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# UNIVERSITÀ DEGLI STUDI DI TORINO 

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# UPGRADED METHODS FOR THE EFFECTIVE COMPUTATION OF MARKED SCHEMES ON A STRONGLY STABLE IDEAL 

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#### Abstract

Let $J \subset S=K\left[x_{0}, \ldots, x_{n}\right]$ be a monomial strongly stable ideal. The collection $\mathcal{M f}(J)$ of the homogeneous polynomial ideals $I$, such that the monomials outside $J$ form a $K$-vector basis of $S / I$, is called a $J$-marked family. It can be endowed with a structure of affine scheme, called a $J$-marked scheme. For special ideals $J, J$-marked schemes provide an open cover of the Hilbert scheme $\mathcal{H i l b}_{p(t)}^{n}$, where $p(t)$ is the Hilbert polynomial of $S / J$. Those ideals more suitable to this aim are the $m$-truncation ideals $\underline{J}_{>m}$ generated by the monomials of degree $\geq m$ in a saturated strongly stable monomial ideal $\underline{J}$. Exploiting a characterization of the ideals in $\mathcal{M}\left(\underline{J}_{\geq m}\right)$ in terms of a Buchberger-like criterion, we compute the equations defining the $\underline{J}_{\geq m}$-marked scheme by a new reduction relation, called superminimal reduction, and obtain an embedding of $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ in an affine space of low dimension. In this setting, explicit computations are achievable in many non-trivial cases. Moreover, for every $m$, we give a closed embedding $\phi_{m}: \mathcal{M} f\left(\underline{J}_{\geq m}\right) \hookrightarrow \mathcal{M f}\left(\underline{J}_{\geq m+1}\right)$, characterize those $\phi_{m}$ that are isomorphisms in terms of the monomial basis of $\underline{J}$, especially we characterize the minimum integer $m_{0}$ such that $\phi_{m}$ is an isomorphism for every $m \geq m_{0}$.


## Introduction

Let $J$ be a monomial ideal of the polynomial ring $S=K\left[x_{0}, \ldots, x_{n}\right]$ in $n+1$ variables over a field $K$. In this paper, we refine and develop the study begun in [7] to characterize the homogeneous polynomial ideals $I \subset S$ such that the monomials outside $J$ form a $K$-vector basis of the $K$-vector space $S / I$. If $J$ is strongly stable, such homogeneous ideals constitute a family $\mathcal{M f}(J)$, that is called a $J$-marked family and that can be endowed in a very natural way with a structure of affine scheme, called a $J$-marked scheme, which turns out to be homogeneous with respect to a non-standard grading and flat at $J$ (see [7]). Moreover, $J$-marked schemes generalize the notion of Gröbner strata [15] because $\mathcal{M f}(J)$ contains all the ideals having $J$ as initial ideal with respect to some term order; however in general $\mathcal{M f}(J)$ contains also ideals which do not belong to a Gröbner stratum.

In this paper we focus on a particular class of strongly stable ideals: letting $\underline{J}$ be a saturated strongly stable ideal, we will consider the truncations $\underline{J}_{\geq m}$, for every positive integer $m$, because in this setting marked schemes give a theoretical and effective alternative to the study of Hilbert schemes as subvarieties of a Grassmannian. Theorem 3.3 and Example 3.4 show the reason for the choice of this special setting. Let $\mathcal{H i l b} b_{p(t)}^{n}$ be the Hilbert scheme that parameterizes all subschemes of $\mathbb{P}^{n}$ with Hilbert polynomial $p(t)$, $r$ be the Gotzmann number of $p(t)$ and $q(t)=\left|S_{t}\right|-p(t)=\binom{n+t}{n}-p(t)$ be the volume polynomial. By theoretical results in $[4,7,6]$ we are able to compute first the set $\mathcal{B}_{p(t)}$ of all saturated strongly stable ideals $\underline{J}$ in $S$, such that $p(t)$ is the Hilbert polynomial of $S / \underline{J}$; then, for every ideal $\underline{J} \in \mathcal{B}_{p(t)}$, we compute explicit equations of degree $\leq \operatorname{deg}(p(t))+2$ defining $\mathcal{M f}\left(\underline{J}_{\geq r}\right)$ as an affine subscheme of $\mathbb{A}^{p(r) q(r)}$. In particular, every $\mathcal{M f}\left(\underline{J}_{\geq r}\right)$ can be embedded in $\mathcal{H} \mathrm{ilb}_{p(t)}^{n}$ as an open subscheme and moreover, as $\underline{J}$ varies in $\mathcal{B}_{p(t)}$, the $\underline{J}_{\geq r}$-marked schemes $\mathcal{M} f\left(\underline{J}_{\geq r}\right)$ form an open cover of $\mathcal{H i l b} b_{p(t)}^{n}$, up to changes of coordinates in $\mathbb{P}^{n}$. Observe that this is not true for $\mathcal{M f}(\underline{J})$, because in

[^0]general $\mathcal{M f}(\underline{J})$ is not isomorphic to an open subset of $\mathcal{H i l b}_{p(t)}^{n}$ (see Example 6.1 and [24, Section 5]). Such computational method is effective because the dimension $p(r) q(r)$ of the affine space in which the $\underline{J}_{\geq r}$-marked schemes $\mathcal{M f}\left(\underline{J}_{\geq r}\right)$ are embedded is significantly lower than the number $\binom{\left|S_{r}\right|}{q(r)}$ of Plücker coordinates. However there is room for further significant improvements.

The present paper is inspired by two questions raised, on the one hand, by similarities of marked schemes with Gröbner strata and, on the other hand, by experimental observations on examples.

First, we observed that we could eliminate a significant number of variables from the equations defining $\mathcal{M}\left(\underline{J}_{\geq m}\right)$ as an affine subscheme of $\mathbb{A}^{p(m) q(m)}$, computed using the method developed in [7]; in this way we obtain equations of higher degree than the starting ones, but often more convenient to use (for example, see [7, Appendix]). This feature has already been observed and studied for Gröbner strata in [15]. The bottleneck is that elimination of variables is too time-consuming. From this we wondered how to obtain this new set of equations using in the computations only necessary variables, avoiding the elimination process.

Our second observation is that, for a fixed $\underline{J} \in \mathcal{B}_{p(t)}$, as the integer $m$ grows, the families parameterized by marked schemes $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ become larger, up to a certain value of $m$ bounded by $r$. The study of relations among marked schemes $\mathcal{M f}\left(\underline{J}_{>m}\right)$ as $m$ varies can improve the efficiency of the computational methods in [7]: indeed, if $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ and $\mathcal{M f}\left(\underline{J}_{\geq m^{\prime}}\right)$ are isomorphic for some integers $m^{\prime}<m$, then we can choose to compute defining equations that involve a lower number of variables, that is equations for $\mathcal{M f}\left(\underline{J}_{>m^{\prime}}\right) \subseteq \mathbb{A}^{p\left(m^{\prime}\right) q\left(m^{\prime}\right)}$. In particular, for applications to the study of Hilbert schemes, we would like to determine a priori the minimum integer $m_{0}$ for which $\mathcal{M f}\left(\underline{J}_{\geq m_{0}}\right)$ is isomorphic to an open subset of $\mathcal{H i l b} b_{p(t)}^{n}$, that is $\mathcal{M f}\left(\underline{J}_{\geq m_{0}}\right) \simeq \mathcal{M f}\left(\underline{J}_{\geq r}\right)$.

In this paper, considering truncated ideals $\underline{J}_{\geq m}$, we answer to both questions by a new reduction algorithm, called superminimal reduction, that uses, for every $I \in \mathcal{M f}\left(\underline{J}_{\geq m}\right)$, its $\underline{J}_{\geq m}$-superminimal basis (see Definition 3.9), a special subset of the $\underline{J}_{\geq m}$-marked basis of $I$.

For every strongly stable monomial ideal $J$, the notion of $J$-marked basis (Definition 1.8) is the main tool for the study of marked schemes in [7] and also the starting point of the present paper. Indeed a homogeneous ideal $I$ belongs to $\mathcal{M f}(J)$ if and only if $I$ is generated by a $J$-marked basis $G$ (Proposition 1.11). This basis resembles a reduced Gröbner basis for $I$, where $J$ plays the role of the initial ideal and the strongly stable property plays the role of the term order.

Indeed, similarly to a reduced Gröbner basis, $G$ is a system of generators of $I$ that contains a polynomial $f_{\alpha}$ for every term $x^{\alpha}$ in the monomial basis of $J: f_{\alpha}=x^{\alpha}-T\left(f_{\alpha}\right)$ where no monomial appearing in $T\left(f_{\alpha}\right)$ belongs to $J$. Moreover, $G$ is characterized by a Buchberger-like criterion (Theorem 2.11) and allows to compute the $J$-reduced form modulo $I$ of every polynomial in $S$, by a Noetherian reduction process (Proposition 2.3).

The $J$-superminimal basis of $I$ introduced in the present paper is a special subset $s G$ of $G$ containing a polynomial for every term in the monomial basis of the saturated ideal $\underline{J}$ (for the details, see Definitions 3.5 and 3.9): the two sets $G$ and $s G$ are equal if and only if $\underline{J}=J$. Using only polynomials in $s G$ and the strongly stable property of $J$, we define a special process of reduction $\xrightarrow{s G *}$ called superminimal reduction.

In the special case when $J$ is a truncation $\underline{J}_{\geq m}$ of a saturated strongly stable ideal $\underline{J}$, the $\underline{J}_{\geq m^{-}}$ superminimal basis $s G$ has very interesting properties. First of all in this case (but not in general) the superminimal reduction $\xrightarrow{s G *}$ turns out to be Noetherian (see Theorem 3.14, (i) and Example 3.13). Moreover, although in general $s G$ is not a system of generators of $I$, it completely determines the ideal $I$ because we can solve the ideal-membership problem by the superminimal reduction process $\xrightarrow{s G *}$ (Theorem 3.14, (iv)). This allows to compute equations for $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ as a subscheme of an affine
space of dimension far lower than $p(m) q(m)$, without any variable elimination process (Theorem 5.4), answering the first question above.

In this new setting, in Theorem 5.7 we compare the $\underline{J}_{\geq m}$-marked schemes $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ for a fixed saturated $\underline{J}$ as $m$ varies, using superminimal bases. We prove that for every $m$ there is a closed scheme-theoretical embedding $\phi_{m}: \mathcal{M f}\left(\underline{J}_{\geq m-1}\right) \hookrightarrow \mathcal{M f}\left(\underline{J}_{\geq m}\right)$. Moreover, we provide an easy criterion on the monomial basis of $J$ to characterize the integers $m$ for which $\phi_{m}$ is an isomorphism. Especially, this criterion allows to determine the minimum integer $m_{0}$ such that $\phi_{m}$ is an isomorphism for every $m \geq m_{0}$, and in particular $\mathcal{M f}\left(\underline{J}_{\geq m_{0}}\right)$ is isomorphic to an open subset of $\mathcal{H i l b}{ }_{p(t)}^{n}$ (see [4]).

Our investigation on marked schemes lies in the framework of the methods and results obtained in the last years by several authors [5,10, 15, 20, 23, 24] about families of ideals with a fixed monomial basis for the quotient. Another close framework is the one in $[2,19]$, where the authors study the collection of all monomial ideals $J$ that are initial ideals of a fixed homogeneous ideal $I$ w.r.t. some term order.

In [4], the results of [7] and the ones of the present paper are applied to study relations among marked schemes and Hilbert schemes; in particular, in [4] the authors study how marked schemes can be used to obtain a computable open cover of $\mathcal{H i l b}_{p(t)}^{n}$ that has also interesting theoretical features. We are confident that these results, both theoretical and computational ones, may be helpful in the solution of some open problems about Hilbert schemes; indeed, they have been already applied in order to investigate the locus of points of the Hilbert scheme with bounded regularity (see [1]); the ideas and strategies used by [11] to study deformations of ACM curves are inspired by the ones in the present paper; in [16] the authors apply the computational strategy to the Hilbert scheme of locally CM curves. Other investigations led by these tools are in progress. In the future, we are interested in deeply comparing marked bases with other kinds of Gröbner-like bases, referring to [18].

In Section 1, we introduce notations and basic results and, in Section 2, we recall the Buchber-gerlike criterion described in [7], with some development that involves the Eliahou and Kervaire syzygies of a strongly stable ideal (Theorem 2.11, (iii) and Corollary 2.13). Moreover, we compute sets of generators of the ideal $\mathfrak{A}_{J}$ that defines the structure of affine scheme of $\mathcal{M f}(J)$ (see Corollary 2.17 and Remark 2.18).

In Section 3, we define the superminimal reduction (Definition 3.11) and investigate its properties. In Section 4 we describe a new Buchberger-like criterion for $\underline{J}_{\geq m}$-marked bases (Theorem 4.5) and some variants of it (Corollary 4.6 and Theorem 4.7). In particular, the second variant leads to a remarkable improvement of the efficiency of explicit computational procedures.

In Section 5, we focus on the ideal that defines the structure of affine scheme of $\mathcal{M}\left(\underline{J}_{\geq m}\right)$ and we characterize the integers $m, m^{\prime}, m>m^{\prime}$, such that the schemes $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ and $\mathcal{M f}\left(\underline{J}_{\geq m^{\prime}}\right)$ are isomorphic (Theorem 5.7).

Finally, in Section 6 we provide examples in which we apply the proved results and we compute the equations defining the affine structure of a $\underline{J}_{\geq m}$-marked scheme in a "small" affine space, using the Algorithm that we describe in the Appendix.

## 1. Notations and generalities

Let $K$ be an algebraically closed field and $S:=K\left[x_{0}, \ldots, x_{n}\right]$ ( $K[x]$ for short) the polynomial ring in $n+1$ variables with $x_{0}<\cdots<x_{n}$. We will denote by $x^{\alpha}=x_{0}^{\alpha_{0}} \cdots x_{n}^{\alpha_{n}}$ every monomial in $S$, where $\alpha=\left(\alpha_{0}, \ldots, \alpha_{n}\right)$ is its multi-index and $|\alpha|$ is its degree.

We say that a monomial $x^{\gamma}$ is divisible by $x^{\alpha}\left(x^{\alpha} \mid x^{\gamma}\right.$ for short) if there exists a monomial $x^{\beta}$ such that $x^{\alpha} \cdot x^{\beta}=x^{\gamma}$. If such monomial does not exist, we will write $x^{\alpha} \nmid x^{\gamma}$. For every monomial $x^{\alpha} \neq 1$, we set $\min \left(x^{\alpha}\right):=\min \left\{x_{i}: x_{i} \mid x^{\alpha}\right\}$ and $\max \left(x^{\alpha}\right):=\max \left\{x_{i}: x_{i} \mid x^{\alpha}\right\}$.

We will denote by $>_{\text {Lex }}$ the usual lexicographic order on the monomials of $S$ : in our setting $x^{\alpha}>_{\text {Lex }}$ $x^{\beta}$ if the last non-null element of $\alpha-\beta$ is positive.

We consider the standard grading on $S=\bigoplus_{m \in \mathbb{Z}} S_{m}$, where $S_{m}$ is the additive group of homogeneous polynomials of degree $m$; we let $S_{\geqslant m}=\bigoplus_{m^{\prime}>m} S_{m^{\prime}}$ and in the same way, for every subset $A \subseteq S$, we let $A_{m}=A \cap S_{m}$ and $A_{\geqslant m}=A \cap S_{\geqslant m}$. Elements and ideals in $S$ are always supposed to be homogeneous.

We will say that a monomial $x^{\beta}$ can be obtained by a monomial $x^{\alpha}$ through an elementary move if $x^{\alpha} x_{j}=x^{\beta} x_{i}$ for some variables $x_{i} \neq x_{j}$. In particular, if $i<j$, we say that $x^{\beta}$ can be obtained by $x^{\alpha}$ through an increasing elementary move and we write $x^{\beta}=\mathrm{e}_{i, j}^{+}\left(x^{\alpha}\right)$, whereas if $i>j$ the move is said to be decreasing and we write $x^{\beta}=\mathrm{e}_{i, j}^{-}\left(x^{\alpha}\right)$. The transitive closure of the relation $x^{\beta}>x^{\alpha}$ if $x^{\beta}=\mathrm{e}_{i, j}^{+}\left(x^{\alpha}\right)$ gives a partial order on the set of monomials of a fixed degree, that we will denote by $>_{B}$ and that is often called Borel partial order:

$$
x^{\beta}>_{B} x^{\alpha} \Longleftrightarrow \exists x^{\gamma_{1}}, \ldots, x^{\gamma_{t}} \text { such that } x^{\gamma_{1}}=\mathrm{e}_{i_{0}, j_{0}}^{+}\left(x^{\alpha}\right), \ldots, x^{\beta}=\mathrm{e}_{i_{t}, j_{t}}^{+}\left(x^{\gamma_{t}}\right)
$$

for suitable indexes $i_{k}, j_{k}$. In analogous way, we can define the same relation using decreasing moves:

$$
x^{\beta}>_{B} x^{\alpha} \Longleftrightarrow \exists x^{\delta_{1}}, \ldots, x^{\delta_{s}} \text { such that } x^{\delta_{1}}=\mathrm{e}_{h_{0}, l_{0}}^{-}\left(x^{\beta}\right), \ldots, x^{\alpha}=\mathrm{e}_{h_{s}, l_{s}}^{-}\left(x^{\delta_{s}}\right)
$$

for suitable indexes $i_{k}, j_{k}$. Note that every term order $\succ$ is a refinement of the Borel partial order $>_{B}$, that is $x^{\beta}>_{B} x^{\alpha}$ implies that $x^{\beta} \succ x^{\alpha}$.
Definition 1.1. An ideal $J \subset K[x]$ is said to be strongly stable if every monomial $x^{\beta}$ such that $x^{\beta}>_{B} x^{\alpha}$, with $x^{\alpha} \in J$, belongs to $J$.

A strongly stable ideal is always Borel fixed, that is fixed by the action of the Borel subgroup of upper triangular matrices of $G L(n+1)$. If $\operatorname{ch}(K)=0$, also the vice versa holds (e.g. [8]) and [13] guarantees that in generic coordinates the initial ideal of an ideal $I$, w.r.t. a fixed term order, is a constant Borel fixed monomial ideal called the generic initial ideal of $I$.

If $J$ is a monomial ideal in $S, B_{J}$ will denote its monomial basis and $\mathcal{N}(J)$ its sous-escalier, that is the set of monomials not belonging to $J$.

An homogeneous ideal $I$ is $m$-regular if the $i$-th syzygy module of $I$ is generated in degree $\leq m+i$, for all $i \geq 0$. The regularity of $I$ is the smallest integer $m$ for which $I$ is $m$-regular; we denote it by $\operatorname{reg}(I)$. The saturation of a homogeneous ideal $I$ is $I^{\text {sat }}=\left\{f \in S \mid \forall j=0, \ldots, n, \exists r \in \mathbb{N}: x_{j}^{r} f \in I\right\}$. The ideal $I$ is saturated if $I^{\text {sat }}=I$ and is $m$-saturated if $\left(I^{\text {sat }}\right)_{t}=I_{t}$ for each $t \geq m$. The satiety of $I$ is the smallest integer $m$ for which $I$ is $m$-saturated; we denote it by $\operatorname{sat}(I)$.

We recall that if $J$ is strongly stable then $\operatorname{reg}(J)=\max \left\{\operatorname{deg} x^{\alpha}: x^{\alpha} \in B_{J}\right\}$ [3, Proposition 2.9] and $\operatorname{sat}(J)=\max \left\{\operatorname{deg} x^{\alpha}: x^{\alpha} \in B_{J}\right.$ and $\left.x_{0} \mid x^{\alpha}\right\}$ (for example, see [14, Corollary 2.10]).
Lemma 1.2. Let $J$ be a strongly stable ideal in $K\left[x_{0}, \ldots, x_{n}\right]$. Then:
(i) $x^{\alpha} \in J \backslash B_{J} \Rightarrow \frac{x^{\alpha}}{\min \left(x^{\alpha}\right)} \in J$;
(ii) $x^{\beta} \in \mathcal{N}(J)$ and $x_{i} x^{\beta} \in J \Rightarrow$ either $x_{i} x^{\beta} \in B_{J}$ or $x_{i}>\min \left(x^{\beta}\right)$.

Proof. Both properties follow from Definition 1.1.
Definition 1.3. For every monomial $x^{\alpha}$ in $S$ we denote by $x^{\alpha}$ the monomial obtained putting $x_{0}=1$. Analogously, if $J$ is a monomial ideal in $K[x]$, we denote by $\underline{J}$ the ideal in $K[x]$ generated by $\left\{x^{\underline{\alpha}}\right.$ : $\left.x^{\alpha} \in B_{J}\right\}$.

If $J$ is strongly stable, then $J^{\text {sat }}=\underline{J}$ (this follows straightforwardly from [14, Corollary 2.10]); in particular, the set $\left\{x^{\underline{\alpha}}: x^{\alpha} \in B_{J}\right\}$ of the monomials $x^{\underline{\alpha}}$, such that $x^{\alpha}=x^{\underline{\alpha}} \cdot x_{0}^{t}$ belongs to $B_{J}$ for a suitable $t \geq 0$, contains the monomial basis $B_{\underline{J}}$.

Many tools we are going to use were introduced in [22] and developed in [7]. For this reason, we now resume some notations and definitions given in those papers.

Definition 1.4. For any non-zero homogeneous polynomial $f \in S$, the support of $f$ is the set $\operatorname{Supp}(f)$ of monomials that appear in $f$ with a non-zero coefficient.

Definition 1.5 ([22]). A marked polynomial is a polynomial $f \in S$ together with a specified monomial of $\operatorname{Supp}(f)$ that will be called head term of $f$ and denoted by $\operatorname{Ht}(f)$.

Remark 1.6. Although in this paper we use the word "monomial", we say "head term" for coherency with the notation introduced by [22]. Anyway, in this paper there will be no possible ambiguity on the meaning of "head term of $f$ ", because we will always consider marked polynomials $f$ such that the coefficient of $\operatorname{Ht}(f)$ in $f$ is 1 .

Definition 1.7 ([7]). Given a monomial ideal $J$ and an ideal $I$, a polynomial is $J$-reduced if its support is contained in $\mathcal{N}(J)$ and a $J$-reduced form modulo $I$ of a polynomial $h$ is a polynomial $h_{0}$ such that $h-h_{0} \in I$ and $\operatorname{Supp}\left(h_{0}\right) \subseteq \mathcal{N}(J)$. If there is a unique $J$-reduced form modulo $I$ of $h$, we call it $J$-normal form modulo $I$ and denote it by $\operatorname{Nf}(h)$.

Note that every polynomial $h$ has a unique $J$-reduced form modulo an ideal $I$ if and only if $\mathcal{N}(J)$ is a $K$-basis for the quotient $S / I$ or, equivalently, $S=I \oplus\langle\mathcal{N}(J)\rangle$ as a $K$-vector space. If moreover $I$ is homogeneous, the $J$-reduced form modulo $I$ of a homogeneous polynomial is supposed to be homogeneous too. These facts motivate the following definitions.

Definition 1.8. A finite set $G$ of homogeneous marked polynomials $f_{\alpha}=x^{\alpha}-\sum c_{\alpha \gamma} x^{\gamma}$, with $\operatorname{Ht}\left(f_{\alpha}\right)=$ $x^{\alpha}$, is called a $J$-marked set if the head terms $\operatorname{Ht}\left(f_{\alpha}\right)$ form the monomial basis $B_{J}$ of a monomial ideal $J$, are pairwise different and every $x^{\gamma}$ belongs to $\mathcal{N}(J)$, i.e. $\left|\operatorname{Supp}\left(f_{\alpha}\right) \cap J\right|=1$. We call tail of $f_{\alpha}$ the polynomial $T\left(f_{\alpha}\right):=\operatorname{Ht}\left(f_{\alpha}\right)-f_{\alpha}$, so that $\operatorname{Supp}\left(T\left(f_{\alpha}\right)\right) \subseteq \mathcal{N}(J)$. A $J$-marked set $G$ is a $J$-marked basis if $\mathcal{N}(J)$ is a basis of $S /(G)$ as a $K$-vector space.

Definition 1.9. The collection of all the homogeneous ideals $I$ such that $\mathcal{N}(J)$ is a basis of the quotient $S / I$ as a $K$-vector space will be denoted by $\mathcal{M f}(J)$ and called a $J$-marked family. If $J$ is a strongly stable ideal, then $\mathcal{M f}(J)$ can be endowed with a natural structure of scheme (see [7, Section 4]) that we call $J$-marked scheme.

Remark 1.10.
(i) The ideal $(G)$ generated by a $J$-marked basis $G$ has the same Hilbert function of $J$, hence $\operatorname{dim}_{K} J_{m}=\operatorname{dim}_{K}(G)_{m}$, by the definition of $J$-marked basis itself. Moreover, note that a $J$ marked basis is unique for the ideal that it generates, by the uniqueness of the $J$-normal forms modulo $I$ of the monomials in $B_{J}$.
(ii) $\mathcal{M f}(J)$ contains every homogeneous ideal having $J$ as initial ideal w.r.t. some term order, but it might also contain other ideals: see [7, Example 3.18].
(iii) When $J$ is a strongly stable ideal, all homogeneous polynomials have $J$-reduced forms modulo every ideal generated by a $J$-marked set $G$ (see [7, Theorem 2.2]).

Proposition 1.11. Let $J$ be a strongly stable ideal, I be a homogeneous ideal generated by a J-marked set $G$. The following facts are equivalent:
(i) $I \in \mathcal{M f}(J)$
(ii) $G$ is a $J$-marked basis;
(iii) $\operatorname{dim}_{k} I_{t}=\operatorname{dim}_{K} J_{t}$, for every integer $t$;
(iv) if $h \in I$ and $h$ is $J$-reduced, then $h=0$.

Proof. For the equivalence among the first three statements, see [7, Corollaries 2.3, 2.4, 2.5]. For the equivalence among (i) and (iv), observe that if $I \in \mathcal{M f}(J)$, then every polynomial has a unique $J$-reduced form modulo $I$; so, the $J$-reduced form modulo $I$ of a polynomial of $I$ must be null. Vice versa, it is enough to show that every polynomial $f$ has a unique $J$-reduced form modulo $I$. Let $\bar{f}$ and $\overline{\bar{f}}$ be two $J$-reduced forms modulo $I$ of $f$. Then, $\bar{f}-\overline{\bar{f}}$ is a $J$-reduced polynomial of $I$ because $f-\bar{f}$ and $f-\overline{\bar{f}}$ belong to $I$ by definition. We are done, because $\bar{f}-\bar{f}$ is null by the hypothesis.

## 2. Background on Buchberger-Like criterion for $J$-marked bases and some DEVELOPMENTS

In this section we recall and develop some results of [7]. Throughout this section, $J$ is a strongly stable ideal and $G$ is a J-marked set.

Definition 2.1. Let $m_{J}:=\min \left\{t: J_{t} \neq(0)\right\}$ be the initial degree of $J$. For every $\ell \geq m_{J}$ we define the set

$$
W_{\ell}:=\left\{x^{\delta} f_{\alpha} \mid f_{\alpha} \in G \text { and }|\delta+\alpha|=\ell\right\}
$$

that becomes a set of marked polynomials by letting $\operatorname{Ht}\left(x^{\delta} f_{\alpha}\right)=x^{\delta+\alpha}$. We set $W=\cup_{\ell} W_{\ell}$. For every $\ell \geq m_{J}$ we also define a special subset of $W_{\ell}$ :

$$
V_{\ell}:=\left\{x^{\delta} f_{\alpha} \in W_{\ell} \mid x^{\delta}=1 \text { or } \max \left(x^{\delta}\right) \leq \min \left(x^{\alpha}\right)\right\} .
$$

We let $V=\cup_{\ell} V_{\ell}$. Moreover, $\langle V\rangle$ denotes the vector space generated by the polynomials in $V$ and $\xrightarrow{V_{\ell}}$ is the reduction relation on homogeneous polynomials of degree $\ell$ defined in the usual sense of Gröbner basis theory (see also [7, Proposition 3.6]).

The above Definition is equivalent to the Definition 3.2 in [7] due to Remark 3.3 of the same paper. Note that $(G)_{\ell}$ is generated by $W_{\ell}$ as a $K$-vector space.

Lemma 2.2. Let $J$ be a strongly stable ideal. An ideal I generated by a J-marked set $G$ belongs to $\mathcal{M f}(J)$ if and only if $\langle W\rangle=\langle V\rangle$ as $K$-vector spaces.

Proof. It is sufficient to observe that for every $\ell \geq m_{J}$, the number of elements in $V_{\ell}$ is equal to the number of monomials in $J_{\ell}$, so $\operatorname{dim}\left\langle V_{\ell}\right\rangle \leq \operatorname{dim} J_{\ell}$. On the other hand, $\operatorname{dim}\left\langle W_{\ell}\right\rangle=\operatorname{dim} I_{\ell} \geq \operatorname{dim} J_{\ell}$ by [7, Corollary 2.3]. By Proposition 1.11 we get the equivalence of the statements.

We have already recalled that, when $J$ is a strongly stable ideal, every homogeneous polynomial has a $J$-reduced form modulo an ideal generated by a $J$-marked set $G$ (Remark 1.10 (iii)). Further, a $J$-reduced form of a homogeneous polynomial can be constructed by the reduction relation $\xrightarrow{V_{l}}$, as it is recalled by next Proposition.

Proposition 2.3. [7, Proposition 3.6] With the above notation, every monomial $x^{\beta} \in J_{\ell}$ can be reduced to a $J$-reduced form modulo $(G)$ in a finite number of reduction steps, using only polynomials of $V_{\ell}$. Hence, the reduction relation $\xrightarrow{V_{\ell}}$ is Noetherian.

The Noetherianity of the reduction relation $\xrightarrow{V_{\ell}}$ provides an algorithm that reduces every homogeneous polynomial of degree $\ell$ to a $J$-reduced form modulo $(G)$ in a finite number of steps. We note that on the one hand it is convenient to substitute the polynomials in $V_{\ell}$ by their $J$-reduced normal forms for an efficient implementation of a reduction algorithm, but, on the other hand, in the proofs it is convenient to use the polynomials of $V_{\ell}$ as constructed in Definition 2.1.
2.1. Order on $W_{\ell}$. Using the Noetherianity of the reduction relation $\xrightarrow{V_{\ell}}$, we can recognize when a $J$-marked set is a $J$-marked basis by a Buchberger-like criterion (see [7, Theorem 3.12]). To this aim we need to set an order on the set $W_{\ell}$.

The order that we are going to define on $W_{\ell}$ in Definition 2.7 is based on the following Definition and Lemma that are inspired by [9] and [17, Lemma 2.11].
Definition 2.4. Given a strongly stable monomial ideal $J$ in $S$, with monomial basis $B_{J}$, and a monomial $x^{\gamma} \in J$, we define

$$
x^{\gamma}=x^{\alpha} *_{J} x^{\eta}, \quad \text { with } \gamma=\alpha+\eta, x^{\alpha} \in B_{J} \text { and } \min \left(x^{\alpha}\right) \geq \max \left(x^{\eta}\right) .
$$

This decomposition exists and is unique (see [9, Lemma 1.1]).
Lemma 2.5. Let $J$ be a strongly stable ideal. If $x^{\epsilon}$ belongs to $\mathcal{N}(J)$ and $x^{\epsilon} \cdot x^{\delta}=x^{\epsilon+\delta}$ belongs to $J$ for some $x^{\delta}$, then $x^{\epsilon+\delta}=x^{\alpha} *_{J} x^{\eta}$ with $x^{\eta}<_{\text {Lex }} x^{\delta}$. Furthermore:
(i) if $|\delta|=|\eta|$, then $x^{\eta}<_{B} x^{\delta}$; and
(ii) $x^{\underline{\eta}}<_{\text {Lex }} x^{\underline{\delta}}$.

Proof. We can assume that $x^{\delta}$ and $x^{\eta}$ are coprime; indeed, if this is not the case, we can divide the involved equalities of monomials by $\operatorname{gcd}\left(x^{\delta}, x^{\eta}\right)$. If $x^{\eta}=1$, all the statements are obvious. If $x^{\eta} \neq 1$, then $\min \left(x^{\delta}\right) \mid x^{\alpha}$ because $x^{\delta}$ and $x^{\eta}$ are coprime, hence $\min \left(x^{\delta}\right) \geq \min \left(x^{\alpha}\right) \geq \max \left(x^{\eta}\right)$ and so $\min \left(x^{\delta}\right)>\max \left(x^{\eta}\right)$ because they cannot coincide. This inequality implies both $x^{\eta}<_{\text {Lex }} x^{\delta}$ and $x^{\eta}<_{\text {Lex }} x \underline{\delta}$. Moreover, if $|\delta|=|\eta|$, this is also sufficient to conclude that $x^{\eta}<_{B} x^{\delta}$.
Remark 2.6. Observe that if $g_{\beta}=x^{\delta} f_{\alpha}$ belongs to $V_{\ell}$, then $x^{\beta}=x^{\alpha} *_{J} x^{\delta}$.
Definition 2.7. Let $\geq$ be any order on $G$ and $x^{\delta} f_{\alpha}, x^{\delta^{\prime}} f_{\alpha^{\prime}}$ be two elements of $W_{\ell}$. We set

$$
x^{\delta} f_{\alpha} \succeq_{\ell} x^{\delta^{\prime}} f_{\alpha^{\prime}} \Leftrightarrow x^{\delta}>_{\text {Lex }} x^{\delta^{\prime}} \text { or } x^{\delta}=x^{\delta^{\prime}} \text { and } f_{\alpha} \geq f_{\alpha^{\prime}} \text {. }
$$

## Lemma 2.8 .

(i) For every two elements $x^{\delta} f_{\alpha}, x^{\delta^{\prime}} f_{\alpha^{\prime}}$ of $W_{\ell}$ we get

$$
x^{\delta} f_{\alpha} \succeq_{\ell} x^{\delta^{\prime}} f_{\alpha^{\prime}} \Rightarrow \forall x^{\eta}: \quad x^{\delta+\eta} f_{\alpha} \succeq_{\ell^{\prime}} x^{\delta^{\prime}+\eta} f_{\alpha^{\prime}},
$$

where $\ell^{\prime}=|\delta+\eta+\alpha|$.
(ii) Every polynomial $g_{\beta} \in V_{\ell}$ is the minimum w.r.t. $\preceq_{\ell}$ of the subset $W_{\beta}$ of $W_{\ell}$ containing all polynomials of $W_{\ell}$ with $x^{\beta}$ as head term.
(iii) If $x^{\delta} f_{\alpha}$ belongs to $W_{\ell} \backslash G_{\ell}$ and $x^{\beta}$ belongs to $\operatorname{Supp}\left(x^{\delta} T\left(f_{\alpha}\right)\right)$ with $g_{\beta} \in V_{\ell}$, then $x^{\delta} f_{\alpha} \succ_{\ell} g_{\beta}$.

Proof.
(i) This follows by the analogous property of the term order $>_{\text {Lex }}$.
(ii) Let $g_{\beta}=x^{\delta^{\prime}} f_{\alpha^{\prime}}$ be the polynomial of $V$ such that $x^{\beta}=x^{\alpha^{\prime}} *_{J} x^{\delta^{\prime}}$ and $x^{\delta} f_{\alpha}$ be another polynomial of $W_{\beta}$. We can assume that $x^{\delta}$ and $x^{\delta^{\prime}}$ are coprime; otherwise, we can divide the involved inequalities of monomials by $\operatorname{gcd}\left(x^{\delta}, x^{\delta^{\prime}}\right)$. By Remark 2.6 and Definition 2.4, we have that $\max \left(x^{\delta^{\prime}}\right) \leq \min \left(x^{\alpha^{\prime}}\right)$ and $\max \left(x^{\delta}\right)>\min \left(x^{\alpha}\right)$. Then, we get $\max \left(x^{\delta}\right)>\max \left(x^{\delta^{\prime}}\right)$ because $x^{\alpha^{\prime}} \nmid x^{\alpha}$ and $x^{\alpha} \nmid x^{\alpha^{\prime}}$. Thus, $x^{\delta}>_{\text {Lex }} x^{\delta^{\prime}}$.
(iii) If $x^{\beta}$ belongs to $B_{J}$ we are done. Otherwise, let $x^{\beta}=x^{\alpha^{\prime}} *_{J} x^{\delta^{\prime}}$ and note that every monomial of $\operatorname{Supp}\left(x^{\delta} f_{\alpha}\right)$ is a multiple of $x^{\delta}$, in particular $x^{\beta}=x^{\delta+\gamma}$ for some $x^{\gamma} \in \mathcal{N}(J)$. By Lemma 2.5, we get $x^{\delta^{\prime}}<$ Lex $x^{\delta}$.
Remark 2.9. We point out that the order defined in [7, Definition 3.9] does not satisfy the conditions listed in Lemma 2.8 and in [7, Lemma 3.10]. These conditions have a crucial role in the proof of [7, Theorem 3.12] and for this reason it has been a mistake to use the order of [7, Definition 3.9] in
that Theorem. So, here we replace such order by that defined in new Definition 2.7. Aside the order, the original reduction and Buchberger criterion are the same, as we will state in Theorem 2.11, (i) and (ii). Also, we give an improvement by Theorem 2.11, (iii) and by Corollary 2.13. Moreover, we observe that for the same reason the results about syzygies of the ideal $I$ generated by a $J$-marked basis proposed in [7, Section 3] hold by using the order on $W_{\ell}$ of Definition 2.7 and do not hold by using the order of [7, Definition 3.9].

### 2.2. Improved Buchberger-like criterion for $J$-marked bases.

Definition 2.10. The $S$-polynomial of two elements $f_{\alpha}, f_{\alpha^{\prime}}$ of a $J$-marked set $G$ is the polynomial $S\left(f_{\alpha}, f_{\alpha^{\prime}}\right):=x^{\gamma} f_{\alpha}-x^{\gamma^{\prime}} f_{\alpha^{\prime}}$, where $x^{\gamma+\alpha}=x^{\gamma^{\prime}+\alpha^{\prime}}=\operatorname{lcm}\left(x^{\alpha}, x^{\alpha^{\prime}}\right)$.
Theorem 2.11. (Buchberger-like criterion) Let $J$ be a strongly stable ideal and $I$ the homogeneous ideal generated by a J-marked set $G$. With the above notation, TFAE:
(i) $I \in \mathcal{M f}(J)$;
(ii) $\forall f_{\alpha}, f_{\alpha^{\prime}} \in G, S\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \xrightarrow{V_{\ell}} 0$;
(iii) $\forall f_{\alpha}, f_{\alpha^{\prime}} \in G, S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x^{\gamma} f_{\alpha}-x^{\gamma^{\prime}} f_{\alpha^{\prime}}=\sum a_{j} x^{\eta_{j}} f_{\alpha_{j}}$, with $x^{\eta_{j}}<_{\text {Lex }} \max _{\text {Lex }}\left\{x^{\gamma}, x^{\gamma^{\prime}}\right\}$ and $x^{\eta_{j}} f_{\alpha_{j}} \in V_{\ell}$.
Proof. For the equivalence between (i) and (ii), we refer to the proof of [7, Theorem 3.12] by using Definition 2.7 instead of [7, Definition 3.9].

Statement (ii) implies (iii) by the definition of the reduction relation $\xrightarrow{V_{\ell}}$ and by Lemma 2.8 (iii). It remains to prove that statement (iii) implies (i).

We want to prove that $I=\langle V\rangle$ or, equivalently by Lemma 2.2, that $\langle V\rangle=\langle W\rangle$. It is sufficient to prove that $x^{\eta} \cdot V \subseteq\langle V\rangle$, for every monomial $x^{\eta}$. We proceed by induction on the monomials $x^{\eta}$, ordered according to Lex. The thesis is obviously true for $x^{\eta}=1$. We then assume that the thesis holds for any monomial $x^{\eta^{\prime}}$ such that $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$.

If $|\eta|>1$, we can consider any product $x^{\eta}=x^{\eta_{1}} \cdot x^{\eta_{2}}, x^{\eta_{1}}$ and $x^{\eta_{2}}$ non-constant. Since $x^{\eta_{i}}<_{\text {Lex }}$ $x^{\eta}, i=1,2$, we immediately obtain by induction

$$
x^{\eta} \cdot V=x^{\eta_{1}} \cdot\left(x^{\eta_{2}} \cdot V\right) \subseteq x^{\eta_{1}}\langle V\rangle \subseteq\langle V\rangle
$$

If $|\eta|=1$, then we need to prove that $x_{i} \cdot V \subseteq\langle V\rangle$. Since $x_{0} V \subseteq V$, it is then sufficient to prove the thesis for $x^{\eta}=x_{i}$, assuming that the thesis holds for every $x^{\eta^{\prime}}<_{\text {Lex }} x_{i}$. We consider $g_{\beta}=x^{\delta} f_{\alpha} \in V$, where $\max \left(x^{\delta}\right) \leq \min \left(x^{\alpha}\right)$. If $x_{i} g_{\beta}$ does not belong to $V$, then $\max \left(x_{i} \cdot x^{\delta}\right)>\min \left(x^{\alpha}\right)$, so $x_{i}>\min \left(x^{\alpha}\right)$. In particular, $x_{i}>\min \left(x^{\alpha}\right) \geq \max \left(x^{\delta}\right)$, so $x_{i}>_{\text {Lex }} x^{\delta}$ : by induction, it is now sufficient to prove the thesis for $x_{i} f_{\alpha}$.

We consider an $S$-polynomial $S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x_{i} f_{\alpha}-x^{\gamma} f_{\alpha^{\prime}}$ such that $x^{\gamma}<_{\text {Lex }} x_{i}$. Such $S$-polynomial always exists: for instance, we can consider $x_{i} x^{\alpha}=x^{\alpha^{\prime}} *_{J} x^{\eta^{\prime}}$. By the hypothesis $x_{i} f_{\alpha}-x^{\eta^{\prime}} f_{\alpha^{\prime}}=$ $\sum a_{j} x^{\eta^{\prime}{ }_{j}} f_{\alpha_{j}}$ where $x^{\eta^{\prime}} f_{\alpha^{\prime}}, x^{\eta^{\prime}}{ }_{j} f_{\alpha_{j}} \in V_{\ell}$ and then $x_{i} f_{\alpha}$ belongs to $\langle V\rangle$.

For any strongly stable ideal $J$, with monomial basis $B_{J}=\left\{x^{\alpha_{1}}, \ldots, x^{\alpha_{r}}\right\}$, we can consider the set of syzygies of the following kind

$$
x_{j} e_{\alpha_{i}}-x^{\eta} e_{\alpha_{k}}, \quad \text { with } x_{j}>\min \left(x^{\alpha_{i}}\right) \text { and } x_{j} x^{\alpha_{i}}=x^{\alpha_{k}} *_{J} x^{\eta}
$$

This set of syzygies is actually a minimal set of generators for the first module of syzygies of $J$; this is due to Eliahou and Kervaire (see [9] and [14, Theorem 1.31]).

Definition 2.12. We call Eliahou-Kervaire couple of the $J$-marked set $G$ any couple of polynomials $f_{\alpha}, f_{\beta}, \operatorname{Ht}\left(f_{\alpha}\right)=x^{\alpha}, \operatorname{Ht}\left(f_{\beta}\right)=x^{\beta}$, such that

$$
x_{j} x^{\alpha}=x^{\beta} *_{J} x^{\eta} \text { for some } x_{j}>\min \left(x^{\alpha}\right)
$$

We call Eliahou-Kervaire $S$-polynomial (EK-polynomial, for short) of $G$ an $S$-polynomial among an Eliahou-Kervaire couple of polynomials $f_{\alpha}$ and $f_{\beta}$. We denote such $S$-polynomial by $S^{E K}\left(f_{\alpha}, f_{\beta}\right)$. Observe that, thanks to the definition, an EK-polynomial is of kind

$$
S^{E K}\left(f_{\alpha}, f_{\beta}\right)=x_{j} f_{\alpha}-x^{\eta} f_{\beta}, \text { for some } x_{j}>\min \left(x^{\alpha}\right), \text { with } x_{j} x^{\alpha}=x^{\beta} *_{J} x^{\eta}
$$

In the proof of Theorem 2.11, it is sufficient to assume that (iii) holds only for EK-polynomials, as stated in the following result.

Corollary 2.13. With the same notation of Theorem 2.11,

$$
I \in \mathcal{M}(J) \Leftrightarrow \text { for every EK-polynomial between elements of } G, S^{E K}\left(f_{\alpha}, f_{\beta}\right) \xrightarrow{V_{\ell}} 0 .
$$

Proof. In the proof of Theorem 2.11 the crucial point is the existence of an $S$-polynomial of kind $x_{i} f_{\alpha}-x^{\eta} f_{\beta}$ with $x^{\eta}<_{\text {Lex }} x_{i}$, and we used exactly an EK-polynomial.
2.3. The scheme structure of $\mathcal{M f}(J)$. Now, we recall and develop some features of the affine scheme structure of $\mathcal{M f}(J)$. Let $p(t)$ the Hilbert polynomial of $S / J$ and $r$ its Gotzmann number. In the following we will denote by $\mathcal{G}$ the $J$-marked set:

$$
\begin{equation*}
\mathcal{G}=\left\{F_{\alpha}=x^{\alpha}-\sum C_{\alpha \gamma} x^{\gamma}: \operatorname{Ht}\left(F_{\alpha}\right)=x^{\alpha} \in B_{J}, x^{\gamma} \in \mathcal{N}(J)_{|\alpha|}\right\} \tag{1}
\end{equation*}
$$

and by $\mathfrak{I}_{J}$ the ideal generated by $\mathcal{G}$ in the ring $K[C, x]$, where $C$ is a compact notation for the set of new variables $C_{\alpha \gamma}$.

For every polynomial $H \in K[C, x]$, we denote by $\operatorname{Sup}_{x}(H)$ the set of monomials in the variables $x_{i}$ that appear in $H$ with non-null coefficients and by $\operatorname{Coeff}_{x}(H) \subset K[C]$ the set of such coefficients, that we call $x$-coefficients.

Let $\mathcal{V}_{\ell}$ and $\mathcal{W}_{\ell}$ be the analogous for $\mathcal{G}$ of $V_{\ell}$ and of $W_{\ell}$, respectively, for any $J$-marked set $G$. We will denote by $\mathfrak{A}_{J}$ the ideal of $K[C]$ generated by the $x$-coefficients of the $J$-reduced forms, obtained by $\xrightarrow{\mathcal{V}_{l}}$, of the $S$-polynomials $S\left(F_{\alpha}, F_{\alpha^{\prime}}\right)$ among elements of $\mathcal{G}$. This ideal does not depend on $\xrightarrow{\mathcal{V}_{l}}$ and defines the subscheme structure of $\mathcal{M f}(J)$ in the affine space $\mathbb{A}^{|C|}$ (see [7, Theorem 4.1]). Let $\mathfrak{A}_{J}^{E K}$ be the ideal of $K[C]$ generated by the $x$-coefficients of the $J$-reduced forms of the EK-polynomials of $\mathcal{G}$ obtained by $\xrightarrow{\mathcal{V}_{\ell}}$.

It is clear that $\mathfrak{A}_{J}^{E K} \subseteq \mathfrak{A}_{J}$. Anyway, we will prove that $\mathfrak{A}_{J}^{E K}$ and $\mathfrak{A}_{J}$ are the same ideal, although $\mathfrak{A}_{J}$ is defined by a set of generators bigger than the set of generators of $\mathfrak{A}_{J}^{E K}$. More precisely, we prove that the ideal $\mathfrak{A}_{J}^{E K}$ contains the $x$-coefficients of every $J$-reduced polynomial in $\mathfrak{I}_{J}$.

## Lemma 2.14.

(i) For every monomial $x^{\beta}=x^{\alpha} *_{J} x^{\delta} \in J$, there is a formula of type

$$
x^{\beta}=\sum a_{i} x^{\gamma_{i}} F_{\alpha_{i}}+H_{\beta},
$$

with $a_{i} \in K[C], x^{\gamma_{i}} F_{\alpha_{i}} \in \mathcal{V}, x^{\gamma_{i}} \leq_{\text {Lex }} x^{\beta}$ and $\operatorname{Supp}_{x}\left(H_{\beta}\right) \subset \mathcal{N}(J)$.
(ii) For every polynomial $x_{i} F_{\alpha} \in \mathcal{W} \backslash \mathcal{V}$, there is a formula of type

$$
x_{i} F_{\alpha}=\sum a_{j} x^{\eta_{j}} F_{\alpha_{j}}+H_{i, \alpha}
$$

with $a_{j} \in K[C], x^{\eta_{j}} F_{\alpha_{j}} \in \mathcal{V}, x^{\eta_{j}}<_{\text {Lex }} x_{i}, \operatorname{Supp}_{x}\left(H_{i, \alpha}\right) \subset \mathcal{N}(J)$ and $\operatorname{Coeff}_{x}\left(H_{i, \alpha}\right) \subset \mathfrak{A}_{J}^{E K}$.
Proof. Statement (i) follows from the existence of $J$-reduced forms obtained by $\xrightarrow{\mathcal{V}_{\ell}}$ and by Lemma 2.8, (iii). Statement (ii) follows also from the definition of $\mathfrak{A}_{J}^{E K}$.

Proposition 2.15. For every polynomial $x^{\delta} F_{\alpha} \in \mathcal{W} \backslash \mathcal{V}$, we have

$$
\begin{equation*}
x^{\delta} F_{\alpha}=\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H_{\delta, \alpha} \tag{2}
\end{equation*}
$$

with $b_{j} \in K[C], x^{\eta_{j}} F_{\alpha_{j}} \in \mathcal{V}, x^{\eta_{j}}<_{\text {Lex }} x^{\delta}, \operatorname{Supp}_{x}\left(H_{\delta, \alpha}\right) \subset \mathcal{N}(J)$ and $\operatorname{Coeff}_{x}\left(H_{\delta, \alpha}\right) \subset \mathfrak{A}_{J}^{E K}$.
Proof. For $|\delta|=1$ it is enough to use Lemma 2.14 (ii). Assume that $|\delta|>1$ and that the thesis holds for every $x^{\delta^{\prime}}<_{\text {Lex }} x^{\delta}$. Let $x_{i}=\min \left(x^{\delta}\right)$ and $x^{\delta^{\prime}}=\frac{x^{\delta}}{x_{i}}$, so that $x^{\delta^{\prime}} F_{\alpha}$ belongs to $\mathcal{W} \backslash \mathcal{V}$.

By the inductive hypothesis, we have $x^{\delta^{\prime}} F_{\alpha}=\sum b_{j}^{\prime} x^{\eta_{j}^{\prime}} F_{\alpha_{j}}+H_{\delta^{\prime}, \alpha}$, with $x^{\eta_{j}^{\prime}}<_{\text {Lex }} x^{\delta^{\prime}}$. So, multiplying by $x_{i}$, we obtain $x^{\delta} F_{\alpha}=\sum b_{j}^{\prime} x_{i} x^{\eta_{j}^{\prime}} F_{\alpha_{j}}+x_{i} H_{\delta^{\prime}, \alpha}$ and the thesis holds for every polynomial $x_{i} x^{\eta_{j}^{\prime}} F_{\alpha_{j}}$ that belongs to $\mathcal{W} \backslash \mathcal{V}$ because $x_{i} x^{\eta_{j}^{\prime}}<_{\text {Lex }} x_{i} x^{\delta^{\prime}}=x^{\delta}$. Then, we replace such polynomials by formulas of type (2) and obtain

$$
x^{\delta} F_{\alpha}=\sum b_{s} x^{\eta_{s}} F_{\alpha_{s}}+H^{\prime}+x_{i} H_{\delta^{\prime}, \alpha}
$$

where the first sum satisfies the conditions of (2) and $H^{\prime}$ is $J$-reduced with $\operatorname{Supp}_{x}\left(H^{\prime}\right) \subset \mathcal{N}(J)$ and $\operatorname{Coeff}_{x}\left(H^{\prime}\right) \subset \mathfrak{A}_{J}^{E K}$.

Note that $\operatorname{Coeff}_{x}\left(x_{i} H_{\delta^{\prime}, \alpha}\right)=\operatorname{Coeff}_{x}\left(H_{\delta^{\prime}, \alpha}\right) \subset \mathfrak{A}_{J}^{E K}$, but we do not know if $\operatorname{Supp}_{x}\left(x_{i} H_{\delta^{\prime}, \alpha}\right) \subset \mathcal{N}(J)$. If $x^{\beta^{\prime}} \in \operatorname{Supp}_{x}\left(H_{\delta^{\prime}, \alpha}\right)$ has $x$-coefficient $b$ in $H_{\delta^{\prime}, \alpha}$ and $x^{\beta}=x_{i} x^{\beta^{\prime}}$ belongs to $J$, then we can use Lemma 2.14 (i) obtaining $b x^{\beta}=\sum b a_{k} x^{\gamma_{i}} F_{\alpha_{k}}+b H_{\beta}$. Moreover, if $x^{\beta}=x^{\alpha^{\prime}} *_{J} x^{\epsilon}$, then $x^{\gamma_{i}} \leq_{\text {Lex }} x^{\epsilon}<_{\text {Lex }}$ $x_{i}<_{\text {Lex }} x^{\delta}$, where the second inequality is due to the fact that $x^{\beta^{\prime}} \in \mathcal{N}(J)$ and to Lemma 2.5, and all the $x$-coefficients of $H_{\beta}$ belong to $\mathfrak{A}_{J}^{E K}$ because they are divisible by $b$. Replacing all such monomials $x^{\beta}$, we obtain the thesis and $H_{\delta, \alpha}$ is $J$-reduced with $x$-coefficients in $\mathfrak{A}_{J}^{E K}$, because it is the sum of $J$-reduced polynomials with $x$-coefficients in $\mathfrak{A}_{J}^{E K}$.

Corollary 2.16. Every polynomial of $\Im_{J}$ can be written in a unique way as $\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H$, with $b_{j} \in K[C], x^{\eta_{j}} F_{\alpha_{j}} \in \mathcal{V}$ and $H$ J-reduced. Moreover, we obtain also that $\operatorname{Coeff}_{x}(H) \subset \mathfrak{A}_{J}^{E K}$.
Proof. By definition, every polynomial of $\mathfrak{I}_{J}$ is a linear combination of polynomials of $\mathcal{V} \cup(\mathcal{W} \backslash \mathcal{V})$ with $x$-coefficients in $K[C]$ and, by Proposition 2.15 , every such polynomial can be written has described in the statement. Hence, we have only to prove the uniqueness of this writing.

Let $\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H=0$ be the difference between two writings of the same polynomial of $\mathfrak{I}_{J}$, with $b_{j} \neq 0, x^{\eta_{j}} F_{\alpha_{j}} \in \mathcal{V}$ pairwise different and $H J$-reduced. Let $x^{\eta_{1}} x^{\alpha_{1}}$ the maximum of the monomials w.r.t. the order for which $x^{\eta_{i}} x^{\alpha_{i}}$ is lower than $x^{\eta_{j}} x^{\alpha_{j}}$ if $x^{\eta_{i}}<_{\text {Lex }} x^{\eta_{j}}$ or $x^{\eta_{i}}=x^{\eta_{j}}$ and $x^{\alpha_{i}}<x^{\alpha_{j}}$, where $<$ is any order fixed on $B_{J}$. By definition of $\mathcal{V}$, the unique polynomial of $\mathcal{V}$ with head term $x^{\eta_{1}} x^{\alpha_{1}}$ is $x^{\eta_{1}} F_{\alpha_{1}}$. Moreover, the monomial $x^{\eta_{1}} x^{\alpha_{1}}$ does not appear with a non-null coefficient in any polynomial of the sum because every other monomial belongs to $\mathcal{N}(J)$ or is lower than it, by construction. Further, $x^{\eta_{1}} x^{\alpha_{1}}$ does not belong to $\operatorname{Supp}_{x}(H)$ because $\operatorname{Supp}_{x}(H) \subset \mathcal{N}(J)$ and $x^{\eta_{1}} x^{\alpha_{1}} \in J$. Thus, we obtain a contradiction to the fact that $b_{j} \neq 0$.
Corollary 2.17. The ideal $\mathfrak{A}_{J}^{E K}$ contains the $x$-coefficients of every J-reduced polynomial of $\mathfrak{I}_{J}$. In particular, $\mathfrak{A}_{J}^{E K}=\mathfrak{A}_{J}$.

Proof. Let $F$ be a $J$-reduced polynomial of $\Im_{J}$ and let $F=\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H$ as in Corollary 2.16. Since $F$ itself is $J$-reduced, also $F=0+F$ is a formula as described in Corollary 2.16 and we obtain that $F=H$, by the uniqueness of this formula. Hence, we have $\operatorname{Coeff}_{x}(F)=\operatorname{Coeff}_{x}(H) \subset \mathfrak{A}_{J}^{E K}$. The last assertion is due to the definition of $\mathfrak{A}_{J}$.

Remark 2.18. Actually, for every ideal $\widehat{\mathfrak{A}}_{J} \subseteq \mathfrak{A}_{J} \subseteq K[C]$ such that condition (ii) of Lemma 2.14 holds, also Corollary 2.17 holds. We are then allowed to choose different sets of $S$-polynomials of $\mathcal{G}$ in order to obtain generators of the ideal $\mathfrak{A}_{J}$.

## 3. Superminimal generators and reduction

In this section we introduce the notion of $m$-truncation ideal and a new polynomial reduction process, that we call superminimal reduction, useful to find a new set of equations to define a marked scheme. From the next section on, we will focus on $J$-marked schemes with $J$ a strongly stable $m$ truncation. The reason is twofold: on the one hand, strongly stable $m$-truncation ideals have a good behavior also from the geometric point of view (Theorem 3.3 and Example 3.4); on the other hand, the superminimal reduction is Noetherian when we take a strongly stable $m$-truncation ideal (Theorem 3.14), but it is not if we just consider a strongly stable ideal (Example 3.13).

### 3.1. Truncation strongly stable ideals.

Definition 3.1. Let $J \subseteq S$ be a monomial ideal. We will say that $J$ is an m-truncation if $J$ is the truncation of $J^{\text {sat }}$ in degree $m$, that is $J=\left(J^{\text {sat }}\right) \geq m$.

We observe that an $m$-truncation ideal $J$ is strongly stable if and only if $J^{\text {sat }}$ is and that if $J$ is strongly stable, then $J^{\text {sat }}=\underline{J}$.

The following Lemma highlights some simple features of $m$-truncation strongly stable ideals, which will turn out to be crucial in the proofs of our main results.
Lemma 3.2. Let $J$ be a strongly stable m-truncation. Then:
(i) $B_{J} \cap B_{\underline{J}}=\left(B_{\underline{J}}\right) \geq m$.
(ii) $\forall x^{\beta} \in B_{\underline{J}} \backslash B_{J}: x^{\beta} x_{0}^{m-|\beta|} \in B_{J}$.
(iii) $\forall x^{\gamma} \in S_{\geq m}, \forall t \in \mathbb{N}: x^{\gamma} x_{0}^{t} \in J \Leftrightarrow x^{\gamma} \in J$.
(iv) $\mathcal{N}(J)_{\geq m}=\mathcal{N}(\underline{J})_{\geq m}$.
(v) $\forall h \in S_{\geq m}$ : $h$ is $J$-reduced $\Leftrightarrow h$ is $\underline{J}$-reduced.
(vi) If I belongs to $\mathcal{M f}(J)$, then for every homogeneous polynomial $h$ of degree $\geq m$, $J$-normal forms modulo I satisfy: $\mathrm{Nf}\left(x_{0}^{t} \cdot h\right)=x_{0}^{t} \cdot \mathrm{Nf}(h)$.

Proof. Facts (i) and (ii) are straightforward consequences of the definition of $m$-truncations.
For (iii), we only prove the non trivial part " $\Rightarrow$ ". If $x^{\gamma} x_{0}^{t} \in J$, then $x^{\gamma}$ belongs to $\underline{J}$. Since $J$ is an $m$-truncation and $x^{\gamma} \in S_{\geq m}$, then $x^{\gamma} \in J$ too.

Statements (iv) and (v) are obviously equivalent to (iii).
For (vi), we recall that the $J$-reduced form modulo $I$ of any polynomial is unique since $I$ belongs to $\mathcal{M f}(J)$. By (iii), both $\operatorname{Nf}\left(x_{0}^{t} \cdot h\right)$ and $x_{0}^{t} \cdot \mathrm{Nf}(h)$ are $J$-reduced forms of $x_{0}^{t} h$ and then they coincide.
Theorem 3.3. Let $J$ be a strongly stable m-truncation ideal. Two different ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{M} f(J)$ give rise to different subschemes of $\mathbb{P}^{n}$, thus they correspond to different points of the Hilbert scheme $\mathcal{H i l b}_{p(t)}^{n}$ with $p(t)$ the Hilbert polynomial of $S / J$.
Proof. By the uniqueness of the reduced form, there is a monomial $x^{\alpha} \in B_{J}$ such that the corresponding polynomials $f_{\alpha}^{\mathfrak{a}}$ and $f_{\alpha}^{\mathfrak{b}}$ of the $J$-marked bases of $\mathfrak{a}$ and $\mathfrak{b}$, respectively, are different and moreover such that $f_{\alpha}^{\mathfrak{a}} \notin \mathfrak{b}$ and $f_{\alpha}^{\mathfrak{b}} \notin \mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ defined the same projective scheme, we would have $\mathfrak{a}_{r}=\mathfrak{b}_{r}$ for some $r \gg 0$. Hence $x_{0}^{r-m} f_{\alpha}^{\mathfrak{a}}-x_{0}^{r-m} f_{\beta}^{\mathfrak{b}}=x_{0}^{r-m}\left(-T\left(f_{\alpha}^{\mathfrak{a}}\right)+T\left(f_{\beta}^{\mathfrak{b}}\right)\right)$ is a non-zero polynomial that belongs (for instance) to $\mathfrak{a}$. Moreover, due to Lemma 3.2, (iii), $x_{0}^{r-m}\left(-T\left(f_{\alpha}^{\mathfrak{a}}\right)+T\left(f_{\beta}^{\mathfrak{b}}\right)\right)$ is $J$-reduced modulo $\mathfrak{a}$ : this is impossible because of Proposition 1.11, (iv).

The following example shows that, if $J$ is a strongly stable ideal but not an $m$-truncation, different ideals in $\mathcal{M f}(J)$ may define the same subscheme in $\mathbb{P}^{n}$. This is the first reason why we will focus mainly on strongly stable $m$-truncations.
Example 3.4. In the ring $S=K\left[x_{0}, x_{1}, x_{2}\right]$, let us consider the strongly stable ideal $J=\left(x_{2}, x_{1}^{2}\right.$, $\left.x_{1} x_{0}\right)$ and for every $c \in K$ the ideal $\mathfrak{a}_{c}=\left(x_{2}+c x_{1}, x_{1}^{2}, x_{1} x_{0}\right)$. An easy computation shows that the
ideals $\mathfrak{a}_{c}$ belong to $\mathcal{M f}(J)$ and are pairwise different. However, $x_{2}^{2}, x_{2} x_{1}, x_{2} x_{0}$ belong to $\mathfrak{a}_{c}$ : indeed $x_{2}^{2}=\left(x_{2}+c x_{1}\right)\left(x_{2}-c x_{1}\right)+c^{2} x_{1}^{2}, x_{2} x_{1}=\left(x_{2}+c x_{1}\right) x_{1}-c x_{1}^{2}, x_{2} x_{0}=\left(x_{2}+c x_{1}\right) x_{0}-c x_{1} x_{0}$; hence the saturation of $\mathfrak{a}_{c}$ is $J$. Then, the subschemes $\operatorname{Proj}\left(S / \mathfrak{a}_{c}\right)$ of $\mathbb{P}^{2}$ coincide. We can observe that the difference between the ideals $\mathfrak{a}_{c}$ disappears if we only consider their homogeneous components of degree $\geq 2$.
3.2. Superminimals. In the following we will use the notation stated in Definition 1.3.

Definition 3.5. Let $J$ be a strongly stable ideal. The set of superminimal generators of $J$ is

$$
s B_{J}=\left\{x^{\beta} \in B_{J} \mid x^{\underline{\beta}} \in B_{\underline{J}}\right\} .
$$

Remark 3.6. Another special set of monomials for a strongly stable ideal $J$ is the so-called set of Borel generators (see [12]), namely the smallest subset of $B_{J}$ such that $J$ is the minimum strongly stable ideal containing them. Although there is a clear analogy between the ideas underlying the definition of superminimal generators and that of Borel generators, however they do not coincide in general.

Example 3.7. Consider $\underline{J}:=\left(x_{2}^{3}, x_{2}^{2} x_{1}, x_{2} x_{1}^{2}, x_{1}^{6}\right) \subseteq K\left[x_{0}, x_{1}, x_{2}\right]$ and its 5 -truncation ideal $J:=\underline{J} \geqslant{ }^{2}$. The set of superminimal generators of $J$ is $s B_{J}=\left\{x_{2}^{3} x_{0}^{2}, x_{2}^{2} x_{1} x_{0}^{2}, x_{2} x_{1}^{2} x_{0}^{2}, x_{1}^{6}\right\}$, while the set of Borel generators of $J$ is $\left\{x_{2} x_{1}^{2} x_{0}^{2}, x_{1}^{6}\right\}$, because $x_{2} x_{1}^{2} x_{0}^{2} \in J$ imposes $x_{2}^{2} x_{1} x_{0}^{2}=\mathrm{e}_{1,2}^{+}\left(x_{2} x_{1}^{2} x_{0}^{2}\right) \in J$ and $x_{2}^{3} x_{0}^{2}=$ $\mathrm{e}_{1,2}^{+} \circ \mathrm{e}_{1,2}^{+}\left(x_{2} x_{1}^{2} x_{0}^{2}\right) \in J$.

Example 3.8. Consider $J:=\left(x_{2}^{2}, x_{2} x_{1}^{2}, x_{2} x_{1} x_{0}, x_{2} x_{0}^{2}\right) \subseteq K\left[x_{0}, x_{1}, x_{2}\right]$ whose saturation is $\underline{J}=\left(x_{2}\right)$. The set of superminimal generators of $J$ is $s B_{J}=\left\{x_{2} x_{0}^{2}\right\}$, while the set of Borel generators of $J$ is $\left\{x_{2}^{2}, x_{2} x_{0}^{2}\right\}$.

Definition 3.9. Let $J$ be a strongly stable ideal. A finite set of marked polynomials $f_{\beta}=x^{\beta}-\sum c_{\beta \gamma} x^{\gamma}$, with $\operatorname{Ht}\left(f_{\beta}\right)=x^{\beta}$, is a $J$-marked superminimal set if the head terms form the set of superminimal generators $s B_{J}$ of $J$, they are pairwise different, and $x^{\gamma} \in \mathcal{N}(J)$. We call tail of $f_{\beta}$ the homogeneous polynomial $T\left(f_{\beta}\right):=x^{\beta}-f_{\beta}$.

Every $J$-marked set $G$ contains a (unique) subset $s G$ of this type, that is called the set of superminimals of $G$; if $G$ is a $J$-marked basis, $s G$ is called $J$-superminimal basis.

Remark 3.10. If $\Gamma$ is a $J$-marked superminimal set, it can always be completed to a (non-unique) $J$-marked set $G$. For instance $G=\Gamma \cup\left(B_{J} \backslash s B_{J}\right)$.

On the other hand, if $I \in \mathcal{M}(J)$, then its $J$-superminimal basis is the only $J$-marked superminimal set contained in $I$. In fact, for every $x^{\beta} \in s B_{J}$, if $f_{\beta}$ belongs to both $I$ and a $J$-marked superminimal set, then $x^{\beta}-f_{\beta}$ has to be a $J$-reduced form of $x^{\beta}$ modulo $I$, which is the unique normal form $\operatorname{Nf}\left(x^{\beta}\right)$ modulo $I$.

Definition 3.11. Consider a strongly stable ideal $J$, a $J$-marked set $G$ and two polynomials $h$ and $h_{1}$. We say that $h$ is in $s G_{*}$-relation with $h_{1}$ if there is a monomial $x^{\gamma} \in \operatorname{Supp}(h) \cap J, c=\operatorname{Coeff}\left(x^{\gamma}\right)$, such that $x^{\gamma}$ is divisible by a superminimal generator $x^{\alpha}$ of $J$, with $x^{\gamma}=x^{\underline{\alpha}} *_{\underline{J}} x^{\eta}=x^{\alpha} \cdot x^{\epsilon}$ and $h_{1}=h-c \cdot x^{\epsilon} f_{\alpha}$, that is $h_{1}$ is obtained by replacing the monomial $x^{\gamma}$ in $h$ by $x^{\epsilon} \cdot T\left(f_{\alpha}\right)$. We call superminimal reduction the transitive closure of the above relation and denote it by $\xrightarrow{s G *}$. Moreover, we say that:

- $h$ can be reduced to $h_{1}$ by $\xrightarrow{s G *}$ if $h \xrightarrow{s G *} h_{1}$;
- $h$ is non-reducible w.r.t. $\xrightarrow{s G *}$ if no step of reduction on $h$ by $\xrightarrow{s G *}$ can be performed;
- $h$ is strongly reduced if for every $t, x_{0}^{t} \cdot h$ is non-reducible w.r.t. $\xrightarrow{s G *}$.

Remark 3.12.
(i) We use the notation $\xrightarrow{s G *}$ to underline that this reduction also involves the decomposition $*_{J}$ of Definition 2.4 and it is not the usual polynomial reduction w.r.t. a set of marked polynomials $s G$. Indeed even if a polynomial $h$ is non-reducible w.r.t. $\xrightarrow{s G *}$, its support can contain some monomial which is multiple of a monomial in $s B_{J}$ (see Example 3.15); hence $h$ would be reducible w.r.t. the usual reduction $\xrightarrow{s G}$.
(ii) A homogeneous polynomial $h$ is strongly reduced if and only if no monomial in $\operatorname{Supp}(h)$ is divisible by a monomial of $B_{\underline{J}}$, that is $h$ is $\underline{J}$-reduced. In fact, if $x^{\gamma} \in \operatorname{Supp}(h) \cap \underline{J}$ then $x^{\gamma}=x^{\underline{\alpha}} * \underline{J} x^{\eta}$ and there is $t$ such that $x^{\alpha}=x_{0}^{t} x^{\underline{\alpha}} \in B_{J}$. Thus $x_{0}^{t} h$ can be reduced by $\xrightarrow{s G *}$ using the polynomial $f_{\alpha}$.
(iii) The polynomials $x^{\epsilon} f_{\alpha}$ that we use for the reduction procedure $\xrightarrow{s G *}$ have head terms pairwise different. Moreover, if $x^{\delta} f_{\alpha^{\prime}}$ is used in the $\xrightarrow{s G *}$ reduction of $x^{\epsilon} T\left(f_{\alpha}\right)$ then $x^{\underline{\delta}}<_{\text {Lex }} x^{\underline{\epsilon}}$.
If we consider a strongly stable ideal $J$ with no further hypothesis, we cannot generalize the properties of the reduction $\xrightarrow{V_{\ell}}$ to $\xrightarrow{s G *}$, as shown in the following example.
Example 3.13. In the ring $S=K\left[x_{0}, x_{1}, x_{2}\right]$ (with $x_{2}>x_{1}>x_{0}$ ) let us consider the strongly stable ideal $J=\left(x_{2}^{3}, x_{2}^{2} x_{1}, x_{2} x_{1}^{2}, x_{2}^{2} x_{0}, x_{2} x_{1} x_{0}, x_{1}^{4}, x_{1}^{3} x_{0}, x_{1}^{2} x_{0}^{2}\right)$ and its saturation $\underline{J}=\left(x_{2}^{2}, x_{2} x_{1}, x_{1}^{2}\right)$. The set of superminimals of $J$ is $s B_{J}=\left\{x_{2}^{2} x_{0}, x_{2} x_{1} x_{0}, x_{1}^{2} x_{0}^{2}\right\}$. Let us consider the $J$-marked superminimal set $s G=\left\{f_{x_{2}^{2} x_{0}}=x_{2}^{2} x_{0}, f_{x_{2} x_{1} x_{0}}=x_{2} x_{1} x_{0}-x_{1}^{3}, f_{x_{1}^{2} x_{0}^{2}}=x_{1}^{2} x_{0}^{2}-x_{2} x_{0}^{3}\right\}$. The superminimal reduction w.r.t. $s G$ is not Noetherian. For instance:

$$
x_{1}^{3} x_{0}^{2} \xrightarrow{s G *} T\left(f_{x_{1}^{2} x_{0}^{2}}\right) \cdot x_{1}=x_{2} x_{0}^{3} x_{1} \xrightarrow{s G *} T\left(f_{x_{2} x_{1} x_{0}}\right) \cdot x_{0}^{2}=x_{1}^{3} x_{0}^{2} .
$$

However, if we assume that the strongly stable ideal $J$ is also an $m$-truncation ideal, then the reduction $\xrightarrow{s G *}$ turns out to be Noetherian and satisfies several good properties, similar to the ones of $\xrightarrow{V_{\ell}}$.

Theorem 3.14. Let $J$ be a strongly stable m-truncation ideal and $s G$ be a $J$-marked superminimal set. Then:
(i) $\xrightarrow{s G *}$ is Noetherian.
(ii) For every homogeneous polynomial $h$ there exist $t$ and a unique polynomial $h(t)$ strongly reduced such that $x_{0}^{t} \cdot h \xrightarrow{s G *} h(t)$. If $\bar{t}$ is the minimum one and $\bar{h}:=h(\bar{t})$, then $h(t)=x_{0}^{t-\bar{t}} \cdot \bar{h}$ for every $t \geq \bar{t}$. There is an effective procedure that computes $\bar{t}$ and $\bar{h}$.
If moreover $s G$ is the superminimal basis of an ideal I of $\mathcal{M f}(J)$, then:
(iii) $\xrightarrow{s G *}$ computes the J-normal forms modulo I. More precisely, for every homogeneous polynomial $h$ :

$$
\operatorname{Nf}(h)= \begin{cases}h, & \text { if } \operatorname{deg}(h)<m \\ \bar{h} / x_{0}^{\bar{t}}, & \text { if } \operatorname{deg}(h) \geq m \text { and } x_{0}^{\bar{t}} \cdot h \xrightarrow{s G *} \bar{h}\end{cases}
$$

(iv) $\xrightarrow{s G *}$ solves the ideal-membership problem for $I$ : for every homogeneous polynomial $h$ :

$$
h \in I \Leftrightarrow \operatorname{deg}(h) \geq m \text { and } x_{0}^{\bar{t}} \cdot h \xrightarrow{s G *} 0
$$

(v) There is a one-to-one correspondence between ideals in $\mathcal{M} f(J)$ and $J$-superminimal bases.

Proof.
(i) Since $J$ is a strongly stable and $m$-truncation ideal, then $\mathcal{N}(J)_{\geq m}=\mathcal{N}(\underline{J})_{\geq m}$ (Lemma 3.2, (iv)). If $\xrightarrow{s G *}$ was not Noetherian, by Lemma 2.5 applied to $\underline{J}$, we would be able to find infinite descending chains of monomials w.r.t. $<_{\text {Lex }}$.
(ii) It is sufficient to prove the thesis for monomials $x^{\gamma}$ in $J$. Let $x^{\gamma}=x^{\underline{\alpha}} *_{\underline{J}} x^{\eta}$. If $x^{\eta}=1$, then $x^{\alpha}=x_{0}^{t_{\alpha}} \cdot x^{\alpha}$ is in $s B_{J}, f_{\alpha}$ belongs to $s G$ and $x_{0}^{t_{\alpha}} \cdot x^{\underline{\alpha}} \xrightarrow{s G *} T\left(f_{\alpha}\right)$, where $\operatorname{Supp}\left(T\left(f_{\alpha}\right)\right) \subseteq \mathcal{N}(J)$. In this case $\bar{h}=T\left(f_{\alpha}\right)$ and $\bar{t}=t_{\alpha}$. If $x^{\eta} \neq 1$, we can assume that the thesis holds for any monomial $x^{\gamma^{\prime}}=x^{\underline{\beta}} *_{\underline{J}} x^{\eta^{\prime}}$, such that $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$.

We perform a first reduction $x_{0}^{t_{\alpha}} \cdot x^{\gamma} \xrightarrow{s G *} x^{\eta} \cdot T\left(f_{\alpha}\right)$. If $x^{\eta} \cdot T\left(f_{\alpha}\right)$ is strongly reduced, we are done. Otherwise, we have $x^{\eta} \neq x_{0}^{|\eta|}$. For every monomial $x^{\gamma^{\prime}} \in \operatorname{Supp}\left(x^{\eta} \cdot T\left(f_{\alpha}\right)\right) \cap \underline{J}$ we have $x^{\gamma^{\prime}}=x^{\underline{\beta}^{\prime}} *_{\underline{J}} x^{\eta^{\prime}}$, with $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$ by Lemma 2.5. So, we have also $x_{0}^{t} \cdot x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$, for every $t$. By the inductive hypothesis we can find a suitable power $t$ of $x_{0}$ such that every monomial in $x_{0}^{t} \cdot x^{\eta} \cdot T\left(f_{\alpha}\right)$ can be reduced by $\xrightarrow{s G *}$ to a strongly reduced polynomial.

It remains to prove the uniqueness of the strongly reduced polynomial $h(t)$. Let us consider two different strongly reduced $\xrightarrow{s G *}$ reductions of $x_{0}^{t} h$ : their difference is again strongly reduced and can be written as $\Sigma a_{i} x^{\eta_{i}} f_{\alpha_{i}}$ with $a_{i} \in K, a_{i} \neq 0$ and $x^{\eta_{i}} f_{\alpha_{i}}$ pairwise different. Let $x^{\eta_{1}} f_{\alpha_{1}}$ be such that for every $i \geq 2$, either $x^{\eta_{1}}>_{\text {Lex }} x^{\eta_{i}}$ or $x^{\eta_{1}}=x^{\eta_{i}}$ and $x^{\alpha_{1}}>_{\text {Lex }} x^{\alpha_{i}}$. Then $x^{\eta_{1}} x^{\alpha_{1}}$ should cancel with a monomial in $\operatorname{Supp}\left(x^{\eta_{i}} T\left(f_{\alpha_{i}}\right)\right)$ for some $i$, but this is impossible as observed in Remark 3.12, (iii).

Observe that, though for a fixed $x^{\gamma}=x^{\underline{\alpha}} *_{J} x^{\eta}$, there are infinitely many monomials $x^{\gamma^{\prime}}=$ $x^{\beta} *_{J} x^{\eta^{\prime}}$ such that $x^{\eta^{\prime}}<$ Lex $x^{\eta}$, we use the inductive hypothesis only with respect to the finite number of them that appear on the support of $\cdot x^{\eta} \cdot T\left(f_{\alpha}\right)$. For this reason our procedure is effective.
From now on we consider $I \in \mathcal{M f}(J)$; therefore if $h$ is a homogeneous polynomial and $h \xrightarrow{s G *} h_{1}$ with $h_{1}$ strongly reduced, then by uniqueness of $J$-normal forms modulo $I$ we have $h_{1}=\operatorname{Nf}(h)$.
(iii) If $\operatorname{deg} h<m$ we are done. Otherwise from (ii) we have that $x_{0}^{\bar{t}} \cdot h \xrightarrow{s G *} \bar{h}$ and $\bar{h}$ is a $J$-reduced form modulo $I$. Thus $x_{0}^{t} \cdot \mathrm{Nf}(h)$ is $J$-reduced too (Lemma 3.2, (iii)) and we get the desired equality by uniqueness of $J$-normal forms modulo $I$.
(iv) This is a consequence of (iii) and of Proposition 1.11 (iv).
(v) This is the straightforward consequence of (iv).

Whenever $J$ is a strongly stable $m$-truncation ideal and $s G$ is the superminimal basis of an ideal $I \in \mathcal{M f}(J)$, then $s G$ is a subset of the set $V$ of Definition 2.1. Nevertheless, it is interesting to notice that not every step of reduction by $\xrightarrow{s G *}$ is also a step of reduction by $\xrightarrow{V_{\ell}}$, as shown in the following example.

Example 3.15. Consider $J=\left(x_{1}^{2}, x_{0} x_{2}, x_{1} x_{2}, x_{2}^{2}\right)$ which is a strongly stable ideal and a 2 -truncation of $\underline{J}=\left(x_{2}, x_{1}^{2}\right)$ in $K\left[x_{0}, x_{1}, x_{2}\right]$. Let $G$ be a $J$-marked set.

- The monomial $x_{2} \cdot x_{1}^{2}$ is non-reducible w.r.t. $s G$, because the only monomial of $s B_{J}$ dividing it is $x_{1}^{2}$, but $x_{2} x_{1}^{2}=x_{2} *_{\underline{J}} x_{1}^{2}$. On the other hand, $x_{2} x_{1}^{2}=x_{2} x_{1} *_{J} x_{1}$, so $x_{2} x_{1}^{2} \xrightarrow{V_{3}} x_{1} T(f)$ where $f \in V_{2}, \operatorname{Ht}(f)=x_{2} x_{1}$.
- The only way to reduce $x_{0} \cdot x_{2}^{2}$ via $\xrightarrow{V_{3}}$ leads to $x_{0} \cdot T\left(f^{\prime}\right)$, where $f^{\prime}$ is the unique polynomial of $V_{2}$ such that $\operatorname{Ht}\left(f^{\prime}\right)=x_{2}^{2}$. Moreover, $x_{0} \cdot T\left(f^{\prime}\right)$ is not further reducible, because all the monomials of its support belong to $\mathcal{N}(J)$. On the other hand, according to Definition 3.11, a first step of reduction of the monomial $x_{0} \cdot x_{2}^{2}$ via $\xrightarrow{s G *}$ is $x_{0} x_{2}^{2} \xrightarrow{s G *} x_{2} \cdot T\left(f^{\prime \prime}\right)$, where $f^{\prime \prime}$ is the polynomial in $s G$ with $\operatorname{Ht}\left(f^{\prime \prime}\right)=x_{0} \cdot x_{2}$. Since $x_{2}$ is a monomial of $B_{\underline{\underline{J}}}$, every monomial appearing in $\operatorname{Supp}\left(x_{2} \cdot T\left(f^{\prime \prime}\right)\right)$ belongs to $J$, and so we will need further steps of reduction via $\xrightarrow{s G *}$ to compute a polynomial non-reducible w.r.t. $s G$.


## 4. Buchberger-Like criterion by superminimal reduction

In the present and following sections, we assume that $J \subseteq S$ is a strongly stable m-truncation ideal, in order to apply the main results of Section 3, mainly those concerning the new reduction process $\xrightarrow{s G *}$ (Theorem 3.14). We will also use the sets of polynomials $V$ and $W$ which are defined from a $J$-marked set $G$ (see Definition 2.1), and the reduction relation $\xrightarrow{V_{\ell}}$.

In Section 2, we proved that $J$-marked bases are characterized by a Buchberger-like criterion on the reduction of $S$-polynomials between elements of $G$ by $\xrightarrow{V_{e}}$ (Theorem 2.11). Afterwards, in Section 3 we showed that every homogeneous ideal $I$ in $\mathcal{M f}(J)$ is completely determined by its superminimal basis $s G$ and that $J$-normal forms modulo $I$ can be computed using $\xrightarrow{s G *}$, that is again using polynomials in the subset $s G$ of $G$ (Theorem 3.14).

Therefore, it is natural to ask whether one can obtain a Buchberger-like criterion only considering $S$-polynomials among elements in $s G$. Unfortunately, the answer is negative, as clearly shown by Example 4.1. However, we can prove a few variants of the Buchberger-like criterion of Theorem 2.11, in which the set of superminimals $s G$ and the superminimal reduction process $\xrightarrow{s G *}$ replace $G$ and $\xrightarrow{V_{e}}$. Using these new criteria in the next section we will be able to obtain sets of equations defining $\mathcal{M f}(J)$ in a smaller set of variables than those of Subsection 2.3.

Example 4.1. We consider the strongly stable 2 -truncation ideal

$$
J=\left(x_{3}^{2}, x_{3} x_{2}, x_{3} x_{1}, x_{3} x_{0}, x_{2}^{2}\right) \subseteq K\left[x_{0}, x_{1}, x_{2}, x_{3}\right]
$$

whose saturation is $\underline{J}=\left(x_{3}, x_{2}^{2}\right)$. In this case, $s B_{J}$ contains only two monomials, $x_{3} x_{0}$ and $x_{2}^{2}$. If $G$ is any $J$-marked set, then $s G=\left\{f_{x_{3} x_{0}}, f_{x_{2}^{2}}\right\}$. The unique $S$-polynomial among superminimal elements is

$$
S\left(f_{x_{3} x_{0}}, f_{x_{2}^{2}}\right)=x_{2}^{2} f_{x_{3} x_{0}}-x_{3} x_{0} f_{x_{2}^{2}}=x_{3} x_{0} \cdot T\left(f_{x_{2}^{2}}\right)-x_{2}^{2} \cdot T\left(f_{x_{3} x_{0}}\right)
$$

Any monomial appearing in $\operatorname{Supp}\left(T\left(f_{x_{3} x_{0}}\right)\right)$ is in $\mathcal{N}(J)_{2}=K\left[x_{0}, x_{1}, x_{2}\right]_{2} \backslash\left\{x_{2}^{2}\right\}$. Then any monomial appearing in $\operatorname{Supp}\left(x_{2}^{2} \cdot T\left(f_{x_{3} x_{0}}\right)\right)$ is further reduced by $f_{x_{2}^{2}}$, obtaining by $\xrightarrow{V_{4}}$ or $\xrightarrow{s G *}$

$$
S\left(f_{x_{3} x_{0}}, f_{x_{2}^{2}}\right)=x_{3} x_{0} \cdot T\left(f_{x_{2}^{2}}\right)-T\left(f_{x_{2}^{2}}\right) \cdot T\left(f_{x_{3} x_{0}}\right)=T\left(f_{x_{2}^{2}}\right) \cdot f_{x_{3} x_{0}} \rightarrow 0 .
$$

Nevertheless, even if the only $S$-polynomial among superminimal generators reduces to 0 , if we consider $s G=\left\{f_{x_{3} x_{0}}, f_{x_{2}^{2}}\right\}$ with $f_{x_{3} x_{0}}=x_{3} x_{0}+x_{1}^{2}$ and $f_{x_{2}^{2}}=x_{2}^{2}$, then for any choice of $f_{x_{3}^{2}}, f_{x_{3} x_{2}}, f_{x_{3} x_{1}}$, the $S$-polynomial among $f_{x_{3} x_{1}}$ and $f_{x_{3} x_{0}}$ does not reduce to 0 :

$$
S\left(f_{x_{3} x_{1}}, f_{x_{3} x_{0}}\right)=x_{0} f_{x_{3} x_{1}}-x_{1} f_{x_{3} x_{0}}=\sum_{x^{\alpha_{i} \in \mathcal{N}(J)_{2}}} a_{i} x^{\alpha_{i}} x_{0}-x_{1}^{3} .
$$

The monomials $x^{\alpha_{i}} x_{0}$ are in $\mathcal{N}(J)_{3}$ and are strongly reduced. Furthermore, $x_{1}^{3}$ does not appear among monomials $x^{\alpha_{i}} x_{0}$, so it is not canceled, and it is strongly reduced too. Therefore, for any choice of coefficients in the tail of $f_{x_{3} x_{1}}$, we have an $S$-polynomial which is not reducible to 0 , and any $J$-marked set containing $f_{x_{3} x_{0}}=x_{3} x_{0}+x_{1}^{2}$ is not a $J$-marked basis.
4.1. Buchberger-like criteria via $\xrightarrow{s G *}$ : first variant. In this subsection we prove that the Buchberger-like criterion of Theorem 2.11 and Corollary 2.13 can be rephrased in terms of the reduction process $\xrightarrow{s G *}$. The involved $S$-polynomials will be all those between elements in $G$ (Theorem 4.5), or only $E K$-polynomials between elements of $G$ (Corollary 4.6). We will need a few lemmas.
Lemma 4.2. Let $J$ be a strongly stable m-truncation ideal, $G$ be a J-marked set and $h$ be a homogeneous polynomial of degree $\ell \geq m$. Then:

$$
h \in\left\langle V_{\ell}\right\rangle \Leftrightarrow x_{0} \cdot h \in\left\langle V_{\ell+1}\right\rangle .
$$

Proof. If $h \in\left\langle V_{\ell}\right\rangle$, then $x_{0} \cdot h \in\left\langle V_{\ell+1}\right\rangle$ by definition of $V$.
Vice versa, assume that $x_{0} \cdot h \in\left\langle V_{\ell+1}\right\rangle$. This is equivalent to $x_{0} \cdot h \xrightarrow{V_{\ell+1}} 0$. Every monomial in $\operatorname{Supp}\left(x_{0} \cdot h\right)$ can be written as $x_{0} \cdot x^{\epsilon}$; observe that $x_{0} \cdot x^{\epsilon} \notin B_{J}$, because $\operatorname{deg}\left(x_{0} \cdot x^{\epsilon}\right)>m$, by Lemma 3.2, (i). Then, if $x_{0} \cdot x^{\epsilon}$ belongs to $J$, we can decompose it as $x_{0} \cdot x^{\epsilon}=x^{\alpha} *_{J} x^{\eta}, x^{\alpha} \in B_{J}$ and $x^{\eta} \neq 1$. Since $\min \left(x^{\alpha}\right) \geq \max \left(x^{\eta}\right)$, we have that $x^{\eta}$ is divisible by $x_{0}$. So $x^{\eta}=x_{0} \cdot x^{\eta^{\prime}}$.

Summing up, in order to reduce the monomial $x_{0} \cdot x^{\epsilon}$ of $\operatorname{Supp}\left(x_{0} \cdot h\right)$ using $V$, we use the polynomial $x_{0} \cdot x^{\eta^{\prime}} \cdot f_{\alpha} \in V, \operatorname{Ht}\left(f_{\alpha}\right)=x^{\alpha}$. If the coefficient of $x_{0} \cdot x^{\epsilon}$ in $x_{0} \cdot h$ is $a$, we obtain

$$
x_{0} \cdot h \xrightarrow{V_{\ell+1}} x_{0} \cdot\left(h-a \cdot x^{\eta^{\prime}} f_{\alpha}\right) .
$$

At every step of reduction, we obtain a polynomial which is divisible by $x_{0}$. In particular,

$$
x_{0} \cdot h \in\left\langle V_{\ell+1}\right\rangle \Rightarrow x_{0} \cdot h=x_{0} \cdot \sum a_{i} x^{\eta_{i}} f_{\alpha_{i}}, \text { where } x_{0} \cdot x^{\eta_{i}} f_{\alpha_{i}} \in V_{\ell+1} .
$$

Then we have that $h=\sum a_{i} x^{\eta_{i}} f_{\alpha_{i}}$ and $x^{\eta_{i}} f_{\alpha_{i}} \in V_{\ell}$, that is $h \in\left\langle V_{\ell}\right\rangle$.
Consider $f_{\alpha}, f_{\alpha^{\prime}} \in G$, the $S$-polynomial $S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x^{\gamma} f_{\alpha}-x^{\gamma^{\prime}} f_{\alpha^{\prime}}$ and assume that $x^{\gamma^{\prime}}<_{\text {Lex }} x^{\gamma}$. By Lemma 2.8, (iii), if $S\left(f_{\alpha}, f_{\alpha}^{\prime}\right) \xrightarrow{V_{\ell}} h$, then $S\left(f_{\alpha}, f_{\alpha}^{\prime}\right)-h=\sum a_{j} x^{\delta_{j}} f_{\beta_{j}}$ with $x^{\delta_{j}} f_{\beta_{j}} \in V_{\ell}, x^{\delta_{j}}<_{\text {Lex }} x^{\gamma}$. Now we show that a similar result holds for the superminimal reduction $\xrightarrow{s G *}$.

Lemma 4.3. Let $J$ be a strongly stable m-truncation ideal, $G$ be a J-marked set and $f_{\alpha}, f_{\alpha^{\prime}}$ be two polynomials belonging to $G$. Consider the $S$-polynomial $S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x^{\gamma} f_{\alpha}-x^{\gamma^{\prime}} f_{\alpha^{\prime}}$, with $x^{\gamma^{\prime}}<_{\text {Lex }} x^{\gamma}$. If $x_{0}^{t} \cdot S\left(f_{\alpha}, f_{\alpha}^{\prime}\right) \xrightarrow{s G *} h$, then $x_{0}^{t} \cdot S\left(f_{\alpha}, f_{\alpha}^{\prime}\right)-h=\sum a_{j} x^{\eta_{j}} f_{\beta_{j}}$ with $f_{\beta_{j}} \in s G, x^{\eta_{j}}<_{\text {Lex }} x^{\gamma}$ and $x_{\underline{\eta_{j}}}<_{\text {Lex }} x \underline{\underline{\gamma}}$.
Proof. Every monomial $x_{0}^{t} \cdot x^{\gamma} \cdot x^{\epsilon}$ in $\operatorname{Supp}\left(x_{0}^{t} \cdot x^{\gamma} \cdot T\left(f_{\alpha}\right)\right) \cap J$ decomposes as $x_{0}^{t} \cdot x^{\gamma} \cdot x^{\epsilon}=x^{\beta} *_{J} x^{\eta}$, $x^{\eta}<_{\text {Lex }} x_{0}^{t} x^{\gamma}$ and $x^{\underline{\eta}}<_{\text {Lex }} x^{\underline{\gamma}}$ by Lemma 3.2, (iv) and Lemma 2.5. The same holds for any further reduction and the same argument applies to monomials appearing in $\operatorname{Supp}\left(x_{0}^{t} \cdot x^{\gamma^{\prime}} \cdot T\left(f_{\alpha^{\prime}}\right)\right)$.

We point out that Lemma 4.3 does not hold without the hypothesis that $J$ is an $m$-truncation ideal, as shown by the following example.

Example 4.4. In $S=K\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$, consider the strongly stable ideal

$$
J=\left(x_{3}^{2}, x_{3} x_{2}, x_{3} x_{1}\right)_{\geq 4}+\left(x_{2}^{2}\right)_{\geq 6},
$$

whose saturation is $\underline{J}=\left(x_{3}^{2}, x_{2} x_{3}, x_{1} x_{3}, x_{2}^{2}\right)$. $J$ is not an $m$-truncation for any $m$. Consider a $J$-marked set $G$ and $f_{\alpha}, f_{\beta} \in G$ such that $\operatorname{Ht}\left(f_{\alpha}\right)=x_{0}^{2} x_{2} x_{3}$ and $\operatorname{Ht}\left(f_{\beta}\right)=x_{0}^{2} x_{1} x_{3}$ and consider $x_{2}^{4} \in \operatorname{Supp}\left(T\left(f_{\beta}\right)\right)$. Then $S\left(f_{\alpha}, f_{\beta}\right)=x_{1} f_{\alpha}-x_{2} f_{\beta}$. If we apply Definition 3.11, we reduce $x_{2}^{4} \in \operatorname{Supp}\left(S\left(f_{\alpha}, f_{\beta}\right)\right)$ by $\xrightarrow{s G *}$, pre-multiplying by $x_{0}^{4}$. We get that $x_{0}^{4} x_{2}^{4}$ belongs to $\operatorname{Supp}\left(x_{0}^{4} S\left(f_{\alpha}, f_{\beta}\right)\right)$ and $x_{0}^{4} x_{2}^{4}=x_{2}^{2} *_{\underline{J}} x_{0}^{4} x_{2}^{2}$. But $x_{2}^{2}>_{\text {Lex }} x_{2}$.

Theorem 4.5. Let $J$ be a strongly stable m-truncation ideal, $G$ be a J-marked set and $I$ be the homogeneous ideal generated by $G$. The followings are equivalent:
(i) $I \in \mathcal{M f}(J)$;
(ii) $\forall f_{\alpha}, f_{\alpha^{\prime}} \in G, \exists t$ such that $x_{0}^{t} \cdot S\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \xrightarrow{s G *} 0$;
(iii) $\forall f_{\alpha}, f_{\alpha^{\prime}} \in G, \exists t$ such that $x_{0}^{t} \cdot S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x_{0}^{t}\left(x^{\gamma} f_{\alpha}-x^{\gamma^{\prime}} f_{\alpha^{\prime}}\right)=\sum a_{j} x^{\eta_{j}} f_{\alpha_{j}}$, with $x^{\eta_{j}}<_{\text {Lex }}$ $\max _{\text {Lex }}\left\{x^{\gamma}, x^{\gamma^{\prime}}\right\}$ and $f_{\alpha_{j}} \in s G$.

Proof. If $I \in \mathcal{M f}(J)$, we can apply Theorem 3.14, (iv) because any $S$-polynomial among elements in $G$ belongs to $I$.

If statement (ii) holds, then we get (iii) by Lemma 4.3.

We now assume that statement (iii) holds and by Lemma 2.2 it is sufficient to prove that $\langle V\rangle=\langle W\rangle$ using an argument analogous to that applied in the proof of Theorem 2.11. It is sufficient to prove that $x^{\eta} \cdot V \subseteq\langle V\rangle$, for every monomial $x^{\eta}$. We proceed by induction on the monomials $x^{\eta}$, ordered according to $>_{\text {Lex }}$. The thesis is obviously true for $x^{\eta}=1$. We then assume that the thesis holds for any monomial $x^{\eta^{\prime}}$ such that $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$.

If $|\eta|>1$, we can consider any product $x^{\eta}=x^{\eta_{1}} \cdot x^{\eta_{2}}, x^{\eta_{1}}$ and $x^{\eta_{2}}$ non-constant. Since $x^{\eta_{i}}<_{\text {Lex }}$ $x^{\eta}, i=1,2$, we immediately obtain by induction

$$
x^{\eta} \cdot V=x^{\eta_{1}} \cdot\left(x^{\eta_{2}} \cdot V\right) \subseteq x^{\eta_{1}}\langle V\rangle \subseteq\langle V\rangle .
$$

If $|\eta|=1$, then we need to prove that $x_{i} \cdot V \subseteq\langle V\rangle$. Since $x_{0} V \subseteq V$, it is then sufficient to prove the thesis for $x^{\eta}=x_{i}, i \geq 1$, assuming that the thesis holds for every $x^{\eta^{\prime}}<_{\text {Lex }} x_{i}$. We consider $g_{\beta}=x^{\delta} f_{\alpha} \in V$, where $\max \left(x^{\delta}\right) \leq \min \left(x^{\alpha}\right)$. If $x_{i} g_{\beta}$ does not belong to $V$, then $\max \left(x_{i} \cdot x^{\delta}\right)>\min \left(x^{\alpha}\right)$, so $x_{i}>\min \left(x^{\alpha}\right)$ because $\max \left(x^{\delta}\right) \leq \min \left(x^{\alpha}\right)$ by construction. In particular, $x_{i}>\min \left(x^{\alpha}\right) \geq \max \left(x^{\delta}\right)$, so $x_{i}>_{\text {Lex }} x^{\delta}$ and it is sufficient to prove the thesis for $x_{i} f_{\alpha}$.

We consider an $S$-polynomial $S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x_{i} f_{\alpha}-x^{\gamma} f_{\alpha^{\prime}}$ such that $x^{\gamma}<_{\text {Lex }} x_{i}$. Such $S$-polynomial always exists: for instance, we can consider $x_{i} x^{\alpha}=x^{\alpha^{\prime}}{ }_{J} x^{\eta^{\prime}}$.

By hypothesis there is $t$ such that $x_{0}^{t} S\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x_{0}^{t}\left(x_{i} f_{\alpha}-x^{\eta^{\prime}} f_{\alpha^{\prime}}\right)=\sum a_{j} x^{\eta^{\prime}}{ }_{j} f_{\alpha_{j}}$ where $x_{0}^{t} x^{\eta^{\prime}}, x^{\eta^{\prime}}{ }_{j}$ are lower than $x_{i}$ w.r.t. Lex. Then $x_{0}^{t} x^{\eta^{\prime}} f_{\alpha^{\prime}}, x^{\eta^{\prime} j} f_{\alpha_{j}}$ belong to $\langle V\rangle$ by induction and we conclude that $x_{i} f_{\alpha} \in\langle V\rangle$, by Lemma 4.2.

The previous theorem is the analogous of Theorem 2.11 for the reduction process $\xrightarrow{s G *}$. As stated in Corollary 2.13 concerning the Buchberger-like criterion for the reduction $\xrightarrow{V_{l}}$, also in Theorem 4.5 it would be sufficient to assume statement (iii) only for EK-polynomials.

Corollary 4.6. With the same notations of Theorem 4.5, the followings are equivalent:
(i) $I \in \mathcal{M f}(J)$
(ii) for every EK-polynomial between elements of $G, \exists t: x_{0}^{t} S^{E K}\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \xrightarrow{s G *} 0$.
(iii) for every EK-polynomial between elements of $G, \exists t$ such that $x_{0}^{t} \cdot S^{E K}\left(f_{\alpha}, f_{\alpha^{\prime}}\right)=x_{0}^{t}\left(x_{i} f_{\alpha}-\right.$ $\left.x^{\gamma^{\prime}} f_{\alpha^{\prime}}\right)=\sum a_{j} x^{\eta_{j}} f_{\alpha_{j}}$, with $x^{\eta_{j}}<_{\text {Lex }} x_{i}$ and $f_{\alpha_{j}} \in s G$.
4.2. Buchberger-like criteria via $\xrightarrow{s G *}$ : second variant. As before, let $J$ be a strongly stable $m$ truncation ideal. By Example 4.1, we have already shown that reductions of $S$-polynomials between elements of $s G$ are not sufficient to characterize ideals of $\mathcal{M f}(J)$; hence some more conditions are necessary. To this aim, we add some further $S$-polynomials.

Indeed, Theorem 4.7 uses the set $L_{1}$ of some couples of polynomials of $s G$ and the set $L_{2}$ of some particular couples of elements of $G$ of minimal degree $m$ to obtain a new characterization of $\mathcal{M f}(J)$. Actually the elements of $L_{1}$ are not all the possible couples of elements in $s G$, but a subset of them, corresponding to a minimal set of generators for the first module of syzygies of the Eliahou-Kervaire resolution of $\underline{J}$.
Theorem 4.7. Consider a strongly stable m-truncation ideal $J$ and $G$ a J-marked set. Let us define the following sets:

$$
\begin{gathered}
L_{1}:=\left\{\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \mid f_{\alpha}, f_{\alpha^{\prime}} \in s G \text { and } x_{i} x^{\underline{\alpha}}=x^{\underline{\alpha}^{\prime}} *_{\underline{J}} x^{\eta}\right\}, \\
L_{2}:=\left\{\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \mid f_{\alpha}, f_{\alpha^{\prime}} \in G_{m} \text { and } x_{i} x^{\alpha^{\prime}}=x_{0} x^{\alpha}, x_{i}=\min _{j>0}\left\{x_{j}: x_{j} \mid x^{\alpha}\right\}\right\} .
\end{gathered}
$$

Then:

$$
I \in \mathcal{M f}(J) \Leftrightarrow \forall\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \in L_{1} \cup L_{2}, \exists t \text { such that } x_{0}^{t} \cdot S\left(f_{\alpha}, f_{\alpha^{\prime}}\right) \xrightarrow{s G *} 0 .
$$

Proof. If $I$ belongs to $\mathcal{M f}(J)$, then it is enough to apply Theorem 4.5, (ii).
Vice versa, by Lemma 2.2 it is sufficient to prove that $\langle V\rangle=\langle W\rangle$, that is $x_{i} \cdot V \subseteq\langle V\rangle$ for every $i=0, \ldots, n$. We proceed by induction on the variables. By construction we have $x_{0} \cdot V \subseteq\langle V\rangle$. We now assume that $\left(x_{0}, \ldots, x_{i-1}\right) V \subseteq\langle V\rangle$ and we prove that $x_{i} \cdot V \subseteq\langle V\rangle$. Consider $x^{\delta} f_{\beta} \in V$. The thesis is that $x_{i} \cdot x^{\delta} f_{\beta}$ is contained in $\langle V\rangle$. If $x_{i} x^{\delta} f_{\beta}$ does not belong to $V$, then $\max \left(x_{i} \cdot x^{\delta}\right)>\min \left(x^{\beta}\right)$, so $x_{i}>\min \left(x^{\beta}\right)$ because $\max \left(x^{\delta}\right) \leq \min \left(x^{\beta}\right)$ by construction. In particular, $x_{i}>\min \left(x^{\beta}\right) \geq \max \left(x^{\delta}\right)$, so that it is sufficient to prove the thesis for $x_{i} f_{\beta}$, because by induction then we have $x^{\delta} x_{i} f_{\beta} \in\langle V\rangle$. Consider $x^{\beta}=x^{\underline{\alpha}}{ }_{\underline{\underline{J}}} x^{\eta}$.

We have a first case when $x^{\eta}=1$. Then $x^{\beta}=x^{\alpha}$ and $f_{\beta}$ belongs to $s G$. We consider $x^{\alpha} x_{i}=$ $x^{\underline{\alpha^{\prime}}} * \underline{J} x^{\eta^{\prime}}$. Observe that since $x_{i}>\min \left(x^{\underline{\alpha}}\right)$ then $x_{i}$ does not divide $x^{\eta^{\prime}}$ and $\max \left(x^{\eta^{\prime}}\right)<x_{i}$. Consider $x^{\alpha^{\prime}}=x^{\underline{\alpha^{\prime}}} \cdot x_{0}^{t_{\alpha^{\prime}}}$, so that we can take the polynomial $f_{\alpha^{\prime}} \in s G$. The pair $\left(f_{\beta}, f_{\alpha^{\prime}}\right)$ belongs to $L_{1}$, hence, by the hypothesis and by Lemma 4.3, there is $t$ such that

$$
x_{0}^{t} S\left(f_{\beta}, f_{\alpha^{\prime}}\right)=x_{0}^{t}\left(x_{0}^{t_{\alpha^{\prime}}} x_{i} f_{\beta}-x^{\eta^{\prime}} f_{\alpha^{\prime}}\right)=\sum a_{j} x^{\eta_{j}} f_{\alpha_{j}},
$$

with $x^{\eta_{j}}<_{\text {Lex }} x_{i}$ and $f_{\alpha_{j}} \in s G$. Hence we obtain that both $x^{\eta_{j}} f_{\alpha_{j}}$ and $x^{\eta^{\prime}} f_{\alpha^{\prime}}$ belong to $\langle V\rangle$ by induction on the variables, and so $x_{i} f_{\beta}$ belongs to $\langle V\rangle$ (by Lemma 4.2).

We have a second case when $x^{\eta}=x_{0}^{t}, t>0$. Then, $|\beta|=m$ and $f_{\beta}$ belongs to $s G$. Let $x_{i} x^{\beta}=$ $x^{\underline{\alpha}^{\prime}} *_{\underline{J}} x^{\eta^{\prime}}$. If $x_{i}>\min \left(x^{\alpha^{\prime}}\right)$, then $x^{\eta^{\prime}}$ is not divisible by $x_{i}$ and we repeat the argument above. Otherwise, $x_{i} \leq \min \left(x^{\alpha^{\prime}}\right)$ and $x_{i}$ does not divide $x^{\eta^{\prime}}$, so that $x_{i}=\min \left(x^{\alpha^{\prime}}\right)$ and $x^{\eta^{\prime}}<$ Lex $x_{i}$. Then, we take $x^{\beta^{\prime}}=\frac{x^{\beta}}{x_{0}} \cdot x_{i}$ that belongs to $B_{J}$ because it has degree $m$. The pair $\left(f_{\beta}, f_{\beta^{\prime}}\right)$ belongs to $L_{2}$ and we repeat the same reasoning above.

We now assume the thesis holds for every $f_{\beta^{\prime}}$ such that $x^{\beta^{\prime}}=x^{\underline{\alpha}^{\prime}} *_{\underline{J}} x^{\eta^{\prime}}$ with $x^{\eta^{\prime}} \ll_{\text {Lex }} x^{\eta}$. By the base of the induction, we can suppose that $x^{\eta} \geq_{\text {Lex }} x_{1}$; so, $f_{\beta}$ does not belong to $s G$ and it has degree $m$. Let $x_{j}:=\min _{l>0}\left\{x_{l}: x_{l} \mid x^{\beta}\right\}$.

Observe that if $x_{0}$ does not divide $x^{\beta}$, then $x_{j}=\min \left(x^{\beta}\right)$; in this case, we have $x_{i}>x_{j}$ because $x_{i}>\min \left(x^{\beta}\right)$. Anyway, first we suppose that $x_{i} \leq x_{j} ; x_{j}>\min \left(x^{\beta}\right)$ and $x_{0}$ divides $x^{\beta}$. We consider $x^{\beta^{\prime}}=\frac{x^{\beta}}{x_{0}} \cdot x_{i}$, the pair $\left(f_{\beta}, f_{\beta^{\prime}}\right)$ that belongs to $L_{2}$ and we repeat the argument of the previous case.

We now assume that $x_{i}>x_{j}$ and consider $x^{\beta^{\prime}}=\frac{x^{\beta}}{x_{j}} \cdot x_{0}=x^{\alpha^{\prime}} *_{\underline{I}} x^{\eta^{\prime}}$. Observe that $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}$ as $x^{\eta^{\prime}}=\frac{x^{\eta}}{x_{j}} \cdot x_{0}$. Therefore the pair $\left(f_{\beta^{\prime}}, f_{\beta}\right)$ belongs to $L_{2}$; by the hypothesis and by Lemma 4.3, there is an integer $t$ such that

$$
\begin{equation*}
x_{0}^{t} S\left(f_{\beta^{\prime}}, f_{\beta}\right)=x_{0}^{t}\left(x_{j} f_{\beta^{\prime}}-x_{0} f_{\beta}\right)=\sum a_{l} x^{\eta_{l}} f_{\alpha_{l}} \tag{3}
\end{equation*}
$$

with $x^{\eta_{l}}<_{\text {Lex }} x_{j}, f_{\alpha_{l}} \in s G$. We now multiply (3) by $x_{i}$. We observe that $x_{i} f_{\alpha_{l}}$ belongs to $\langle V\rangle$, because $f_{\alpha_{l}} \in s G$ and by the first two cases. Also $x_{i} f_{\beta^{\prime}}$ belongs to $\langle V\rangle$ because $x^{\eta^{\prime}}<_{\text {Lex }} x^{\eta}<_{\text {Lex }} x_{i}$. Moreover, $x_{j} x_{i} f_{\beta^{\prime}}$ belongs to $\langle V\rangle$ by induction on the variables. Finally $x_{i} f_{\beta}$ belongs to $\langle V\rangle$ thanks to Lemma 4.2.

## 5. Embedding of $\mathcal{M}(f)$ in affine linear spaces of low dimension

In this section we continue to consider a strongly stable $m$-truncation ideal $J=\underline{J}_{\geq m}$ and, as in Subsection 2.3, we work again with $J$-marked sets $\mathcal{G}$ where the coefficients of the monomials in the tails are considered as parameters.
Definition 5.1. If $\mathcal{G}$ is the set of marked polynomials given as in (1) for the ideal $J$, we will call set of superminimals, and denote it by $s \mathcal{G}$, the subset of $\mathcal{G}$ made up of $F_{\alpha} \in \mathcal{G}$ with $\operatorname{Ht}\left(F_{\alpha}\right) \in s B_{J}$. We will denote by $C$ the set of variables appearing in the tails of the polynomials in $\mathcal{G}$ and by $\widetilde{C}$ the set of
variables appearing in the tails of the polynomials in $s \mathcal{G} . \mathfrak{A}_{J}$ is the ideal defining the affine subscheme $\mathcal{M f}(J)$ in the ring $K[C]$.

Observe that the $J$-marked basis $G$ of every $I \in \mathcal{M f}(J)$ is obtained by specializing in a suitable way the variables $C$ in $\mathcal{G}$ and that the set of superminimals $s G$ of $I$ is obtained in the same way by $s \mathcal{G}$ through the same specialization of the variables $\widetilde{C}$.
5.1. The new embedding of $\mathcal{M f}(J)$. In this subsection we answer to the first question raised in the Introduction. In Theorem 5.4 we prove that the set of equations in $K[\widetilde{C}]$ defining $\mathcal{M f}(J)$ allows the elimination of a large number of parameters, more precisely those of $C \backslash \widetilde{C}$. Furthermore, using results of previous sections about the superminimal reduction, we are able to determine a set of equations defining $\mathcal{M f}(J)$ in $K[\widetilde{C}]$ avoiding at all the introduction of parameters in $C \backslash \widetilde{C}$. This fact combined with the choice of a small set of $S$-polynomials (according to Corollary 4.6 or Theorem 4.7) will turn out to be significantly useful in projecting an effective algorithm for the computation of such equations. Furthermore, this new sets of equations turns out to be more suitable in order to compare marked schemes of $m$-truncation ideals of a strongly stable saturated ideal $\underline{J}$ as $m$ varies.

Definition 5.2. Let $x^{\alpha} \in B_{J}$ and $t$ be an integer such that $x_{0}^{t} \cdot x^{\alpha} \xrightarrow{s \mathcal{G} *} H_{\alpha} \in K[\tilde{C}, x]$, with $H_{\alpha}$ strongly reduced (the integer $t$ exists by Theorem 3.14). We can write $H_{\alpha}=H_{\alpha}^{\prime}+x_{0}^{t} \cdot H_{\alpha}^{\prime \prime}$, where no monomial appearing in $H_{\alpha}^{\prime}$ is divisible by $x_{0}^{t}$. We will denote by:

- $\mathfrak{B}=\left\{C_{\alpha \gamma}-\phi_{\alpha \gamma}: x^{\alpha} \in B_{J} \backslash s B_{J}, x^{\gamma} \in \mathcal{N}(J)_{|\alpha|}\right\}$ the set of the $x$-coefficients of $T\left(F_{\alpha}\right)-H_{\alpha}^{\prime \prime}$ for every $x^{\alpha} \in B_{J}$;
- $\mathfrak{D}_{1} \subset K[\widetilde{C}]$ the set of the $x$-coefficients of $H_{\alpha}^{\prime}$ for every $x^{\alpha} \in B_{J} \backslash s B_{J}$;
- $\mathfrak{D}_{2}$ the set of the $x$-coefficients of the strongly reduced polynomials in $(s \mathcal{G}) K[\widetilde{C}, x]$.

Remark 5.3. Observe that not only $\mathfrak{D}_{2}$ but also $\mathfrak{B}$ and $\mathfrak{D}_{1}$ are well-defined thanks to the uniqueness of $H_{\alpha}$, by Theorem 3.14, (ii).

Theorem 5.4. The J-marked scheme $\mathcal{M f}(J)$ is defined by the ideal $\widetilde{\mathfrak{A}}_{J}:=\mathfrak{A}_{J} \cap K[\widetilde{C}]$ as subscheme of the affine space $\mathbb{A}^{|\widetilde{C}|}$, where $|\widetilde{C}|=\sum_{x^{\alpha} \in s B_{J}}\left|\mathcal{N}(J)_{|\alpha|}\right|$. Moreover $\mathfrak{A}_{J}=\left(\mathfrak{B} \cup \mathfrak{D}_{1} \cup \mathfrak{D}_{2}\right) K[C]$ and $\widetilde{\mathfrak{A}}_{J}=\left(\mathfrak{D}_{1} \cup \mathfrak{D}_{2}\right) K[\widetilde{C}]$.

Proof. For the first part it is sufficient to prove that $\mathfrak{A}_{J}$ contains $\mathfrak{B}$ and so it contains an element of the type $C_{\alpha \gamma}-\phi_{\alpha \gamma}$, for every $C_{\alpha \gamma} \in C \backslash \widetilde{C}$, where $\phi_{\alpha \gamma} \in K[\widetilde{C}]$, that allows the elimination of the variables $C_{\alpha \gamma} \in C \backslash \widetilde{C}$.

It is clear by the construction in Definition 5.2 that $H_{\alpha}$ belongs to $K[\widetilde{C}, x]$ and that both $x_{0}^{t} \cdot T\left(F_{\alpha}\right)$ and $H_{\alpha}$ are strongly reduced. Thus their difference $x_{0}^{t} \cdot T\left(F_{\alpha}\right)-H_{\alpha}$ is strongly reduced and moreover it belongs to $\Im_{J}$, because $x_{0}^{t} \cdot T\left(F_{\alpha}\right)-H_{\alpha}=-x_{0}^{t} \cdot F_{\alpha}+\left(x_{0}^{t} \cdot x^{\alpha}-H_{\alpha}\right)$. Hence, by Corollary 2.17, its $x$-coefficients belong to $\mathfrak{A}_{J}$ and in particular the coefficient of $x_{0}^{t} \cdot x^{\gamma}$ is of the type $C_{\alpha \gamma}-\phi_{\alpha \gamma}$, with $\phi_{\alpha \gamma} \in K[\widetilde{C}]$. Then $\mathfrak{A}_{J} \supseteq \mathfrak{B}$ and $\mathfrak{A}_{J}$ is generated by $\mathfrak{B} \cup \widetilde{\mathfrak{A}}_{J}$.

To prove the second part, it is sufficient to show that $\mathfrak{A}_{J} \cap K[\widetilde{C}]=\left(\mathfrak{D}_{1} \cup \mathfrak{D}_{2}\right) K[\widetilde{C}]$.
" $\supseteq$ " Taking the $x$-coefficients in $x_{0}^{t} \cdot T\left(F_{\alpha}\right)-H_{\alpha}$ of monomials that are not divisible by $x_{0}^{t}$, we see that $\mathfrak{A}_{J}$ contains the $x$-coefficients of $H_{\alpha}^{\prime}$. Then $\mathfrak{A}_{J} \cap K[\widetilde{C}] \supseteq \mathfrak{D}_{1}$, because $H_{\alpha}^{\prime} \in K[\widetilde{C}, x]$.

Moreover we recall that $\mathfrak{A}_{J}$ is made by all the $x$-coefficients in the polynomials of $\mathfrak{I}_{J}$ that are strongly reduced. Indeed, $\mathfrak{A}_{J}$ is made by all the $x$-coefficients of the polynomials of $\mathfrak{I}_{J}$ that are $J$ reduced. Since the degree of the monomials in the variables $x$ of every polynomial in $\mathfrak{I}_{J}$ is $\geq m$, then " $J$-reduced" is equivalent to " $\underline{J}$-reduced", that it is strongly reduced, by Lemma 3.2, (iv). Then $\mathfrak{A}_{J} \cap K[\widetilde{C}] \supseteq \mathfrak{D}_{2}$, because $(s \mathcal{G}) K[\widetilde{C}, x] \subset \mathfrak{I}_{J}$.
" $\subseteq$ " For every polynomial $F \in K[C, x]$, let us denote by $F^{\phi}$ the polynomial in $K[\widetilde{C}, x]$ obtained substituting every $C_{\alpha \gamma} \in C \backslash \widetilde{C}$ by $\phi_{\alpha \gamma}$; if $F$ is strongly reduced, then $F^{\phi}$ is strongly reduced too. Observe that for every $x^{\alpha} \in B_{J}$ we have $F_{\alpha}^{\phi}=x^{\alpha}-H_{\alpha}^{\prime \prime}$ and moreover $x_{0}^{t}\left(x^{\alpha}-H_{\alpha}^{\prime \prime}\right)-H_{\alpha}^{\prime} \in(s \mathcal{G}) K[\widetilde{C}, x]$. In particular $x_{0}^{t} F_{\alpha}^{\phi}$ and $x_{0}^{t}\left(x^{\alpha}-H_{\alpha}^{\prime \prime}\right)-H_{\alpha}^{\prime}$ are equal modulo $\mathfrak{D}_{1}$.

It remains to prove that every element $w \in \mathfrak{A}_{J} \cap K[\widetilde{C}]$ can be obtained modulo $\mathfrak{D}_{1}$ as a $x$-coefficient in some strongly reduced polynomial of the ideal $(s \mathcal{G}) \subset K[\widetilde{C}]$. We know that $w$ is a $x$-coefficient in a strongly reduced polynomial $D \in \mathfrak{I}_{J}$.

If $D=\sum D_{\alpha} F_{\alpha} \in \Im_{J}$, then for a suitable $t$,

$$
x_{0}^{t} \cdot D^{\phi}=\sum D_{\alpha}^{\phi} \cdot\left(x_{0}^{t} \cdot\left(x^{\alpha}-H_{\alpha}^{\prime \prime}\right)-H_{\alpha}^{\prime}\right) \quad \bmod \mathfrak{D}_{1}
$$

and the polynomial in the right-hand side of the equality is strongly reduced and it belongs to $(s \mathcal{G}) K[\widetilde{C}, x]$. Therefore $w$ is still one of the $x$-coefficients of $D^{\phi}$ since it does not contain any variable in $C \backslash \widetilde{C}$ and it remains unchanged. Then $w \in\left(\mathfrak{D}_{1} \cup \mathfrak{D}_{2}\right) K[\widetilde{C}]$.
Proposition 5.5. Let $\widetilde{\mathfrak{A}}_{J}$ be as in Theorem 5.4 and let $\mathfrak{U}$ be any ideal in $K[\widetilde{C}]$. Assume that $\mathfrak{U} \subseteq \widetilde{\mathfrak{A}}_{J}$ and that the following conditions hold:
(i) For every monomial $x^{\beta} \in B_{J} \backslash s B_{J}, x^{\beta}=x^{\underline{\alpha}}{ }_{{ }_{\underline{J}}} x^{\delta} \in J$, there exists $t$ such that we have a formula of type

$$
x_{0}^{t} \cdot x^{\beta}=\sum b_{i} x^{\eta_{i}} F_{\alpha_{i}}+H_{\beta},
$$

with $a_{i} \in K[\widetilde{C}], F_{\alpha_{i}} \in s \mathcal{G}, x^{\eta_{i}} \leq_{\text {Lex }} x^{\delta}, x^{\underline{\eta}_{j}+\underline{\alpha}_{j}}=x^{\underline{\alpha}_{j}} *_{\underline{J}} x^{\underline{\eta}_{j}}$ and $H_{\beta}=H_{\beta}^{\prime}+x_{0}^{t} \cdot H_{\beta}^{\prime \prime}$, with $H_{\beta}$ strongly reduced, $x_{0}^{t}$ does not divide any monomial in $\operatorname{Supp}\left(H_{\beta}^{\prime}\right)$ and $\operatorname{Coeff}_{x}\left(H_{\beta}^{\prime}\right) \subseteq \mathfrak{U}$.
(ii) For every polynomial $F_{\alpha} \in s \mathcal{G}$ and for every $x_{i}>\min \left(x^{\underline{\alpha}}\right)$ there exists $t$ such that we have a formula of type

$$
x_{0}^{t} x_{i} F_{\alpha}=\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H_{i, \alpha}
$$

where $b_{j} \in K[\widetilde{C}], F_{\alpha_{j}} \in s \mathcal{G}, x^{\eta_{j}}<_{\text {Lex }} x_{i}, x^{\underline{\eta}_{j}+\underline{\alpha}_{j}}=x^{\underline{\alpha}_{j}} *_{\underline{J}} x^{\underline{\eta}_{j}}, H_{i, \alpha}$ strongly reduced and $\operatorname{Coeff}_{x}\left(H_{i, \alpha}\right) \subseteq \mathfrak{U}$.
Then $\mathfrak{U}=\left(\mathfrak{D}_{1} \cup \mathfrak{D}_{2}\right)=\widetilde{\mathfrak{A}}_{J}$.
Proof. Thanks to (i), we immediately have that $\mathfrak{D}_{1} \subseteq \mathfrak{U}$.
For the inclusion $\mathfrak{D}_{2} \subseteq \mathfrak{U}$, observe that if (i) and (ii) hold for $\mathfrak{U}$, then we can use the same arguments of Proposition 2.15 and obtain that:
for every $F_{\alpha} \in s \mathcal{G}$, for every $x^{\delta}$, there exists $t$ such that

$$
\begin{equation*}
x_{0}^{t} x^{\delta} F_{\alpha}=\sum b_{j} x^{\eta_{j}} F_{\alpha_{j}}+H \tag{4}
\end{equation*}
$$

with $b_{j} \in K[\widetilde{C}], F_{\alpha_{j}} \in s \mathcal{G}, x^{\eta_{j}}<_{\text {Lex }} x^{\delta}, x^{\eta_{j}+\underline{\alpha}_{j}}=x^{\alpha_{j}{ }_{\underline{J}}} x^{\eta_{j}} \operatorname{Supp}_{x}\left(H_{\delta, \alpha}\right) \subseteq \mathcal{N}(J)$ and $\operatorname{Coeff}_{x}\left(H_{\delta, \alpha}\right) \subseteq \mathfrak{U}$.
We can also prove the uniqueness of such a rewriting: thanks to the uniqueness of the decomposition by $*_{\underline{J}}$, the polynomials $x^{\eta_{j}} F_{\alpha_{j}}$ that can appear in (4) have pairwise different head terms. So an analogous of Corollary 2.16 holds for this setting.

Thanks to this uniqueness, as in Corollary 2.17, we get the non trivial inclusion of the thesis.
Proposition 5.5 is very important from the computational point of view. Indeed, its condition (i) allows to explicitely construct the set of polynomials $\mathfrak{B}$, namely to write a $J$-marked set $\widetilde{\mathcal{G}}$ in $K[\widetilde{C}, x]$, whose superminimal set is $s \mathcal{G}$. Using such a $J$-marked set in $K[\widetilde{C}, x]$, we can use either Theorem 4.5 or Corollary 4.6 or Theorem 4.7 to obtain a set of generators for $\widetilde{\mathfrak{A}}_{J}$. For instance, the algorithm presented in the Appendix is based on Theorem 4.7 and the proof of its correctness on Proposition 5.5. In the future, we will investigate which is the best set of polynomials to start from in order to
get a performing algorithm for the computation of equations for $\mathcal{M f}(J)$. The correctness of such an algorithm will be verified by the conditions of Proposition 5.5.
5.2. Relations among $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ as $m$ varies. In this subsection we will compare the marked schemes constructed from different truncations of a saturated strongly stable ideal $\underline{J}$. Let us consider two integers $m^{\prime}, m,\left(m^{\prime}<m\right)$. If $I$ is an ideal in the $\underline{J}_{\geq m^{\prime}}$-marked family $\mathcal{M f}\left(\underline{J}_{\geq m^{\prime}}\right)$, then it is not difficult to show that $I_{\geq m}$ belongs to the marked family $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$, namely that there is a injective map of sets $\mathcal{M f}\left(\underline{J}_{\geq m^{\prime}}\right) \rightarrow \mathcal{M f}\left(\underline{J}_{\geq m}\right)$. Aim of the present subsection is a scheme theoretical version of this fact; indeed we will prove that there is a closed embedding of schemes $\mathcal{M f}\left(\underline{J}_{\geq m^{\prime}}\right) \hookrightarrow \mathcal{M f}\left(\underline{J}_{\geq m}\right)$ that induces the previous one on the sets of closed points. It is sufficient to prove the existence of such a closed embedding for $m^{\prime}=m-1$; in this case we denote the embedding map by $\phi_{m}$. Furthermore, we characterize the cases in which $\phi_{m}$ is a isomorphism.

To this purpose, the main tool we will use is the set of defining equations for a $J$-marked scheme obtained by superminimal reduction, namely the ideal $\widetilde{\mathfrak{A}}_{J}$; moreover we will consider at the Zariski tangent space of $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ at the origin, denoted by $T_{0}\left(\mathcal{M} f\left(\underline{J}_{\geq m}\right)\right)$.

Remark 5.6. As for any affine variety, if $\mathcal{M f}(J)$ is defined by an ideal $\mathfrak{U}$ as a subscheme of an affine space $\mathbb{A}^{N}$, then the Zariski tangent space $T_{0}(\mathcal{M f}(J))$ is defined by the linear part of a set generators of $\mathfrak{U}$ so that it can be identified to a linear subspace of $\mathbb{A}^{N}$. In the special case of marked schemes, it is quite easy to compute a set of generators for $T_{0}(\mathcal{M f}(J))$, using the properties and techniques of [15, Definition 3.4 and Proposition 4.3], [24, Proposition 3.4] and [10, Theorem 3.2].

Theorem 5.7 is inspired by an analogous result proved for Gröbner Strata in [15, Theorem 4.7]. Given a monomial ideal $J$, the Gröbner $\operatorname{Stratum} \operatorname{St}(J, \prec)$ of $J$ w.r.t. a term order $\prec$ can be isomorphically projected in its Zariski tangent space at the origin $T_{0}(\mathcal{S t}(J, \prec))$ (see [15, Proposition 4.3]). Furthermore, if the origin is a smooth point, then $\mathcal{S} t(J, \prec)$ is isomorphic to this tangent space. Unluckily, it is not true that for every strongly stable ideal $J$ there exists a term order $\prec$ such that $\mathcal{M f}(J) \simeq \mathcal{S t}(J, \prec)$, as shown in [7, Appendix], so in general we cannot project isomorphically $\mathcal{M f}(J)$ into $T_{0}(\mathcal{M f}(J))$.

We introduce some useful notations: once fixed a saturated strongly stable ideal $\underline{J}$ and a positive integer $m$, we denote by

- $\mathcal{G}^{[m]}$ a $\underline{J}_{\geq m}$-marked set as in (1) and with $F_{\beta}^{[m]}$ a marked polynomial belonging to $\mathcal{G}^{[m]}$;
- $C^{[m]}$ the set of parameters $C_{\alpha \gamma}^{[m]}$ appearing in the tails of the marked polynomials in $\mathcal{G}^{[m]}$;
- $s \mathcal{G}^{[m]}$ the set of superminimals of $\mathcal{G}^{[m]}$;
- $\widetilde{C}^{[m]}$ the subset of $C^{[m]}$ containing only the parameters $\widetilde{C}_{\alpha \gamma}^{[m]}$ appearing in the tails of marked polynomials in $s \mathcal{G}^{[m]}$;
- $\widetilde{\mathfrak{A}}^{[m]}$ is the ideal in $K\left[\widetilde{C}^{[m]}\right]$ defining $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ as a subscheme in $\mathbb{A}^{\left|\widetilde{C}^{[m]}\right|}$ (defined in Theorem 5.4).

Theorem 5.7. Let $\underline{J}$ be a saturated strongly stable ideal and let $m$ be any integer. With the previous notations, the followings hold:
(i) $\underset{\sim}{\mathcal{M}}\left(\underline{J}_{\geq m-1}\right)$ is a closed subscheme of $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ cut out by a suitable linear space. More precisely, $\widetilde{C}^{[m-1]}$ can be identified with a suitable subset of $\widetilde{C}^{[m]}$ so that the following diagram of schemes commutes:

(ii) Let $\Omega$ be the number of monomials $x^{\alpha} \in B_{\underline{J}}$ of degree $m+1$ divisible by $x_{1}$ and $\Theta:=\left|B_{\underline{J}} \cap S_{\leq m-1}\right|$; then,

$$
\operatorname{dim} T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m}\right)\right) \geq \operatorname{dim} T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m-1}\right)\right)+\Omega \cdot \Theta .
$$

(iii) $\mathcal{M f}\left(\underline{J}_{\geq m-1}\right) \simeq \mathcal{M f}\left(\underline{J}_{\geq m}\right)$ if and only if either $\underline{J}_{\geq m-1}=\underline{J}_{\geq m}$ or no monomial of degree $m+1$ in $B_{J}$ is divisible by $x_{1}$.
In particular:

$$
\mathcal{M f}\left(\underline{J}_{\geq \rho-1}\right) \simeq \mathcal{M f}\left(\underline{J}_{\geq m}\right), \text { for every } m \geq \rho
$$

where $\rho$ is the maximal degree of monomials divisible by $x_{1}$ in $B_{\underline{J}}$.
Proof.
(i) Thanks to Theorem 5.4, a $\underline{J}_{\geq m}$-marked scheme is defined by an ideal generated by polynomials of $K\left[\widetilde{C}^{[m]}\right]$ that are constructed using only the superminimals. So, now it is enough to prove that the set of superminimals $s \mathcal{G}^{[m-1]}$ corresponds to $s \mathcal{G}^{[m]}$ modulo a subset of the variables $\widetilde{C}^{[m]}$, in the following sense.

Consider $x^{\alpha} \in s B_{\underline{J}_{\geq m-1}}$. If $|\alpha| \geq m$, then $x^{\alpha}$ belongs to $s B_{\underline{J_{\geq m}}}$ and we can identify $F_{\alpha}^{[m]} \in$ $s \mathcal{G}^{[m]}$ and $F_{\alpha}^{[m-1]} \in s \mathcal{G}^{[m-1]}$ (and in particular the variables in their tails: $\widetilde{C}_{\alpha \gamma}^{[m]}=\widetilde{C}_{\alpha \gamma}^{[m-1]}$ ).

If $|\alpha|=m-1$, then we can consider the corresponding superminimal element $F_{\beta}^{[m]} \in s \mathcal{G}^{[m]}$, with $x^{\beta}=x_{0} \cdot x^{\alpha}$. Then we identify the variable $\widetilde{C}_{\beta \delta^{\prime}}^{[m]}$, which is the coefficient of a monomial in $\operatorname{Supp}_{x}\left(F_{\beta}^{[m]}\right)$ of kind $x^{\delta^{\prime}}=x_{0} \cdot x^{\delta}$, with the variable $\widetilde{C}_{\alpha \delta}^{[m-1]}$ which is the coefficient of the monomial $x^{\delta}$ in $\operatorname{Supp}_{x}\left(F_{\alpha}^{[m-1]}\right)$.

We repeat this identifications for all $x^{\alpha} \in s B_{\underline{J} \geq m-1}$ and we denote by $\bar{C}^{[m]}$ the subset of $\widetilde{C}^{[m]}$ containing the variables non-identified with variables of $\widetilde{C}^{[m-1]}$, that is the variables appearing as coefficients of monomials not divisible by $x_{0}$ in the tails of polynomials in $s \mathcal{G}^{[m]} \backslash s \mathcal{G}^{[m-1]}$. Now, every polynomial in $s \mathcal{G}^{[m]} \bmod \left(\bar{C}^{[m]}\right)$ either belongs to $s \mathcal{G}^{[m-1]}$ or is a polynomials of $s \mathcal{G}^{[m-1]}$ multiplied by $x_{0}$. Thanks to Theorem 5.4, we have that

$$
\widetilde{\mathfrak{A}}^{[m]}+\left(\bar{C}^{[m]}\right) \simeq \widetilde{\mathfrak{A}}^{[m-1]}
$$

This relation among the ideals induces the embeddings of scheme of diagram (5).
(ii) We now consider $x^{\gamma} \in B_{\underline{J}},|\gamma|=m+1, x^{\gamma}$ divisible by $x_{1}$. We define $x^{\beta}:=x^{\gamma} / x_{1}$; observe that $x^{\beta} \in \mathcal{N}(\underline{J})$. Furthermore, $x^{\beta}$ is not divisible by $x_{0}$, otherwise $x^{\gamma}$ would be too.

Then, for every $x^{\underline{\alpha}} \in B_{\underline{J}}$ with $|\underline{\alpha}| \leq m-1$, there is $F_{\alpha}^{[m]}=x^{\underline{\alpha}} x_{0}^{m-|\underline{\alpha}|}-T\left(F_{\alpha}^{[m]}\right) \in s \mathcal{G}^{[m]}$ such that $x^{\beta} \in \operatorname{Supp}_{x}\left(T\left(F_{\alpha}^{[m]}\right)\right)$. We focus on the coefficient $\widetilde{C}_{\alpha \beta}^{[m]}$ of $x^{\beta}$. Since $x^{\beta}$ is not divisible by $x_{0}, \widetilde{C}_{\alpha \beta}^{[m]}$ cannot be identified with a coefficient appearing in $F_{\alpha}^{[m-1]}=x^{\underline{\alpha}} x_{0}^{m-|\underline{\alpha}|-1}-T\left(F_{\alpha}^{[m-1]}\right) \in s \mathcal{G}^{[m-1]}$. So $\widetilde{C}_{\alpha \beta}^{[m]}$ belongs to the subset of variables $\bar{C}^{[m]}$ defined in the proof of (i).

We now use the construction of $T_{0}\left(\mathcal{M f}\left(\mathcal{J}_{\geq m}\right)\right)$ recalled in Remark 5.6. If we think about syzygies of the ideal $\underline{J}_{\geq m}$, we can see that in a $S$-polynomial, $F_{\alpha}^{[m]}$ is multiplied by a monomial $x^{\delta}$
divisible by $x_{i}, i>0$. In particular, $x^{\delta} \cdot x^{\beta}$ belongs to $\underline{J}_{\geq m}$ : if $x_{i}=x_{1}$ we are done by construction, otherwise we apply the strongly stable property because $x_{1} x^{\beta} x^{\delta}=\frac{x^{\gamma}}{x_{i}} x_{1} x^{\delta}$ belongs to $\underline{J}_{\geq m}$. This means that the coefficient $\widetilde{C}_{\alpha \beta}^{[m]}$ does not appear in any equation defining $T_{0}\left(\mathcal{M f}\left(J_{\geq m}\right)\right)$.

Applying this argument to the $\Omega$ monomials in $B_{\underline{J}}$ of degree $m+1$ which are divisible by $x_{1}$ and to the $\Theta$ monomials in $B_{J}$ of degree $\leq m-1$, we obtain the result.
(iii) If $\underline{J}_{\geq m}=\underline{J}_{\geq m-1}$, obviously $\overline{\mathcal{M}}\left(\underline{J}_{\geq m}\right)=\mathcal{M f}\left(\underline{J}_{\geq m-1}\right)$. We now assume that $\underline{J}_{\geq m} \neq \underline{J}_{\geq m-1}$ and no monomial of degree $m+1$ in the monomial basis of $\underline{J}$ is divisible by $x_{1}$; we prove that every polynomial in $s \mathcal{G}^{[m]}$ either belongs to $s \mathcal{G}^{[m-1]}$ or it is the product of $x_{0}$ by the "corresponding" polynomial in $s \mathcal{G}^{[m-1]}$.

If $x^{\alpha} \in s B_{\underline{J}_{\geq m-1}}$ and $|\alpha| \geq m$, then $F_{\alpha}^{[m]} \in s \mathcal{G}^{[m]}$ and $F_{\alpha}^{[m-1]} \in s \mathcal{G}^{[m-1]}$ have the same shape and we can identify them letting $\widetilde{C}_{\alpha \gamma}^{[m]}=\widetilde{C}_{\alpha \gamma}^{[m-1]}$, as done in the proof of (i). If $|\alpha|=m-1$, then $x^{\beta}=x_{0} \cdot x^{\alpha} \in s B_{\underline{J}_{\geq m}}$ and all the monomials in the support of $x_{0} \cdot F_{\alpha}^{[m-1]}$ appear in the support of $F_{\beta}^{[m]}$ (and we identify their coefficients as above). In the support of $F_{\beta}^{[m]}$ there are also some more monomials that are not divisible by $x_{0}$. We will prove now that the coefficients of these last monomials in fact belong to $\widetilde{\mathfrak{A}}^{[m]}$.

Consider the monomial $x_{0} \cdot x_{1} \cdot x^{\alpha}$. If we perform its reduction using $s \mathcal{G}^{[m]}$, the first step of reduction will lead to

$$
x_{0} \cdot x_{1} \cdot x^{\alpha} \xrightarrow{s \mathcal{G}^{[m]} *} x_{1} T\left(F_{\beta}^{[m]}\right) .
$$

Let $x^{\gamma}$ be a monomial of $\operatorname{Supp}\left(T\left(F_{\beta}^{[m]}\right)\right)$. If $x_{1} \cdot x^{\gamma} \in \underline{J}_{\underline{ }}$, then $x_{1} \cdot x^{\gamma}=x^{\underline{\alpha}^{\prime}}{ }_{\underline{\underline{J}}} x^{\eta}$, with $x \underline{\underline{\alpha}}^{\prime} \in B_{\underline{J}}$ and $x^{\eta}<_{\text {Lex }} x_{1}$. If $x^{\eta}=1$, then $\left|\underline{\alpha}^{\prime}\right|=m+1$ and $x^{\underline{\alpha}^{\prime}}$ is divisible by $x_{1}$, against the hypothesis. Then $x^{\eta}=x_{0}^{\bar{t}}$, with $t>0$, and so the monomial $x_{1} \cdot x^{\gamma} \in \underline{J}_{\geq m}$ is actually divisible by $x_{0}$. If $x_{1} \cdot x^{\gamma} \in \mathcal{N}\left(\underline{J}_{\geq m}\right)$, then this monomial is not further reducible, so that its coefficient belongs to $\widetilde{\mathfrak{A}}^{[m]}$.

Vice versa, by contradiction suppose now that $\underline{J}_{\geq m-1} \neq \underline{J}_{\geq m}$ and that there exists $x^{\alpha} \in B_{\underline{J}}$ divisible by $x_{1},|\alpha|=m+1$. Using (ii), we have that $T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m-1}\right)\right) \not \nsim T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m}\right)\right)$ because $\operatorname{dim} T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m-1}\right)\right)<\operatorname{dim} T_{0}\left(\mathcal{M f}\left(\underline{J}_{\geq m}\right)\right)$, and so $\mathcal{M f}\left(\underline{J}_{\geq m-1}\right) \nsucceq \mathcal{M f}\left(\underline{J}_{\geq m}\right)$.
For the last part of the statement, note that if $\rho$ is the maximal degree of a monomial divisible by $x_{1}$ in the monomial basis of $\underline{J}$, for every $m \geq \rho$, applying iteratively (iii) we obtain

$$
\mathcal{M f}\left(\underline{J}_{\geq \rho-1}\right) \simeq \mathcal{M f}\left(\underline{J}_{\geq m}\right) .
$$

In the above setting, if $p(t)$ is the Hilbert polynomial of $S / \underline{J}$ and $r$ is its Gotzmann number, it is worth considering the $r$-truncation of $\underline{J}$. Indeed, in [4] the authors prove that $\mathcal{M f}\left(\underline{J}_{\geq r}\right)$ is naturally isomorphic to an open subset of the Hilbert scheme $\mathcal{H i l b} b_{p(t)}^{n}$. We recall that $r$ is the maximum among the regularities of ideals that are closed points of $\mathcal{H i l b}_{p(t)}^{n}$; hence, $r \geq \operatorname{reg}(\underline{J}) \geq \rho-1$. Then Theorem 5.7 allows us to study such an open subset of $\mathcal{H i l b}_{p(t)}^{n}$ embedded in an affine space of lower dimension than the expected one. More precisely:

Corollary 5.8. Let $\underline{J}$ be a saturated strongly stable ideal, and let $\rho$ be the maximal degree of monomials divisible by $x_{1}$ in $B_{\underline{J}}$. For every $m \geq \rho-1, \mathcal{M f}\left(\underline{J}_{\geq m}\right)$ can be embedded in an affine space of dimension

$$
\begin{equation*}
\left|\widetilde{C}^{[\rho-1]}\right|=\sum_{x^{\alpha} \in s B_{\underline{J_{\geq ㇒ ~}^{\prime}}}}\left|\mathcal{N}(\underline{J})_{|\alpha|}\right| \leq\left|B_{\underline{J}}\right| \cdot p\left(r^{\prime}\right), \tag{6}
\end{equation*}
$$

where $r^{\prime}=\operatorname{reg}(\underline{J})$ and $p(t)$ is the Hilbert polynomial of $S / \underline{J}$.

Proof. The equality of (6) directly follows from Theorem 5.7. For the inequality, we simply need to observe that the regularity $r^{\prime}$ of a strongly stable ideal is simply the maximum of the degrees of its monomial generators; hence every monomial in $s B_{\underline{J}_{\geq \rho-1}}$ has degree $\leq r^{\prime}$. Furthermore $r^{\prime}$ is greater than or equal to the regularity of the Hilbert function of $S / \underline{J}$, thus $\left|\mathcal{N}(\underline{J})_{|\alpha|}\right| \leq \mathcal{N}\left(\underline{J}_{r^{\prime}}\right)=p\left(r^{\prime}\right)$.

An equivalent proof follows from the diagram (5) of Theorem 5.7.

## 6. Examples

In the hypothesis that the field $K$ has characteristic 0 , the methods of computations developed in the previous sections can be applied to the study of Hilbert schemes: indeed, for $m$ big enough, $\mathcal{M} \mathrm{f}\left(\underline{J}_{>m}\right)$ corresponds to an open subset of the Hilbert scheme parameterizing the ideals having the same Hilbert polynomial as $S / \underline{J}$ (see [4]).

Now we give some examples for applications of the obtained results, mainly Theorem 4.5, Theorem 4.7 and Theorem 5.7. We keep on using the notations introduced before Theorem 5.7.

Example 6.1. Let $\underline{J}$ be the saturated strongly stable ideal $\left(x_{n}, \ldots, x_{2}, x_{1}^{\mu}\right) \subseteq S=K\left[x_{0}, \ldots, x_{n}\right]$. Observe that $\underline{J}$ is a Lex-segment, the Hilbert polynomial of $S / \underline{J}$ is $p(t)=\mu$, the regularity of $\underline{J}$ is $r^{\prime}=\mu$, and also $\rho=\mu$. By Corollary 5.8, $\mathcal{M f}(\underline{J})$ can be embedded into an affine space of dimension $2 n-2+\mu$. Using the criterion of Theorem 4.5, we can see that actually $\mathcal{M} \mathfrak{f}(\underline{J}) \simeq \mathbb{A}^{2 n-2+\mu}$, as shown also in [24]. By Theorem 5.7, (iii), $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ is isomorphic to $\mathcal{M f}(\underline{J})$ for every $m \leq \mu-2$.

It is well-known that $\operatorname{Proj}(S / \underline{J})$ is the Lex-point of $\mathcal{H} i l b_{\mu}^{n}$ and lies on a component of dimension $n \mu$ (see [21]). Then $\mathcal{M f}\left(\underline{J}_{\geq m}\right)$ is not isomorphic to an open subset of $\mathcal{H i l b} b_{\mu}^{n}$ for every $m \leq \mu-2$.

On the other hand, the same reasonings above leads to $\mathcal{M f}\left(\underline{J}_{\geq \mu}\right) \simeq \mathbb{A}^{n \mu}$ so that $\mathcal{M f}\left(\underline{J}_{\geq \mu}\right)$ is an open subset of $\mathcal{H} i l b_{\mu}^{n}$. This is shown also in [24].
Example 6.2. We consider the strongly stable saturated ideal

$$
\underline{J}=\left(x_{2}^{3}, x_{1} x_{2}^{2}, x_{1}^{2} x_{2}, x_{1}^{5}\right) \subseteq K\left[x_{0}, x_{1}, x_{2}\right] .
$$

It corresponds to a point of $\mathcal{H i l b} b_{8}^{2}$, with Gotzmann number $r=8$, the regularity $r^{\prime}$ of $\underline{J}$ is 5 , and the same value for $\rho$. By Theorem 5.7 we have that $\mathcal{M f}\left(\underline{J}_{\geq 4}\right) \simeq \mathcal{M f}\left(\underline{J}_{\geq r}\right)$. Observe that $\underline{J}_{\geq 4}$ is not segment w.r.t. any term order (see [7, Appendix]), hence in this case the results of [15] do not apply.

In [7, Appendix], the authors first consider $\mathcal{M f}\left(\underline{J}_{\geq 4}\right)$ as an affine subscheme of $A^{64}$ and then show that 45 of the variables can be eliminated, but using a time-consuming process of elimination of variables. By Corollary 5.8 , we can directly embed $\mathcal{M f}\left(\underline{J}_{\geq r}\right)$ in an affine space of dimension 32 , and we have to eliminate only 13 of the remaining variables.

Example 6.3. We take $p(t)=4 t, n=3, q(t)=\binom{3+t}{3}-p(t)=\binom{3+t}{3}-4 t$; the Gotzmann number of $p(t)$ is $r=6$. The Hilbert scheme $\mathcal{H i l b}_{4 t}^{3}$ can be considered as a subscheme of the Grassmannian $\mathbb{G}=\mathbb{G}\left(q(6), K[x]_{6}\right)$ of linear spaces of dimension $q(6)=60$ in the vector space $K[x]_{6}$ of dimension $\binom{3+6}{3}=84$ (see [4, Section 1] for some details about this construction). Therefore equations for $\mathcal{H i l b}{ }_{4 t}^{3}$ involve $E=\binom{84}{60}-1 \sim 6 \cdot 10^{20}$ Plücker coordinates. We can obtain an open cover of $\mathcal{H} \mathrm{ilb}_{4 t}^{3}$ by the non-vanishing of each Plücker coordinate of $\mathbb{G}$ : we get $E$ open subsets, each of them isomorphic to a subscheme of $\mathbb{A}^{1440}$.

In [4] the authors consider a different open cover (up to the action of $P G L(4))$ of $\mathcal{H} \mathrm{ilb}_{4 t}^{3}$, formed by 4 open subsets only, isomorphic to a marked scheme of a suitable truncation of the saturated strongly stable monomial ideals $\underline{J}_{i}, i=1,2,3,4$ in $K[x]$. We can choose for every $i$ the truncation $m=r=6$, nevertheless in order to perform computations with a lower number of variables, it is better to choose the truncations according to Theorem 5.7.

We denote the cardinality of the monomial basis of $\underline{J}_{i}$ by $\sigma_{i}$. In the following table we list the dimensions of the different affine spaces where we can embed the marked schemes, using Theorem 5.7
and Corollary 5.8.

|  | Monomial basis of $\underline{J}_{i}$ | $\operatorname{reg}\left(\underline{J}_{i}\right)$ | $\sigma_{i}$ | $\rho_{i}-1$ | $\sigma_{i} p\left(\operatorname{reg}\left(\underline{J}_{i}\right)\right)$ | $\widetilde{C}^{\left[\rho_{i}-1\right]}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\underline{J}_{1}$ | $x_{3}^{2}, x_{3} x_{2}, x_{2}^{3}$ | 3 | 3 | -1 | 36 | 28 |
| $\underline{J}_{2}$ | $x_{3}^{2}, x_{3} x_{2}, x_{3} x_{1}^{2}, x_{2}^{4}$ | 4 | 4 | 2 | 64 | 44 |
| $\underline{J}_{3}$ | $x_{3}^{2}, x_{3} x_{2}, x_{3} x_{1}, x_{2}^{5}, x_{2}^{4} x_{1}$ | 5 | 5 | 4 | 100 | 88 |
| $\underline{J}_{4}$ | $x_{3}, x_{2}^{5}, x_{2}^{4} x_{1}^{2}$ | 6 | 3 | 5 | 72 | 64 |

Observe that for $\underline{J}_{1}$ and $\underline{J}_{2}$, the truncation giving an open subset of $\mathcal{H i l b}{ }_{4 t}^{3}$ is exactly the saturated ideal.
$\underline{J}_{4}$ is the Lex-segment ideal: $\mathcal{M f}\left(\left(\underline{J}_{4}\right)_{\geq 5}\right)$ is isomorphic to $\mathbb{A}^{23}$ (see [15, Theorem 7.3]). In this case we should further eliminate 41 variables. This means that our bounds are not in general "sharp", however the computational consequences of Theorem 5.7 are significant and our results allow the treatment of non-trivial cases that cannot be handled with "classical" techniques.

## Appendix. A pseudocode description of the algorithm for computing a $J$-marked SCHEME

We now describe a prototype of the algorithm for computing $J$-marked families based on Proposition 5.5 and on the analogous of the set $L_{1}$ defined in Theorem 4.7. The ideal $J$ is always supposed to be a strongly stable $m$-truncation ideal.

Let us suppose that the following functions are made available.

- generators $(J)$. It determines the monomial basis of $J$.
- superminimalGenerators( $J$ ). It determines the superminimal generators of $J$.
- SuperminimalReduction $(H, s G)$. Given a $J$-marked superminimal set $s G$ and a polynomial $H$, it returns a pair $(t, h)$ where $t$ is the minimal power of $x_{0}$ such that there is a superminimal reduction of $x_{0}^{t} H$ to a strongly reduced polynomial and $h$ is such polynomial, namely $x_{0}^{t} H \xrightarrow{s G *} h$ (as in Theorem 3.14, (ii)).
- quotientAndRemainder $(H, t)$. Given a polynomial $H$ and a non-negative integer $t$, it returns the pair of polynomials $\left(H^{\prime}, H^{\prime \prime}\right)$ such that $H=H^{\prime}+H^{\prime \prime} x_{0}^{t}$.
- PairsL1 $(s G)$. Given a $J$-marked superminimal set $s G$, it computes the pairs of polynomials belonging to the set $\mathcal{L}_{1}$ (analogous of the set $L_{1}$ of Theorem 4.7).
- $\operatorname{CoEff}\left(H, x^{\alpha}\right)$. It returns the coefficient of the monomial $x^{\alpha}$ in the polynomial $H$ (obviously 0 if $x^{\alpha} \notin \operatorname{Supp}(H)$ ).

```
1: MARKEDSChEme \((J)\)
Input: \(J \subset K\left[x_{0}, \ldots, x_{n}\right]\) strongly stable \(m\)-truncation ideal.
Output: an ideal defining the marked scheme \(\mathcal{M f}(J)\).
    \(B_{J} \leftarrow \operatorname{Generators}(J)\);
    \(s B_{J} \leftarrow \operatorname{SUPERMInimaLGEnERATORS}(J)\);
    \(\widetilde{\mathcal{G}} \leftarrow \emptyset ; \quad s \mathcal{G} \leftarrow \emptyset ;\)
    for all \(x^{\alpha} \in s B_{J}\) do
        \(F_{\alpha} \leftarrow x^{\alpha} ;\)
        for all \(x^{\beta} \in \mathcal{N}(J)_{|\alpha|}\) do
            \(F_{\alpha} \leftarrow F_{\alpha}+\widetilde{C}_{\alpha \beta} x^{\beta} ;\)
        end for
        \(\widetilde{\mathcal{G}} \leftarrow \widetilde{\mathcal{G}} \cup\left\{F_{\alpha}\right\} ;\)
        \(s \mathcal{G} \leftarrow s \mathcal{G} \cup\left\{F_{\alpha}\right\} ;\)
    end for
    equations \(\leftarrow \emptyset\);
    \(\bar{B}_{J} \leftarrow B_{J} \backslash s B_{J} ;\)
    for all \(x^{\alpha} \in \bar{B}_{J}\) do
        \((t, H) \leftarrow \operatorname{SUPERminimalREDUCTION}\left(x^{\alpha}, s \mathcal{G}\right)\);
        \(\left(H^{\prime}, H^{\prime \prime}\right) \leftarrow\) QUotientAndREmainder \((H, t)\);
        for all \(x^{\eta} \in \operatorname{Supp}\left(H^{\prime}\right)\) do
            equations \(\leftarrow\) equations \(\cup\left\{\operatorname{CoEFF}\left(H^{\prime}, x^{\eta}\right)\right\} ;\)
        end for
        \(\widetilde{\mathcal{G}} \leftarrow \widetilde{\mathcal{G}} \cup\left\{x^{\alpha}-H^{\prime \prime}\right\} ;\)
    end for
    \(\mathcal{L}_{1} \leftarrow \operatorname{PairsL} 1(s \mathcal{G}) ;\)
    for all \(\left(F_{\alpha}, F_{\alpha^{\prime}}\right) \in \mathcal{L}_{1}\) do
        \((t, H) \leftarrow \operatorname{SuperminimalReduction}\left(S\left(F_{\alpha}, F_{\alpha^{\prime}}\right), s \mathcal{G}\right)\);
        for all \(x^{\eta} \in \operatorname{Supp}(H)\) do
            equations \(\leftarrow\) equations \(\cup\left\{\operatorname{CoEFF}\left(H, x^{\eta}\right)\right\} ;\)
        end for
    end for
    return (equations);
```

Theorem A.1. The algorithm markedScheme is correct.
Proof. To prove that the algorithm terminates it is sufficient to recall that the superminimal reduction is Noetherian (Theorem 3.14 (i)).

Now we show that the algorithm markedScheme returns a set of generators for the ideal defining $\mathcal{M f}(J)$. The starting point is the $J$-marked superminimal set $s \mathcal{G}$ given in Definition 5.1, having parameters in $\widetilde{C}$ as coefficients of every monomial in the tails and get a set equations of polynomials in $K[\widetilde{C}]$. We claim that the ideal $\mathfrak{U}$ generated by equations coincides with the ideal $\widetilde{\mathfrak{A}}_{J}$ of Theorem 5.4, by Proposition 5.5.

Indeed in the first part (lines 15-22), the algorithm computes the superminimal reduction $H$ of each monomial $x^{\alpha} \in B_{J} \backslash s B_{J}$ and it imposes the conditions required by Proposition 5.5 (i), in other words the algorithm computes the set $\mathfrak{D}_{1} \subseteq \widetilde{\mathfrak{A}}_{J}$ of Definition 5.2. At the same time, the algorithm constructs the $J$-marked set $\widetilde{\mathcal{G}} \subset K[\widetilde{C}, x]$.

In the second part (lines 23-29), the algorithm considers pairs of superminimal generators ( $F_{\alpha}, F_{\alpha^{\prime}}$ ) such that $x_{i} x^{\alpha}=x^{\underline{\alpha}^{\prime}} *_{\underline{J}} x^{\eta}$. Recall that $x^{\eta}<_{\text {Lex }} x_{i}$ by Lemma 2.5. These couples of polynomials in $s \mathcal{G}$ correspond to the couples of the set $L_{1}$ in Theorem 4.7.

At line 25 of the algorithm we compute the superminimal reduction of the associated $S$-polynomial

$$
x_{0}^{t} S\left(F_{\alpha}, F_{\alpha}^{\prime}\right)=x_{0}^{t}\left(x_{i} x_{0}^{t^{\prime}} F_{\alpha}-x^{\eta} x_{0}^{t^{\prime \prime}} F_{\alpha^{\prime}}\right) \xrightarrow{s \mathcal{G} *} H \quad x_{i} x_{0}^{t^{\prime}}=\frac{\operatorname{lcm}\left(x^{\alpha}, x^{\alpha^{\prime}}\right)}{x^{\alpha}}, x^{\eta} x_{0}^{t^{\prime \prime}}=\frac{\operatorname{lcm}\left(x^{\alpha}, x^{\alpha^{\prime}}\right)}{x^{\alpha^{\prime}}}
$$

that is applying Lemma 4.3

$$
x_{0}^{t}\left(x_{0}^{t^{\prime}} x_{i} F_{\alpha}-x_{0}^{t^{\prime \prime}} x^{\eta} F_{\alpha^{\prime}}\right)-\sum b_{j} x^{\eta_{j}} F_{\beta_{j}}=H
$$

with $b_{j} \in K[\widetilde{C}], F_{\beta_{j}} \in s \mathcal{G}, x^{\eta_{j}}<_{\text {Lex }} x_{i} x_{0}^{t^{\prime}}$ and $x^{\eta_{j}}<_{\text {Lex }} x_{i}$, so that

$$
x_{0}^{\bar{t}} x_{i} F_{\alpha}=x^{\bar{\eta}} F_{\alpha^{\prime}}+\sum b_{j} x^{\eta_{j}} F_{\beta_{j}}+H .
$$

The polynomial $H$ is strongly reduced and it belongs to the ideal $(s \mathcal{G}) \subseteq K[\widetilde{C}, x]$, then its $x$-coefficients belong to $\mathfrak{D}_{2} \subseteq \widetilde{\mathfrak{A}}_{J}$.

Then by construction (lines $26-28$ ), $\mathfrak{U}$ is contained in $\widetilde{\mathfrak{A}}_{J}$ and it satisfies the condition required by Proposition 5.5 (ii), hence $\mathfrak{U}=\widetilde{\mathfrak{A}}_{J}$.

We are convinced that this version can be strongly strengthened drawing inspiration from some of the improvements studied for the computation of Gröbner bases and border bases. In this direction, we have already developed a first prototype which is giving good and promising results. In the following table, we report the results of the computation of the marked schemes considered in Example 6.3. The algorithm has been run on a MacBook Pro with a $2,4 \mathrm{GHz}$ Intel Core 2 Duo processor.

|  | Parameters | Equations | Time |
| :---: | :---: | :---: | :---: |
| $\mathcal{M f}\left(\underline{J}_{1}\right)$ | 28 | 28 | 0.165568 seconds |
| $\mathcal{M f}\left(\underline{J}_{2}\right)$ | 44 | 64 | 0.295802 seconds |
| $\mathcal{M f}\left(\left(\underline{J}_{3}\right)_{\geq 4}\right)$ | 88 | 228 | 110050 seconds |

The prototype of the algorithm is available at www.personalweb.unito.it/paolo.lella/ HSC/Documents/MarkedSchemes.m2

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