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On the (in-)dependence between financial and actuarial risks

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Abstract

Probability statements about future evolutions of financial and actuarial risks are expressed in terms of the 'real-world' probability measure $\mathbb{P}$, whereas in an arbitrage-free environment, the prices of these traded risks can be expressed in terms of an equivalent martingale measure $\mathbb{Q}$. The assumption of independence between financial and actuarial risks in the real world may be quite reasonable in many situations. Making such an independence assumption in the pricing world however, may be convenient but hard to understand from an intuitive point of view. In this pedagogical paper, we investigate the conditions under which it is possible (or not) to transfer the independence assumption from $\mathbb{P}$ to $\mathbb{Q}$. In particular, we show that an independence relation that is observed in the $\mathbb{P}$-world can often not be maintained in the $\mathbb{Q}$-world.

Keywords: Independence, real-world probability measure $\mathbb{P}$, risk-neutral probability measure $\mathbb{Q}$, financial risks, actuarial risks, insurance securitization.

1 Introduction

'Insurance securitization' can be defined as the transfer of underwriting risk of the insurance industry to investors in capital markets through the issuance of financial securities of which the payoffs depend on the outcome of quantities related to this underwriting risk, see e.g. Gorvett (1999). Examples of such financial securities are longevity bonds and catastrophe bonds. Modeling and pricing these insurance-related instruments involve

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both financial and actuarial considerations. In this note, we will investigate the assumption of independence between pure financial and pure actuarial risks that is often made in this context. In particular, we will focus on the differences between this independence assumption when it is made in the physical world versus the pricing world. As this note is of a pedagogical nature, it is to a large extent written in a self-contained way.

As usual, we model the financial world with the help of a filtered probability space. Instantaneous interest rates and stock prices are stochastic processes adapted to the filtration in this probability space. Actuarial risks are described via adapted stochastic processes in a second filtered probability space. Hereafter, we will restrict actuarial risks to biometrical risks, such as remaining lifetimes of individuals or survival indices of populations, but our findings can immediately be applied to other actuarial risks as well, such as catastrophic loss indices. The combined financial and biometrical world is described via the product space of the two above-mentioned filtered measurable spaces. Real-world probabilities in this combined world are described by a measure $\mathbb{P}$, of which the projections to the financial and the biometrical subworlds coincide with the respective probability measures attached to these subworlds. Notice that in general the measure $\mathbb{P}$ is not the product of the measures attached to the subworlds, meaning that stochastic processes in the financial and in the biometrical world are not necessarily mutually independent.

We assume a perfectly liquid and frictionless (no transaction costs, no trading constraints) market, as well as an arbitrage-free pricing framework. In this case, the physical probability measure $\mathbb{P}$ in the product space under consideration goes along with the existence of a (not necessarily unique) equivalent martingale measure $\mathbb{Q}$. Prices of exchange traded financial-biometrical risks are then given by discounted expectations, where expectations are taken with respect to $\mathbb{Q}$.

Hereafter, we will always assume that under the real-world measure $\mathbb{P}$, the dynamics of financial risks and biometrical risks are mutually independent, unless explicitly stated otherwise. This independence assumption may be quite reasonable and also intuitive in many cases. In the literature, one often makes the assumption that under the equivalent martingale measure $\mathbb{Q}$, the dynamics of financial risks and biometrical risks are also mutually independent. The latter assumption is very convenient as it allows us to separate the pricing of biometrical risk from the pricing of financial risk, but the intuitive idea behind this assumption is hard, if not impossible, to explain. In this paper, we focus on the meaning of independence in the pricing world. In particular, we investigate whether there is any relation between $\mathbb{P}$-independence and $\mathbb{Q}$-independence.

The remainder of this paper is structured as follows. In Section 2, we consider a combined financial-biometrical world with two possible scenarios in every subworld. In this simple world, we investigate the independence property of financial and biometrical risks by considering several examples. We start with a market which is home to traded assets of which the payoffs only depend on the outcome of one of both subworlds. In this incomplete world, an assumed independence which holds between financial and biometrical risks under the real-world probability measure $\mathbb{P}$ does not necessarily lead to an independence under the pricing measure $\mathbb{Q}$ that is chosen by the market. Here, 'chosen by the market' means that it follows implicitly from the prices of traded assets. Next, we complete this market by adding a combined financial-biometrical security. We show that, depending on the current price of the combined asset, it may be possible or not to find a pricing measure $\mathbb{Q}$ under which financial and biometrical risks are mutually independent. In order to prove that the non-existence of such a pricing measure is not related to the completeness of the market, we end Section 2 with an example of a combined incomplete market where it is impossible to find a pricing measure for which the independence property holds. In Section 3, we consider a general continuous-time combined financial-biometrical world and analyze pricing of traded mortality-linked derivatives, of which the payoffs depend on financial and biometrical evolutions. We investigate the relation between $\mathbb{P}$- and $\mathbb{Q}$-world independence among financial and biometrical risks. In Section 4, we consider an arbitrage-free bivariate Black & Scholes model. We show that under this model, independence relations between asset prices can be translated from the real world to the pricing world, and vice versa. Section 5 concludes the paper.

2 A simple combined financial-biometrical world

2.1 Financial and biometrical risks

In this section, we consider a combined financial-biometrical world in a discrete single period setting. This world is called 'combined' as it is hosting pure financial risks (such as stocks), as well as pure biometrical risks (such as a survival index related to a given population). Some (combinations) of the risks encountered in the combined world are traded (bought and sold) in a market. Throughout, we will assume that the market of these traded risks is arbitrage-free. Several of the observations that we will make concerning the theoretical example explored in this section will be formalized in a more realistic setting in Section 3.

Consider a *financial world* $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)$, containing a risk-free bank account with interest rate equal to 0 (for notational and computational convenience) and a traded stock with initial price $S^{(1)}(0) = 100$. Eventual dividend payments can only occur at time 1. The price (cum dividend) of the stock in 1 years time is equal to either $S^{(1)}(1) = 50$ or $S^{(1)}(1) = 150$. The financial universe $\Omega^{(1)}$, which describes all possible evolutions of the financial world, is given by

$$\Omega^{(1)} = \{50, 150\},$$

where the different elements stand for the different possible values of the stock price at time 1. The σ-algebra $\mathcal{F}^{(1)}$ is the set of all subsets of $\Omega^{(1)}$. The elements of $\mathcal{F}^{(1)}$ are the events which may or may not occur in the financial world in the time interval $[0,1]$. The probability measure $\mathbb{P}^{(1)}$, which attaches the 'real-world' probability to any event in $\mathcal{F}^{(1)}$, is characterized by the positive real numbers $\mathbb{P}^{(1)}\left[\{50\}\right] > 0$ and $\mathbb{P}^{(1)}\left[\{150\}\right] = 1 - \mathbb{P}^{(1)}\left[\{50\}\right] > 0$.

In the *biometrical world* $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)}\right)$, we observe a *survival index* which gives information about the survival experience of a given population, e.g. the population consisting of all persons in a given country. For simplicity, let us assume that the index

$I(1)$ equals 0 in case 'few' persons survive during the experience year $[0,1]$, whereas $I(1)$ equals 1 in case 'many' persons survive this year. The biometrical universe $\Omega^{(2)}$ describes all possible biometrical evolutions:

$$\Omega^{(2)} = \{0, 1\},$$

where the different elements stand for different values of the biometrical index at time 1. The σ-algebra $\mathcal{F}^{(2)}$ is the set of all subsets of $\Omega^{(2)}$. The elements of $\mathcal{F}^{(2)}$ are the events which may or may not occur in the biometrical world in the time interval $[0,1]$. The probability measure $\mathbb{P}^{(2)}$ which attaches the 'real-world' probability to any event in the biometrical world, is characterized by the positive real numbers $\mathbb{P}^{(2)}[\{0\}]$ and $\mathbb{P}^{(2)}[\{1\}] = 1 - \mathbb{P}^{(2)}[\{0\}]$.

Next, we consider the *combined financial-biometrical world* $(\Omega, \mathcal{F}, \mathbb{P})$ which is the Cartesian product of the financial and the biometrical world. The universe Ω, generated by elements of the form (ω_1, ω_2) with $\omega_1 \in \Omega^{(1)}$ and $\omega_2 \in \Omega^{(2)}$, is given by

$$\Omega = \Omega^{(1)} \times \Omega^{(2)} = \{(50, 0), (150, 0), (50, 1), (150, 1)\}.$$

The σ-algebra $\mathcal{F}$ is the set of all events in the combined world. It is the set of all subsets of Ω:

$$\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \sigma\left(A \times B \mid A \in \mathcal{F}^{(1)}, B \in \mathcal{F}^{(2)}\right).$$

The probability measure $\mathbb{P}$ attaches the 'real-world' probability to any event in the combined world. Throughout this section, we will assume that financial and biometrical risks are independent in the following sense:

$$\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}, \tag{1}$$

where $\mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$ is the probability measure defined by

$$\mathbb{P}[\{\omega_1, \omega_2\}] = \mathbb{P}^{(1)}[\{\omega_1\}] \times \mathbb{P}^{(2)}[\{\omega_2\}], \qquad \text{for any } \{\omega_1, \omega_2\} \in \mathcal{F}. \tag{2}$$

For ease of notation, hereafter we will denote $\mathbb{P}[\{\omega_1, \omega_2\}]$ as $\mathbb{P}[\omega_1, \omega_2]$, and $\mathbb{P}[\{\omega_i\}]$ as $\mathbb{P}[\omega_i]$, $i = 1, 2$. The independence assumption (2) immediately leads to

$$\begin{cases} \mathbb{P}[50, 0] = \mathbb{P}^{(1)}[50] \times \mathbb{P}^{(2)}[0] \\ \mathbb{P}[150, 0] = \mathbb{P}^{(1)}[150] \times \mathbb{P}^{(2)}[0] \\ \mathbb{P}[50, 1] = \mathbb{P}^{(1)}[50] \times \mathbb{P}^{(2)}[1] \\ \mathbb{P}[150, 1] = \mathbb{P}^{(1)}[150] \times \mathbb{P}^{(2)}[1]. \end{cases}$$

Hereafter, we consider the pricing of exchange traded securities in the combined world, of which the payoff at time 1 depends on the stock price $S^{(1)}(1)$ and/or the survival index $I(1)$. For that purpose, we introduce the notion of an 'equivalent martingale measure'. Recall that in our one-period discrete setting, a probability measure $\mathbb{Q}$ defined on the combined measurable space $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure (or a risk-neutral measure) for the combined world if it fulfills the following two conditions:

1. $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures.

2. For any traded asset in the combined world, one has that its future payoff, discounted at the risk-free rate, is a martingale with respect to $\mathbb{Q}$.

Taking into account that $\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$, the first condition is equivalent to the condition that $\mathbb{Q}$ has a positive probability mass on each element of Ω. Taking into account that the risk-free interest rate is equal to 0, the second condition states that the current price of any traded asset in the combined world is equal to the expected value of the payoff of this asset at time 1, where the expectation is taken with respect to $\mathbb{Q}$.

It is well-known that in our discrete setting, the no-arbitrage condition is equivalent to the existence of an equivalent martingale measure, whereas completeness of the arbitrage-free market is equivalent to the existence of a unique equivalent martingale measure, see e.g. Shiryaev et al. (1994).

Starting from a given equivalent martingale measure $\mathbb{Q}$ for the combined world, one can construct the following probability measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ for the financial and the biometrical world, respectively:

$$\begin{cases} \mathbb{Q}^{(1)}\left[50\right] = \mathbb{Q}\left[50, 0\right] + \mathbb{Q}\left[50, 1\right] \\ \mathbb{Q}^{(1)}\left[150\right] = \mathbb{Q}\left[150, 0\right] + \mathbb{Q}\left[150, 1\right], \end{cases}$$

and

$$\begin{cases} \mathbb{Q}^{(2)}\left[0\right] = \mathbb{Q}\left[50, 0\right] + \mathbb{Q}\left[150, 0\right] \\ \mathbb{Q}^{(2)}\left[1\right] = \mathbb{Q}\left[50, 1\right] + \mathbb{Q}\left[150, 1\right]. \end{cases}$$

The measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ are called the *projections* of $\mathbb{Q}$ to the financial and the biometrical world, respectively. Based on these projections, we introduce the probability measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ on the combined measurable space $(\Omega, \mathcal{F})$, which is defined by

$$\left(\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}\right)\left[\omega_1, \omega_2\right] = \mathbb{Q}^{(1)}\left[\omega_1\right] \times \mathbb{Q}^{(2)}\left[\omega_2\right], \qquad \text{for any } \{\omega_1, \omega_2\} \in \mathcal{F}. \tag{3}$$

Financial and biometrical risks are said to be independent under the measure $\mathbb{Q}$ in case the following condition holds:

$$\mathbb{Q} \equiv \mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}, \tag{4}$$

which can also be expressed as

$$\mathbb{Q}\left[\omega_1, \omega_2\right] = \mathbb{Q}^{(1)}\left[\omega_1\right] \times \mathbb{Q}^{(2)}\left[\omega_2\right], \qquad \text{for any } \{\omega_1, \omega_2\} \in \mathcal{F}.$$

Although the measures $\mathbb{P}$ and $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ agree on sure events, and hence, are equivalent, $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ is not necessarily a martingale measure for the combined world. Furthermore, from the definitions (1) and (4) of independence in the $\mathbb{P}$- and in the $\mathbb{Q}$-world, respectively, we see that there is no link between these two notions of independence. This is due to the fact that these definitions are about two equivalent, but further unrelated probability measures.

2.2 A market consisting of two financial securities

Let us first assume that the only traded securities in the combined world are the risk-free bank account with zero interest rate and the stock with current price $S^{(1)}(0) = 100$ and price $S^{(1)}(1)$ at time 1. The survival index $I(1)$ is not traded.

In this particular setting, the combined financial-biometrical market is arbitrage-free if and only if there exists a vector $(\mathbb{Q}[50,0], \mathbb{Q}[150,0], \mathbb{Q}[50,1], \mathbb{Q}[150,1])$ with positive components which satisfies the following system of equations:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{Q}[50,0] + \mathbb{Q}[150,0] + \mathbb{Q}[50,1] + \mathbb{Q}[150,1] = 1. \end{cases} \tag{5}$$

The first equation expresses the martingale requirement, while the second one guarantees that $\mathbb{Q}$ is a probability measure.

The system of equations (5) can be transformed into the following equivalent system:

$$\begin{cases} \mathbb{Q}^{(1)}[50] = 0.5 \\ \mathbb{Q}^{(1)}[150] = 0.5. \end{cases} \tag{6}$$

Due to the absence of traded biometrical securities in this combined world, the no-arbitrage condition only leads to restrictions on the projected measure $\mathbb{Q}^{(1)}$ of the financial world, while there are no restrictions on the projected measure $\mathbb{Q}^{(2)}$ of the biometrical world. As a consequence, there are infinitely many equivalent martingale measures $\mathbb{Q}$ satisfying (5), which means that the combined market is arbitrage-free but incomplete. This result is obvious, as this market leaves no room for arbitrage opportunities, while it is incomplete due to the irreplicability of the biometrical index $I(1)$.

A particular solution of the system of equations (6) is given by

$$\begin{cases} \overline{\mathbb{Q}}[50,0] = 0.2 \\ \overline{\mathbb{Q}}[150,0] = 0.1 \\ \overline{\mathbb{Q}}[50,1] = 0.3 \\ \overline{\mathbb{Q}}[150,1] = 0.4. \end{cases} \tag{7}$$

The related projected probability measures $\overline{\mathbb{Q}}^{(1)}$ and $\overline{\mathbb{Q}}^{(2)}$ of the financial and the biometrical world are given by

$$\begin{cases} \overline{\mathbb{Q}}^{(1)}[50] = 0.5 \\ \overline{\mathbb{Q}}^{(1)}[150] = 0.5, \end{cases}$$

and

$$\begin{cases} \overline{\mathbb{Q}}^{(2)}[0] = 0.3 \\ \overline{\mathbb{Q}}^{(2)}[1] = 0.7, \end{cases}$$

respectively.

The product measure $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$ related to $\overline{\mathbb{Q}}$ is specified by

$$\begin{cases} \overline{\mathbb{Q}}^{(1)}[50] \times \overline{\mathbb{Q}}^{(2)}[0] = 0.15 \\ \overline{\mathbb{Q}}^{(1)}[150] \times \overline{\mathbb{Q}}^{(2)}[0] = 0.15 \\ \overline{\mathbb{Q}}^{(1)}[50] \times \overline{\mathbb{Q}}^{(2)}[1] = 0.35 \\ \overline{\mathbb{Q}}^{(1)}[150] \times \overline{\mathbb{Q}}^{(2)}[1] = 0.35. \end{cases} \tag{8}$$

Observe that $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$ is equivalent to $\mathbb{P}$ and satisfies the restrictions in (6), which makes it a valid equivalent martingale measure under which financial and biometrical risks are independent. Moreover, the requirements (6) which have to hold in this market allow for an infinite number of equivalent martingale measures under which financial and biometrical risks are independent.

Let us now assume that the market chooses the pricing measure $\overline{\mathbb{Q}}$. Taking into account that $\overline{\mathbb{Q}} \neq \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$, we can conclude that although financial and biometrical risks are independent under the physical measure $\mathbb{P}$, this independence relation is not maintained under the pricing measure $\overline{\mathbb{Q}}$.

This example shows that independence between financial and biometrical risks which holds under the real-world probability measure does not necessarily translate into an independence relation between these risks under the pricing measure. Notice that the results of this section are obvious: as there are no biometrical risks traded, one can restrict to the financial world for pricing the traded security.

2.3 A market consisting of two financial and one biometrical security

In the previous subsection, we considered a combined world with a market in which only financial securities are traded. We observed that the independence under the real-world measure $\mathbb{P}$ does not necessarily lead to an independence under the pricing measure $\mathbb{Q}$ that is chosen by the market. Let us now enlarge the market of traded assets by assuming that, apart from the securities considered above, also a biometrical security with current price $S^{(2)}(0) = 70$ and payoff at time 1 given by

$$S^{(2)}(1) = 100 \times I(1), \tag{9}$$

is traded. Assuming that basis risk can be ignored or is sufficiently low, the insurer of an annuity portfolio can buy this instrument in order to partially hedge against longevity risk: If more insureds survive, the insurer will have to pay more pensions, but will also receive a higher investment return.

This enlarged financial-biometrical market is arbitrage-free if and only if there exists a vector $(\mathbb{Q}[50,0], \mathbb{Q}[150,0], \mathbb{Q}[50,1], \mathbb{Q}[150,1])$ of positive probabilities, which satisfies the following system of equations:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{Q}[50,0] + \mathbb{Q}[150,0] + \mathbb{Q}[50,1] + \mathbb{Q}[150,1] = 1. \end{cases} \tag{10}$$

Taking into account the equivalence of (5) and (6), it is easy to see that the system of equations (10) is equivalent to the following system:

$$\begin{cases} \mathbb{Q}^{(1)}[50] = 0.5 \\ \mathbb{Q}^{(1)}[150] = 0.5 \\ \mathbb{Q}^{(2)}[0] = 0.3 \\ \mathbb{Q}^{(2)}[1] = 0.7. \end{cases} \tag{11}$$

Obviously, the system of equations (11) only fixes the pricing probabilities in the financial and the biometrical subworld, but does not specify the dependence structure of the pricing measure. This implies that there still exist infinitely many measures $\mathbb{Q}$ that satisfy (10). Two particular solutions of (11) are given by $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$, which were defined in (7) and (8), respectively. We can conclude that the enlarged market is again arbitrage-free but still incomplete. From $\overline{\mathbb{Q}} \neq \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$, we observe that although under the physical measure $\mathbb{P}$ financial and biometrical risks are independent, this independence relation is not necessarily maintained under the pricing measure that is chosen by the market. Taking into account the conditions (11) which have to hold in this market, we find that the unique equivalent measure for which the independence property holds is given by the measure $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$.

Comparing these observations with the results of the previous section, we can conclude that the existence of an equivalent martingale measure under which financial and biometrical risks are independent will in general be less feasible in a market that is closer to completeness (due to the availability of more traded assets) as such a measure will have to be found in a smaller set of admissible measures. In this sense, when the market is 'sufficiently close to being complete', an equivalent martingale measure that has the independence property may even not exist. We will illustrate this phenomenon in the two following subsections.

2.4 A market consisting of two financial, one biometrical and one combined security

As long as no combined financial-biometrical securities are traded, pricing in the financial and the biometrical markets can be considered separately, and the discussion about independence under $\mathbb{Q}$ is in some sense artificial as this independence has not any practical relevance. In this subsection, we further enlarge the market of traded assets by introducing a combined financial-biometrical security with current price $S(0) \in (10, 25)$ and payoff at time 1 given by

$$S(1) = \left(100 - S^{(1)}(1)\right)_+ \times I(1),$$

where $(x)_+$ stands for $\max(0, x)$. This security provides a put-option type of payoff at time 1. The payoff equals 50 in case $S^{(1)}(1) = 50$ and $I(1) = 1$. In all other cases, the payoff is equal to 0. Hence, this asset produces a positive payoff only if the stock market performs poorly and in addition, many people in the observed population survive until time 1. An insurer can invest in this product to partially hedge against the joint risk of increased longevity and a bearish stock market.

This financial-biometrical market is arbitrage-free if and only if there exists a vector $(\mathbb{Q}[50,0], \mathbb{Q}[150,0], \mathbb{Q}[50,1], \mathbb{Q}[150,1])$ of positive probabilities, which satisfies the following system of equations:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[S(1)\right] = S(0) \\ \mathbb{Q}[50,0] + \mathbb{Q}[150,0] + \mathbb{Q}[50,1] + \mathbb{Q}[150,1] = 1. \end{cases} \tag{12}$$

This system of equations allows a unique solution, which is given by

$$\begin{cases} \widetilde{\mathbb{Q}}[50,0] = \frac{25-S(0)}{50} \\ \widetilde{\mathbb{Q}}[150,0] = \frac{-10+S(0)}{50} \\ \widetilde{\mathbb{Q}}[50,1] = \frac{S(0)}{50} \\ \widetilde{\mathbb{Q}}[150,1] = \frac{35-S(0)}{50}. \end{cases} \tag{13}$$

The probabilities in (13) are positive provided that $S(0) \in (10, 25)$. Notice that in case $S(0) \notin (10, 25)$, there exists no equivalent measure for $\mathbb{P}$, which means that the market is not arbitrage-free. Consider e.g. the situation where $S(0) = 10$. Then it is easy to show that the investment strategy consisting of borrowing an amount of 50 from the risk-free bank account, buying 1 unit of the financial security $S^{(1)}$, selling 1 unit of the biometrical security $S^{(2)}$ and buying 2 units of the combined security S, while closing the position at time 1, is an arbitrage opportunity. Hereafter, we will assume that $S(0) \in (10, 25)$. In this case, we can conclude that the market is arbitrage-free.

The uniqueness of the solution of (12) means that the introduction of the financial-biometrical security completes the combined market.

Taking into account that $\widetilde{\mathbb{Q}}^{(1)}[50] = 0.5$, $\widetilde{\mathbb{Q}}^{(1)}[150] = 0.5$, $\widetilde{\mathbb{Q}}^{(2)}[0] = 0.3$ and $\widetilde{\mathbb{Q}}^{(2)}[1] = 0.7$, we have that the product measure $\widetilde{\mathbb{Q}}^{(1)} \times \widetilde{\mathbb{Q}}^{(2)}$ is given by

$$\begin{cases} \widetilde{\mathbb{Q}}^{(1)}[50] \times \widetilde{\mathbb{Q}}^{(2)}[0] = 0.15 \\ \widetilde{\mathbb{Q}}^{(1)}[150] \times \widetilde{\mathbb{Q}}^{(2)}[0] = 0.15 \\ \widetilde{\mathbb{Q}}^{(1)}[50] \times \widetilde{\mathbb{Q}}^{(2)}[1] = 0.35 \\ \widetilde{\mathbb{Q}}^{(1)}[150] \times \widetilde{\mathbb{Q}}^{(2)}[1] = 0.35. \end{cases} \tag{14}$$

Comparing the systems of equations (13) and (14), we find that financial and biometrical risks are independent, i.e. $\widetilde{\mathbb{Q}} \equiv \widetilde{\mathbb{Q}}^{(1)} \times \widetilde{\mathbb{Q}}^{(2)}$, if and only if $S(0) = 17.5$.

In this enlarged market, the price setting of the combined security determines whether $\mathbb{Q}$-world independence between financial and biometrical risks is possible or not. The only admissible price for the independence property to hold turns out to be 17.5. Any other price will lead to a unique martingale measure which does not satisfy the independence property.

We can conclude that one has to be extremely careful when assuming an equivalent martingale measure under which financial and biometrical risks are independent. Indeed, the example considered in this subsection shows that the procedure where one first postulates a theoretical pricing measure $\mathbb{Q}$ under which financial and biometrical risks are independent, and then calibrates the model to observed market prices, may lead to inconsistencies.

2.5 An incomplete market where $\mathbb{Q}$-world independence between financial and biometrical risks is impossible

In the previous subsection, we considered an arbitrage-free and complete market with a unique pricing measure $\mathbb{Q}$, where independence between financial and biometrical risks is only possible for a specific value of the combined security's price. In this section, we will show that also in an incomplete combined market, it may happen that independence under the $\mathbb{Q}$-measure is impossible.

Consider the financial world $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)$, in which we observe a 'barometer of the economy'. In 1 year time, this barometer can attain three values, according to the state of the economy at that time: B in case of a *booming* economy, M in case of *moderate growth* of the economy, and R in case of an economy in *recession*. The financial universe $\Omega^{(1)}$, describing the possible evolutions in the financial world, is therefore defined by

$$\Omega^{(1)} = \{B, M, R\}\,.$$

The σ-algebra $\mathcal{F}^{(1)}$ is the set of all subsets of $\Omega^{(1)}$, while the 'real-world' probability measure $\mathbb{P}^{(1)}$ is described by the positive probabilities $\mathbb{P}^{(1)}[B]$, $\mathbb{P}^{(1)}[M]$ and $\mathbb{P}^{(1)}[R]$. Further, we consider the biometrical world $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)}\right)$ which was defined in Section 2.1.

As before, the combined financial-biometrical world $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as the Cartesian product of the financial and the biometrical world. The universe $\Omega = \Omega^{(1)} \times \Omega^{(2)}$ is now given by

$$\Omega = \{(B,0)\,, (M,0), (R,0), (B,1)\,, (M,1)\,, (R,1)\}\,.$$

Again, we assume that financial and biometrical risks are independent in the real world, which means that the real-world probability $\mathbb{P}$ is characterized by the product of the measures attached to the subworlds, i.e.

$$\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}.$$

Apart from the risk-free bank account with interest rate equal to 0, there are three traded securities in this combined market. Their payoffs depend either on the outcomes of the barometer and/or the survival index.

The financial asset has current price $S^{(1)}(0) = 50$. Its payoff $S^{(1)}(1)$ at time 1 equals 100 when the economy is booming, i.e. the barometer value is B, while it is 0 in all other cases. Furthermore, the biometrical security is the one with payoff $S^{(2)}(1)$ at time 1 given by (9), i.e. $S^{(2)}(1) = 100 \times I(1)$. Its current price is equal to $S^{(2)}(0) = 70$. Finally, the payoff $S(1)$ at time 1 of the combined financial-biometrical security equals

$$S(1) = S^{(1)}(1) \times (1 - I(1))\,.$$

The payoff $S(1)$ is positive only in case the barometer at time 1 equals B and the index $I(1)$ is equal to 0, that is in case the economy is 'booming' and 'few' persons survive during the experience year $[0,1]$. In any other case, the payoff of the combined asset equals 0. The current price of this combined security is equal to $S(0) \in (0,30)$.

The financial-biometrical market is arbitrage-free if and only if there exist positive probabilities $\mathbb{Q}[B,0]$, $\mathbb{Q}[M,0]$, $\mathbb{Q}[R,0]$, $\mathbb{Q}[B,1]$, $\mathbb{Q}[M,1]$ and $\mathbb{Q}[R,1]$, which satisfy the following system of equations:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 50 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[S(1)\right] = S(0) \\ \mathbb{Q}[B,0] + \mathbb{Q}[M,0] + \mathbb{Q}[R,0] + \mathbb{Q}[B,1] + \mathbb{Q}[M,1] + \mathbb{Q}[R,1] = 1. \end{cases} \tag{15}$$

The solutions to this system will depend on $S(0)$. Straightforward calculations lead to the following equivalent system of equations:

$$\begin{cases} \mathbb{Q}[B,0] = \frac{S(0)}{100} \\ \mathbb{Q}[B,1] = \frac{50-S(0)}{100} \\ \mathbb{Q}[M,0] + \mathbb{Q}[R,0] = \frac{30-S(0)}{100} \\ \mathbb{Q}[M,1] + \mathbb{Q}[R,1] = \frac{20+S(0)}{100}. \end{cases} \tag{16}$$

For any value of $S(0)$ in $(0,30)$, the system of equations (16) has infinitely many solutions of which the components take values in $(0,1)$. As an example, suppose that $S(0) = 20$. In this case, two particular solutions of (16) are given by

$$\begin{cases} \overline{\mathbb{Q}}[B,0] = 0.20 \\ \overline{\mathbb{Q}}[M,0] = 0.05 \\ \overline{\mathbb{Q}}[R,0] = 0.05 \\ \overline{\mathbb{Q}}[B,1] = 0.30 \\ \overline{\mathbb{Q}}[M,1] = 0.05 \\ \overline{\mathbb{Q}}[R,1] = 0.35 \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\mathbb{Q}}[B,0] = 0.20 \\ \widetilde{\mathbb{Q}}[M,0] = 0.05 \\ \widetilde{\mathbb{Q}}[R,0] = 0.05 \\ \widetilde{\mathbb{Q}}[B,1] = 0.30 \\ \widetilde{\mathbb{Q}}[M,1] = 0.35 \\ \widetilde{\mathbb{Q}}[R,1] = 0.05. \end{cases}$$

We can conclude that for any value of $S(0)$ in $(0,30)$, the combined market is arbitrage-free but incomplete. Notice that there exists no equivalent martingale measure if $S(0) \notin (0,30)$. E.g. in case $S(0) = 60$, it is easy to verify that the investment strategy consisting of depositing an amount of 200 on the risk-free bank account, selling 2 units of the biometrical security $S^{(2)}$ and selling 1 unit of the combined security S has a zero initial cost but leads to an arbitrage opportunity at time 1. Hereafter, we assume that $S(0) \in (0,30)$.

Let $\mathbb{Q}$ be an equivalent martingale measure which is a solution of the system of equations (16). From these equations, we find that

$$\mathbb{Q}^{(1)}[B] = 0.5 \qquad \text{and} \qquad \mathbb{Q}^{(2)}[0] = 0.3.$$

Taking into account that $\mathbb{Q}[(B,0)] = \frac{S(0)}{100}$, we find that the following equivalence relation holds:

$$\mathbb{Q}[B,0] = \mathbb{Q}^{(1)}[B] \times \mathbb{Q}^{(2)}[0] \Longleftrightarrow S(0) = 15.$$

This equivalence implies that in the arbitrage-free and incomplete market described in this subsection, it is impossible to find an equivalent martingale measure $\mathbb{Q}$ under which financial and biometrical risks are independent if $S(0) \neq 15$.

Let us now suppose that $S(0) = 15$. In this case, one can easily verify that the probability measures $\widehat{\mathbb{Q}}$ and $\widehat{\widehat{\mathbb{Q}}}$ defined by

$$\begin{cases} \widehat{\mathbb{Q}}\left[B,0\right] = 0.15 \\ \widehat{\mathbb{Q}}\left[M,0\right] = 0.09 \\ \widehat{\mathbb{Q}}\left[R,0\right] = 0.06 \\ \widehat{\mathbb{Q}}\left[B,1\right] = 0.35 \\ \widehat{\mathbb{Q}}\left[M,1\right] = 0.21 \\ \widehat{\mathbb{Q}}\left[R,1\right] = 0.14 \end{cases} \quad \text{and} \quad \begin{cases} \widehat{\widehat{\mathbb{Q}}}\left[B,0\right] = 0.15 \\ \widehat{\widehat{\mathbb{Q}}}\left[M,0\right] = 0.06 \\ \widehat{\widehat{\mathbb{Q}}}\left[R,0\right] = 0.09 \\ \widehat{\widehat{\mathbb{Q}}}\left[B,1\right] = 0.35 \\ \widehat{\widehat{\mathbb{Q}}}\left[M,1\right] = 0.14 \\ \widehat{\widehat{\mathbb{Q}}}\left[R,1\right] = 0.21 \end{cases}$$

are equivalent martingale measures satisfying the conditions (15). Moreover, financial and biometrical risks are independent under these measures.

We can again conclude that the current price of the combined asset, which is chosen by the market, determines whether a pricing measure with the independence property is possible or not. This means that we have to be careful when using an asset price model where financial and biometrical risks are independent under the pricing measure. As is shown in the example above, such a model might be impossible in an arbitrage-free and incomplete market.

3 A general combined financial-biometrical world in a continuous-time framework

3.1 Financial and biometrical risks

Consider a finite time horizon of T years and suppose that we are currently at time 0. In order to describe the financial world, we introduce the filtered probability space $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}, \mathbb{P}^{(1)}\right)$. The non-empty set $\Omega^{(1)}$ is called the financial universe. It has to be interpreted as the set of all possible evolutions of the financial world in the time interval $[0,T]$. Hence, any $\omega_1 \in \Omega^{(1)}$ corresponds to a feasible scenario of randomness concerning the evolution of this world in the time interval under consideration. If the financial world consists of a stock and a bond that are traded (bought and sold) in the market, any ω_1 describes a feasible scenario for the prices of these assets in $[0,T]$. The σ-algebra $\mathcal{F}^{(1)}$ is to be understood as the set consisting of subsets of $\Omega^{(1)}$, including the empty set, describing all events which may or may not occur in the financial world in the time interval $[0,T]$. The filtration $\left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}$ is an increasing sequence of σ-algebras: $\mathcal{F}_s^{(1)} \subseteq \mathcal{F}_t^{(1)} \subseteq \mathcal{F}^{(1)}$, for any $0 \leq s \leq t \leq T$. For any t in $[0,T]$, the σ-algebra $\mathcal{F}_t^{(1)}$

describes all information available about the financial world up to and including time t. Any element of $\mathcal{F}_t^{(1)}$ is an event of which we will know at time t whether it has occurred or not. Finally, $\mathbb{P}^{(1)}$ is the 'real-world' probability measure, also called the objective or physical measure, which attaches the 'real-world' probability $\mathbb{P}^{(1)}\left[A^{(1)}\right]$ to any financial event $A^{(1)} \in \mathcal{F}^{(1)}$. The filtered probability space is often assumed to satisfy the "usual conditions" of completeness and right-continuity. Completeness means that all subsets of zero-probability events of $\mathcal{F}^{(1)}$ are also contained in $\mathcal{F}^{(1)}$ and moreover, that $\mathcal{F}_0^{(1)}$ contains all zero-probability sets of $\mathcal{F}^{(1)}$. Right-continuity of the filtration $\left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}$ means that for all t, $\mathcal{F}_t^{(1)} = \mathcal{F}_{t+}^{(1)}$, where $\mathcal{F}_{t+}^{(1)}$ is the intersection of $\mathcal{F}_s^{(1)}$ over all $s > t$.

Interest rates, stock and bond prices, as well as prices of other financial assets are described by stochastic processes (which are assumed to be semi-martingales[1]) on the filtered measurable space $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}\right)$. Let us denote such a stochastic process by $\left(S^{(1)}(t)\right)_{0\leq t\leq T}$, where for each value of t, the r.v. $S^{(1)}(t)$, which is defined on the measurable space $\left(\Omega^{(1)}, \mathcal{F}^{(1)}\right)$, is the value of the underlying asset at time t. The process $\left(S^{(1)}(t)\right)_{0\leq t\leq T}$ is supposed to be adapted to the filtration $\left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}$, meaning that for any time t, the r.v. $S^{(1)}(t)$ is $\mathcal{F}_t^{(1)}$-measurable; i.e. the value $S^{(1)}(t)$ of the asset at time t is known at time t. The increase of the sequence of σ-algebras implies that at any time t, all asset prices $S^{(1)}(s), 0 \leq s \leq t$, will be known as well. We call stochastic processes such as $\left(S^{(1)}(t)\right)_{0\leq t\leq T}$ *financial stochastic processes,* while we call the underlying r.v.'s $S^{(1)}(t)$ *financial risks.*

We assume that the financial world contains a bank account where any market participant can borrow or lend cash. We introduce the notation $r(t)$ for the (random) instantaneous risk-free interest rate, also called the short rate, at time t. We consider the financial stochastic price process $\left(e^{\int_0^t r(\tau)\, d\tau}\right)_{0\leq t\leq T}$, where $e^{\int_0^t r(\tau)\, d\tau}$ is the value at time t of an investment of amount 1 made in the bank account at time 0.

Next, we introduce a second filtered probability space $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}, \mathbb{P}^{(2)}\right)$ for describing the relevant biometrical evolutions in the time interval $[0, T]$. We call this space the biometrical world. In case we are interested in the future mortality of a group of insureds e.g., we define $\Omega^{(2)}$ as the non-empty set of all possible scenarios concerning the evolution of the mortality of the group under consideration in the time interval $[0, T]$. The σ-algebra $\mathcal{F}^{(2)}$ is the set consisting of the empty set and all events which may or may not occur in the biometrical world, or equivalently, the set of all statements which can be made about the biometrical evolution in $[0, T]$. The filtration $\left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}^{(2)}$, where any $\mathcal{F}_t^{(2)}$ describes all relevant information about death and survival of the considered population up to and including time t. The physical probability measure $\mathbb{P}^{(2)}$ attaches the 'real-world' probability $\mathbb{P}^{(2)}\left[A^{(2)}\right]$ to any biometrical

[1]These stochastic processes are assumed to be semi-martingales in order to be able to define self-financing strategies rigorously, see e.g. Delbaen and Schachermayer (2008).

event $A^{(2)} \in \mathcal{F}^{(2)}$. Again, it is convenient to assume that the filtered probability space satisfies the "usual conditions" of completeness and right-continuity.

Stochastic processes (which are assumed to be semi-martingales) adapted to the filtered measurable space $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}\right)$ are called *biometrical stochastic processes,* while the underlying r.v.'s are called *biometrical risks.* E.g., for each person (x) we can define an adapted stochastic process $\left(S_{(x)}^{(2)}(t)\right)_{0\leq t\leq T}$ on the filtered measurable space $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}\right)$, consisting of indicator variables $S_{(x)}^{(2)}(t)$ which are 1 for any time t that the person (x) is alive and 0 from the moment this person passes away.

In order to describe the combined financial and biometrical evolution over time, we introduce the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$, which is defined as the product space of the financial and the biometrical spaces:

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right) = \left(\Omega^{(1)} \times \Omega^{(2)}, \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(1)} \otimes \mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}, \mathbb{P}\right).$$

Any $(\omega_1, \omega_2) \in \Omega = \Omega^{(1)} \times \Omega^{(2)}$ corresponds to a feasible scenario of randomness concerning the financial and the biometrical evolution in the time interval under consideration. The σ-algebra $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ is the set consisting of the empty set and all events which may or may not occur in the combined financial and biometrical world in the time interval $[0, T]$. Furthermore, the filtration $(\mathcal{F}_t)_{0\leq t\leq T} = \left(\mathcal{F}_t^{(1)} \otimes \mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$, where any $\mathcal{F}_t$ describes all financial and biometrical information available up to and including time t.[2] Finally, the physical probability measure $\mathbb{P}$ is such that

$$\mathbb{P}\left[A^{(1)} \times \Omega^{(2)}\right] = \mathbb{P}^{(1)}\left[A^{(1)}\right] \text{ for any } A^{(1)} \in \mathcal{F}^{(1)}$$

and

$$\mathbb{P}\left[\Omega^{(1)} \times A^{(2)}\right] = \mathbb{P}^{(2)}\left[A^{(2)}\right] \text{ for any } A^{(2)} \in \mathcal{F}^{(2)}.$$

This means that the projections of $\mathbb{P}$ to the financial and the biometrical world coincide with $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$, respectively. In case both subworlds satisfy the "usual conditions" of completeness and right-continuity, then the same conditions apply for the combined financial-biometrical world.

Any financial risk $X^{(1)}$ defined on $\left(\Omega^{(1)}, \mathcal{F}^{(1)}\right)$ can be considered as a r.v. on the combined space $(\Omega, \mathcal{F})$ by letting

$$X^{(1)}(\omega_1, \omega_2) \equiv X^{(1)}(\omega_1), \qquad \text{for all } (\omega_1, \omega_2) \in \Omega.$$

Similarly, any biometrical risk $X^{(2)}$ defined on $\left(\Omega^{(2)}, \mathcal{F}^{(2)}\right)$ can be considered as an r.v. on $(\Omega, \mathcal{F})$ by letting

$$X^{(2)}(\omega_1, \omega_2) \equiv X^{(2)}(\omega_2), \qquad \text{for all } (\omega_1, \omega_2) \in \Omega.$$

[2] The σ-algebra $\mathcal{F}_t$ contains all sets $A^{(1)} \times A^{(2)}$ with $A^{(1)} \in \mathcal{F}_t^{(1)}$ and $A^{(2)} \in \mathcal{F}_t^{(2)}$, but is wider than the class consisting of these product sets. To be more precise, $\mathcal{F}_t$ is the completion w.r.t. $\mathbb{P}$ of the smallest σ-algebra generated by the set of all $A^{(1)} \times A^{(2)}$ with $A^{(1)} \in \mathcal{F}_t^{(1)}$ and $A^{(2)} \in \mathcal{F}_t^{(2)}$. A similar remark holds for the σ-algebra $\mathcal{F}$.

Apart from pure financial and pure biometrical risks, we will also consider combined financial-biometrical risks defined on $(\Omega, \mathcal{F})$, as well as combined financial-biometrical stochastic processes which are adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.[3]

Hereafter, we will consider the pricing of mortality-linked securities, i.e. financial-biometrical contracts, of which the payoffs depend on the evolution of interest rates and bond and stock prices as well as on the biometrical evolution (e.g. the survival experience of a group of persons under consideration) and which are traded (i.e. bought and sold) in the combined market. The price processes of these assets are described by stochastic processes which are adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ (and assumed to be semi-martingales).

Recall that a probability measure $\mathbb{Q}$ defined on the combined measurable space $(\Omega, \mathcal{F}_T)$ is said to be an equivalent martingale measure (or a risk-neutral measure) for the combined world if it fulfills the following two conditions:

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$.
2. For any traded asset in the combined world, one has that its discounted gain process is a martingale with respect to $\mathbb{Q}$.

The first condition means that $\mathbb{P}$ and $\mathbb{Q}$ agree on the events in $\mathcal{F}_T$ which cannot take place (i.e. $\mathbb{P}$ and $\mathbb{Q}$ have the same null sets in $\mathcal{F}_T$). The second condition means that the price at time t of any traded asset is equal to the conditional expectation of the value of its discounted gain process at time $u > t$, given the information available at time t. Here, the gain process of an asset is the sum of its price process and the process describing its accumulated dividends. Furthermore, the expectation is taken with respect to $\mathbb{Q}$, while accumulating and discounting are performed at the risk-free rate.

Let us assume that the combined market as described above is perfectly liquid, frictionless and arbitrage-free. From the First Fundamental Theorem of Asset Pricing we know that the no-arbitrage condition is 'essentially equivalent' to the existence of an equivalent martingale measure $\mathbb{Q}$. For a detailed and rigorous discussion of this fundamental theorem in mathematical finance, we refer to Delbaen and Schachermayer (2008).

The price at time t of a traded mortality-linked payoff $H(u)$ at some fixed term $u \geq t$ is then equal to

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^u r(\tau)\, d\tau} \; H(u) \mid \mathcal{F}_t \right],$$

where the superscript $\mathbb{Q}$ in $\mathbb{E}^{\mathbb{Q}}$ is used to indicate that the expectation is taken with respect to the pricing measure $\mathbb{Q}$. An example of a such a contract is the one with payoff at time u given by $S^{(1)}(u) \times S^{(2)}(u)$, where $S^{(1)}$ is a non-dividend paying financial security

[3] For any Borel measurable function $f(x, y)$, any financial risk $S^{(1)}(t)$ and any biometrical risk $S^{(2)}(t)$, the combined risk $f\left(S^{(1)}(t), S^{(2)}(t)\right)$ is well-defined on the measurable space $(\Omega, \mathcal{F})$. In particular, we will consider r.v.'s of the form $S^{(1)}(t) \times S^{(2)}(t)$ and $\frac{S^{(2)}(t)}{S^{(1)}(t)}$, which correspond to the continuous, and hence Borel measurable functions $f(x, y) = xy$ and $f(x, y) = \frac{y}{x}$, respectively. A similar argument tells us that the r.v. $e^{-\int_0^t r(\tau)\, d\tau} \times S^{(1)}(t) \times S^{(2)}(t)$ is also well-defined on $(\Omega, \mathcal{F})$.

and $S^{(2)}$ a non-dividend paying security in the biometrical world. In case this combined security is traded in the market, its no-arbitrage price at time t is given by

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^u r(\tau)\,d\tau}\left(S^{(1)}(u)\times S^{(2)}(u)\right)\mid\mathcal{F}_t\right].$$

The equivalent measure $\mathbb{Q}$ for the real-world probability measure $\mathbb{P}$ naturally leads to the following projected measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ defined on $\left(\Omega^{(1)},\mathcal{F}^{(1)}\right)$ and $\left(\Omega^{(2)},\mathcal{F}^{(2)}\right)$, respectively:

$$\mathbb{Q}^{(1)}\left[A^{(1)}\right]=\mathbb{Q}\left[A^{(1)}\times\Omega^{(2)}\right]\text{ for any }A^{(1)}\in\mathcal{F}^{(1)}$$

and

$$\mathbb{Q}^{(2)}\left[A^{(2)}\right]=\mathbb{Q}\left[\Omega^{(1)}\times A^{(2)}\right]\text{ for any }A^{(2)}\in\mathcal{F}^{(2)}.$$

The probability measures $\mathbb{Q}^{(1)}$ and $\mathbb{P}^{(1)}$ are equivalent. Indeed, for any $A^{(1)}\in\mathcal{F}^{(1)}$ it holds that

$$\mathbb{Q}^{(1)}\left[A^{(1)}\right]=0\Leftrightarrow\mathbb{Q}\left[A^{(1)}\times\Omega^{(2)}\right]=0\Leftrightarrow\mathbb{P}\left[A^{(1)}\times\Omega^{(2)}\right]=0\Leftrightarrow\mathbb{P}^{(1)}\left[A^{(1)}\right]=0.$$

In the same way, one can prove that $\mathbb{Q}^{(2)}$ and $\mathbb{P}^{(2)}$ are equivalent measures.

Consider a financial asset with price process $\left(S^{(1)}(t)\right)_{0\leq t\leq T}$ defined on the financial space $\left(\Omega^{(1)},\mathcal{F}^{(1)},\left(\mathcal{F}_t^{(1)}\right)_{0\leq t\leq T}\right)$. For any $0\leq s\leq t\leq T$, the r.v. $e^{-\int_s^t r(\tau)\,d\tau}\,S^{(1)}(t)$ is a function of the financial scenario $\omega_1\in\Omega^{(1)}$ that will unfold in $[0,T]$, but does not depend on the biometrical scenario $\omega_2\in\Omega^{(2)}$. Therefore, we have that

$$S^{(1)}(s)=\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_s^t r(\tau)\,d\tau}\;S^{(1)}(t)\mid\mathcal{F}_s\right]=\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\int_s^t r(\tau)\,d\tau}\;S^{(1)}(t)\mid\mathcal{F}_s^{(1)}\right],\tag{17}$$

which holds a.s. for any $0\leq s\leq t\leq T$.

From the considerations made above we find that the existence of an equivalent martingale measure in the combined market (which implies that this market is arbitrage-free) implies that also in the financial market there exists an equivalent martingale measure (which implies that also this sub-market is arbitrage-free). Indeed, the equivalent martingale measure $\mathbb{Q}$ for the combined world immediately leads to the equivalent martingale measure $\mathbb{Q}^{(1)}$ for the financial world. From (17), we see that for traded financial assets defined in the financial world, we can either price them in the combined world (using $\mathbb{Q}$ and $\mathcal{F}_s$) or price them in the financial world (using $\mathbb{Q}^{(1)}$ and $\mathcal{F}_s^{(1)}$).

Consider now a traded biometrical risk process $\left(S^{(2)}(t)\right)_{0\leq t\leq T}$ defined on the biometrical space $\left(\Omega^{(2)},\mathcal{F}^{(2)},\left(\mathcal{F}_t^{(2)}\right)_{0\leq t\leq T}\right)$. Then we have that

$$S^{(2)}(s)=\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_s^t r(\tau)\,d\tau}\;S^{(2)}(t)\mid\mathcal{F}_s\right]\tag{18}$$

holds a.s. for any $0\leq s\leq t\leq T$. Due to the presence of the random discount factor $e^{-\int_s^t r(\tau)\,d\tau}$ in (18), the price of traded biometrical risks can in general only be determined in the combined world. We remark that this conclusion only holds in case the risk-free interest rate is stochastic. In case of a deterministic risk-free interest rate, the equivalent martingale measure $\mathbb{Q}$ immediately leads to the equivalent martingale measure $\mathbb{Q}^{(2)}$ for the biometrical world, and the biometrical security can be priced in the biometrical subworld as well.

3.2 From real-world independence to pricing independence

In the remainder of this paper, we make the following convenient independence assumption:

$$\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}, \tag{19}$$

where $\mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$ is the probability measure defined by[4]

$$\left(\mathbb{P}^{(1)} \times \mathbb{P}^{(2)}\right)\left[A^{(1)} \times A^{(2)}\right] = \mathbb{P}^{(1)}\left[A^{(1)}\right] \times \mathbb{P}^{(2)}\left[A^{(2)}\right] \text{ for any } A^{(1)} \in \mathcal{F}^{(1)} \text{ and } A^{(2)} \in \mathcal{F}^{(2)}. \tag{20}$$

The assumption (19) implies that financial and biometrical risks or stochastic processes are mutually independent[5] under $\mathbb{P}$. This means that remaining lifetimes on the one hand and interest rates and stock prices on the other hand are mutually independent.

Starting from the equivalent martingale measure $\mathbb{Q}$ in the combined world and its projections $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ on the corresponding subworlds, we consider the probability measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ on the combined measurable space $(\Omega, \mathcal{F})$, which is defined by

$$\left(\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}\right)\left[A^{(1)} \times A^{(2)}\right] = \mathbb{Q}^{(1)}\left[A^{(1)}\right] \times \mathbb{Q}^{(2)}\left[A^{(2)}\right] \text{ for any } A^{(1)} \in \mathcal{F}^{(1)} \text{ and } A^{(2)} \in \mathcal{F}^{(2)}. \tag{21}$$

The independence assumption (20) implies that the measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ is equivalent to the physical measure $\mathbb{P}$.

In the literature, it is often assumed that under the pricing measure $\mathbb{Q}$, the dynamics of financial risks on the one hand and the dynamics of biometrical risks on the other hand are mutually independent in the sense that

$$\mathbb{Q} \equiv \mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}. \tag{22}$$

This assumption simplifies the discussion as it allows us to express the price of a security as the product of an expectation under the financial measure $\mathbb{Q}^{(1)}$ and an expectation under the biometrical measure $\mathbb{Q}^{(2)}$. As an illustration of this separation property, let us consider the asset with biometrical payoff $S^{(2)}(u)$ at time $u \geq t \geq 0$. Under the assumption (22), we find that its price at time t can be expressed as

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^u r(\tau)\,d\tau}\ S^{(2)}(u) \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\int_t^u r(\tau)\,d\tau} \mid \mathcal{F}_t^{(1)}\right] \times \mathbb{E}^{\mathbb{Q}^{(2)}}\left[S^{(2)}(u) \mid \mathcal{F}_t^{(2)}\right]. \tag{23}$$

The first term on the right hand side of this expression corresponds to the price at time t of the risk-free zero-coupon bond with payoff 1 at time u, while the second term corresponds

[4] Although $\mathcal{F}$ is broader than the class of all elements $A^{(1)} \times A^{(2)}$ where $A^{(1)} \in \mathcal{F}^{(1)}$ and $A^{(2)} \in \mathcal{F}^{(2)}$, it can be proven that $\mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$ is uniquely determined by only defining it on this class of events.

[5] Indeed, for any Borel measurable sets $B^{(1)}$ and $B^{(2)}$, we have that

$$\mathbb{P}\left[S^{(1)} \in B^{(1)}, S^{(2)} \in B^{(2)}\right] = \mathbb{P}\left[A^{(1)} \times A^{(2)}\right] = \mathbb{P}^{(1)}\left[A^{(1)}\right] \times \mathbb{P}^{(2)}\left[A^{(2)}\right] = \mathbb{P}\left[S^{(1)} \in B^{(1)}\right] \times \mathbb{P}\left[S^{(2)} \in B^{(2)}\right]$$

with

$$A^{(i)} = \left\{\omega_i \mid S^{(i)}\left(\omega_i\right) \in B^{(i)}\right\}.$$

to the expected value of the biometrical payoff, determined under the measure $\mathbb{Q}^{(2)}$ and taking into account the biometrical information that is available at time t.

Intuitively stated, incompleteness of the combined world means that there are not enough different types of assets traded to be able to hedge the monetary consequences of any risk that exists in this world. Loosely speaking, the Second Fundamental Theorem of Asset Pricing states that the condition of completeness of the arbitrage-free combined world is 'essentially equivalent' to the existence of a unique equivalent martingale measure. For a rigorous formulation of the Second Fundamental Theorem, we again refer to Delbaen and Schachermayer (2008). Notice that the completeness property of the combined world is highly questionable. This is, in particular, true because of the lack of traded assets with a biometrical payoff.

In an incomplete market, several pricing measures are feasible, implying that $\mathbb{P}$-world independence will not necessarily lead to $\mathbb{Q}$-world independence. As $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, but further unrelated probability measures, there is almost no link between the two notions of independence. This implies that not only in an incomplete but also in a complete combined world, it may happen that the $\mathbb{Q}$-measure that is chosen by the market does not exhibit the independence property between financial and biometrical risks, although the $\mathbb{P}$-measure does.

Hereafter, we will look for conditions under which a pricing measure with independence between financial and biometrical risks is feasible. Therefore, let $\mathbb{Q}$ be an equivalent martingale measure for the combined world. From the real-world independence assumption (20), we know that the measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ is equivalent to $\mathbb{P}$. Let us now additionally assume that the risk-free interest rate is deterministic and that there are no combined assets traded in the market. These assumptions imply that $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ is also a martingale measure for the combined world. Indeed, consider e.g. the non-dividend paying security with biometrical payoff $S^{(2)}(u)$ at time $u \geq t \geq 0$. Its price $S^{(2)}(t)$ at time t can be expressed as

$$\begin{aligned}
S^{(2)}(t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^u r(\tau)\,d\tau}\; S^{(2)}(u) \mid \mathcal{F}_t\right] \\
&= e^{-\int_t^u r(\tau)\,d\tau}\; \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(u) \mid \mathcal{F}_t\right] \\
&= e^{-\int_t^u r(\tau)\,d\tau}\; \mathbb{E}^{\mathbb{Q}^{(2)}}\left[S^{(2)}(u) \mid \mathcal{F}_t^{(2)}\right] \\
&= e^{-\int_t^u r(\tau)\,d\tau}\; \mathbb{E}^{\mathbb{Q}^{(1)}\times\mathbb{Q}^{(2)}}\left[S^{(2)}(u) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}^{(1)}\times\mathbb{Q}^{(2)}}\left[e^{-\int_t^u r(\tau)\,d\tau} \times S^{(2)}(u) \mid \mathcal{F}_t\right].
\end{aligned}$$

Hence, the stochastic process $\left(e^{-\int_0^t r(\tau)\,d\tau}\; S^{(2)}(t)\right)_{0\leq t\leq T}$ is a martingale with respect to $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$. A similar argument holds for non-dividend paying securities with financial payoff $S^{(1)}(u)$. We can conclude that under the stated assumptions, the measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$ is a martingale measure for the combined world. We summarize this result in the following theorem.

Theorem 1 *Consider a combined world where the following assumptions hold:*
(1) Under the real-world measure $\mathbb{P}$, financial and biometrical risks are independent in the sense that $\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$.
(2) The risk-free interest rate is deterministic.
(3) No combined assets are traded in the market.
(4) The market is arbitrage-free in the sense that there exists an equivalent martingale measure $\mathbb{Q}$.
The stated assumptions imply that the measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$, derived from $\mathbb{Q}$, is an equivalent martingale measure for the combined world, under which financial and biometrical risks are independent.

Under the conditions of the theorem, the observed asset prices only reveal information about the projections $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$, while the $\mathbb{Q}$-world dependence structure between financial and biometrical risks remains unspecified. As a consequence, there are infinitely many pricing measures possible, implying that the combined market is incomplete.

Often in the literature, one starts from the observation that in the combined world, it may be reasonable to assume that financial and biometrical risks are independent under the physical probability measure $\mathbb{P}$. In a next step, one then postulates a pricing measure $\mathbb{Q}$ for the combined world under which financial and biometrical risks are independent. From the discussion above, we know that there is no implicative relation between $\mathbb{P}$- and $\mathbb{Q}$-world independence, so that the occurrence of the first type of independence cannot be used as a valuable argument for making a $\mathbb{Q}$-world independence assumption. Notice however that this wrong deduction is implicitly made here and there in the literature. Moreover, a pricing measure $\mathbb{Q}$ under which financial and biometrical risks are independent may even be non-existent, so that one has to be careful about making such an assumption. We refer to the examples discussed in Sections 2.4 and 2.5 to illustrate this phenomenon in a discrete setting.

The discussions made above about non-dividend paying securities can be generalized to the dividend paying case by considering the appropriate gain processes when taking expectations under the $\mathbb{Q}$-measure. In the following section, we consider dividend paying securities in a Black & Scholes setting.

4 A combined financial-biometrical world in a Black & Scholes-setting

The idea of maintaining the dependence structure between risks when moving from the real world to the pricing world is not without any foundation. Indeed, in a multivariate Black & Scholes-setting, independence relations between stock values under $\mathbb{P}$ translate in independence relations under $\mathbb{Q}$, and vice versa.

In order to illustrate this phenomenon, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a correlated Brownian motion process $\left\{ \left(B^{(1)}(t), B^{(2)}(t) \right) \mid 0 \leq t \leq T \right\}$. Conditioned on time 0, the dependence structure between the standard Brownian motion processes $\left\{B^{(i)}(t) \mid t \geq 0\right\}$ is captured by

$$\mathrm{Cov}_{\mathbb{P}}\left[B^{(1)}(t), B^{(2)}(t+s)\right] = \rho t, \qquad \text{for any } t, s \geq 0,$$

with $\rho \in [-1, +1]$ being the correlation coefficient of the random couple $\left(B^{(1)}(t), B^{(2)}(t)\right)$.

The probability space is equipped with the 'natural filtration' $(\mathcal{F}_t)_{0 \leq t \leq T}$ induced by $\left\{\left(B^{(1)}(t), B^{(2)}(t)\right) \mid 0 \leq t \leq T\right\}$. This means that $\mathcal{F}_t$, $0 \leq t \leq T$, is the σ-algebra $\sigma\left(\left(B^{(1)}(s), B^{(2)}(s)\right), s \leq t\right)$, completed with the $\mathbb{P}$-null sets of $\mathcal{F}$. Intuitively stated, $\mathcal{F}_t$ is the sub-σ-algebra of $\mathcal{F}$ generated by all values of the correlated Brownian motions up to and including time t, and the natural filtration records the 'past behavior' of the bivariate Brownian motion process. In order to avoid trivialities, we will always assume that $\mathcal{F}$ (i.e. the original σ-algebra of all random events) is identical to $\mathcal{F}_T$.

Suppose that the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right)$ is home to a market with a constant risk-free interest rate r, in which a financial asset (denoted by superscript $^{(1)}$) and a biometrical asset (denoted by superscript $^{(2)}$) are traded. The real-world or $\mathbb{P}$-dynamics of these assets are described by

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = \left(\mu^{(i)} - \delta^{(i)}\right) dt + \sigma^{(i)} dB^{(i)}(t), \qquad \text{for } i = 1, 2,$$

for $t > 0$, while $S^{(i)}(0)$ is the given price of asset i at time 0. The parameters in these equations are the drifts $\mu^{(i)} > 0$, volatilities $\sigma^{(i)} > 0$ and dividend payment rates $\delta^{(i)} > 0$ of the respective assets.

Let us now assume that this Black & Scholes market is arbitrage-free. Then it is well-known that there exists a unique equivalent martingale measure on $(\Omega, \mathcal{F}_T)$ and that the risk-neutral pricing dynamics of the traded assets under this measure $\mathbb{Q}$ are given by

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = \left(r - \delta^{(i)}\right) dt + \sigma^{(i)} dW^{(i)}(t), \qquad \text{for } i = 1, 2,$$

see e.g. Dhaene et al. (2013). Here, the process $\left\{\left(W^{(1)}(t), W^{(2)}(t)\right) \mid 0 \leq t \leq T\right\}$ is a two-dimensional correlated Brownian motion process defined on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q}\right)$, with the unconditional dependence structure between the standard Brownian motion processes $\left\{W^{(i)}(t) \mid t \geq 0\right\}$ captured by

$$\mathrm{Cov}_{\mathbb{Q}}\left[W^{(1)}(t), W^{(2)}(t+s)\right] = \mathrm{Cov}_{\mathbb{P}}\left[B^{(1)}(t), B^{(2)}(t+s)\right] = \rho t, \qquad \text{for any } t, s \geq 0.$$

Since $\left(B^{(1)}(t), B^{(2)}(t)\right)$ and $\left(W^{(1)}(t), W^{(2)}(t)\right)$ are both bivariate Brownian motions, under the measures $\mathbb{P}$ and $\mathbb{Q}$ respectively, they have the same bivariate normal distribution for any $t > 0$. Furthermore, one has that

$$\begin{aligned}\mathrm{Corr}_{\mathbb{P}}\left[S^{(1)}(t), S^{(2)}(t)\right] &= \mathrm{Corr}_{\mathbb{Q}}\left[S^{(1)}(t), S^{(2)}(t)\right] \\ &= \frac{e^{\rho\sigma^{(1)}\sigma^{(2)}t} - 1}{\sqrt{e^{\left(\sigma^{(1)}\right)^2 t} - 1}\sqrt{e^{\left(\sigma^{(2)}\right)^2 t} - 1}}\end{aligned}$$

holds for any $t > 0$.

Notice that for a given $t > 0$, the independence of $S^{(1)}(t)$ and $S^{(2)}(t)$ under $\mathbb{P}$ (resp. $\mathbb{Q}$) is equivalent to independence of $B^{(1)}(t)$ and $B^{(2)}(t)$ under $\mathbb{P}$ (resp. of $W^{(1)}(t)$ and $W^{(2)}(t)$ under $\mathbb{Q}$), which in turn is equivalent to $\rho = 0$. Hence, the statements

$$\mathbb{P}\left[S^{(1)}(t) \leq s_1, S^{(2)}(t) \leq s_2\right] = \mathbb{P}\left[S^{(1)}(t) \leq s_1\right] \times \mathbb{P}\left[S^{(2)}(t) \leq s_2\right]$$

and

$$\mathbb{Q}\left[S^{(1)}(t) \leq s_1, S^{(2)}(t) \leq s_2\right] = \mathbb{Q}\left[S^{(1)}(t) \leq s_1\right] \times \mathbb{Q}\left[S^{(2)}(t) \leq s_2\right]$$

are equivalent, which means that independence of $S^{(1)}(t)$ and $S^{(2)}(t)$ under $\mathbb{P}$ is equivalent to independence of $S^{(1)}(t)$ and $S^{(2)}(t)$ under $\mathbb{Q}$. Moreover, as the condition $\rho = 0$ does not depend on t, independence of $S^{(1)}(t)$ and $S^{(2)}(t)$ for a particular value of $t > 0$ is equivalent to independence of $S^{(1)}(t)$ and $S^{(2)}(t)$ for any $t > 0$.

The above-mentioned statements about the independence case, i.e. the independent copula, can easily be generalized to any dependence structure or copula. Indeed, we have that

$$\left(S^{(1)}(t), S^{(2)}(t)\right) \stackrel{\mathbb{P}}{=} \left(f_t^{(1)}\left(B^{(1)}(t)\right), f_t^{(2)}\left(B^{(2)}(t)\right)\right),$$

where $\stackrel{\mathbb{P}}{=}$ stands for 'equality in distribution under the measure $\mathbb{P}$'. The functions $f_t^{(i)}$, which are given by

$$f_t^{(i)}(s) = S^{(i)}(0) \exp\left\{\left(\mu^{(i)} - \delta^{(i)} - \frac{1}{2}\left(\sigma^{(i)}\right)^2\right) t + \sigma^{(i)} s\right\}, \qquad \text{for } i = 1, 2,$$

are strictly increasing and continuous functions of s. Therefore, under the measure $\mathbb{P}$, the vector $\left(S^{(1)}(t), S^{(2)}(t)\right)$ has the same copula as the vector $\left(B^{(1)}(t), B^{(2)}(t)\right)$, see e.g. Proposition 4.4.4 in Denuit et al. (2005). Furthermore,

$$\left(S^{(1)}(t), S^{(2)}(t)\right) \stackrel{\mathbb{Q}}{=} \left(g_t^{(1)}\left(W^{(1)}(t)\right), g_t^{(2)}\left(W^{(2)}(t)\right)\right),$$

where $\stackrel{\mathbb{Q}}{=}$ stands for 'equality in distribution under the measure $\mathbb{Q}$'. The functions $g_t^{(i)}$, which are given by

$$g_t^{(i)}(s) = S^{(i)}(0) \exp\left\{\left(r - \delta^{(i)} - \frac{1}{2}\left(\sigma^{(i)}\right)^2\right) t + \sigma^{(i)} s\right\}, \qquad \text{for } i = 1, 2,$$

are again strictly increasing and continuous functions of s. This means that under the measure $\mathbb{Q}$, the vector $\left(S^{(1)}(t), S^{(2)}(t)\right)$ has the same copula as the vector $\left(W^{(1)}(t), W^{(2)}(t)\right)$. We can conclude that the copula of $\left(S^{(1)}(t), S^{(2)}(t)\right)$ is not changed when moving from the real world to the pricing world.

We remark that

$$\left(B^{(1)}(t), B^{(2)}(t)\right) \stackrel{\mathbb{P}}{=} \left(\sqrt{t}B^{(1)}(1), \sqrt{t}B^{(2)}(1)\right)$$

and

$$\left(W^{(1)}(t), W^{(2)}(t)\right) \stackrel{\mathbb{Q}}{=} \left(\sqrt{t}W^{(1)}(1), \sqrt{t}W^{(2)}(1)\right).$$

As a consequence, the copula connecting the marginals of $\left(S^{(1)}(t), S^{(2)}(t)\right)$ is the same for any $t > 0$, in both the $\mathbb{P}$- and the $\mathbb{Q}$-world.

Notice that although correlations and copulas of $\left(S^{(1)}(t), S^{(2)}(t)\right)$ are unchanged in the Black & Scholes model when moving from the real world to the pricing world, we have that $\mathrm{Var}_{\mathbb{P}}\left[S^{(1)}(t)\right]$ is in general different from $\mathrm{Var}_{\mathbb{Q}}\left[S^{(1)}(t)\right]$. However, the corresponding variances, correlations and copulas of the log prices $\ln S^{(1)}(t)$ and $\ln S^{(2)}(t)$ are unaffected when changing the real-world measure into the pricing measure.

As shown in the previous sections, the equality of the $\mathbb{P}$- and the $\mathbb{Q}$-copula of a random couple $\left(S^{(1)}(t), S^{(2)}(t)\right)$ which holds in the Black & Scholes model is certainly not maintained for general continuous- or discrete-time asset pricing models. An exception is the comonotonic copula. Indeed, consider a general asset pricing model and suppose that the couple $\left(S^{(1)}(t), S^{(2)}(t)\right)$ is $\mathbb{P}$-comonotonic. This statement is equivalent to the existence of a comonotonic set A such that $\mathbb{P}\left[\left(S^{(1)}(t), S^{(2)}(t)\right) \in A\right] = 1$, see e.g. Dhaene et al. (2002a) and Dhaene et al. (2002b). As a support in the $\mathbb{P}$-world is also a support in the equivalent $\mathbb{Q}$-world, we can conclude that $\left(S^{(1)}(t), S^{(2)}(t)\right)$ is also $\mathbb{Q}$-comonotonic.

5 Conclusion

In this note, we investigated mutual independence between financial and biometrical risks, which live in a combined financial-biometrical world. Although the independence between such risks in the physical world with measure $\mathbb{P}$ may often be reasonable and has an intuitively clear meaning, it is not obvious what is meant by mutual independence under a pricing measure $\mathbb{Q}$. In a Black & Scholes-setting, independence relations between r.v.'s under $\mathbb{P}$ go along with independence relations between these r.v.'s under $\mathbb{Q}$, and vice versa. In a general pricing model however, this equivalence relation is not maintained.

The examples and results considered in this paper were presented in a combined financial-biometrical world, although they can immediately be restated in a more general setting of a combined financial-actuarial world, where the financial world is as described above and the actuarial world is home to general actuarial risks, such as catastrophic non-life insurance risk.

We did not discuss the appropriateness of the real-world independence assumption. The assumption of independence between interest rates and mortality might be reasonable for a relatively short time horizon. On the other hand, Jalen and Mamon (2009) state that "in the long run, interest rates can be influenced by the relative size of the population, which in turn, is influenced by mortality development (as well as fertility)". Concerning the independence assumption between financial and catastrophic risks, these authors mention that "in the short term, a catastrophic event that seriously affects the size of the population, such as major natural disasters or a nuclear war, can also affect interest rates".

In this study, we only considered $\mathbb{P}$- versus $\mathbb{Q}$-world independence in a combined financial-actuarial world. General dependence structures or copulas in the $\mathbb{P}$-world and their relation with the dependence structure or copula under a corresponding $\mathbb{Q}$-measure are a topic for future research.

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References

Delbaen, F. and Schachermayer, W. (2008), *The mathematics of arbitrage*, Springer. correcter second printing, pp. 373.

Denuit, M., Dhaene, J., Goovaerts, M. J. and Kaas, R. (2005), *Actuarial theory for dependent risks: measures, orders and models*, Wiley.

Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vyncke, D. (2002a), 'The concept of comonotonicity in actuarial science and finance: theory', *Insurance: Mathematics & Economics* **31**(1), 3–33.

Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vyncke, D. (2002b), 'The concept of comonotonicity in actuarial science and finance: applications', *Insurance: Mathematics & Economics* **31**(2), 133–161.

Dhaene, J., Kukush, A. and Linders, D. (2013), 'The multivariate Black & Scholes market: conditions for completeness and no-arbitrage', *Theory of Probability and Mathematical Statistics (in press)* .

Gorvett, R. W. (1999), 'Insurance securitization: The development of a new asset class', *Casualty Actuarial Society* pp. 133 – 172.

Jalen, L. and Mamon, R. (2009), 'Valuation of contingent claims with mortality and interest rate risks', *Mathematical and Computer Modelling* **49**(9-10), 1893–1904.

Shiryaev, A. N., Kabanov, Y. M., Kramkov, O. D. and Melnikov, A. V. (1994), 'Towards the theory of pricing of options of both European and American types. I. Discrete time', *Theory of Probability and Its Applications* **39**(1), 23–79.