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A BOREL OPEN COVER OF THE HILBERT SCHEME

CRISTINA BERTONE, PAOLO LELLA, AND MARGHERITA ROGGERO

Abstract. Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ of degree d. The Hilbert scheme $\mathcal{H}\text{ilb}^n_{p(t)}$ can be realized as a closed subscheme of a suitable Grassmannian $\mathbb{G}$, hence it could be globally defined by homogeneous equations in the Plücker coordinates of $\mathbb{G}$ and covered by open subsets given by the non-vanishing of a Plücker coordinate, each embedded as a closed subscheme of the affine space $\mathbb{A}^D$, $D = \dim(\mathbb{G})$. However, the number E of Plücker coordinates is so large that effective computations in this setting are practically impossible. In this paper, taking advantage of the symmetries of $\mathcal{H}\text{ilb}^n_{p(t)}$, we exhibit a new open cover, consisting of marked schemes over Borel-fixed ideals, whose number is significantly smaller than E. Exploiting the properties of marked schemes, we prove that these open subsets are defined by equations of degree $\leq d+2$ in their natural embedding in $\mathbb{A}^D$. Furthermore we find new embeddings in affine spaces of far lower dimension than D, and characterize those that are still defined by equations of degree $\leq d+2$. The proofs are constructive and use a polynomial reduction process, similar to the one for Gröbner bases, but are term order free. In this new setting, we can achieve explicit computations in many non-trivial cases.

Introduction

The Hilbert scheme, associated to the projective space $\mathbb{P}^n$ and to an admissible Hilbert polynomial $p(t)$, was first introduced by [15] and parametrizes the set of all subschemes Z of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$. We use $\mathcal{H}\text{ilb}^n_{p(t)}$ for $\mathcal{H}\text{ilb}_{p(t)}(\mathbb{P}^n)$ throughout the paper. The aim of this paper, and of some other related ones (see [8, 9, 20]), is to find effective methods that allow explicit computations on $\mathcal{H}\text{ilb}^n_{p(t)}$. We choose the following definition of Hilbert scheme since it is the one that best suits this purpose.

Let $S = K[x_0, \ldots, x_n]$ be a polynomial ring in $n+1$ variables with coefficients in a field K of characteristic 0 and let S_m be the homogeneous component of S of degree m. We denote by r the Gotzmann number of $p(t)$, whose definition and properties were first established by [13], and by $q(t)$ the Gotzmann (or volume) polynomial $N(t) - p(t) = \dim_K(S_t) - p(t)$. Every subscheme Z of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ is defined by a homogeneous saturated ideal $\mathcal{I}_Z \subset S$ whose homogeneous component of degree r is a k-vector space of dimension $s := q(r)$. It can be proved that the vector space $\mathcal{I}_Z \cap S_r$ generates $(\mathcal{I}_Z)_{\geq r}$ and then completely determines Z; by an abuse of notation we will write $I \in \mathcal{H}\text{ilb}^n_{p(t)}$ meaning that $\text{Proj}\,(S/I) \in \mathcal{H}\text{ilb}^n_{p(t)}$. Further, using the correspondence $\mathcal{I}_Z \mapsto (\mathcal{I}_Z)_r$, the Hilbert scheme can be realized as a closed subscheme of the Grassmannian $\mathbb{G} = \mathbb{G}(s, S_r)$ of linear spaces of dimension s in S_r. A set of generators for the ideal defining $\mathcal{H}\text{ilb}^n_{p(t)}$ in the ring of Plücker coordinates of $\mathbb{G}$ can be obtained by imposing on an s-dimensional subspace V of S_r the condition that the ideal $(V) \subset S$ has the prescribed Hilbert polynomial $q(t)$. Explicitly, by Gotzmann theorems of minimal growth [13], it is sufficient to impose that $\dim_K S_1V$ is exactly $q(r+1)$. This setting dates back to [15], [22] (regularity) and [13] (regularity and persistence), and it is fully described in [17] and also in [7] and [2].

This way to define the Hilbert scheme is constructive, because it relies on linear algebra. Nevertheless, explicit computations were not carried out until now, except in trivial cases, because the number

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E of Plücker coordinates is almost always huge. For instance, for $n = 3$ and $p(t) = 4t$ the Gotzmann number is 6 and $\mathcal{H}\text{ilb}^3_{4t}$ becomes a subscheme of $\mathbb{G}(60, 84)$; a rough computation shows that the number E of Plücker coordinates is $\sim 6 \cdot 10^{20}$.

In spite of these difficulties, many authors have dealt with the problem of determining a set of explicit equations for $\mathcal{H}\text{ilb}^n_{p(t)}$ as a subscheme of $\mathbb{G}$. For instance in [17], [3], [16], and more recently in [1] and [6] the authors determine upper bounds for the degree of special sets of equations in the Plücker coordinates of $\mathbb{G}$ defining scheme-theoretically $\mathcal{H}\text{ilb}^n_{p(t)}$. In particular in [6] the authors obtain a set of equations for $\mathcal{H}\text{ilb}^n_{p(t)}$ of degree $\leq \deg(p(t)) + 2$, lower than the previous ones. However, an explicit and computable presentation of $\mathcal{H}\text{ilb}^n_{p(t)}$ is not yet easily achieved from this setting.
A simpler way to address this issue is to consider an open cover of $\mathbb{G}$, consisting of cells of maximal dimension $D = p(r)q(r)$, and study the induced open cover of $\mathcal{H}\text{ilb}^n_{p(t)}$. Each Plücker coordinate can be naturally coupled to a monomial ideal $J \subseteq S$ generated by s linearly independent monomials of degree r. If Δ_J denotes the Plücker coordinate corresponding to J and $\mathcal{U}_J \simeq \mathbb{A}^D$ is the open cell of $\mathbb{G}$ given by $\Delta_J \neq 0$, it will be sufficient to compute equations for $\mathcal{H}_J = \mathcal{H}\text{ilb}^n_{p(t)} \cap \mathcal{U}_J$ as a closed subscheme of $\mathbb{A}^D$.

However, in order to make this strategy truly effective, we must overcome two main difficulties: the first one is finding an open cover of $\mathcal{H}\text{ilb}^n_{p(t)}$ consisting of a reasonably small number of open sets $\mathcal{H}_J$; the second one is finding an efficient way to compute equations for $\mathcal{H}_J$. We are able to achieve both goals using monomial ideals which have strong combinatorial properties and a long history in the theory of Hilbert schemes: the *Borel-fixed ideals* (see Definition 1.1).

In Section 1 we prove, for monomial ideals J of this type, the following result (Theorem 1.13) that establishes a close relation between the open subset $\mathcal{H}_J$ of $\mathcal{H}\text{ilb}^n_{p(t)}$ and the *J-marked scheme* $\mathcal{M}\text{f}(J)$ introduced in [9]:

Theorem. *Let J be a Borel-fixed ideal, such that S/J has Hilbert polynomial $p(t)$, let r be the Gotzmann number of $p(t)$. Furthermore, assume that $J = {J^{\text{sat}}}_{\geq r}$. Then there is a commutative diagram of morphisms of schemes:*

$$\begin{array}{ccc} \mathcal{H}_J & \xrightarrow{\sim} & \mathcal{M}\text{f}(J) \\ \downarrow & & \downarrow \\ \mathcal{U}_J & \xrightarrow{\sim} & \mathbb{A}^D \end{array}$$

The main results of the present paper strongly rely on the features and properties of marked schemes proved in [9] and [5].

In Section 2, we show that the open subsets $\mathcal{H}_J$ corresponding to Borel-fixed ideals J with Hilbert polynomial $q(t)$, are sufficient to define an open cover of $\mathcal{H}\text{ilb}^n_{p(t)}$, up to changes of coordinates in $\mathbb{P}^n$, while if J is Borel but its Hilbert polynomial is not $q(t)$, then $\mathcal{H}_J = \emptyset$ (cf. Theorem 2.5). Furthermore we present an explicit algorithm that, given an ideal I defining a point of $\mathcal{H}\text{ilb}^n_{p(t)}$, returns a suitable change of coordinates g of $\mathbb{P}^n$ and a Borel-fixed ideal J such that $I^g \in \mathcal{H}_J$. The algorithm exploits results of [8] and of [19] to compute the list of Borel-fixed ideals belonging to $\mathcal{H}\text{ilb}^n_{p(t)}$.

In Section 3 we investigate the scheme structure of $\mathcal{H}_J$ based on the isomorphism with $\mathcal{M}\text{f}(J)$ given in Theorem 1.13. The results about the theory of marked schemes obtained in [9] allow us to prove that $\mathcal{H}_J$ can be scheme theoretically defined by equations of degree $\leq \deg(p(t)) + 2$ as a subscheme of $\mathbb{A}^D$. The improvements introduced in [5] lead to the definition of new embeddings of $\mathcal{H}_J$ in affine spaces of dimension lower than D (cf. Theorem 3.1). Among them, we determine those defined by equations of degree $\leq \deg(p(t)) + 2$ and compute upper bounds for the number of such defining equations (cf.

Theorem 3.3); however the embedding corresponding to an affine space $\mathbb{A}^{D'}$ of lowest dimension D' may be given by equations of very high degree.

Summing up, we obtain an open cover of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ such that each open subset can be realized as a subscheme of a suitable affine space by a bounded number of equations of degree $\leq \deg(p(t)) + 2$ (see Theorem 3.3) .

We emphasize that our proofs are constructive because they are based on the features of some special polynomial reduction processes introduced in [9] and in [5]. These are similar to ones for Gröbner bases, but are term order free: hence we can achieve explicit computations in many non-trivial cases. We apply our results to some examples in the final Section 4.

1. Hilbert scheme, Borel-fixed ideals and marked schemes

In this section we first introduce the notations used throughout the paper and present the three protagonists of the paper: first of all the Hilbert scheme parameterizing subschemes of $\mathbb{P}^n$ with a prescribed Hilbert polynomial, then the Borel-fixed ideals with their main features and finally the J-marked scheme, with J a Borel ideal. At the end of the section (Theorem 1.13) we will establish a first relation among them.

Throughout the paper K is a field of characteristic 0, and $S := K[x_0, \ldots, x_n]$ is the polynomial ring over K in the set of variables $x_0, \ldots, x_n$. The elements and ideals that we consider are always homogeneous.

An ideal J in S is *monomial* if it is generated by monomials. If $J \subset S$ is a monomial ideal, then it has a unique minimal set of monomial generators [21, Lemma 1.2] that we call the *monomial basis of* J and denote by B_J.

1.1. **The Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$.** In the following, we consider a degree d Hilbert polynomial $p(t)$ in the projective space $\mathbb{P}^n = \mathrm{Proj}\,(S)$. $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ will denote the *Hilbert scheme* parameterizing all subschemes $Z \subset \mathbb{P}^n$ with Hilbert polynomial $p(t)$. We will denote by r the Gotzmann number of $p(t)$, that is the highest Castelnuovo-Mumford regularity for the subschemes parameterized by $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ (see [14, Theorem 3.11]), by $N(t)$ the binomial $\binom{n+t}{n}$ and by $q(t)$ the volume polynomial (or Gotzmann polynomial) $N(t) - p(t)$. Moreover, we set $s = q(r)$, $s' = q(r+1)$.

We now recall one of the ways to embed the Hilbert scheme as a closed subscheme of a Grassmannian of suitable dimension. This setting dates back to [15] and [13] and it is fully described in [17]; it can also be found in [7] and [2, Chapter IX]. This is only one of the several ways to embed the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ in a Grassmannian $\mathbb{G}(p(r), S_r)$ [15, 17, 16, 1, 6] or product of Grassmannians $\mathbb{G}(p(r), S_r) \times \mathbb{G}(p(r+1), S_{r+1})$ [13, 17]. For a comparison between the different sets of equations defining such embeddings of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, see [6].

Every subscheme $Z \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ can be defined by many different ideals I in S such that S/I has Hilbert polynomial $p(t)$; among them, there is the saturated ideal $\mathcal{I}_Z$, whose regularity is less than or equal r. We now consider $Z \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ as defined by the truncated ideal $I := (\mathcal{I}_Z)_{\geqslant r}$, which is generated by s linearly independent forms of degree r, therefore it is uniquely determined by a linear subspace of dimension s in the K-vector space S_r of dimension $N(r)$. Thus, $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ can be set-theoretically embedded in the Grassmannian $\mathbb{G} := \mathbb{G}(s, S_r)$ of all s-dimensional subspaces of S_r.

By abuse of notation, we will write $I \in \mathbb{G}$ if I is the ideal generated by the vector space $I_r \in \mathbb{G}$ and we will write $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ if $I \in \mathbb{G}$ and its Hilbert polynomial is the volume polynomial $q(t)$. We will denote by D the dimension of $\mathbb{G}$.

The scheme structure of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is defined by imposing to $I \in \mathbb{G}$ that the dimension of the vector space I_{r+1} is s' (see [17, Theorems C.17]). Moreover, by Macaulay's lower bound on growth, the

inequality $\dim_K(I_{r+1}) \geqslant s'$ is always true and therefore, the condition $\dim_K(I_{r+1}) = s'$ is in fact equivalent to $\dim_K(I_{r+1}) \leqslant s'$.

If we fix bases for the K-vector spaces S_m, $m \in \mathbb{N}$ (for instance the bases of monomials), every vector space V in S_m can be represented by a (not unique) matrix $M(V, m)$ whose rows contain the coefficients w.r.t. the fixed basis of S_m of a set of polynomials that generate V. In particular, every ideal $I \in \mathbb{G}$ can be represented by a $s \times N(r)$ matrix $M(I_r, r)$, whose minors of maximal order s are the Plücker coordinates of I. Moreover, by Macaulay's growth theorem, the rank of the matrix $M(I_{r+1} = S_1 I_r, r+1)$ in S_{r+1} is $\geq s'$. By the Gotzmann's Hilbert Scheme Theorem, equality holds if and only if $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$. In this way $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ can be defined by a homogeneous ideal in the ring of Plücker coordinates $K[\Delta]$ (see [6] for more details).

In the following we fix the set of monomials as basis of S_m for every $m \in \mathbb{N}$. Every Plücker coordinate Δ of $I \in \mathbb{G}$ corresponds to a subset of s elements of the fixed basis of S_r, so there is a one-to-one correspondence between the Plücker coordinates of the Grassmannian $\mathbb{G}$ and sets of s monomials of degree r: we will omit "r" and denote by $\Delta_J(I)$ (instead of $\Delta_{J_r}(I_r)$) the Plücker coordinate of $I \in \mathbb{G}$ corresponding to the monomial ideal $J \in \mathbb{G}$. In this paper Plücker coordinates corresponding to *Borel-fixed monomial ideals* (introduced in the next section) will take center stage.

1.2. **Borel-fixed ideals.** We will use the compact notation x for the set of variables $x_0, \ldots, x_n$ and x^α for monomials in S, where α represents a multi-index $(\alpha_0, \ldots, \alpha_n)$ of non-negative integers, that is $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. The symbol $x^\alpha \mid x^\gamma$ means that x^α divides x^γ, that is there exists a monomial x^β such that $x^\alpha \cdot x^\beta = x^\gamma$. If such monomial does not exist, we will write $x^\alpha \nmid x^\gamma$.

We consider the standard grading on the polynomial ring $S = \bigoplus_{m \in \mathbb{N}} S_m$, where S_m denotes the homogeneous component of degree m; let $S_{\geqslant m} = \oplus_{m' \geqslant m} (S_{m'})$ and in the same way, for every subset $\mathfrak{a} \subseteq S$, we will denote by $\mathfrak{a}_m$ and $\mathfrak{a}_{\geqslant m}$ the intersections of $\mathfrak{a}$ with S_m and $S_{\geqslant m}$ respectively.

We always order the variables of S in the following way: $x_0 < x_1 < \cdots < x_n$.

For every monomial $x^\alpha \neq 1$, we set $\min(x^\alpha) = \min\{x_i : x_i \mid x^\alpha\}$ and $\max(x^\alpha) = \max\{x_i : x_i \mid x^\alpha\}$. We will say that a monomial x^β can be obtained by a monomial x^α through an *elementary move* if $x^\alpha x_j = x^\beta x_i$ for some variables $x_i \neq x_j$ or equivalently if there exists a monomial x^δ such that $x^\alpha = x^\delta x_i$ and $x^\beta = x^\delta x_j$. If $i < j$, we say that x^β can be obtained by x^α through an *increasing* elementary move and we write $x^\beta = \mathrm{e}^+_{i,j}(x^\alpha)$. The transitive closure of the relation $x^\beta > x^\alpha$ if $x^\beta = \mathrm{e}^+_{i,j}(x^\alpha)$ gives a partial order on the set of monomials of any fixed degree (often called *Borel partial order*), that we will denote by $>_B$:

$$x^\beta >_B x^\alpha \iff \exists\, x^{\gamma_1}, \ldots, x^{\gamma_t} \text{ such that } x^{\gamma_1} = \mathrm{e}^+_{i_0,j_0}(x^\alpha),\ \ldots\ , x^\beta = \mathrm{e}^+_{i_t,j_t}(x^{\gamma_t})$$

Note that every term order $\succ$ is a refinement of the Borel partial order $>_B$, that is $x^\beta >_B x^\alpha \Rightarrow x^\beta \succ x^\alpha$.

Definition 1.1. A monomial ideal $J \subset K[x_0, \ldots, x_n]$ is said to be *strongly stable* if every monomial x^α such that $x^\alpha >_B x^\beta$, with $x^\beta \in J$, belongs to J.

Under the hypothesis that $\mathrm{char}(K) = 0$, a monomial ideal J is strongly stable if, and only if, J is fixed by the action of the Borel subgroup of $\mathrm{GL}(n+1)$ of lower triangular matrices (see for instance [14, Proposition 1.25] for a proof). Using this equivalence, we apply to the *Borel-fixed ideals* or, for short, *Borel ideals*, results proved in [9] and [5] for strongly stable ideals. We consider the usual action of $\mathrm{GL}(n+1)$ on the ring S given by the change of coordinates $x_i \mapsto \sum_j g_{ij} x_j$ for any invertible matrix $g = (g_{ij})$. For any polynomial $f(x_0, \ldots, x_n)$ and any change of coordinates g, we denote by f^g the result of the action of g over f, i.e. $f(\ldots, \sum_j g_{ij} x_j, \ldots)$. In the same way, given an ideal $I \subset S$, we denote by I^g the ideal $(f^g \mid f \in I)$.

The main reason why we assume that K has characteristic 0, is Galligo's Theorem, which is a key point for our investigations (mainly in the proof of Lemma 2.1).

Theorem 1.2. *([12]) Assume that* $\operatorname{char}(K) = 0$ *and fix a term order* $\prec$ *on the monomials of* S*. For every ideal* $I \subset S$ *there exists a Zariski open subset* $U \subset \mathrm{GL}(n+1)$ *such that for every* $g \in U$*, the initial ideal* $\operatorname{in}_\prec(I^g)$ *is constant and strongly stable.*

Remark 1.3. Since K contains the rationals $\mathbb{Q}$, and $\mathrm{GL}(n+1,\mathbb{Q})$ is Zariski-dense in $\mathrm{GL}(n+1,K)$, we may restrict g to a general change of coordinates with coefficients in $\mathbb{Q}$. This can be useful in practical applications, as in direct computation the use of rational numbers is more efficient.

Thanks to Galligo's Theorem, any component, and any intersection of components on the Hilbert scheme, contains a point corresponding to a Borel ideal. Therefore, it is natural to investigate the structure of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ using the Borel ideals and the action of the linear group. This will be our approach and was recently considered for instance by [23] to prove a first order infinitesimal version of Galligo's Theorem.

A homogeneous ideal I is *m-regular* if the i-th syzygy module of I is generated in degree $\leq m+i$, for all $i \geq 0$. The *regularity* $\operatorname{reg}(I)$ of I is the smallest integer m for which I is m-regular. The *saturation* of a homogeneous ideal I is $I^{\mathrm{sat}} = \{f \in S \mid \forall\, j = 0,\dots,n, \exists\, r \in \mathbb{N} : x_j^r f \in I\}$. The ideal I is *saturated* if $I^{\mathrm{sat}} = I$.

We recall that if J is Borel then $\operatorname{reg}(J) = \max\{\deg x^\alpha \mid x^\alpha \in B_J\}$ [4, Proposition 2.9] and its saturation J^{sat} is $(J : x_0^\infty)$ (for example, see [14, Corollary 2.10]). Hence $\operatorname{reg}(J^{\mathrm{sat}}) \leq \operatorname{reg}(J)$. Furthermore if B_J is the monomial basis of a Borel ideal J, then the saturation of J is generated by $B_J|_{x_0=1}$, that is one deletes the variable x_0 in each monomial of B_J [23, Theorem 3.2].

For a monomial ideal $J \subset S$, we denote by $\mathcal{N}(J)$ the *sous-escalier* of the monomial J, that is the set of monomials not belonging to J.

Lemma 1.4. *Let* J *be a Borel ideal in* S*. Then:*

(i) $x^\alpha \in J \setminus B_J \;\Rightarrow\; \dfrac{x^\alpha}{\min(x^\alpha)} \in J$*;*

(ii) $x^\beta \in \mathcal{N}(J)$ *and* $x_i x^\beta \in J \;\Rightarrow$ *either* $x_i x^\beta \in B_J$ *or* $x_i > \min(x^\beta)$*.*

(iii) if $J \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$*, then* $K[x_0,\dots,x_d] \subset \mathcal{N}(J^{\mathrm{sat}})$ *and* $K[x_{d+1},\dots,x_n]_{\geq m} \subset J^{\mathrm{sat}}$ *for some* $m \leq \operatorname{reg}(J^{\mathrm{sat}})$*.*

Proof.

(i) We consider $x^\alpha \in J \setminus B_J$. Then $x^\alpha = x^\gamma \cdot x_i = x^\delta \cdot x_j$, with $x^\gamma \in J$, $x_j = \min(x^\alpha)$. Then $x^\delta = e^+_{j,i}(x^\gamma)$, and then $x^\delta \in J$, since J is Borel.

(ii) If x^β belongs to $\mathcal{N}(J)$, and $x_i x^\beta \in J \setminus B_J$, then $x_i > \min(x^\beta)$ by the previous item.

(iii) If x_i is the maximal variable that is not nilpotent in S/J^{sat}, then $K[x_0,\dots,x_i] \subset \mathcal{N}(J^{\mathrm{sat}})$ by the strongly stable property of J^{sat}. Hence $K[x_0,\dots,x_i] \hookrightarrow S/J^{\mathrm{sat}}$ so that $i \leqslant \dim(S/J) = d$.
On the other hand, some power x_{i+1}^m belongs to B_J; hence, again by the strongly stable property, $K[x_{i+1},\dots,x_n]_{\geq m} \subset J$, so that $i+1 > \dim(S/J^{\mathrm{sat}}) = d$.

□

1.3. **Marked schemes over Borel ideals.** In this subsection, we introduce the main definitions and properties concerning marked sets of polynomials and marked bases. These special sets were investigated in [9] and [5], highlighting their interesting features under the hypothesis they are marked over a Borel ideal.

Definition 1.5. For any non-zero polynomial $f \in S$, the *support* of f is the set of monomials $\operatorname{Supp}(f)$ that appear in f with a non zero coefficient. By definition $\operatorname{Supp}(0) = \emptyset$.

A *marked polynomial* is a polynomial $f \in S$ together with a specified monomial of $\mathrm{Supp}\,(f)$ that will be called *head term* of f and denoted by $\mathrm{Ht}(f)$.
Given a monomial ideal J and an ideal I such that $\mathcal{N}(J)$ generates S/I as a K-vector space, a *J-reduced form modulo I* of a polynomial h is a polynomial h_0 such that $h - h_0 \in I$ and $\mathrm{Supp}\,(h_0) \subseteq \mathcal{N}(J)$. If the J-reduced form modulo I is unique, we call it *J-normal form modulo I*.

Note that every polynomial h has a unique J-reduced form modulo I if, and only if, $\mathcal{N}(J)$ is a K-basis for the quotient S/I. In this case, the J-reduced form modulo the homogeneous ideal I of a homogeneous polynomial h turns out to be homogeneous too of the same degree. If we only assume that $\mathcal{N}(J)$ generates S/I, there could be several J-reduced forms modulo I of h, but among them we can always find at least one that is homogeneous. In fact, if $h \in S_m$ and h' is a J-reduced form modulo I of h, then $(h - h')_m = h - h'_m$ belongs to I too and $\mathrm{Supp}\,(h'_m) \subseteq \mathrm{Supp}\,(h') \subseteq \mathcal{N}(J)$. Then h'_m is a homogeneous J-reduced form modulo I of h. Thus, we can always consider homogeneous J-reduced forms of homogeneous polynomials.

Definition 1.6. A finite set G of homogeneous marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $\mathrm{Ht}(f_\alpha) = x^\alpha$, is called *$J$-marked set* if the head terms $\mathrm{Ht}(f_\alpha)$ are pairwise different, they form the monomial basis B_J of the monomial ideal J and every x^γ belongs to $\mathcal{N}(J)$, i.e. $|\mathrm{Supp}\,(f) \cap J| = 1$. We call *tail* of f_α the polynomial $T(f_\alpha) := \mathrm{Ht}(f_\alpha) - f_\alpha$, so that $\mathrm{Supp}\,(T(f_\alpha)) \subseteq \mathcal{N}(J)$. A J-marked set G is a *J-marked basis* if $\mathcal{N}(J)$ is a basis of $S/(G)$ as a K-vector space.
The family of all homogeneous ideals I such that $\mathcal{N}(J)$ is a basis of the quotient S/I as a K-vector space will be denoted by $\mathcal{M}\mathrm{f}(J)$.

If J is a Borel ideal, then $\mathcal{M}\mathrm{f}(J)$ can be endowed with a natural structure of affine scheme (shown in [9, Section 4] and recalled in Remark 1.12 below) called *J-marked scheme*, that can be explicitly computed, using a polynomial reduction process, similar to the one for Gröbner bases, but term order free. Observe that $\mathcal{M}\mathrm{f}(J)$ contains every homogeneous ideal having J as initial ideal with respect to some term order, but in general it can also contain other ideals [**?**, see]Example 3.18]CR.
As a straightforward consequence of the definition, the ideals of $\mathcal{M}\mathrm{f}(J)$ define points on the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ containing S/J. However, two different ideals of $\mathcal{M}\mathrm{f}(J)$ may correspond to the same point [5, Example 3.4], while we get a one-to-one correspondence between $\mathcal{M}\mathrm{f}(J)$ and a subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ assuming that J is an *m-truncation ideal* of its saturation. [5, Theorem 3.3].

Definition 1.7. Let $I \subseteq S$ be a homogeneous ideal. We will say that I is an *m-truncation* if I is the truncation of I^{sat} in degree m, that is $I = {I^{\mathrm{sat}}}_{\geq m}$.

Remark 1.8. Observe that not every homogeneous ideal I generated by its degree m component is an m-truncation. For instance the ideal $I = (x_0^2, \ldots, x_i^2, \ldots, x_n^2)$ is not a 2-truncation, since $I^{\mathrm{sat}} = (1)$ and $I \neq S_2$. Nevertheless, in our setting of Section 1.1 every ideal $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ is an r-truncation. In fact, of course $I \subseteq (I^{\mathrm{sat}})_{\geq r}$. Moreover the two ideals I and I^{sat} have the same Hilbert polynomial $p(t)$ and their Hilbert functions coincide with $p(t)$ for every $t \geq r \geq \mathrm{reg}(I^{\mathrm{sat}})$.
Moreover, every Borel ideal $J \in \mathbb{G}$, as in Section 1.1, is an r-truncation. Indeed, $J^{\mathrm{sat}} = (B_J|_{x_0=1})$ and then for every $x^\gamma \in B_{J^{\mathrm{sat}}}$, the monomial $x^\gamma x_0^{r-|\gamma|} \in B_J$. If x^β belongs to ${J^{\mathrm{sat}}}_{\geq r}$, then $x^\beta = x^\gamma x^\delta$ for some $x^\gamma \in B_{J^{\mathrm{sat}}}$. Hence, $x^\beta >_B x^\gamma x_0^m$, $m \geq r - |\gamma|$, and x^β belongs to J by the Borel property.

1.4. **Marked schemes as open subsets of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$.** In the present subsection, we will consider a Borel ideal $J \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ and show that the corresponding J-marked scheme is scheme theoretically isomorphic to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$.

We denote by $\mathcal{Q}$ the set of monomial ideals in $\mathbb{G} = \mathbb{G}(s, S_r)$ and by $\mathcal{B}$ the set of those that are Borel. Moreover for every $J \in \mathcal{Q}$, $\mathcal{U}_J$ will be the open subset of $\mathbb{G}$ containing the ideals I such that their

Plücker coordinate $\Delta_J(I)$ is not 0, and $\mathcal{H}_J := \mathcal{U}_J \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}$ will be the corresponding open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$. As is well known, $\mathcal{U}_J$ is a Schubert cell of maximal dimension $D := p(r)q(r) = \dim(\mathbb{G})$ and is isomorphic to the affine space $\mathbb{A}^D$.

Lemma 1.9. *Let I be an ideal in $\mathbb{G}$, and let B_J be the monomial basis of an ideal $J \in \mathcal{Q}$. Then the following are equivalent:*

(i) $\Delta_J(I) \neq 0$*:*

(ii) I_r *can be represented by a matrix of the form* $\left(\begin{array}{c|c} \mathrm{Id} & \mathcal{C} \end{array}\right)$*, where the left block is the $s \times s$ identity matrix and corresponds to the monomials in B_J and the entries of the right block $\mathcal{C}$ are constants $-c_{\alpha\gamma} \in K$, where $x^\alpha \in B_J$ and $x^\gamma \in \mathcal{N}(J)_r$;*

(iii) I is generated by a J-marked set:

$$G = \left\{ f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma : Ht(f_\alpha) = x^\alpha \in B_J \right\}$$

Proof.
(i) $\Rightarrow$ (ii). Up to rearranging the columns, it is sufficient to multiply any matrix $M(I, r)$ (as in Section 1.1) by the inverse of its submatrix made up of the columns corresponding to B_J, since its determinant is $\Delta_J(I) \neq 0$.
(ii) $\Rightarrow$ (i) is obvious.
(ii) $\Leftrightarrow$ (iii). The generators of I given by the rows of $M(I, r)$ as in (ii) are indeed a J-marked set and, conversely, the matrix containing the coefficients of the polynomials f_α has precisely the shape required in (ii). □

Definition 1.10. Let J be a Borel ideal, let $C = \{C_{\alpha\gamma} \mid x^\alpha \in B_J,\ x^\gamma \in \mathcal{N}(J)_{|\alpha|}\}$ denote a set of new variables and consider the ring $K[C, x]$. We denote by $\mathcal{G} \subset K[C, x]$ the J-marked set:

$$\mathcal{G} = \left\{ F_\alpha = x^\alpha - \sum C_{\alpha\gamma} x^\gamma : Ht(F_\alpha) = x^\alpha \in B_J, x^\gamma \in \mathcal{N}(J) \right\} \tag{1}$$

and by $\mathfrak{I}_J$ the ideal generated by $\mathcal{G}$ in the ring $K[C, x]$.

Corollary 1.11. *Under the hypotheses of Lemma 1.9, there is an isomorphism between $\mathcal{U}_J$ and $\mathbb{A}^D = \mathrm{Spec}\,(K[C])$ such that the (closed) points in $\mathcal{U}_J$ correspond to all ideals that we obtain from $\mathfrak{I}_J$ by specializing the variables $C_{\alpha\gamma}$ to $c_{\alpha\gamma} \in K$.*

Proof. Under the hypotheses of Lemma 1.9, it is sufficient to fix an isomorphism $\mathcal{U}_J \cong \mathbb{A}^D$ such that the constants $c_{\alpha\gamma}$ are the coordinates of I in $\mathbb{A}^D$. □

Remark 1.12. $\mathcal{M}\mathrm{f}(J)$ is naturally endowed of the structure of an affine subscheme of $\mathbb{A}^D$. More specifically, as shown by [9, Lemma 4.2], we can obtain a set of generators for the ideal $\mathfrak{A}_J$ in $K[C]$ defining $\mathcal{M}\mathrm{f}(J)$ by imposing conditions on the rank of some matrices. We obtain the same structure by a polynomial reduction process similar to the Gröbner one, but term order free [9, Theorem 3.12, Appendix].

We now fix $J \in \mathcal{B} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}$ and show that $\mathcal{H}_J$ is nothing but the J-marked scheme $\mathcal{M}\mathrm{f}(J)$.

Theorem 1.13. *For $J \in \mathcal{B} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}$, there are scheme theoretic isomorphisms ϕ_1 and ϕ_2 and embeddings such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}_J & \xrightarrow{\phi_1} & \mathcal{M}\mathrm{f}(J) \\ \downarrow & & \downarrow \\ \mathcal{U}_J & \xrightarrow{\phi_2} & \mathbb{A}^D \end{array} \tag{2}$$

Proof. By Gotzmann's Hilbert Scheme Theorem, we can give to the open subset $\mathcal{H}_J$ the structure of closed affine subscheme of $\mathcal{U}_J \simeq \mathbb{A}^D$. Indeed, $I \in \mathcal{U}_J$ belongs to $\mathcal{H}_J$ if $\dim I_{r+1} = q(r+1)$. This can be obtained by considering the matrix of the coefficients of the generators of I_{r+1} and imposing that its rank is $q(r+1) = \dim J_{r+1}$. This gives the scheme-theoretical embedding of $\mathcal{H}_J$ in $\mathcal{U}_J$ of the diagram (see also [17, Proposition C.30]).

As recalled by Remark 1.12, $\mathcal{M}\mathrm{f}(J)$ can be considered as an affine subscheme of $\mathbb{A}^D$: this gives the second embedding of the diagram. In the present hypothesis, we obtain a set of generators for the ideal $\mathfrak{A}_J$ in $K[C]$ defining $\mathcal{M}\mathrm{f}(J)$ by imposing that the rank of the matrix corresponding to the degree $r+1$ is $\leq \dim J_{r+1}$. This matrix turns out to be indeed $M(\mathfrak{I}_J, r+1)$ and $\dim J_{r+1} = q(r+1)$. This gives the isomorphism ϕ_1 between $\mathcal{H}_J$ and $\mathcal{M}\mathrm{f}(J)$ of the diagram.

Finally the isomorphism $\mathcal{U}_J \simeq \mathbb{A}^D$ is the one of Corollary 1.11. □

In the next section, we will further investigate the open subsets $\mathcal{H}_J$ of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ for a Borel ideal $J \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$, showing that they are sufficient to give a cover of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ (up to the action of the group $\mathrm{PGL}(n+1)$). In Section 3 we will use the tools and results of [5] to obtain embeddings of $\mathcal{H}_J$ in affine spaces of dimension smaller than $D = p(r)q(r)$ and find upper bounds for the number and degrees of the corresponding equations.

2. The Borel cover

In this section we will investigate the families of open subsets $\mathcal{U}_J \subseteq \mathbb{G}$ and $\mathcal{H}_J \subseteq \mathcal{H}\mathrm{ilb}^n_{p(t)}$ and deduce from them a open cover of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$. For a different approach giving a cover by locally closed subschemes of the Hilbert scheme of points see [18].

It is quite obvious that they cover respectively $\mathbb{G}$ and $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ as J varies in $\mathcal{Q}$. However $\mathcal{Q}$ in general has a large number of elements. In the study of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, it is quite natural to consider Borel ideals, since every component and any intersection of components of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ contains a point corresponding to a Borel ideal. Further $\mathcal{B}$ is a comparatively small subset of $\mathcal{Q}$; then it would be convenient to find a way to cover $\mathbb{G}$ and $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ by subfamilies indexed by $\mathcal{B}$.

The open subsets $\mathcal{U}_J$, as J varies in $\mathcal{B}$, do not cover the whole Grassmannian: for instance, any monomial ideal $J' \in \mathcal{Q} \setminus \mathcal{B}$ does not belong to $\mathcal{U}_J$, for every $J \in \mathcal{B}$. However, thanks to Galligo's Theorem (Theorem 1.2), we get a complete cover of $\mathbb{G}$ up to the action on $\mathbb{G}$ induced by the projective linear group $\mathrm{PGL}(n+1)$ (PGL for short) of changes of coordinates in $\mathbb{P}^n$. As usual we denote by a superscript g the action of an element g of the group.

Lemma 2.1. *The open subsets $\mathcal{U}_J$, $J \in \mathcal{B}$, up to the action of* PGL, *cover $\mathbb{G}$, that is:*

$$\bigcup_{\substack{J\in\mathcal{B}\\ g\in\mathrm{PGL}}} \mathcal{U}_J^g = \mathbb{G}.$$

Proof. Let $I \in \mathbb{G}$ be any ideal and let $\preceq$ be any term order on the monomials of S. Due to Galligo's Theorem 1.2, in generic coordinates the initial ideal J' of I w.r.t $\prec$ is Borel, and then $J := (J'_r)$ is Borel too. Moreover, by construction J is generated by s monomials of degree r and then $J \in \mathcal{B}$. Hence for a general $g \in \mathrm{PGL}$ we have $\Delta_J(I^g) \neq 0$ that is $I^g \in \mathcal{U}_J$, so that $I \in \mathcal{U}_J^{g^{-1}}$. □

Remark 2.2. In the proof of Lemma 2.1 we deal with the generic initial ideal J', that may also have minimal generators of degree $> r$, so that sometimes $J' \notin \mathbb{G}$. To avoid this problem, we consider instead the ideal J generated by J'_r, which is Borel and certainly belongs to $\mathbb{G}$ because a basis for J'_r is given by s monomials in degree r. The same expedient is used in [6].

We can rephrase Lemma 2.1 saying that for every $I \in \mathbb{G}$ there are: a Borel ideal $J \in \mathcal{B}$ and a generic linear change of coordinates $g \in \mathrm{PGL}$, such that I^g is generated by a J-marked set G: for instance we

can choose $J = (\mathrm{in}_\prec(I^g)_r)$, for any fixed term order $\prec$ and a general g. Nevertheless, G does not need to be a J-marked basis for I^g, namely I and J may have different Hilbert polynomials.

Example 2.3. Let us consider the ideal $I = (x_2^2, x_1^2)$ in $K[x_0, x_1, x_2]$ belonging to $\mathbb{G}(2, 6)$. After the change of coordinates g: $x_2 \to x_2$, $x_1 \to x_2 + x_1$, $x_0 \to x_0$, we get $I^g = (x_2^2, x_2^2 + 2x_2x_1 + x_1^2)$ whose initial ideal with respect to any term order such that $x_2 > x_1 > x_0$ is $J' = (x_2^2, x_2x_1, x_1^3)$. In fact, K being a field of characteristic 0, the reduced Grobner basis of I^g is the set of three polynomials $\{f_1, f_2, f_3\}$, where $f_1 = x_2^2$, $f_2 = x_2x_1 + x_1^2/2$ and $f_3 = x_1^3 = 4x_1f_1 - (4x_2 - 2x_1)f_2$. Obviously J' does not belong to $\mathbb{G}(2, 6)$. Following the line of the proof of Lemma 2.1 we then consider $J = (J'_2) = (x_2^2, x_2x_1)$: now $J \in \mathbb{G}(2, 6)$ and moreover $I^g \in \mathcal{U}_J$, as one can see by the coefficient matrix of $\{f_1, f_2\}$, (3), even though the Hilbert polynomial of S/I is 4 and that of S/J is $t + 2$.

$$\left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 0 \end{array}\right) \tag{3}$$

We now restrict the above open cover to the the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ and denote by $\mathcal{B}_{p(t)}$ the Borel ideals J such that S/J has Hilbert polynomial $p(t)$. Since $\mathcal{H}\mathrm{ilb}^n_{p(t)} \hookrightarrow \mathbb{G}$ is a closed embedding, the intersection $\mathcal{H}_J = \mathcal{U}_J \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}$ is an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, which may possibly be empty. If $J \in \mathcal{B}_{p(t)}$, then of course $\mathcal{H}_J$ is non-empty because it contains J itself. We show now that the converse is also true and this justifies the following:

Definition 2.4. The *Borel cover* of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is the family of all the open subsets $\mathcal{H}^g_J$ with $J \in \mathcal{B}_{p(t)}$ and $g \in \mathrm{PGL}$.

Theorem 2.5. *If $J \in \mathcal{B}$, then:*

$$\mathcal{H}_J \neq \emptyset \iff J \in \mathcal{B}_{p(t)}.$$

As a consequence, we get:

$$\mathcal{H}\mathrm{ilb}^n_{p(t)} = \bigcup_{\substack{g \in \mathrm{PGL} \\ J \in \mathcal{B}_{p(t)}}} \mathcal{H}^g_J.$$

Proof. We prove only the non-trivial part ($\Rightarrow$) of the first statement. Assume that $J \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}$. By Gotzmann's Hilbert Scheme Theorem, this is equivalent to $\dim_K(J_{r+1}) > q(r + 1)$. If I is any ideal in $\mathcal{U}_J$, then it has a set of generators as those given in Lemma 1.9 (iii), so that $\dim_K(I_{r+1}) \geqslant \dim_K(J_{r+1}) > q(r + 1)$ (see [9, Corollary 2.3]). Hence $I \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}$.

The other statement is a direct consequence of the first one and of Lemma 2.1. □

Corollary 2.6. *If J belongs to $\mathcal{B}$, then:*

- *the ideal I belongs to $\mathcal{U}_J$ if, and only if, I is generated by a J-marked set;*
- *the ideal I belongs to $\mathcal{H}_J$ if, and only if, $J \in \mathcal{B}_{p(t)}$ and I is generated by a J-marked basis.*

There are several strategies that we can follow in order to compute for a given ideal $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ a change of coordinates $g \in \mathrm{PGL}$ and a Borel ideal $J \in \mathcal{B}_{p(t)}$ such that $I^g_r \in \mathcal{H}_J$.

First of all, as observed in Remark 1.3, we can consider a generic change of coordinates with rational coefficients.

The method used in the proof of Lemma 2.1 may be used to determine $J \in \mathcal{B}_{p(t)}$. Note that we do not need to perform the Buchberger's Algorithm on the generators of I^g in order to derive a complete Gröbner basis of I^g , but we can simply compute, by means of a Gaussian reduction, $s = q(r)$ distinct initial terms with respect to any term order $\preceq$ of degree r forms in I^g. The ideal J generated by these monomials is indeed the initial ideal J' of I^g with respect to $\preceq$. In fact, using Lemma 1.9, I^g is generated by a J-marked set, hence $\dim J_{r+1} \leq \dim {I^g}_{r+1} = q(r + 1)$, by [9, Corollary 2.3].

On the other hand, by Macaulay's growth theorem, $\dim J_{r+1} \geq q(r+1)$. Then equality holds and by Gotzmann's Hilbert Scheme Theorem J has the same Hilbert polynomial $q(t)$ as I^g and J'. By construction $J = (J_r) \subseteq J'$; furthermore, for all $t < r$, we have that $\dim J_t = \dim J'_t = 0$, while for all $t \geq r$, $\dim J_t = \dim J'_t = q(t)$; thus $J = J'$.

Another strategy to determine a Borel ideal J, such that $\mathcal{H}_J$ contains the point I^g (avoiding to compute the generic initial ideal), is that of computing the Plücker coordinate $\Delta_J(I)$ for every $J \in \mathcal{B}_{p(t)}$: given n and $p(t)$, the complete list $\mathcal{B}_{p(t)}$ can be explicitly computed by the algorithms presented by [8] or the ones presented by [19]. Starting from this list, the following algorithm returns a Borel ideal J in $\mathcal{B}_{p(t)}$ with minimum regularity such that $I^g \in \mathcal{H}_J$. The computational advantage of the condition about the regularity will be discussed in Section 3. Further, an easy variant of the algorithm finds $g \in \mathrm{PGL}$ and the maximal subset $\mathcal{B}(I^g) \subseteq \mathcal{B}_{p(t)}$ such that $I^g \in \mathcal{H}_J$ for every $J \in \mathcal{B}(I^g)$.

Let us suppose that the following functions are made available.

- SuitableGrasmannian(I). It computes the suitable Grassmannian (in the sense of Section 1.1) in which the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is embedded, where $p(t)$ is the Hilbert polynomial of S/I. It returns the pair $(\tilde{I}, \mathbb{G})$ such that $\tilde{I}$ is the point of $\mathbb{G}$ corresponding to I.
- QGenericChangeOfCoordinates(I). It performs a random linear change of coordinates g with rational coefficients on the generators of the ideal I, returning the pair (g, I^g).
- BorelIdeals$(n, p(t))$. It returns the full list of Borel ideals J in $\mathcal{B}_{p(t)}$; this list is sorted according to encreasing regularity of J^{sat}.
- PlückerCoordinate$(I, J, \mathbb{G})$. Given I, J homogeneous ideals in $\mathbb{G}$, computes $\Delta_J(I)$.
- GetElement(L, j). It returns the j-th element of the list L.

```
 1: BorelOpenSet(I)
Require: I homogeneous ideal in K[x_0, ..., x_n].
Ensure: the pair (g, J), with g ∈ PGL and J a Borel ideal defining an open subset H_J of the Hilbert
    scheme containing the point corresponding to Proj(S/I^g).
 2: p(t) ← Hilbert polynomial of S/I;
 3: (Ĩ, G) ← SuitableGrasmannian(I);
 4: B_{p(t)} ← BorelIdeals(n, p(t));
 5: openSetFound ← false;
 6: output ← ∅;
 7: while not openSetFound do
 8:    (g, Ĩ^g) ← QGenericChangeOfCoordinates(Ĩ);
 9:    for i = 1, ..., |B_{p(t)}| and not openSetFound do
10:       J ← GetElement(B_{p(t)}, i);
11:       if PlückerCoordinate(Ĩ^g, J, G) ≠ 0 then
12:          openSetFound ← true;
13:          output ← (g, J);
14:       end if
15:    end for
16: end while
17: return output;
```

Algorithm 1: Algorithm computing $g \in \mathrm{PGL}$ and $J \in \mathcal{B}_{p(t)}$ such that $\mathcal{H}_J$ is an open subset of the Hilbert scheme containing the point $\mathrm{Proj}(S/I^g)$

Example 2.7. We now execute Algorithm BorelOpenSet on the ideal $I = (x_2^2, x_1^2) \subset S = K[x_0, x_1, x_2]$ of Example 2.3. The Hilbert polynomial of S/I is $p(t) = 4$ and the Gotzmann number is $r = 4$. We

now find a generic change of coordinates $g \in \mathrm{PGL}$ and ideal $J \in \mathcal{B}_{p(t)}$ such that $(I_4)^g \in \mathcal{H}_J$. There are only two Borel ideals J in $\mathcal{B}_4$:

$$J_1 = (x_2^2, x_2x_1, x_1^3)_{\geq 4}, \quad J_2 = (x_2, x_1^4)_{\geq 4}$$

After a generic linear change of coordinates g (for instance the one used in Example 2.3), we compute the coordinate $\Delta_{J_1}((I_4)^g)$ by writing down the matrix of the coefficients of the 14 homogeneous forms of degree 4 generating $(I_4)^g$. The minor corrresponding to the monomial basis of J_1 is non-zero, so $(I_4)^g$ belongs to $\mathcal{H}_{J_1}$. For g as in Example 2.3, $(I_4)^g$ is generated by the J_1-marked basis:

$$G = \left\{ x_2^4, x_2^3x_1, x_2^3x_0, x_2^2x_1^2, x_2^2x_1x_0, x_2^2x_0^2, x_1^4, x_2x_1^3, x_1^3x_0, x_2x_1^2x_0, x_2x_1x_0^2 + \frac{1}{2}x_1^2x_0^2 \right\}.$$

Theorem 2.5 does not extend to monomial ideals $J \in \mathcal{Q} \setminus \mathcal{B}$, as the following example shows.

Example 2.8. Let us consider the constant Hilbert polynomial $p(t) = 2$ and the Hilbert scheme $\mathcal{H}\mathrm{ilb}_2^2$ parameterizing 0-dimensional subschemes in $\mathbb{P}^2$ of degree 2: in this case $r = 2$ and $s = 4$. The monomial ideal $J = (x_0^2, x_1^2, x_2^2, x_0x_1)$ is generated by s monomials of degree 2, it is not Borel and obviously does not belong to $\mathcal{H}\mathrm{ilb}_2^2$ since it is a primary ideal over the irrelevant maximal ideal (x_0, x_1, x_2). Nevertheless, $\mathcal{H}_J$ is not empy, as it contains the ideal $(x_0^2 - x_0x_2, x_1^2 - x_1x_2, x_2^2 - x_0x_2 - x_1x_2, x_0x_1)$ corresponding to the set of points $\{[1:0:1], [0:1:1]\}$ and, more generally, all the ideals corresponding to pairs of distinct points outside the line $x_2 = 0$ and not on the same line through $[0:0:1]$.

Theorem 2.5 not only allows us to define a cover of $\mathcal{H}\mathrm{ilb}_{p(t)}^n$ from the Borel ideals of $\mathcal{B}_{p(t)}$, but also gives an interesting role to the ideals belonging to $\mathcal{B} \setminus \mathcal{B}_{p(t)}$: indeed, $\mathcal{H}\mathrm{ilb}_{p(t)}^n$ lies on every hypersurface of $\mathbb{G}$ defined by the vanishing of the Plücker coordinate corresponding to such a Borel ideal (and on every hypersurface obtained from these by the action of PGL).

Lemma 2.9. *For every distinct $J, J' \in \mathcal{Q}$, $\mathcal{U}_J \setminus \mathcal{U}_{J'}$ is a hypersurface of $\mathcal{U}_J \simeq \mathbb{A}^D$ of degree $|B_J \setminus B_{J'}|$.*

Proof. We simply need to observe that the equation of $\mathcal{U}_J \setminus \mathcal{U}_{J'}$ in $\mathcal{U}_J$ is defined by $\Delta_{J'}/\Delta_J$, whose degree is exactly $|B_J \setminus B_{J'}|$ by Lemma 1.9. □

Corollary 2.10. *In the previous setting, we have the set-theoretical inclusion:*

$$\mathcal{H}\mathrm{ilb}_{p(t)}^n \subseteq \bigcap_{\substack{g \in \mathrm{PGL} \\ J \in \mathcal{B} \setminus \mathcal{B}_{p(t)}}} \Pi_J^g$$

where Π_J is the hypersurface of $\mathbb{G}$ given by $\Delta_J = 0$.

Proof. For every $J \in \mathcal{B} \setminus \mathcal{B}_{p(t)}$, $\mathcal{H}_J$ is empty by Theorem 2.5. This means that the hypersurface Π_J contains $\mathcal{H}\mathrm{ilb}_{p(t)}^n$. Furthermore, since $\left(\mathcal{H}\mathrm{ilb}_{p(t)}^n\right)^g = \mathcal{H}\mathrm{ilb}_{p(t)}^n$ for every $g \in \mathrm{PGL}$, we also have $\mathcal{H}\mathrm{ilb}_{p(t)}^n \subseteq \Pi_J^g$ for every $J \in \mathcal{B} \setminus \mathcal{B}_{p(t)}$ and every $g \in \mathrm{PGL}$. □

If the Hilbert polynomial $p(t)$ is the constant r, then every Borel ideal $J \in \mathbb{G}$ belongs to $\mathcal{H}\mathrm{ilb}_r^n$ i.e. $\mathcal{B} = \mathcal{B}_{p(t)} \subset \mathcal{H}\mathrm{ilb}_{p(t)}^n$ (see [8, Theorem 3.13]). Then in the 0-dimensional case the family of hypersurfaces Π_J considered in Theorem 2.5 is in fact empty. If $\deg p(t) = d \geq 1$, $\mathcal{B} \setminus \mathcal{B}_{p(t)}$ in general is not empty and its elements define subschemes of $\mathbb{P}^n$ of dimension equal to or lower than the one of the subschemes parametrized by $\mathcal{H}\mathrm{ilb}_{p(t)}^n$. Indeed, if $I \in \mathcal{B}$ and S/I has Hilbert polynomial $\widetilde{p}(t) \neq p(t)$, then by Macaulay's lower bound and Gotzmann's persistence, $\dim_K(I_t) > q(t)$ for $t \geq r$. Hence for $t \gg 0$, $\widetilde{q}(t) > q(t)$ and $\widetilde{p}(t) < p(t)$. Therefore $\deg \widetilde{p}(t) \leq d$.

Example 2.11. Let us consider the Hilbert polynomial $p(t) = 3t$ in $\mathbb{P}^3$. The closed points of $\mathcal{H}\text{ilb}^3_{3t}$ correspond to curves in $\mathbb{P}^3$ of degree 3 and arithmetic genus 1, hence it contains all the smooth plane elliptic curves and also some singular or reducible or non-reduced curves. The Gotzmann number of $p(t)$ is $r = 3$, then $s = q(3) = 11$. We embed $\mathcal{H}\text{ilb}^n_{p(t)}$ in the Grassmannian $\mathbb{G}(11, K[x]_3)$. The only Borel ideal in $\mathcal{H}\text{ilb}^3_{3t}$ is the $\mathtt{Lex}$-segment ideal:

$$J_{\mathtt{Lex}} = \left(x_3^3, x_3^2x_2, x_3x_2^2, x_2^3, x_3^2x_1, x_3x_2x_1, x_3x_1^2, x_3^2x_0, x_3x_2x_0, x_3x_1x_0, x_3x_0^2\right).$$

The Borel cover of $\mathcal{H}\text{ilb}^3_{3t}$ is then $\cup_{g\in\mathrm{PGL}}\mathcal{H}^g_{J_{Lex}}$, with $\mathcal{H}_{J_{Lex}} := \mathcal{U}_{J_{\mathtt{Lex}}} \cap \mathcal{H}\text{ilb}^3_{3t}$.

We can compute the Borel ideals J_i in $\mathbb{G}(11, K[x]_3)$ that do not belong to $\mathcal{H}\text{ilb}^3_{3t}$ and the Hilbert polynomial of S/J_i:

$J_1 = (x_2^3, x_3^2, x_2^2x_1, x_3x_1, x_3x_2)_{\geq 3}$, Hilbert polynomial of S/J_1: $2t + 3$;
$J_2 = (x_2^2, x_3^2, x_3x_1^2, x_3x_2)_{\geq 3}$, Hilbert polynomial of S/J_2: $2t + 3$;
$J_3 = (x_2^3, x_3^2, x_2x_1^2, x_2^2x_1, x_3x_1^2, x_3x_2)_{\geq 3}$, Hilbert polynomial of S/J_3: $t + 6$;
$J_4 = (x_1^3, x_2^3, x_3^2, x_2x_1^2, x_2^2x_1, x_3x_1^2, x_3x_2^2, x_3x_2x_1)_{\geq 3}$, Hilbert polynomial of S/J_4: 9.

Then $\mathcal{H}\text{ilb}^3_{3t}$ as a subscheme of $\mathbb{G}(11, K[x]_3)$ is contained set-theoretically in the hypersurfaces Π_{J_i} given by $\Delta_{J_i} = 0$, $i = 1, \ldots, 4$ (and in every hypersurface obtained from these by the action of PGL).

3. Equations defining $\mathcal{H}_J$

The aim of the present section is to find suitable affine subspaces of $\mathbb{A}^D \simeq \mathcal{U}_J$ in which $\mathcal{H}_J$ can be isomorphically projected, and furthermore to study in which cases we can control the number and the degree of a set of generators of the ideals defining $\mathcal{H}_J$ as a closed subscheme of such affine subspaces.

The results we obtain are very similar to the ones for Gröbner strata showed by [20], but are more general (see [5, Example 6.2]). Here the isomorphism of Theorem 1.13 is crucial, because it allows the use of the techniques presented in [5] for marked schemes over a m-truncation $J_{\geq m}$ of a Borel ideal J.

We now consider a Borel ideal J belonging to $\mathcal{B}_{p(t)}$: as pointed out in Remark 1.8, in our setting J is an r-truncation ideal, in other words $J = {J^{\mathrm{sat}}}_{\geq r}$. We denote by r' the regularity of J^{sat}, $r' \leq r$, and by ρ the maximal degree of a monomial in $B_{J^{\mathrm{sat}}}$ divisible by x_1; if there are no such monomials in $B_{J^{\mathrm{sat}}}$, we set $\rho := 0$.

We consider the m-truncation Borel ideal ${J^{\mathrm{sat}}}_{\geq m}$. Finally, we will denote by $\phi_{J,r}$ the embedding $\mathcal{H}_J \hookrightarrow \mathbb{A}^{p(r)q(r)}$ given by Theorem 1.13 and by $\mathfrak{A}_J \subset K[C]$ the ideal defining $\mathcal{H}_J$ as a subscheme of $\mathbb{A}^{p(r)q(r)}$.

Theorem 3.1. *In the above setting, the following statements hold:*

(i) if $m \geq r$, then $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq m}) \simeq \mathcal{H}_J$;
(ii) if $m < r$, then $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq m})$ is a closed subscheme of $\mathcal{H}_J$, possibly equal. If we consider the embedding $\phi_{J,r}(\mathcal{H}_J) \subset \mathbb{A}^{p(r)q(r)}$, then $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq m})$ is cut out by a suitable linear space;
(iii) $\mathcal{H}_J \simeq \mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq m})$ if, and only if, either ${J^{\mathrm{sat}}}_{\geq m} = J$ or $m \geq \rho - 1$.

In particular, if $\rho > 0$, then $\rho - 1$ is the smallest integer m such that:

$$\mathcal{H}_J \simeq \mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq \rho-1})$$

The isomorphism $\mathcal{H}_J \simeq \mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq r'})$ induces an embedding $\phi_{J,r'}$ of $\mathcal{H}_J$ in an affine space of dimension $|B_{J^{\mathrm{sat}}}| \cdot p(r')$ and the isomorphism $\mathcal{H}_J \simeq \mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq \rho-1})$ induces an embedding $\phi_{J,\rho-1}$ of $\mathcal{H}_J$ in an affine space of dimension

$$\sum_{x^\alpha \in {B_{J^{\mathrm{sat}}}}_{\geq \rho-1}} \left|\mathcal{N}({J^{\mathrm{sat}}}_{\geq \rho-1})_{|\alpha|}\right|. \tag{4}$$

Proof. Thanks to the isomorphism $\mathcal{H}_J \simeq \mathcal{M}\mathrm{f}(J)$ of Theorem 1.13, the statements are straightforward consequences of [5, Theorem 5.7]. □

The embeddings $\phi_{J,\rho-1}$ (or more generally $\phi_{J,m}$ with $\rho-1 \leq m < r'$) of $\mathcal{H}_J$ in affine spaces defined in Theorem 3.1 are computationally advantageous, because in order to compute equations for $\mathcal{H}_J$ we deal with a number of variables smaller than $p(r)q(r)$. Furthermore, the ideal defining the affine scheme $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq m})$ can be explicitly computed using the algorithm in [5, Appendix]: this algorithm does not perform any elimination of variables in order to pass from the ideal defining $\mathcal{H}_J$ in $\mathbb{A}^{p(r)q(r)}$ to the ideal defining it as a subscheme of a smaller affine space.

Example 3.2. We consider again the ideal J_1 of Example 2.7. We can compute equations for the ideal defining $\mathcal{H}_{J_1}$ in the affine space of dimension given by formula (4), namely in this case $12 < p(r)q(r) = 44$. Observe that $\rho - 1$ in this case is 2, hence $\mathcal{H}_{J_1}$ is isomorphic to $\mathcal{M}\mathrm{f}({J_1}^{\mathrm{sat}})$: by the algorithm presented in [5, Appendix], we obtain for $\mathfrak{A}_{{J_1}^{\mathrm{sat}}}$ a set of 8 generators of degree 3 in 12 variables for the ideal. Starting from the ${J_1}^{\mathrm{sat}}$-marked set

$$\begin{aligned} &x_2^2 - C_{1,1}x_1^2 - C_{1,2}x_2x_0 - C_{1,3}x_1x_0 - C_{1,4}x_0^2,\\ &x_2x_1 - C_{2,1}x_1^2 - C_{2,2}x_2x_0 - C_{2,3}x_1x_0 - C_{2,4}x_0^2,\\ &x_1^3 - C_{3,1}x_1^2x_0 - C_{3,2}x_2x_0^2 - C_{3,3}x_1x_0^2 - C_{3,4}x_0^3 \end{aligned}$$

we obtain the following generators

$$\begin{aligned} &- C_{2,1}C_{2,2}C_{2,4} - C_{2,2}C_{1,4} + C_{1,2}C_{2,4} + C_{1,1}C_{3,4} - {C_{2,1}}^2C_{3,4} - C_{2,3}C_{2,4},\\ &- C_{2,3}C_{2,2} - C_{2,1}C_{2,2}^2 - C_{2,4} + C_{1,1}C_{3,2} - C_{2,1}^2C_{3,2},\\ &C_{1,4} - C_{2,1}C_{2,2}C_{2,3} - C_{2,1}C_{2,4} - C_{2,1}^2C_{3,3} - C_{2,3}^2 + C_{1,1}C_{3,3} + C_{1,2}C_{2,3} - C_{2,2}C_{1,3},\\ &- C_{2,1}^2C_{3,1} + C_{1,3} - C_{2,2}C_{1,1} + C_{1,2}C_{2,1} - C_{2,1}^2C_{2,2} + C_{1,1}C_{3,1} - 2\,C_{2,3}C_{2,1},\\ &C_{2,2}^2C_{2,4} - C_{3,3}C_{2,4} + C_{2,1}C_{3,2}C_{2,4} + C_{2,1}C_{2,2}C_{3,4} + C_{2,3}C_{3,4} - C_{3,2}C_{1,4} - C_{3,1}C_{2,2}C_{2,4},\\ &2\,C_{2,1}C_{3,2}C_{2,2} - C_{3,4} - C_{3,3}C_{2,2} - C_{3,1}C_{2,2}^2 + C_{2,3}C_{3,2} + C_{2,2}^3 - C_{3,2}C_{1,2},\\ &C_{2,1}C_{3,2}C_{2,3} + C_{2,2}C_{2,4} + C_{2,1}C_{2,2}C_{3,3} - C_{3,1}C_{2,2}C_{2,3} - C_{3,1}C_{2,4} + C_{2,1}C_{3,4} - C_{3,2}C_{1,3} + C_{2,2}^2C_{2,3},\\ &C_{2,1}^2C_{3,2} - C_{1,1}C_{3,2} + C_{2,1}C_{2,2}^2 + C_{2,4} + C_{2,3}C_{2,2}. \end{aligned}$$

By Theorem 3.1, we have embedded $\mathcal{H}_{J_1} \simeq \mathcal{M}\mathrm{f}({J_1}^{\mathrm{sat}})$ in $\mathbb{A}^{12}$: however, this is not the smallest affine space in which $\mathcal{M}\mathrm{f}({J_1}^{\mathrm{sat}})$ can be embedded. Indeed, we can see that from the above set of generators of the ideal $\mathfrak{A}_{{J_1}^{\mathrm{sat}}}$, we can easily eliminate the variables $C_{1,3}$, $C_{1,4}$, $C_{2,4}$ and $C_{3,4}$ (each of them appears linearly in a different generator); eliminating them, we obtain the ideal (0), hence $\mathcal{M}\mathrm{f}({J_1}^{\mathrm{sat}})$ is isomorphic to $\mathbb{A}^8$ and this is clearly the smallest affine space in which $\mathcal{H}_{J_1} \simeq \mathcal{M}\mathrm{f}({J_1}^{\mathrm{sat}})$ can be embedded. Furthermore, since $\mathcal{H}_{J_1} \simeq \mathbb{A}^8$, J_1 is a smooth point of $\mathcal{H}\mathrm{ilb}_2^4$.

From the computational point of view, it would be useful and interesting to bound the number and degree of a set of generators of the ideal defining $\mathcal{H}_J$ given by the embeddings $\phi_{J,m}$ of Theorem 3.1.

For what concerns a bound on the degree, we might try to derive it, at least for $\phi_{J,r}$, from other bounds in literature on the degree of the generators for an ideal Γ in $K[\Delta]$ defining the Hilbert scheme $\mathcal{H}\mathrm{ilb}_{p(t)}^n$ in $\mathbb{G}$. In [17] it is proved that there is such an ideal Γ generated in degree $q(r+1)+1$. In [3] it is conjectured, and later on in [16] it is proved that, there is such a Γ generated in the far lower degree $n+1$. More recently, in the case of Hilbert schemes of points, that is $d = 0$, in [1] the authors improved the previous ones, showing that there is a Γ generated in degree 2. Finally, in the recent paper [6] the authors proved that the Hilbert scheme can be defined by an ideal Γ generated in degree $\leq d+2$.

However, when we restrict $\Delta_{J'}$, for every $J' \in \mathcal{Q}$, to $\mathbb{A}^{p(r)q(r)} \simeq \mathcal{U}_J$, $J \in \mathcal{Q}$, $\Delta_{J'}$ becomes a polynomial of degree $\leq q(r)$ in $K[C]$. Thus, restricting anyone of the above ideals Γ of $K[\Delta]$ defining $\mathcal{H}\mathrm{ilb}_{p(t)}^n$ to $K[C]$, the previous bounds are multiplied by a factor $q(r)$.

We can also try to get a bound for the number of generators for the ideal $\mathfrak{A}_J$ defining $\mathcal{H}_J$ as a subscheme of $\mathbb{A}^{p(r)q(r)}$, considering the setting of [17, Proposition C.30]: it is enough count the number of $(s'+1)\times(s'+1)$ minors of the coefficient matrix of $(\mathfrak{I}_J)_{r+1}$, $\mathfrak{I}_J$ generated by a J-marked set as in (1). Unfortunately, this number is $\binom{(n+1)s}{s'+1}\cdot\binom{N(r+1)}{s'+1}$, where $N(r+1)=\binom{n+r+1}{n}$.

We now show that if we consider an embedding $\phi_{J,m}$ for $m\geq r'=\operatorname{reg}(J^{\operatorname{sat}})$, we can get a set of generators for the ideal defining $\mathcal{H}_J$ in $\mathbb{A}^{p(m)q(m)}$ such that its cardinality is bounded by a far smaller number only depending on $p(t)$, n and m; moreover, we get that the degrees of such generators are $\leq d+2$ for every $m\geq r'$ and for every n.

Theorem 3.3. *If $J\in\mathcal{B}_{p(t)}$, for every $m\geq r'$, $\mathcal{H}_J$ is isomorphic to a closed subscheme of $\mathbb{A}^{p(m)q(m)}$, defined by at most $(q(m)(n+1)-q(m+1))\cdot p(m+1)$ polynomials of degree $\leqslant d+2$.*

Proof. Using Theorem 1.13, $\mathcal{H}_J=\mathcal{M}\mathrm{f}(J)$. Furthermore, thanks to Theorem 3.1 , we have that $\mathcal{H}_J\simeq\mathcal{M}\mathrm{f}({J^{\operatorname{sat}}}_{\geq m})$ for every $m\geq r'$.

We next show how to compute a specific set of generators for the ideal $\mathfrak{A}_{{J^{\operatorname{sat}}}_{\geq m}}$ defining the scheme structure of $\mathcal{M}\mathrm{f}({J^{\operatorname{sat}}}_{\geq m})$. We can obtain a set of generators for this ideal using a special procedure of reduction. Given the ${J^{\operatorname{sat}}}_{\geq m}$-marked set $\mathcal{G}^{(m)}$ (similar to the one in (1) of Definition 1.10), we consider $V_{m+1}=\{x_iF_\alpha \mid F_\alpha\in\mathcal{G}^{(m)},\ x_i\leq\min x^\alpha\}$ and we now use the procedure $\xrightarrow{V_{m+1}}$ of [9, Definition 3.2, Proposition 3.6] to reduce S-polynomials of elements in $\mathcal{G}^{(m)}$. We use a Buchberger-like criterion on S-polynomials analogous to the one for Gröbner bases in order to obtain a set of generators for the ideal $\mathfrak{A}_{{J^{\operatorname{sat}}}_{\geq m}}$, by [9, Theorem 3.12, Theorem 4.1].

It is enough to consider S-polynomials corresponding to a basis of the syzygies of the ideal ${J^{\operatorname{sat}}}_{\geq m}$ (see [9, Remark 3.16]). As $m\geq r'=\operatorname{reg}(J^{\operatorname{sat}})$, there exists such a basis given by pairs of variables (for instance the one in [10]). Thus, we consider S-polynomials of the type $x_iF_\alpha-x_jF_{\alpha'}$ with $x_ix^\alpha=x_jx^{\alpha'}$. If x_ix^γ is a monomial of ${J^{\operatorname{sat}}}_{\geq m+1}$ that appears in $x_iF_\alpha-x_jF_{\alpha'}$, then $x^\gamma\in\mathcal{N}(J)_m$. By Lemma 1.4 (ii), we can show that x_ix^γ is equal to x_hx^β, for $x_h=\min(x_ix^\gamma)<x_i$ and $x^\beta\in B_{{J^{\operatorname{sat}}}_{\geq m}}$. Then we can reduce $x_ix^\gamma=x_hx^\beta$ by rewriting it as $x_h(x^\beta-F_\beta)$ [9, Theorem 3.12]. If some monomial of $x_hx^\beta-x_hF_\beta$ belongs to ${J^{\operatorname{sat}}}_{\geq m}$, then again we can reduce it using some polynomial $x_{h'}F_{\beta'}$ with $x_{h'}<x_h$.

At every step of reduction a monomial is replaced by a sum of other monomials multiplied by one of the variables C. Thus, at every step of reduction the degree of the coefficients in $K[C]$ directly involved increases by 1. If $x^{\eta_0},x^{\eta_1}\ldots,x^{\eta_s}$ is a sequence of monomials of degree $m+1$ in ${J^{\operatorname{sat}}}_{\geq m}$ such that $x^{\eta_{i+1}}$ appears in the reduction of x^{η_i}, then $\min(x^{\eta_{i+1}})<\min(x^{\eta_i})$. Since $m\geq r'=\operatorname{reg}(J^{\operatorname{sat}})$, by Lemma 1.4, (iii) the minimal variable of any monomial in $\mathcal{N}(J^{\operatorname{sat}})_{m+1}$ is smaller than x_{d+1}, so the length of any such chain is at most $d+1$. Thus, in the complete reduction of an S-polynomial by $\xrightarrow{V_{m+1}}$ (which is a polynomial whose support is contained in $\mathcal{N}(J^{\operatorname{sat}})_{m+1}$), the final degree of the coefficients in $K[C]$ of each monomial x^γ is at most $1+1\cdot(d+1)=d+2$.

As said before, it is sufficient to reduce the S-polynomials corresponding to a special basis of the syzygies of ${J^{\operatorname{sat}}}_{\geq m+1}$. If we consider the *Eliahou-Kervaire S-polynomials* [5, Definition 2.12], we obtain $(q(m)(n+1)-q(m+1))$ polynomials. For each of them, using the polynomial reduction process just described, we obtain at most $p(m+1)$ generators for $\mathfrak{A}_{{J^{\operatorname{sat}}}_{\geq m}}$, since the monomials x^γ of the complete reduction of a polynomial by $\xrightarrow{V_{m+1}}$ are contained in $\mathcal{N}(J^{\operatorname{sat}})_{m+1}$ and $m+1>r'$. □

Example 3.4. We consider again $p(t)=4$ in $\mathbb{P}^2$ as in Example 2.7. In this case we have $r=4$, $s'=q(5)=17$, $N(5)=21$. The number of 18×18 minors of the coefficient matrix of $(\mathfrak{I}_J)_{r+1}$ (which has 33 rows and 21 columns), for every $J\in\mathcal{Q}$, is 1379420565600 and their degree is ≤ 18. If we consider J_1 as in Examples 2.7 and 3.2, and use Theorem 3.3, we obtain that $\mathcal{H}_{J_1}$ is isomorphic to a subscheme of $\mathbb{A}^{p(3)q(3)}=\mathbb{A}^{24}$ defined by at most 28 equations of degree ≤ 2.

Theorem 3.3 allows us to explicitly compute $\phi_{J,m}(\mathcal{H}_J)$ with a good compromise between the number of involved variables and the degree of the defining equations. This is the reason why in Algorithm BorelOpenSet$(I, p(t), n)$ we sort the list $\mathcal{B}_{p(t)}$ according to increasing regularity of the saturated ideals. In fact, given $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}$ and a generic $g \in \mathrm{PGL}$, the algorithm returns $J \in \mathcal{B}_{p(t)}$ such that $I^g \in \mathcal{H}_J$ with minimum $\mathrm{reg}(J^{\mathrm{sat}})$. This turns out to be computationally convenient: in fact, $\mathcal{H}_J$, among those containing I^g, can be embedded by equations of degree $\leq d+2$ in an affine space of smallest dimension. Furthermore, by Theorems 3.1 and 3.3, we can improve Algorithm BorelOpenSet$(I, p(t), n)$ by replacing the computation of $\Delta_J(I)$ with an analogous test concerning $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq \rho-1})$ or $\mathcal{M}\mathrm{f}({J^{\mathrm{sat}}}_{\geq \mathrm{reg}(J^{\mathrm{sat}})})$. This would be very useful in a practical implementation of this algorithm since it allows to deal with coefficient matrices of a significantly smaller size.

Furthermore, we are able to study the glueing of any pair of subsets of the Borel cover, by choosing a convenient ambient space among those given by the embeddings studied in Theorems 3.1 and 3.3.

Corollary 3.5. *If $\phi_{J_i,r} : \mathcal{H}_{J_i} \to \mathbb{A}^{p(r)q(r)}$ are the embeddings for the open subsets corresponding to two Borel ideals J_1 and J_2 belonging to $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, then:*

$$\phi_{J_1,r}(\mathcal{H}_1 \cap \mathcal{H}_2) = \phi_{J_1,r}(\mathcal{H}_1) \setminus F_1, \quad \phi_{J_2,r}(\mathcal{H}_1 \cap \mathcal{H}_2) = \phi_{J_2,r}(\mathcal{H}_2) \setminus F_2,$$

where F_1 e F_2 are hypersurfaces in $\mathbb{A}^{p(r)q(r)}$ of the same degree $|B_{J_1} \setminus B_{J_2}|$.
If $\overline{m} \geq \max\{\mathrm{reg}({J_1}^{\mathrm{sat}}), \mathrm{reg}({J_2}^{\mathrm{sat}})\}$ the same statement holds for $\phi_{J_i,\overline{m}} \colon \mathcal{H}_{J_i} \hookrightarrow \mathbb{A}^{p(\overline{m})q(\overline{m})}$.

Proof. F_1 is defined by the equation of $\frac{\Delta_{J_2}}{\Delta_{J_1}}$ in $\mathbb{A}^{p(r)q(r)}$, hence its degree corresponds to $|B_{J_1} \setminus B_{J_2}|$, by Lemma 1.9. The statement follows observing that since J_1 and J_2 both belong to $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, $|B_{J_1} \setminus B_{J_2}| = |B_{J_2} \setminus B_{J_1}|$.
The last statement is a straightforward consequence of Theorem 3.3. □

4. Examples

We now present a few examples to illustrate the twofold interest of the above results, mainly Theorems 1.13, 2.5, 3.1, and 3.3. On the one hand, we can perform explicit computations (by the algorithm in [5, Appendix]) to embed an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ in a suitable affine space. On the other hand, we can also use the results of the present paper to investigate some properties and features of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$.

Example 4.1. We consider the Hilbert scheme of μ points in $\mathbb{P}^n$, $\mathcal{H}\mathrm{ilb}^n_\mu$. For any monomial ideal J , we have that the open subset of the Grassmannian $\mathcal{U}_J$ is isomorphic to $\mathbb{A}^{p(r)q(r)}$, where $r = \mu$, $p(r)q(r) = \mu\left(\binom{n+\mu}{n} - \mu\right)$. We also consider the monomial saturated ideal $L = (x_n, \ldots, x_2, x_1^\mu)$, which is a `Lex`-segment, its regularity is $r = \mu$ and $L_{\geq\mu}$ is in $\mathcal{H}\mathrm{ilb}^n_\mu$. Obviously, the open subset $\mathcal{H}_{L_{\geq\mu}}$, the `Lex`-component, contains all the subschemes of $\mathbb{P}^n$ of μ distinct points, thus it has dimension $\geq n\mu$. By Theorem 3.1, $\mathcal{H}_{L_{\geq\mu}}$ is a closed subscheme of $\mathbb{A}^{n\mu}$, where $n\mu = |B_L| \cdot p(r)$. Therefore, $\mathcal{H}_{L_{\geq\mu}} \simeq \mathbb{A}^{n\mu}$.

Example 4.2. We can now easily study some features of $\mathcal{H}\mathrm{ilb}^3_{3t}$ that we have already investigated in Example 2.11. The open Borel cover of $\mathcal{H}\mathrm{ilb}^3_{3t}$ is made up of the open subsets $\mathcal{H}^g_{J_{\mathtt{Lex}}}$, $g \in \mathrm{PGL}$ and $J_{\mathtt{Lex}} = (x_3, x_2^3)_{\geq 3}$, as already pointed out in Example 2.11, using Theorem 2.5 and Theorem 1.13. Since no monomial in the basis of (x_3, x_2^3) is divisible by x_1, using Theorem 3.1, we have that $\mathcal{M}\mathrm{f}((x_3, x_2^3)) \simeq \mathcal{H}_{J_{\mathtt{Lex}}} \hookrightarrow \mathbb{A}^{12}$. By explicit computations, we obtain that the embedding is actually an isomorphism, that is $\mathcal{H}_{J_{\mathtt{Lex}}} \simeq \mathbb{A}^{12}$.

Indeed, every ideal I in $\mathcal{M}\mathrm{f}((x_3, x_2^3))$ is generated by a linear form $x_3 + L$ and a cubic form Q, $L, Q \in K[x_0, x_1, x_2]$, hence it depends on 12 free parameters.

Example 4.3. Consider $n = 2$ and $p(t) = 7$; in this case, we consider the set $\mathcal{B}_{p(t)}$ in $K[x_0, x_1, x_2]$:

- $J_1 = (x_2, x_1^7)_{\geq 7}$;

- $J_2 = (x_2^2, x_2x_1, x_1^6)_{\geq 7}$;
- $J_3 = (x_2^2, x_2x_1^2, x_1^5)_{\geq 7}$;
- $J_4 = (x_2^2, x_2x_1^3, x_1^4)_{\geq 7}$;
- $J_5 = (x_2^3, x_2^2x_1, x_2x_1^2, x_1^4)_{\geq 7}$.

$\mathcal{H}\mathrm{ilb}^n_{p(t)}$ in this case can be covered by the open subsets $\mathcal{H}_{J_i}$, $i = 1, \ldots, 5$ (up to changes of coordinates, as shown in Theorem 2.5), while the cover obtained from the complete list of monomial ideals in $\mathbb{G}(29, 36)$ consists of 8347680 open subsets, each one isomorphic to $\mathbb{A}^{203}$.

If we want to compute the equations defining $\mathcal{H}_{J_i} \simeq \mathcal{M}\mathrm{f}(J_i)$ as an affine scheme, we can use the techniques developed in [5]. Hence we can choose whether we are interested in embedding $\mathcal{H}_{J_i}$ in the smallest affine space (by considering the isomorphism with $\mathcal{M}\mathrm{f}({J_i^{\mathrm{sat}}}_{\geq m})$, for some m greater than or equal to $\rho_i - 1$, ρ_i as in Theorem 3.1) or wheter we also want to keep control on the degree of the equations. Indeed in a larger affine space, $\mathcal{H}_{\mathfrak{j}_i}$ is defined by equations of degree ≤ 2 (see Theorem 3.3).

It is interesting to point out that J_4 does not fullfill the hypotheses of results by [20, Section 6], so these results for Gröbner Strata do not apply to $\mathcal{H}_{J_4}$, while the techniques of [5] do. Similarly, in [9, Appendix], the authors exhibit an open subset $\mathcal{H}_J$ of $\mathcal{H}\mathrm{ilb}^2_8$, showing by explicit computations that it is not a Gröbner stratum.

Finally, the points of $\mathcal{H}\mathrm{ilb}^2_7$ are the hyperplane sections of the curves of $\mathbb{P}^3$ parametrized by the Hilbert polynomial $7t - 5$. This Hilbert scheme is investigated in detail by [11] using the techniques of [5] and of the present paper. The Borel monomial ideals in $K[x_0, x_1, x_2, x_3]$ giving the Borel cover of the Hilbert scheme $\mathcal{H}\mathrm{ilb}^3_{7t-5}$ is 112. Among these, some do not match the hypothesis of [20], while all of them can be handled with the strategies presented in the present paper and in [5].

For other examples, including the explicit computation of equations for marked schemes, we refer to [5, Example 6.3].

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