rho_i \bar{\rho}_j &= \sum_{i,j=1}^N \left(\int_{\mathbb{R}^n} f(x_i - x_j + y) \overline{f(y)} dy \right) \rho_i \bar{\rho}_j \\ &= \sum_{i,j=1}^N \int_{\mathbb{R}^n} \rho_i f(x_i - x_j + y) \overline{\rho_j f(y)} dy. \end{aligned}$$

By the changes of variable $y \rightarrow y + x_j$, the last expression in (B.5) turns into the integral

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\sum_{i,j=1}^N \rho_i f(x_i + y) \cdot \overline{\rho_j f(x_j + y)} \right) dy \\ = \int_{\mathbb{R}^n} \left\langle (\rho_i f(x_i + y))_{i=1,\dots,N}, (\rho_i f(x_i + y))_{i=1,\dots,N} \right\rangle_{M_{n,1}} dy, \end{aligned}$$

which is non-negative² and gives the desired result. \square

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²Remember that the $(n \times n)$ -dimensional matrix $M_{n,1}$ with all entries equal to 1 is positive semi-definite, since one of its eigenvalues is equal to n , while all the others vanish. Such a matrix defines the bilinear form evaluated at (v, v) , $v = (\rho_i f(x_i + y))_{i=1,\dots,N}$, which is present in the last integral.

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