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ON GENERALIZED KUMMER OF RANK 3 VECTOR BUNDLES OVER A GENUS 2 CURVE

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1. Introduction.

Let X be a smooth projective complex curve and let $U_X(r, d)$ be the moduli space of semi-stable vector bundles of rank r and degree d on X (see [8]). It contains an open Zariski subset $U_X(r, d)^s$ which is the coarse moduli space of stable bundles, i.e. vector bundles satisfying inequality

$$\frac{d_F}{r_F} < \frac{d_E}{r_E}.$$

The complement $U_X(r, d) \setminus U_X(r, d)^s$ parametrizes certain equivalence classes of strictly semi-stable vector bundles which satisfy the equality

$$\frac{d_F}{r_F} = \frac{d_E}{r_E}.$$

Each equivalence class contains a unique representative isomorphic to the direct sum of stable bundles. Furthermore one considers subvarieties $SU_X(r, L) \subset U_X(r, d)$ of vector bundle of rank r with determinant isomorphic to a fixed line bundle L of degree d. In this work we study the variety of strictly semi-stable bundles in $SU_X(3, \mathcal{O}_X)$, where X is a genus 2 curve. We call this variety the generalized Kummer variety of X and denote it by $\mathrm{Kum}_3(X)$. Recall that the classical Kummer variety of X is defined as the quotient of the Jacobian variety $\mathrm{Jac}(X) = U_X(1, 0)$ by the involution $L \mapsto L^{-1}$. It turns out that our $\mathrm{Kum}_3(X)$ has a similar description as a quotient of $\mathrm{Jac}(X) \times \mathrm{Jac}(X)$ which justifies the name. We will see that the first definition allows one to define a natural embedding of $\mathrm{Kum}_3(X)$ in a projective space (see section 4). The second approach is useful in order to give local description of $\mathrm{Kum}_3(X)$ by following the theory developed in [1] (section 3).

We point out the use of [4] for local computations.

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2. Generalized Kummer variety.

Let A be an $s-$dimensional abelian variety, A^r the r-Cartesian product of A, and $A^{(r)} := A^r / \Sigma_r$ be the r-symmetric power of A. We can define the usual map $a_r : A^{(r)} \to A$ such that $a_r(\{x_1, \dots x_r\}) = x_1 + \cdots + x_r$ [1]. This surjective map is just a morphism of varieties since there is no group structure on $A^{(r)}$. However, all fibers of a_r are naturally isomorphic.

Definition 2.1. *The generalized $Kummer_r$ variety associated to an abelian variety A is*

$$\mathrm{Kum}_r(A) := a_r^{-1}(0).$$

It is easy to see that

$$\dim(\mathrm{Kum}_r(a)) = s(r-1).$$

When $\dim A > 1$, $A^{(r)}$ is singular. If $\dim A = 2$, $A^{(r)}$ admits a natural desingularization isomorphic to the Hilbert scheme $A^{[r]} := \mathrm{Hilb}(A)^{[r]}$ of 0-dimensional subschemes of A of length r (see [5]). Let $pr : A^{[r]} \to A^{(r)}$ be the usual projection. It is known that the restriction of pr over $\mathrm{Kum}_r(A)$ is a resolution of singularities. Also $\widetilde{\mathrm{Kum}}_r(A)$ admits a structure a holomorphic symplectic manifold (see [1]).

2.1 The Kummer variety of Jacobians.

Let X be a smooth connected projective curve of genus g and $\mathrm{SU}_X(r, L)$ be the set of semi-stable vector bundles on X of rank r and determinant which is

[1] here $\{x_1, \dots x_r\}$ mean an unordered set of r elements.

isomorphic to a fixed line bundle L. Let $\mathrm{Jac}(X)$ be the Jacobian variety of X which parametrizes isomorphism classes of line bundles on X of degree 0, or, equivalently the divisor classes of degree 0. We have a natural embedding:

$$\mathrm{Kum}_r(\mathrm{Jac}(X)) \hookrightarrow \mathrm{SU}_X(r, \mathcal{O}_X)$$

$$\{a_1, \ldots, a_r\} \mapsto (L_{a_1} \oplus \ldots \oplus L_{a_r})$$

where $L_{a_i} := \mathcal{O}_X(a_i)$. Obviously, the condition $a_1 + \ldots + a_r = 0$ means that $\det(L_{a_1} \oplus \ldots \oplus L_{a_r}) = 0$ and $\deg(L_{A_i}) = 0$ for all $i = 1, \ldots, r$. Consequently the Kummer variety $\mathrm{Kum}_r(\mathrm{Jac}(X))$ describes exactly the completely decomposable bundles in $\mathrm{SU}_X(r)$ (from now on we'll write only $\mathrm{SU}_X(r)$ instead of $\mathrm{SU}_X(r, \mathcal{O}_X)$).

In this paper we restrict ourselves with the case $g = 2$ and rank $r = 3$. In this case $\mathrm{Kum}_3(\mathrm{Jac}(X))$ is a 4-fold.

3. Singular locus of $\mathrm{Kum}_3(\mathrm{Jac}(X))$.

From now we let A denote $\mathrm{Jac}(X)$. Let us define the following map:

$$\begin{array}{rcl} \pi : A^{(2)} & \to & \mathrm{Kum}_3(A) \\ \{a, b\} & \mapsto & L_a \oplus L_b \oplus L_{-a-b}. \end{array}$$

This map is well defined and it is a $(3:1) - covering$ of $\mathrm{Kum}_3(A)$. Let now $\rho : A^2 \to A^{(2)}$ be the $(2:1)-$map which sends $(x, y) \in A^2$ to $\{x, y\} \in A^{(2)}$. If we consider the map:

$$p := (\pi \circ \rho) : A^2 \to A^{(2)} \to \mathrm{Kum}_3(A) \subset A^{(3)} \tag{1}$$

we get a $(6:1)-$covering of $\mathrm{Kum}_3(A)$.

Notations: Let X and Y be two varieties and $f : X \to Y$ be a finite morphism. We let $\mathrm{Sing}(X)$ denote the singular locus of X, $B_f \subseteq Y$ the branch locus of f and $R_f \subseteq X$ the ramification locus of f.

Observation: $B_\pi = \pi(B_\rho)$.

Proof. Since $B_\rho = \{\{x, y\} \in A^{(2)} | x = y\}$ and $\pi(\{x, x\}) = \{x, x, -2x\} \in B_\pi$ we obviously get that $\pi(B_\rho) \subset B_\pi$.
Conversely, for any point $\{x, y, z\}$ of B_π, at least two of the three elements x, y, z are equal to some t. Therefore $\pi(\{t, t\}) = \{x, y, z\}$, and hence $B_\pi \subset \pi(B_\rho)$. □

Since A^2 is smooth, we have $\mathrm{Sing}(A^{(2)}) \subset B_\rho$. Obviously $B_\rho \subset R_\pi$, hence $\mathrm{Sing}(\mathrm{Kum}_3(A)) \subset B_\pi$. As a consequence we obtain that $\mathrm{Sing}(\mathrm{Kum}_3(A)) \subseteq B_\pi$. Therefore we have to study the $(3:1)-$covering $\pi : A^{(2)} \to \mathrm{Kum}_3(A)$.

Since π is not a Galois covering, in order to give the local description at every point $Q \in \mathrm{Kum}_3(A)$, we have to consider the following three cases separately:

1. $Q \in \mathrm{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is just a point;
2. $Q \in \mathrm{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of two different points;
3. $Q \in \mathrm{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of exactly three points.

Let us begin studying these cases.

Case 3. When $Q \in \mathrm{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we have that $Q \notin B_\pi$. Since $\pi(B_\rho) = B_\pi$ any point of $\pi^{-1}(Q)$ is smooth in $A^{(2)}$. Then Q is a smooth points of the Kummer variety.

Case 2. When $Q \in \mathrm{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 2$ we fix the two points $P_1, P_2 \in A^{(2)}$ s.t $\pi(P_1) = \pi(P_2) = Q$. In this case $Q = \{x, x, -2x\}$ with $x \neq -2x$; let us fix $P_1 = \{x, x\}$, $P_2 = \{x, -2x\}$. Let $U \subset \mathrm{Kum}_3(A)$ be a sufficiently small analytic neighborhood of Q such that $\pi^{-1}(U) = U_1 \sqcup U_2$ where U_1 and U_2 are respectively analytic neighborhoods of P_1 and P_2 and also $U_1 \cap U_2 = \emptyset$. Let $\widetilde{Q}$ a generic point of U, so $\widetilde{Q} = \{x + \epsilon, x + \delta, -2x - \epsilon - \delta\}$; the preimage of $\widetilde{Q}$ by π is $\pi^{-1}(\widetilde{Q}) = \{\{x+\epsilon, x+\delta\}, \{x+\epsilon, -2x-\epsilon-\delta\}, \{x+\delta, -2x-\epsilon-\delta\}\}$, but $\{x+\epsilon, x+\delta\} \in U_1$ and $\{x+\epsilon, -2x-\epsilon-\delta\}, \{x+\delta, -2x-\epsilon-\delta\} \in U_2$, it means that P_1 has ramification order equal to 1 and P_2 has ramification order equal to 2. Therefore there is an analytic neighborhood of P_1 which is isomorphic by π to an analytic neighborhood of Q. This allows us to study a generic point of B_ρ instead of a generic point of B_π.

Case 1. When $Q \in \mathrm{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we consider a point $P \in A^{(2)}$ s.t. $\pi^{-1}(Q) = P \Rightarrow Q = \{x, x, x\}$ s.t. $3x = 0 \Rightarrow x$ is a $3-$torsion point of A. Now our abelian variety is a complex torus of dimension 2, so we have exactly $3^{2g} = 3^4 = 81$ such points.

Proposition 3.1. *The singular locus of* $\mathrm{Kum}_3(A)$ *is a surface which coincides with the branch locus* B_π *of the projection* $\pi : A^{(2)} \to Kum_3(A)$ *and it is locally isomorphic at a generic point to* $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ *where* Q *is a cone over a rational normal curve and* o *is the vertex of such a cone (see [1]).*

Moreover there are exactly 81 *points of Sing(Kum* $_3(X))$ *whose local tangent cone is isomorphic to the spectrum of:*

$$\frac{\mathbb{C}[[u_1, \dots, u_7]]}{I}$$

where I *is the ideal generated by the following polynomials :*

$u_5^2 - u_4u_6$
$u_4u_7 - u_5u_6$
$u_6^2 - u_5u_7$
$u_3u_4 + u_2u_5 + u_1u_6$
$u_3u_5 + u_2u_6 + u_1u_7.$

Proof. According to what we saw in Case 2, an analytic neighborhood of $Q \in \mathrm{Kum}_3(A)$ such that $\sharp(\pi^{-1}(Q)) = 2$ is isomorphic to a generic element of B_ρ. We have to study the $(2:1)-$covering $A^2 \to A^{(2)}$.
Since $A = \mathrm{Jac}(X)$, A is a smooth abelian variety, this means that A is a complex torus $(\mathbb{C}^g/\mathbb{Z}^{2g})$ where g is the genus of X; in our case X is a genus 2 curve, $A \simeq (\mathbb{C}^2/\mathbb{Z}^4)$. Thus, in local coordinates at $P \in A$, $\widehat{\mathcal{O}_P} \simeq \mathbb{C}[[z_1, z_2]]$, so we consider U_P (a neighborhood of $P \in A$) isomorphic to $\mathbb{C}^2$. Therefore we obtain that locally at $Q \in A^2$, $\widehat{\mathcal{O}}_Q \simeq \widehat{\mathcal{O}}_P \otimes \widehat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2; z_3, z_4]]$.
We fix a coordinate system $(z_1, z_2; z_3, z_4)$ in A^2 such that $A^2 \supset U_P \ni P = (0, 0; 0, 0)$. Let Q be a point in U_P, in the fixed coordinate system $Q = (z_1, z_2; z_3, z_4)$. Since P is such that $\rho(P) \in B_\rho$, by definition of ρ, we have: $A^{(2)} = A^2/ < i >$, where i is the following involution of U_P:

(2) $$i : U_P \to U_P$$

$$i : (z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; z_1, z_2).$$

The involution i is obviously linear and its associated matrix is $M = e_{1,3}+e_{3,1}+e_{2,4}+e_{4,2}$ (where $e_{i,j}$ is the matrix with 1 in the i, j position and 0 otherwhere).

Its eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ have both multiplicity 2, so its diagonal form is:

$$\widetilde{M} = (1, 1, -1, -1)$$

which in a new coordinate system:

$$\begin{cases} x_1 = \frac{z_1+z_3}{2} \\ x_2 = \frac{z_2+z_4}{2} \\ x_3 = \frac{z_1-z_3}{2} \\ x_4 = \frac{z_2-z_4}{2} \end{cases}.$$

corresponds to the linear transformation:

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4).$$

The algebra of invariant polynomials with respect to this actions is generated by the homogeneous forms $(x_1, x_2, x_3^2, x_4^2, x_3x_4)$. Let us now consider these forms as local coordinates $(s_1, s_2, s_3, s_4, s_5)$ around $\rho(P)$, here we have that the completion of the local ring is isomorphic to the following one:

$$\left(\frac{\mathbb{C}[[s_1, \dots s_5]]}{(s_1^2 - s_2 s_3)}\right).$$

Therefore B_ρ at a generic point is locally isomorphic to $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ where Q is a cone over a rational normal curve (we can see this rational normal curve as the image of $\mathbb{P}^1$ in $\mathbb{P}^3$ by the Veronese map $\nu_2 : (\mathbb{P}^1)^* \to (\mathbb{P}^3)^*$, $\nu_2(L) = L^2$) and o the vertex of this cone. (What we have just proved in our particular case of $\mathrm{Kum}_3(A)$ can be found in a more general form in [1].) Therefore we have the same local description of singularity of $\mathrm{Kum}_3(A)$ out of the correspondent points of the 81 three-torsion points of A.

Now we have to study what happens at those 3-torsion. Let Q_0 be one of them, we already know that $p^{-1}(Q_0) = (x, x) := P_0$ is such that $3x = 0$. Let us fix $(z_1, z_2; z_3, z_4) \in \mathbb{C}^2 \times \mathbb{C}^2$ a local coordinate system around P_0 in order to describe locally the $(6:1)-$covering $p : A^2 \to \mathrm{Kum}_3(A)$. We observe that for a generic P in that neighborhood, the pre-image of $p(P)$ is the set of the following 6 points:

$$P_1 := (z_1, z_2; z_3, z_4),$$

$$P_2 := (z_3, z_4; z_1, z_2),$$

$$P_3 := (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)),$$

$$P_4 := ((-z_1 - z_3), (-z_2 - z_4); z_3, z_4),$$

$$P_5 := ((-z_1 - z_3), (-z_2 - z_4); z_1, z_2),$$

$$P_6 := (z_1, z_2; (-z_1 - z_3), (-z_2 - z_4)).$$

Observe that $i(P_1) = P_2$, $i(P_3) = P_4$, $i(P_5) = P_6$ where i is the involution defined in (2). We now define a trivolution τ of $\mathbb{C}^2 \times \mathbb{C}^2$ as follows:

$$\tau : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2 \tag{3}$$

$$(z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)).$$

It is easy to see that:

$$P_1 \xrightarrow{\tau} P_3 \xrightarrow{\tau} P_5 \xrightarrow{\tau} P_1,$$

$$P_2 \xrightarrow{\tau} P_6 \xrightarrow{\tau} P_4 \xrightarrow{\tau} P_2$$

The matrices that represent i and τ are respectively:

$$i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

furthermore $< \tau, i > \simeq \Sigma_3$, then the local description of $\mathrm{Kum}_3(X)$ around Q_0 is isomorphic to A^2/Σ_3.

In what follows we have used [4] program in order to do computations. First we recall Noether's theorem ([3] pag. 331)

Theorem 3.2. *Let $G \subset GL(n, \mathbb{C})$ be a given finite matrix group, we have:*

$$\mathbb{C}[z_1, \dots, z_n]^G = \mathbb{C}[R_G(z^\beta) : |\beta| \leq |G|].$$

where R_G is the Reynolds operator.

In other words, the algebra of invariant polynomials with respect to the action of G is generated by the invariant polynomials whose degree is at most the order of the group. In our case the order of G is 6, so it is not hard to compute $\mathbb{C}[z_1, z_2, z_3, z_4]^G$. Then, after reducing the generators, we obtain that $\mathbb{C}[z_1, z_2, z_3, z_4]^G$ is generated by:

$f_1 := z_2^2 + z_2 z_4 + z_4^2$, $f_2 := 2z_1z_2 + z_2z_3 + z_1z_4 + 2z_3z_4$,
$f_3 := z_1^2 + z_1z_3 + z_3^2$, $f_4 := -3z_2^2z_4 - 3z_2z_4^2$,
$f_5 := z_2^2z_3 + 2z_1z_2z_4 + 2z_2z_3z_4 + z_1z_4^2$,
$f_6 := -2z_1z_2z_3 - z_2z_3^2 - z_1^2z_4 - 2z_1z_3z_4$, $f_7 := 3z_1^2z_3 + 3z_1z_3^2$.

Let us now write $\mathbb{C}[z_1, \dots, z_4]^G = \mathbb{C}[f_1, \dots, f_7]$ as:

$$\mathbb{C}[u_1, \dots, u_7]/I_G,$$

where I_G is the syzygy ideal. It is easy to obtain that I_G is generated by the following polynomials:

$$\begin{gathered} u_1(u_2^2 - 4u_1u_3) + 3(u_5^2 - u_4u_6) \\ u_2(u_2^2 - 4u_1u_3) + 3(u_4u_7 - u_5u_6) \\ u_3(u_2^2 - 4u_1u_3) + 3(u_6^2 - u_5u_7) \\ u_3u_4 + u_2u_5 + u_1u_6 \\ u_3u_5 + u_2u_6 + u_1u_7cr \end{gathered}$$

and so we have the completion of the local ring at P:

$$\widehat{\mathcal{O}}_P \simeq \frac{\mathbb{C}[[u_1,\ldots,u_7]]}{I_G}.$$

Let now calculate the tangent cone in Q_0 in order to understand which kind of singularity occurs in Q_0. With [4] aid we find that this local cone is:

$$Spec\left(\frac{\mathbb{C}[[u_1,\ldots,u_7]]}{I}\right)$$

where I is the ideal generated by the following polynomials:

$$u_5^2 - u_4u_6$$

$$u_4u_7 - u_5u_6$$

$$u_6^2 - u_5u_7$$

$$u_3u_4 + u_2u_5 + u_1u_6$$

$$u_3u_5 + u_2u_6 + u_1u_7.$$

The degree of the variety $V(I) \subset \mathbb{P}^6$ is 5, this means that Q_0 is a singular point of multiplicity 5.

What we want to do now is to describe the singular locus of the local description. Let us start to calculate the Jacobian of $V(I_G)$, what we find is the following 5×7 matrix:

$$J_G := \begin{pmatrix} u_2^2 - 8u_1u_3 & 2u_1u_2 & -4u_1^2 & -3u_6 & 6u_5 & -3u_4 & 0 \\ -4u_2u_3 & 3u_2^2 - 4u_1u_3 & -4u_1u_2 & 3u_7 & -3u_6 & -3u_5 & 3u_4 \\ -4u_3^2 & 2u_2u_3 & u_2^2 - 8u_1u_3 & 0 & -3u_7 & 6u_6 & -3u_5 \\ u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & 0 \\ u_7 & u_6 & u_5 & 0 & u_3 & u_2 & u_1 \end{pmatrix}$$

Local equations define a fourfold, so we have to find the locus where the dimension of $Ker(J_G)$ is at least 5. In order to do it we calculate the minimal system of generators of all 3×3 minors of J_G, we intersect the corresponding variety with $V(I_G)$, we find a minimal base of generators of the ideal corresponding to this intersection and we compute its radical; the polynomials we find define, after suitable change of coordinates, the (reduced) variety of singular locus

$V(I_S)$, where $I_S = (u_6^2 - u_5u_7, u_5u_6 - u_4u_7, u_5^2 - u_4u_6, u_3u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_6 - u_1u_7, u_3u_4 - u_1u_6, u_2u_5 - u_1u_6, u_2u_4 - u_1u_5, u_2^2 - u_1u_3, u_3^3 - u_7^2, u_2u_3^2 - u_6u_7, u_1u_3^2 - u_5u_7, u_1u_2u_3 - u_4u_7, u_1^2u_3 - u_4u_6, u_1^2u_2 - u_4u_5, u_1^3 - u_4^2)$. We verified that the only one singular point of $V(I_S)$ is the origin. Now, let us consider the map from $\mathbb{C}^2$ to $\mathbb{C}^7$ such that:

$$(t, s) \mapsto (t^2, ts, s^2, t^3, t^2s, ts^2, s^3). \tag{4}$$

This is the parametrization of $V(I_S)$; as we have already done we can find relations between these polynomials and verify that the ideal we get is equal to I_S. Now we can consider the following smooth parametrization from $\mathbb{C}^2$ to $\mathbb{C}^9$:

$$(t, s) \mapsto (t, s, t^2, ts, s^2, t^3, t^2s, ts^2, s^3)$$

(which is nothing but the graph of (4)) whose projective closure is the Veronese surface $\nu_3(\mathbb{P}^2) = V_{2,3}$ where $\nu_3 : (\mathbb{P}^2)^* \to (\mathbb{P}^9)^*$, $\nu_3(L) = L^3$.
What we want to find now is the tangent cone in Q_0 seen inside the singular locus. Using [4] we find that its corresponding ideal $\widetilde{I_C}$ is generated by following polynomials:

$$\begin{array}{ccccc}
u_7^2 & u_6u_7 & u_5u_7 & u_4u_7 & u_4u_6 \\
u_4u_5 & u_4^2 & u_6^2 & u_5u_6 & u_5^2 \\
u_3u_6 - u_2u_7 & u_3u_5 - u_1u_7 & u_2u_6 - u_1u_7 & u_3u_4 - u_1u_6 & u_2u_5 - u_1u_6 \\
u_2u_4 - u_1u_5 & u_2^2 - u_1u_3. & & &
\end{array}$$

The ideal $\widetilde{I_C}$ has multiplicity 4 (the corresponding variety has degree four) and its radical is the following ideal:

$$I_C = (u_2^2 - u_3u_1, u_4, u_5, u_6, u_7).$$

Then $V(I_C)$ is a cone and $V(\widetilde{I_C})$ is a double cone.

This gives the description of the singularity at one of the 81 3-torsion points. □

4. Degree of $\mathbf{Kum_3}(A)$.

To find the degree of $\mathrm{Kum}_3(A)$, we have to recall some general facts about theta divisors.

4.1 The Riemann theta divisor.

Let X be a curve of genus g and $\Theta_{\mathrm{Jac}(X)}$ is the *Riemann theta divisor*. It is known that it is an ample divisor and

$$\dim|r\Theta_{\mathrm{Jac}(X)}| = r^g - 1$$

(see [6] Theorem p. 317). Recall that for any fixed point $q_0 \in X$ there exists an isomorphism:

$$\psi_{g-1,0} : \mathrm{Pic}^{g-1}(X) \to \mathrm{Jac}(X) = \mathrm{Pic}^0(X).$$

The set W_{g-1} of effective line bundles of degree $g-1$ is a divisor in $\mathrm{Pic}^{g-1}(X)$ denoted by $\Theta_{\mathrm{Pic}^{g-1}(X)}$. By Riemann's Theorem there exists a divisor k of degree 0 such that:

$$\psi_{g-1,0}(\Theta_{\mathrm{Pic}^{g-1}(X)}) = \Theta_{\mathrm{Jac}(X)} - k.$$

In a similar way we can define the *generalized theta divisor* as follows:

$$\Theta^{gen}_{\mathrm{SU}_X(r,L)} = \{E \in \mathrm{Pic}^{g-1}(X) : h^0(E \otimes L) > 0\}.$$

It is known that

$$\mathrm{Pic}(\mathrm{SU}_X(r, L)) = \mathbb{Z}\Theta^{gen}_{\mathrm{SU}_X(r,L)},$$

and there exists a canonical isomorphism:

$$|r\Theta_{\mathrm{Pic}^{g-1}(X)}| \simeq |\Theta^{gen}_{\mathrm{SU}_X(r)}|^*$$

(see [2]).

4.2 Degree of $\mathrm{Kum}_3(A)$

Let us consider the $(2:1)-$map

$$\phi_3 : \mathrm{SU}_3(X) \longrightarrow |3\Theta_{\mathrm{Pic}^1(X)}| \simeq |\Theta^{gen}_{\mathrm{SU}_X(3)}|^*$$

$$E \longmapsto D_E = \{L \in \mathrm{Pic}^1(X) : h^0(E \otimes L) > 0\}.$$

Definition 4.1. $\Theta_\eta := \{E \in \mathrm{SU}_X(3) : h^0(E \otimes \eta) > 0\} \subset \mathrm{SU}_X(3)$ *where η is a fixed divisor in Pic$^1(X)$.*

Observation: $\phi_3(\Theta_\eta) = H_\eta \subset |3\Theta_{\mathrm{Pic}^1(X)}|$ and H_η is a hyperplane. Since $\phi_3\big|_{\mathrm{Kum}_3(A)} : \mathrm{Kum}_3(A) \to \phi_3(\mathrm{Kum}_3(A))$ is a $(1:1)-$map (it is a well known fact but we will see it in the next section), we have that $\Theta_\eta \cap \mathrm{Kum}_3(X) \simeq H_\eta \cap \phi_3(\mathrm{Kum}_3(X))$. In order to study the degree of $\mathrm{Kum}_3(A)$ we have to take four generic divisors $\eta_1, \ldots, \eta_4 \in \mathrm{Pic}^1(X)$ and consider the respective $\Theta_{\eta_1}, \ldots, \Theta_{\eta_4} \subset \mathrm{SU}_X(3)$. The intersection $\Theta_{\eta_i} \cap \mathrm{Kum}_3(A)$ is equal to $\{L_a \oplus L_b \oplus L_{-a-b} \in \mathrm{Kum}_3(X) : h^0(L_a \oplus L_b \oplus L_{-a-b} \otimes \eta_i) > 0\} = \{L_a \oplus L_b \oplus L_{-a-b} \in \mathrm{Kum}_3(A) : h^0(L_a \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \mathrm{Kum}_3(A) : h^0(L_b \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \mathrm{Kum}_3(A) : h^0(L_{-a-b} \otimes \eta_i) > 0\}$ for all $i = 1, \ldots, 4$. If $L_a \oplus L_b \oplus L_{-a-b}$ is a generic element of $\mathrm{Kum}_3(A)$ and p is the $(6-1)$-covering of $\mathrm{Kum}_3(A)$ defined as in (1), then $p^{-1}(L_a \oplus L_b \oplus L_{a-b}) \subset A^2$ is a set of 6 points. It's easy to see that $p((a,b)) \in \Theta_{\eta_i} \cap \mathrm{Kum}_3(X)$ if and only if or $h^0(L_a \otimes \eta_i) > 0$ or $h^0(L_b \otimes \eta_i) > 0$ or $h^0(L_{-a-b} \otimes \eta_i) > 0$ where $(a,b) \in A^2$ and $L_a, L_b, L_{-a-b} \in \mathrm{Pic}^0(X)$ are three line bundles respectively associated to $a, b, -a-b \in A$.

Let us recall Jacobi's Theorem ([6] page: 235):

Jacobi's Theorem: *Let X be a curve of genus g, $q_0 \in X$ and $\omega_1, \ldots, \omega_g$ a basis for $H^0(X, \Omega^1)$. For any $\lambda \in \mathrm{Jac}(X)$ there exist g points $p_1, \ldots, p_g \in X$ such that*

$$\mu(\sum_{i=1}^{g}(p_i - q_0)) = \lambda,$$

where

$$\mu : Div^0(X) \to \mathrm{Jac}(X)$$

$$\sum_i (p_i - q_i) \mapsto \left(\sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g \right).$$

Since $\mathrm{Jac}(X)$ is isomorphic to $\mathrm{Pic}^0(X)$, this theorem has the following two corollaries:

1. if q_0 is a fixed point of C, then for all $L_a \in \mathrm{Pic}^0(X)$, there are two points P_1, P_2 in X such that $L_a \simeq \mathcal{O}_X(P_1 + P_2 - 2q_0)$;
2. Consider the isomorphism

$$\psi_{1,0} : \mathrm{Pic}^1(X) \xrightarrow{\sim} \mathrm{Pic}^0(X)$$

$$\eta \mapsto \eta \otimes \mathcal{O}_X(-q_0).$$

For every $i = 1, \ldots, 4$ there are $q_{i_1}, q_{i_2} \in C$ such that $\eta_i \simeq \mathcal{O}_X(q_{i_1} + q_{i_2} - q_0)$.

Now these two facts imply that $h^0(L_a \otimes \eta_i) > 0$ if and only if $h^0(\mathcal{O}_X(P_1 + P_2 - 2q_0) \otimes \mathcal{O}_X(q_{i,1} + q_{i,2} - q_0)) > 0$, and this happens if and only if $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) > 0$.

Notations: Θ_{-k} is a translate of theta divisor by $k \in \mathrm{Pic}^0(X)$.

By Riemann's Singularity Theorem (see [6], p. 348) the dimension $h^0(\mathcal{O}_X(P_1+P_2+q_{i,1}+q_{i,2}-3q_0))$ is equal to the multiplicity of $\psi_{1,0}(P_1+P_2+q_{i,1}+q_{i,2}-3q_0)$ in Θ_{-k} (by a suitable $k \in \mathrm{Pic}^0(X)$), i.e. it is equal to the multiplicity of $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0)$ in Θ_{-k}. It follows from this fact that $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$ is greater than zero if and only if $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$.

Notations:

$\Theta_i := \Theta_{-k-\eta_i+q_0}$;
$R_i := \{(a, b) \in A^2 : (a + b) \in \{-\Theta_i\}\}$;
$\Xi_i := (\Theta_i \times A) \cup (A \times \Theta_i) \cup R_i$.

Now $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$ iff $P_1 + P_2 - 2q_0 \in \Theta_i$ which is equivalent to say that L_a belongs to Θ_i, but this implies that $p((a, b)) \in \Theta_{\eta_i} \cap \mathrm{Kum}_3(A)$ if and only if $L_a \in \Theta_i$ or $L_b \in \Theta_i$ or $L_{-a-b} \in \Theta_i$ (or equivalently L_{a+b} belongs to $\{-\Theta_i\}$), i.e. $(a, b) \in \Xi_i$.

Therefore we can conclude:

$(a, b) \in A^2$ is such that $p((a, b)) \in \mathrm{Kum}_3(A) \cap \Theta_{\eta_i}$, $i = 1, \ldots, 4$ if and only if $(a, b) \in \Xi_i$.

The last conclusion together with the observation that $\sharp(pr^{-1}(L_a \oplus L_b \oplus L_{-a-b})) = 6$ gives the following proposition:

Proposition 4.2. $\deg(\mathrm{Kum}_3(A)) = \frac{1}{6}(\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4))$.

Proof. $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 6 \cdot \sharp(\mathrm{Kum}_3(A) \cap \Theta_{\eta_1} \cap \Theta_{\eta_2} \cap \Theta_{\eta_3} \cap \Theta_{\eta_4}) = 6 \cdot deg(\mathrm{Kum}_3(A))$. □

Notations:

$R_j^{a,i} = \{(a, b) \in A^2 : a \in \Theta_i$ and $(a + b) \in \{-\Theta_j\}\}$,
$R_j^{b,i} = \{(a, b) \in A^2 : b \in \Theta_i$ and $(a + b) \in \{-\Theta_j\}\}$ and
$R_{1,2} = \{(a, b) \in A^2 : (a + b) \in \{-\Theta_1\} \cap \{-\Theta_2\}\}$.

Instead of computing directly $\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4$, we will compute $(\Xi_1 \cap \Xi_2) \cap$

$(\Xi_3 \cap \Xi_4)$:

$$\Xi_1 \cap \Xi_2 = ((\Theta_1 \cap \Theta_2) \times A) \cup (A \times (\Theta_1 \cap \Theta_2)) \cup (\Theta_1 \times \Theta_2) \cup (\Theta_2 \times \Theta_1) \cup (R_b^{a,1}) \cup (R_2^{b,1}) \cup (R_1^{a,2}) \cup (R_1^{b,2}) \cup (R_{1,2}).$$
$$\Xi_3 \cap \Xi_4 = ((\Theta_3 \cap \Theta_4) \times A) \cup (A \times (\Theta_3 \cap \Theta_4)) \cup (\Theta_3 \times \Theta_4) \cup (\Theta_4 \times \Theta_3) \cup (R_b^{a,3}) \cup (R_4^{b,3}) \cup (R_3^{a,4}) \cup (R_3^{b,4}) \cup (R_{3,4}).$$

At the end we will obtain that $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 216$ (see also tables 1. and 2.) and so:

Proposition 4.3. $\deg(\mathrm{Kum}_3(A)) = 36$.

Proof. In the following two tables we write at place (i, j) the cardinality of intersection of the subset of $\Xi_1 \cap \Xi_2$ which we write at the place $(0, j)$, with the subset of $\Xi_3 \cap \Xi_4$ which we write at the place $(i, 0)$.

$\cap$	$(\Theta_1 \cap \Theta_2) \times A$	$A \times (\Theta_1 \cap \Theta_2)$	$\Theta_1 \times \Theta_2$	$\Theta_2 \times \Theta_1$
$(\Theta_3 \cap \Theta_4) \times A$	0	4	0	0
$A \times (\Theta_3 \cap \Theta_4)$	4	0	0	0
$\Theta_3 \times \Theta_4$	0	0	4	4
$\Theta_4 \times \Theta_3$	0	0	4	4
$R_4^{a,3}$	0	4	4	4
$R_3^{a,4}$	0	4	4	4
$R_4^{b,3}$	4	0	4	4
$R_3^{b,4}$	4	0	4	4
$R_{3,4}$	4	4	4	4

Table 1.

In order to be more clear we show some cases:

$\mathbf{R}_2^{a,1} \cap \mathbf{R}_4^{b,3}$: $R_2^{a,1} \cap R_4^{b,3} = \{(a, b) \in A^2 : a \in \Theta_1$ and $b \in \Theta_3$ and $(a + b) \in \{-\Theta_2\} \cap \{-\Theta_4\}\}$. Recall that $\Theta_i \cdot \Theta_j = 2$. So $(a + b) \in \{k_1, k_2\}$ where $\{k_1, k_2\} = \{-\Theta_2\} \cap \{-\Theta_4\}$. Fix for a moment $(a+b) = k_1$. If we translate Θ_1 and Θ_3 by $-k_1$ we get that $a \in (\Theta_1)_{-k_1}$, $b \in (\Theta_3)_{-k_1}$ and $a + b = 0$, then b must be equal to $-a$ and $a \in ((\Theta_1)_{-k_1}) \cap ((-\Theta_3)_{+k_1})$. Then for

$\cap$	$R_2^{a,1}$	$R_1^{a,2}$	$R_2^{b,1}$	$R_1^{b,2}$	$R_{1,2}$
$(\Theta_3 \cap \Theta_4) \times A$	0	0	4	4	4
$A \times (\Theta_3 \cap \Theta_4)$	4	4	0	0	4
$\Theta_3 \times \Theta_4$	4	4	4	4	4
$\Theta_4 \times \Theta_3$	4	4	4	4	4
$R_4^{a,3}$	4	4	4	4	0
$R_3^{a,4}$	4	4	4	4	0
$R_4^{b,3}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_{3,4}$	0	0	0	0	0

Table 2.

fixed $a + b$ the couple (a, b) has to belong to $\{(h_1, -h_1), (h_2, -h_2)\}$ where $((\Theta_1)_{+k_1}) \cap ((-\Theta_3)_{-k_1}) = \{h_1, h_2\}$. Therefore $\sharp(R_2^{a,1} \cap R_4^{b,3}) = 2 \cdot 2 = 4$.

$(\boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_2) \cap \mathbf{R}_{3,4}$: $(\Theta_1 \times \Theta_2) \cap R_{3,4} = \{(a, b) \in A^2 : a \in \Theta_1, b \in \Theta_2$ and $(a + b) \in \{-\Theta_3\} \cap \{-\Theta_4\}\}$. Then, as in the previous case, we have $\sharp((\Theta_1 \times \Theta_2) \cap R_{3,4}) = 4$.

$\mathbf{R}_2^{a,1} \cap ((\boldsymbol{\Theta}_3 \cap \boldsymbol{\Theta}_4) \times \mathbf{A})$: $R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = \{(a, b) \in A^2 : a \in \Theta_1 \cap \Theta_3 \cap \Theta_4, (a + b) \in \{-\Theta_2\}\}$, but since Θ_i are generic curves on a surface, their intersection two by two is the empty set, then $\sharp(R_2^{a,1}) \cap ((\Theta_3 \cap \Theta_4) \times A) = 0$. □

4.3 The degree of Sing(Kum$_3(A)$)

As we have already seen, the singular locus of $\mathrm{Kum}_3(A)$ is a surface. What we want to do now is to compute its degree. We use the notation from the previous section.

Let us fix two divisors Ξ_1 and Ξ_2 in A^2. We denote by Δ the diagonal of $A \times A$.

Proposition 4.4. $\deg(\mathrm{Sing}(\mathrm{Kum}_3(A))) = \sharp(\Xi_1 \cap \Xi_2 \cap \Delta)$.

Proof. It is sufficient to consider the restriction to Δ of the map p defined as in (1) and get out the $(1 : 1)$-map $p|_\Delta : \Delta \to \mathrm{Sing}(\mathrm{Kum}_3(A))$. □

Proposition 4.5. $\deg(\mathrm{Sing}(\mathrm{Kum}_3(A))) = 42$.

Proof. The following table is used in the same way as we used Table 1 and Table 2 in the previous section:

$\cap$	Δ
$(\Theta_1 \cap \Theta_2) \times A$	2
$A \times (\Theta_1 \cap \Theta_2)$	/
$\Theta_1 \times \Theta_2$	/
$\Theta_2 \times \Theta_1$	/
$R_2^{a,1}$	4
$R_1^{a,2}$	4
$R_2^{b,1}$	/
$R_1^{b,2}$	/
$R_{1,2}$	32

Table 3.

The following list describes Table 3:

$\mathbf{\Delta} \cap \mathbf{A} \times (\mathbf{\Theta}_1 \cap \mathbf{\Theta}_2)$: we have not considered the intersection points between Δ and $A \times (\Theta_1 \cap \Theta_2)$, $\Theta_1 \times \Theta_2$, $\Theta_2 \times \Theta_1$ because we have already counted them in $((\Theta_1 \cap \Theta_2) \times A) \cap \Delta$.

$\mathbf{\Delta} \cap \mathbf{R}_2^{b,1}$: the previous argument can be used for $\Delta \cap R_2^{b,1}$ and $\Delta \cap R_1^{b,2}$: we have already counted these intersection points respectively in $R_2^{a,1}$ and in $R_1^{a,2}$.

$\mathbf{R_2^{a,1}} \cap \mathbf{\Delta}$: we have now to show that $\sharp(R_2^{a,1} \cap \Delta) = 4$. The set $R_2^{a,1} \cap \Delta$ is $\{(a, a) \in A \times A \mid a \in \Theta_1,\ 2a \in (-\Theta_2)\}$ which is equal to $\{(a, a) \in A \times A : 2a \in ((-\Theta_2) \cap (2 \cdot \Theta_1))$ and $a \in \Theta_1\}$. Let now L_1 be the line bundle on A associated to Θ_1. The line bundle L_1^2 is associated to $(2 \cdot \Theta_1)$ and its divisor is linearly equivalent to $2\Theta_1$. As a consequence of this fact we have that $2a \in (2\Theta_1 \cap (-\Theta_2))$ then $\sharp\{2\Theta_1 \cap (-\Theta_2)\} = 4$. Now, since the map from Θ_1 to $(2 \cdot \Theta_1)$ is $1 : 1$ we get the conclusion.

$\mathbf{R_{1,2}} \cap \mathbf{\Delta}$: finally we have that $(R_{1,2} \cap \Delta)$ is equivalent to the set $\{a \in A | 2a \in ((-\Theta_1) \cap (-\Theta_2))\}$ whose cardinality is 32. □

5. On action of the hyperelliptic involution and $\mathrm{Kum}_3(A)$.

Let X be a curve of genus 2. Consider the degree 2 map:

$$\phi_3 : \mathrm{SU}_X(3) \xrightarrow{2:1} \mathbb{P}^8 = |3\Theta_{\mathrm{Pic}^1(X)}|$$

$$E \longmapsto D_E = \{L \in \mathrm{Pic}^1(X)/h^0(E \otimes L) > 0\}$$

(see [7]). Let τ' be the involution on $\mathrm{SU}_X(3)$ acting by the duality:

$$\tau'(E) = E^*$$

and τ the hyperelliptic involution on $\mathrm{Pic}^1(X)$:

$$\tau(L) = \omega_X \otimes L^{-1}.$$

We will use the following well known relation:

$$\tau \circ \phi_3(E) = \phi_3 \circ \tau'(E).$$

On $\mathrm{SU}_X(3)$ there is also the hyperelliptic involution h^*:

$$E \mapsto h^*(E)$$

induced by the hyperelliptic involution h of the curve X. We define $\sigma := \tau' \circ h^*$. It is the involution which gives the double covering of $\mathrm{SU}_X(3)$ on $\mathbb{P}^8$. The fixed locus of σ is obviously contained in $\mathrm{SU}_X(3)$ and we recall:

(5) $$\phi_3(\mathrm{Fix}(\sigma)) = \text{ Coble sextic hypersurface}$$

(see [7]). By definition, the strictly semi-stable locus $\mathrm{SU}_X(3)^{ss}$ of $\mathrm{SU}_X(3)$ consists of isomorphism classes of split rank 3 semi-stable vector bundles of determinant $\mathcal{O}_X$. Its points can be represented by the vector bundles of the form $F \oplus L$ or $L_a \oplus L_b \oplus L_c$ with trivial determinant where L, L_a, L_b, L_c are line bundles and F is a rank 2 vector bundle. We want to consider the elements of the form $L_a \oplus L_b \oplus L_c$ (those belonging to $\mathrm{Kum}_3(A)$) and actions of previous involutions on them:

- $\tau'(L_a \oplus L_b \oplus L_c) = (L_a \oplus L_b \oplus L_c)^* = L_{-a} \oplus L_{-b} \oplus L_{-c}$;
- $\tau'(h^*(L_a \oplus L_b \oplus L_c)) = L_a \oplus L_b \oplus L_c$.

This implies that $\sigma(\mathrm{Kum}_3(A)) = \mathrm{Kum}_3(A) \subset \mathrm{SU}_X(3)$ which means that $\mathrm{Kum}_3(A) \subset \mathrm{Fix}(\sigma)$ and then $\phi_3(\mathrm{Kum}_3(X)) \subset$ Coble sextic (see 5).

Let us now consider rank 2 semistable vector bundles of trivial determinant: $\mathrm{SU}_X(2)$. If we take its symmetric square, we obtain a semisable rank three vector bundle with trivial determinant:

$$\mathrm{SU}_X(2) \to \mathrm{SU}_X(3); \quad E \mapsto \mathrm{Sym}^2(E).$$

We want to study the action of involutions defined on the beginning of this paragraph on $\mathrm{Sym}^2(E)$ with $E \in \mathrm{SU}_X(2)$. Since $\mathrm{Sym}^2(E)^* = \mathrm{Sym}^2(E) = h^*(\mathrm{Sym}^2(E))$, then $\sigma(\mathrm{Sym}^2(E)) = \mathrm{Sym}^2(E) \subset \mathrm{SU}_X(3)$, so $\mathrm{Sym}^2(\mathrm{SU}_X(2)) \subset \mathrm{Fix}(\sigma)$, and, again by (5), $\phi_3(\mathrm{Sym}^2(\mathrm{SU}_X(2))) \subset$ Coble sextic.

Now we want to see the action of τ on $|3\Theta_{\mathrm{Pic}^1(X)}|$. It is known that $\mathrm{Fix}(\tau) = \mathbb{P}^4 \sqcup \mathbb{P}^3$.

Notations: We denote by $\mathbb{P}^3_\tau$ and $\mathbb{P}^4_\tau$, respectively, the $\mathbb{P}^3$ and the $\mathbb{P}^4$ which are fixed by action of τ.

Since the image of $\mathrm{Sym}^2(\mathrm{SU}_X(2))$ by ϕ_3 in $\mathbb{P}^8$ has dimension 3 and also $\phi_3(\mathrm{Sym}^2(\mathrm{SU}_X(2))) \subset \mathrm{Fix}(\tau)$, we obtain

$$\phi_3(\mathrm{Sym}^2(\mathrm{SU}_X(2))) \subset \mathbb{P}^4_\tau.$$

Let $L_a \oplus L_{-a}$ be an element of $\mathrm{Kum}_2(X) \subset \mathrm{SU}_X(2)$, then $\mathrm{Sym}^2(L_a \oplus L_{-a}) = L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \mathrm{Kum}_3(A) \subset \mathrm{SU}_X(3)$. It means that $\mathrm{Sym}^2(\mathrm{Kum}_2(A)) \subset \mathrm{Kum}_3(A)$.

Observation: Since $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \mathrm{SU}_X(3)\}$ is isomorphic to $S^2(\{L_a \oplus L_{-a}\})$, we can view $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \mathrm{SU}_X(3)\}$ as the image of $\mathrm{Kum}_2(A)$ inside $\mathrm{SU}_X(3)$ under the symmetric square map. Moreover it follows from the surjectivity of the multiplication by 2 map $[2] : A \to A$ that the image of $\mathrm{Kum}_2(A)$ in $\mathrm{SU}_X(3)$ is isomorphic to $\mathrm{Kum}_2(A)$.

We have already observed that $\phi_3\big|_{\mathrm{Kum}_3(A)}$ is a $(1:1)-$map on the image; this fact allows us to view $\phi_3(\mathrm{Kum}_3(A))$ as the $\mathrm{Kum}_3(A)$ in $|3\Theta_{\mathrm{Pic}^1(X)}|$. For the same reason we can view $\phi_3(\mathrm{Sym}^2(\mathrm{SU}_X(2)))$ as $\mathrm{Kum}_2(A) \subset |3\Theta_{\mathrm{Pic}_1(X)}|$. Using this language we can say that $\mathrm{Kum}_2(A)$ is left fixed by the action of τ in $\mathrm{Kum}_3(A) \subset |3\Theta_{\mathrm{Pic}_1(X)}|$ because $|3\Theta_{\mathrm{Pic}^1(X)}| \supset \phi_3(\mathrm{Kum}_3(A)) \supset \phi_3(\mathrm{Sym}^2(\mathrm{SU}_X(2)) = \mathrm{Kum}_2(A) \subset \mathbb{P}^4 \subset \mathrm{Fix}(\tau) \subset |3\Theta^1_{\mathrm{Pic}}(X)|$.

Proposition 5.1. $\mathrm{Fix}(\tau) \cap \phi_3(\mathrm{Kum}_3(A)) = \phi_3(\mathrm{Sym}^2(\mathrm{Kum}_2(A)))$.

Proof. By definition $\tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c}$ then $L_a \oplus L_b \oplus L_c$ belongs to $\mathrm{Fix}(\tau)$ if and only if $\{a, b, c\} = \{-a, -b, -c\}$. Let P belong to $\{-a, -b, -c\}$ and $a = P$.

- If P is different from $-a$, suppose that $P = -c$, then $\{-a, -b, -c\} = \{-a, -b, a\}$; moreover $a+b+c = 0$ because $L_a \oplus L_b \oplus L_c \in \mathrm{Kum}_3(A)$, then $b = 0$.
- Now, if $P = -a$ or, equivalently $a = -a$, then $a = 0$ and $b = -c$.

In both cases $L_a \oplus L_b \oplus L_c \in \mathrm{Kum}_3(A)$ such that $\tau(L_a \oplus L_b \oplus L_c) = L_a \oplus L_b \oplus L_c$ are of the form $L_a \oplus L_{-a} \oplus L_0$. This means that they belong to $\mathrm{Kum}_2(A) \subset |3\Theta_{\mathrm{Pic}^1(X)}|$. $\square$

The previous proposition tells us also that $\mathbb{P}^3_\tau \cap \mathrm{Kum}_3 A = \emptyset$. So the projection of $\mathrm{Kum}_3(A) \subset |3\Theta_{\mathrm{Pic}^1(X)}|$ from $\mathbb{P}^3_\tau$ to $\mathbb{P}^4_\tau$ is a morphism. It would be interesting to find its degree.

Our final observation is the following.

Proposition 5.2. $\mathrm{Sing}(\mathrm{Kum}_3(A)) \cap \mathrm{Kum}_2(A) = \mathrm{Sing}(\mathrm{Kum}_2(A))$

Proof. Points of $\mathrm{Kum}_2(A) \subset \mathrm{Kum}_3(A)$ are of the form $(P, -P, 0)$. Singular points of $\mathrm{Kum}_3(A)$ are those which have at least two equal components, then $\mathrm{Sing}(\mathrm{Kum}_3(A)) \cap \mathrm{Kum}_2(A) = \{(P, -P, 0)\}$ where $2P = 0$ that are exactly the 15 points of $2-$torsion and one more point $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$ which are singularities of the usual $\mathrm{Kum}_2(A)$. This implies that $\sharp(\mathrm{Sing}(\mathrm{Kum}_3(A)) \cap \mathrm{Kum}_2(A)) = 16$ and $\mathrm{Sing}(\mathrm{Kum}_3(A)) \cap \mathrm{Kum}_2(A) = \mathrm{Sing}(\mathrm{Kum}_2(A))$. $\square$

REFERENCES

[1] A. Beauville, *Variété Käheleriennes dont la première classe de Chern est nulle,* J. Didderential Geom, 18 (1983), pp. 755–782.

[2] A. Beauville - M.S. Narasimhan - S. Ramanan, *Spectral curves and the generalised theta divisor,* J. reine angew. Math., 398 (1989), pp. 169–179.

[3] D. Cox - J. Little - O'Shea, *Ideals, Varieties, and Algorithms,* Springer, 1998.

[4] A. Capani - G. Niesi - L. Robbiano, *CoCoA, A system for doing Computations in Commutative Algebra,* Available via anonymous ftp from: cocoa.dima.unige.it..

[5] L. Göttsche, *Hilbert schemes of points on surfaces,* Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), pp. 483–494.

[6] Griffiths - Harris, *Principles of algebraic geometry,* J. Wiley and sons, 1978.

[7] Y. Lazlo, *Local structure of the moduli space of vector bundles over curves*, Comment. Math. Helvetici, 71 (1996), pp. 373–401.

[8] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque, 96 (1982).

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