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# Global $L^p$ estimates for degenerate Ornstein-Uhlenbeck operators with variable coefficients<sup>\*</sup>

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#### Abstract

We consider a class of degenerate Ornstein-Uhlenbeck operators in  $\mathbb{R}^N,$  of the kind

$$\mathcal{A} \equiv \sum_{i,j=1}^{p_0} a_{ij}(x) \partial_{x_i x_j}^2 + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j}$$

where  $(a_{ij})$  is symmetric uniformly positive definite on  $\mathbb{R}^{p_0}$   $(p_0 \leq N)$ , with uniformly continuous and bounded entries, and  $(b_{ij})$  is a constant matrix such that the frozen operator  $\mathcal{A}_{x_0}$  corresponding to  $a_{ij}$   $(x_0)$  is hypoelliptic. For this class of operators we prove global  $L^p$  estimates (1 ofthe kind:

$$\left\| \partial_{x_i x_j}^2 u \right\|_{L^p(\mathbb{R}^N)} \le c \left\{ \| \mathcal{A} u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, ..., p_0.$$

<sup>\*</sup>**Keywords:** Ornstein-Uhlenbeck operators. Global  $L^p$  estimates. Hypoelliptic operators. Singular integrals. Nondoubling spaces. **Mathematics subject classification (2000)**: Primary 35H10. Secondary 35B45 - 35K70 - 42B20.

We obtain the previous estimates as a byproduct of the following one, which is of interest in its own:

$$\left\|\partial_{x_i x_j}^2 u\right\|_{L^p(S_T)} \le c \left\{ \|Lu\|_{L^p(S_T)} + \|u\|_{L^p(S_T)} \right\}$$

for any  $u \in C_0^{\infty}(S_T)$ , where  $S_T$  is the strip  $\mathbb{R}^N \times [-T, T]$ , T small, and L is the Kolmogorov-Fokker-Planck operator

$$L \equiv \sum_{i,j=1}^{p_0} a_{ij} \left( x, t \right) \partial_{x_i x_j}^2 + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_{x_i} \partial_{x_j} + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_{x_i} \partial_{x_i} - \partial_{x_i} \partial_{x_j} - \partial_{x_i} \partial_{x_i} - \partial_{x_i} \partial_{x_i}$$

with uniformly continuous and bounded  $a_{ij}$ 's.

## 1 Introduction

Let us consider the following kind of Ornstein-Uhlenbeck operators:

$$\mathcal{A} = \sum_{i,j=1}^{p_0} a_{ij}(x) \,\partial_{x_i x_j}^2 + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j}, \qquad (1.1)$$

where:

 $A_0 = (a_{ij}(x))_{i,j=1}^{p_0}$  is a  $p_0 \times p_0$   $(p_0 \le N)$  symmetric, bounded and uniformly positive definite matrix:

$$\frac{1}{\Lambda} |\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}(x) \,\xi_i \xi_j \le \Lambda \,|\xi|^2 \tag{1.2}$$

for all  $\xi \in \mathbb{R}^{p_0}$ ,  $x \in \mathbb{R}^N$  and for some constant  $\Lambda \ge 1$ ;

the entries  $a_{ij}$  are supposed to be uniformly continuous functions on  $\mathbb{R}^N$ , with a modulus of continuity

$$\omega\left(r\right) = \max_{\substack{i,j=1,\dots,p_0\\|x-y|\leq r}} \sup_{\substack{x,y\in\mathbb{R}^N\\|x-y|\leq r}} \left|a_{ij}\left(x\right) - a_{ij}\left(y\right)\right|;\tag{1.3}$$

the constant matrix  $B = (b_{ij})_{i,j=1}^N$  has the following structure:

$$B = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix}$$
(1.4)

where  $B_j$  is a  $p_{j-1} \times p_j$  block with rank  $p_j, j = 1, 2, ..., r, p_0 \ge p_1 \ge ... \ge p_r \ge 1$ ,  $p_0 + p_1 + ... + p_r = N$  and the symbols \* denote completely arbitrary blocks.

If the  $a_{ij}$ 's are constant, the above assumptions imply that the operator  $\mathcal{A}$  is hypoelliptic (although degenerate, as soon as  $p_0 < N$ ), see [12]. If the  $a_{ij}$ 's

are just uniformly continuous,  $\mathcal{A}$  is a nonvariational degenerate elliptic operator with continuous coefficients, structured on a hypoelliptic operator. For this class of operators, we shall prove the following global  $L^p$  estimates:

**Theorem 1.1** For every  $p \in (1, \infty)$  there exists a constant c > 0, depending on  $p, N, p_0$ , the matrix B, the number  $\Lambda$  in (1.2) and the modulus  $\omega$  in (1.3) such that for every  $u \in C_0^{\infty}(\mathbb{R}^N)$  one has:

$$\sum_{j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(\mathbb{R}^N)} \le c \left\{ \| \mathcal{A} u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\}, \tag{1.5}$$

$$\left\| \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u \right\|_{L^p(\mathbb{R}^N)} \le c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}.$$
(1.6)

In [4] we have proved this result in the case of constant coefficients  $a_{ij}$ . Here we show that exploiting results and techniques contained in [4], together with a careful inspection of the quantitative dependence of some bounds proved in [12] and [9], we can get Theorem 1.1. The striking feature of our result is twofold. On the one side, the merely uniform continuity of the coefficients  $a_{ij}(x)$ ; on the other side the lack of a Lie group structure making translation invariant the frozen operator

$$\mathcal{A}_{x_0} = \sum_{i,j=1}^{p_0} a_{ij}\left(x_0\right) \partial_{x_i x_j}^2 + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j}, \qquad x_0 \in \mathbb{R}^N.$$

As in [4], we overcome this last difficulty by considering the operator  $\mathcal{A}$  as the stationary counterpart of the corresponding evolution operator  $\mathcal{A} - \partial_t$  and looking for the estimates (1.5) and (1.6) as a consequence of analogous estimates for  $\mathcal{A} - \partial_t$  on a suitable strip  $S_T = \mathbb{R}^N \times [-T, T]$ .

There exists a quite extensive literature related to global  $L^p$  estimates for non-degenerate elliptic and parabolic equations on the whole space with unbounded lower order coefficients and variable coefficients  $a_{ij}$ . The considered  $L^p$ -spaces are defined with respect to Lebesgue measure or with respect to an invariant measure which has also a probabilistic interpretation (see, for instance, [6], [7], [8], [10], [11], [14], [16] and the references therein).

On the other hand, to the best of our knowledge, only the papers [2], [3] and [5] deal with  $L^p$  estimates for classes of *degenerate* operators with both unbounded first order coefficients and bounded variable coefficients  $a_{ij}$ . However, we want to stress that the estimates there proved are only of *local* type.

We also mention that global  $L^p$  estimates like (1.5) are crucial in establishing weak uniqueness theorems for associated stochastic differential equations, see [15] and the references therein. Finally, a priori estimates in non-isotropic Hölder spaces for operators like (1.1) with Hölder continuous  $a_{ij}$  were proved by A. Lunardi in [13].

## 2 Notations and preliminary results

## The operator L

Let us introduce the evolution operator

$$Lu(z) = \sum_{i,j=1}^{p_0} a_{ij}(z) \,\partial_{x_i x_j}^2 u(z) + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = \sum_{i,j=1}^{p_0} a_{ij}(z) \,\partial_{x_i x_j}^2 u(z) + \langle x, B \nabla u(z) \rangle - \partial_t u(z)$$
(2.1)

with z = (x, t) in  $\mathbb{R}^{N+1}$ , where now the coefficients  $a_{ij}$  possibly depend also on t. When the  $a_{ij}$ 's are time independent, we get  $L = \mathcal{A} - \partial_t$ . Let

$$A(z) = \begin{bmatrix} A_0(z) & 0\\ 0 & 0 \end{bmatrix}$$

be an  $N \times N$  matrix where  $A_0(z) = (a_{ij}(z))_{i,j=1}^{p_0}$  is a  $p_0 \times p_0$   $(p_0 \le N)$  symmetric and uniformly positive definite matrix for all z, satisfying

$$\frac{1}{\Lambda} |\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}(z) \,\xi_i \xi_j \le \Lambda \,|\xi|^2 \tag{2.2}$$

for all  $\xi \in \mathbb{R}^{p_0}$  and for some constant  $\Lambda \geq 1$ .

Moreover, we assume that the functions  $a_{ij}$  are uniformly continuous in  $\mathbb{R}^{N+1}$  with modulus of continuity

$$\omega(r) = \max_{\substack{i,j=1,\dots,p_0\\|z_1,z_2\in\mathbb{R}^{N+1}\\|z_1-z_2|\leq r}} \sup_{\substack{a_{ij}(z_1) - a_{ij}(z_2)|}.$$
(2.3)

#### The operator $L_{z_0}$

For a fixed  $z_0 \in \mathbb{R}^{N+1}$  we consider the operator  $L_{z_0}$  that differs from L only for the coefficients  $a_{ij}$ 's, that now are constant coefficients:

$$L_{z_0}u(z) = \sum_{i,j=1}^{p_0} a_{ij}\left(z_0\right)\partial_{x_ix_j}^2 u(z) + \langle x, B\nabla u(z) \rangle - \partial_t u(z),$$

where, as above, z = (x, t).

This operator is hypoelliptic; actually it can be proved (see [12]) that this fact is equivalent to the validity of the condition  $C(z_0;t) > 0$  for every t > 0, where

$$C(z_{0};t) = \int_{0}^{t} E(s) A(z_{0}) E^{T}(s) ds, \text{ where } E(s) = \exp(-sB^{T}).$$

Moreover, it is proved in [12] that  $L_{z_0}$  is left-invariant with respect to the composition law

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau).$$

Note that

$$(\xi, \tau)^{-1} = (-E(-\tau)\xi, -\tau).$$

We explicitly note that such a composition law is independent of  $z_0$ , since only the matrix B is involved.

The operator  $L_{z_0}$  has a fundamental solution  $\Gamma(z_0; \cdot, \cdot)$ ,

$$\Gamma(z_0; z, \zeta) = \gamma(z_0; \zeta^{-1} \circ z) \text{ for } z, \zeta \in \mathbb{R}^{N+1},$$

with

$$\gamma\left(z_{0};z\right) = \begin{cases} 0 & \text{for } t \leq 0\\ \frac{\left(4\pi\right)^{-N/2}}{\sqrt{\det C(z_{0};t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}\left(z_{0};t\right)x,x\right\rangle - t\text{Tr}B\right) & \text{for } t > 0 \end{cases}$$

where z = (x, t).

## The operator $K_{z_0}$

By principal part of  $L_{z_0}$  we mean the operator

$$K_{z_0} = \sum_{i,j=1}^{p_0} a_{ij} (z_0) \partial_{x_i x_j}^2 + \langle x, B_0 \nabla \rangle - \partial_t,$$

where the matrix in the drift term is now  $B_0$ , obtained by annihilating every \* block in (1.4):

$$B_0 = \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

The fundamental solution of the principal part operator  $K_{z_0}$  is  $\Gamma_0(z_0; z, \zeta) = \gamma_0(z_0; \zeta^{-1} \circ z)$ ; namely, for t > 0

$$\gamma_0(z_0; z) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C_0(z_0; t)}} \exp\left(-\frac{1}{4} \left\langle C_0^{-1}(z_0; t) \, x, x \right\rangle\right)$$

with

$$C_0(z_0;t) = \int_0^t E_0(s) A(z_0) E_0^T(s) \, ds, \text{ where } E_0(s) = \exp\left(-sB_0^T\right).$$
(2.4)

#### Homogeneous dimension, norm and distance

For every  $\lambda > 0$ , let us define the matrix of *dilations on*  $\mathbb{R}^N$ ,

$$D(\lambda) = \operatorname{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}\right)$$

where  $I_{p_j}$  denotes the  $p_j \times p_j$  identity matrix, and the matrix of *dilations* on  $\mathbb{R}^{N+1}$ ,

$$\delta(\lambda) = (D(\lambda), \lambda^2) = \operatorname{diag} \left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2\right).$$

Note that

$$\det\left(D\left(\lambda\right)\right) = \lambda^{Q}, \qquad \det\left(\delta\left(\lambda\right)\right) = \lambda^{Q+2}$$

with  $Q = p_0 + 3p_1 + ... + (2r+1)p_r$ ; Q and Q + 2 are called the *homogeneous* dimension of  $\mathbb{R}^N$  and  $\mathbb{R}^{N+1}$ , respectively. The operator  $K_{z_0}$  is homogeneous of degree two with respect to these dilations.

There is a natural homogeneous norm in  $\mathbb{R}^{N+1}$ , induced by these dilations:

$$|(x,t)|| = \sum_{j=1}^{N} |x_j|^{1/q_j} + |t|^{1/2}$$

where  $q_j$  are positive integers such that  $D(\lambda) = \text{diag}(\lambda^{q_1}, ..., \lambda^{q_N})$ . Clearly, we have

 $\|\delta(\lambda) z\| = \lambda \|z\|$  for every  $\lambda > 0, z \in \mathbb{R}^{N+1}$ .

A key geometrical object is the local quasisymmetric quasidistance d. Namely,

$$d(z,\zeta) = \left\| \zeta^{-1} \circ z \right\|.$$

Note that the homogeneous norm involved in the definition of d is related to the principal part operator  $K_{z_0}$ , while the group law  $\circ$  is related to the original operator  $L_{z_0}$ . Hence this function d is not a usual quasidistance on a homogeneous group. The function  $d(z,\zeta)$  satisfies the quasisymmetric and quasitriangle inequalities only for  $d(z,\zeta)$  bounded (see Lemma 2.1 in [9]); this happens for instance on a fixed d-ball  $B_{\rho}(z)$ , where

$$B_{\rho}(z) = \left\{ \zeta \in \mathbb{R}^{N+1} : d(z,\zeta) < \rho \right\}.$$

### 3 Estimates on a strip for evolution operators

Let  $S_T$  be the strip  $\mathbb{R}^N \times [-T, T]$ . We use c to denote constants that may vary from line to line.

Our main result in this section is the following:

**Theorem 3.1** Let L be as in (2.1), with the matrix B satisfying (1.4) and with uniformly continuous coefficients  $a_{ij}$  satisfying (2.2).

For every  $p \in (1, \infty)$  there exist constants c, T > 0 depending on  $p, N, p_0$ , the matrix B, the number  $\Lambda$  in (2.2), c also depending on the modulus of continuity  $\omega$  in (2.3) such that

$$\sum_{i,j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(S_T)} \le c \left\{ \| L u \|_{L^p(S_T)} + \| u \|_{L^p(S_T)} \right\}$$
(3.1)

for every  $u \in C_0^{\infty}(S_T)$ .

¿From Theorem 3.1 one obtains Theorem 1.1 proceeding as follows.

**Proof of Theorem 1.1.** If  $u: \mathbb{R}^N \to \mathbb{R}$  is a  $C_0^{\infty}$  function, we define

$$U\left( x,t\right) =u\left( x\right) \psi \left( t\right) ,$$

where

 $\psi \in C_0^\infty \left( \mathbb{R} \right)$ 

is a cutoff function with  $\operatorname{sprt} \psi \subset [-T,T]$ ,  $\int_{-T}^{T} \psi(t) dt > 0$ . Then (3.1) applied to U gives (1.5) for u. Moreover, inequality (1.6) immediately follows by difference.

The crucial step toward the proof of Theorem 3.1 is a local estimate contained in the following:

**Proposition 3.2** There exist constants  $c, r_0$  such that for every  $z_0 \in S_T$ ,  $r \leq r_0$ ,  $u \in C_0^{\infty}(B_r(z_0))$ , we have

$$\sum_{i,j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(B_r(z_0))} \le c \left\| L u \right\|_{L^p(B_r(z_0))}.$$
(3.2)

**Proof.** Let  $z_0 \in S_T$  and  $\rho_0 \in (0,T]$  be fixed and choose a cutoff function  $\eta \in C_0^{\infty}(\mathbb{R}^{N+1})$  such that

$$\eta(z) = 1 \text{ for } ||z|| \le \rho_0/2;$$
  
 $\eta(z) = 0 \text{ for } ||z|| \ge \rho_0.$ 

Then, by [9, Proposition 2.11] and (25) in [4], we have, for every  $u \in C_0^{\infty}(B_r(z_0))$ ,

$$\partial_{x_{i}x_{j}}^{2} u = -PV\left(L_{z_{0}}u * \left(\eta \partial_{x_{i}x_{j}}^{2}\gamma(z_{0};\cdot)\right)\right) - L_{z_{0}}u * \left((1-\eta) \partial_{x_{i}x_{j}}^{2}\gamma(z_{0};\cdot)\right) + c_{ij}(z_{0}) L_{z_{0}}u$$
$$\equiv -PV\left(L_{z_{0}}u * k_{0}(z_{0};\cdot)\right) - L_{z_{0}}u * k_{\infty}(z_{0};\cdot) + c_{ij}(z_{0}) L_{z_{0}}u \qquad (3.3)$$

having set:

$$k_0(z_0; \cdot) = \eta \partial_{x_i x_j}^2 \gamma(z_0; \cdot)$$
  

$$k_\infty(z_0; \cdot) = (1 - \eta) \partial_{x_i x_j}^2 \gamma(z_0; \cdot)$$
(3.4)

and

$$c_{ij}(z_0) = -\int_{\|\zeta\|=1} \partial_{x_i} \gamma_0(z_0;\zeta) \nu_j(\zeta) \, d\sigma(\zeta),$$

where  $\nu_j$  denotes the *j*-th component of the exterior normal  $\nu$  to the boundary of  $\{\|\zeta\| < 1\}$ . In (3.3) \* denotes the convolution with respect to the composition law  $\circ$ .

Writing

$$L_{z_0}u(z) = (L_{z_0} - L)u(z) + Lu(z)$$
  
=  $\sum_{i,j=1}^{p_0} (a_{ij}(z_0) - a_{ij}(z))\partial_{x_ix_j}^2 u(z) + Lu(z)$ 

we get, by (3.3),

$$\partial_{x_{i}x_{j}}^{2}u = -PV\left(Lu * k_{0}\left(z_{0};\cdot\right)\right) - PV\left(\sum_{h,k=1}^{p_{0}}\left(a_{hk}\left(z_{0}\right) - a_{hk}\left(\cdot\right)\right)\partial_{x_{h}x_{k}}^{2}u * k_{0}\left(z_{0};\cdot\right)\right)\right)$$
$$-Lu * k_{\infty}\left(z_{0};\cdot\right) - \sum_{h,k=1}^{p_{0}}\left(a_{hk}\left(z_{0}\right) - a_{hk}\left(\cdot\right)\right)\partial_{x_{h}x_{k}}^{2}u * k_{\infty}\left(z_{0};\cdot\right)$$
$$+ c_{ij}\left(z_{0}\right)Lu + c_{ij}\left(z_{0}\right)\sum_{h,k=1}^{p_{0}}\left(a_{hk}\left(z_{0}\right) - a_{hk}\left(\cdot\right)\right)\partial_{x_{h}x_{k}}^{2}u$$
$$= I_{1} + I_{2} + J_{1} + J_{2} + A_{1} + A_{2}.$$

$$(3.5)$$

We now split the remaining part of the proof into several steps.

**Step 1.**  $L^p$ -estimate of  $A_1$  and  $A_2$ . We obviously have

$$||A_1||_{L^p(B_r(z_0))} \le |c_{ij}(z_0)|||Lu||_{L^p(B_r(z_0))}.$$

On the other hand, by Theorem 4.1 and Remark 4.2 in Appendix, there exists an absolute constant c such that

$$|c_{ij}(z_0)| \le \int_{\|\zeta\|=1} |\partial_{x_i} \gamma_0(z_0;\zeta)| \, d\sigma(\zeta) \le c \int_{\|\zeta\|=1} \frac{1}{\|\zeta\|^{Q+1}} \, d\sigma(\zeta).$$

Therefore

$$\|A_1\|_{L^p(B_r(z_0))} \le c \|Lu\|_{L^p(B_r(z_0))}.$$
(3.6)

Analogously, using the uniform continuity of the coefficients  $a_{ij}$ 's, we get

$$\|A_2\|_{L^p(B_r(z_0))} \le c\,\omega(r)\sum_{h,k=1}^{p_0} \|\partial_{x_h x_k}^2 u\|_{L^p(B_r(z_0))}.$$
(3.7)

**Step 2.**  $L^p$ -estimate of  $J_1$  and  $J_2$ .

Without loss of generality we can assume  $B_r(z_0) \subseteq S_{2T}$  for every  $z_0 \in S_T$ . Then

$$\|J_1\|_{L^p(B_r(z_0))} \le c \int_{S_{2T}} |k_{\infty}(z_0;\zeta)| \, d\zeta \, \|Lu\|_{L^p(B_r(z_0))}$$

where the presence of the constant c depends on the fact that our group is not unimodular. On the other hand, just proceeding as in [4], pages 799-800, and using the estimates in Appendix (see Proposition 4.6) we get

$$\int_{S_{2T}} |k_{\infty}(z_0;\zeta)| \, d\zeta \le c,$$

where c is independent of  $z_0 \in S_T$ . Therefore

$$\|J_1\|_{L^p(B_r(z_0))} \le c \|Lu\|_{L^p(B_r(z_0))}.$$
(3.8)

Analogously, using the uniform continuity of the  $a_{ij}$ 's, we get

$$\|J_2\|_{L^p(B_r(z_0))} \le c\omega(r) \sum_{h,k=1}^{p_0} \|\partial_{x_h x_k}^2 u\|_{L^p(B_r(z_0))}.$$
(3.9)

**Step 3.**  $L^p$ -estimate of  $I_1$  and  $I_2$ .

To estimate the  $L^p$ -norm of  $I_1$  and  $I_2$ , we can use Theorem 3.3, getting:

$$\|I_{1}\|_{L^{p}(B_{r}(z_{0}))} + \|I_{2}\|_{L^{p}(B_{r}(z_{0}))}$$

$$\leq c \left\{ \|Lu\|_{L^{p}(B_{r}(z_{0}))} + \left\| \sum_{h,k=1}^{p_{0}} \left[a_{hk}\left(z_{0}\right) - a_{hk}\left(\cdot\right)\right]\partial_{x_{h}x_{k}}^{2}u\right\|_{L^{p}(B_{r}(z_{0}))} \right\}$$

$$\leq c \left\{ \|Lu\|_{L^{p}(B_{r}(z_{0}))} + \omega\left(r\right)\sum_{h,k=1}^{p_{0}} \left\|\partial_{x_{h}x_{k}}^{2}u\right\|_{L^{p}(B_{r}(z_{0}))} \right\}$$

$$(3.10)$$

with c independent of r and  $z_0$ .

Step 4. Conclusion.

By (3.5) and the estimates (3.6)-(3.10) in the previous steps, we get

$$\sum_{i,j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(B_r(z_0))} \le c \left\{ \| L u \|_{L^p(B_r(z_0))} + \omega\left(r\right) \sum_{h,k=1}^{p_0} \left\| \partial_{x_h x_k}^2 u \right\|_{L^p(B_r(z_0))} \right\}$$

with c independent of r and  $z_0$ .

We now fix once and for all  $r_0$  small enough so that  $c\omega(r_0) < 1$ , getting

$$\sum_{i,j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(B_r(z_0))} \le c \left\| L u \right\|_{L^p(B_r(z_0))}$$

for every  $u \in C_0^{\infty}(B_r(z_0))$  with  $r \leq r_0$ , with  $c, r_0$  independent of u and  $z_0 \in S_T$ .

Next, we have to prove the following crucial ingredient which has been used in the previous proof:

**Theorem 3.3** Let  $k_0(z_0; \cdot)$  be the singular kernel defined in (3.4). For every  $p \in (1, \infty)$  there exists a positive constant c, independent of  $z_0$ , such that

$$\|PV(f * k_0(z_0; \cdot))\|_{L^p(B_r(z_0))} \le c \|f\|_{L^p(B_r(z_0))}$$
(3.11)

for every  $f \in C_0^{\infty}(B_r(z_0))$ ,  $z_0 \in S_T$  and r > 0 such that  $B_r(z_0) \subseteq S_{2T}$ .

**Proof.** This theorem is analogous to Theorem 22 in [4], the novelty being the uniformity of the bound with respect to the point  $z_0$  in the kernel  $k_0(z_0; \cdot)$ . As in [4], this theorem follows applying the abstract result contained in [1, Thm 3]. Without recalling the general setting of nondoubling spaces considered in [1], here we just list, for convenience of the reader, the assumptions that need to be checked on our kernel, in order to derive Theorem 3.3 from [1, Thm 3]. The constant c in (3.11) will depend only on the constants involved in the following bounds.

Let

$$k(z_0; w^{-1} \circ z) = a(z) k_0(z_0; w^{-1} \circ z) b(w)$$

where  $a, b \in C_0^{\infty}(\mathbb{R}^{N+1})$  with sprt a, sprt  $b \subset B_r(z_0)$ . Then the required properties are the following:

$$\left|k(z_0; w^{-1} \circ z)\right| + \left|k(z_0; z^{-1} \circ w)\right| \le \frac{c}{\|w^{-1} \circ z\|^{Q+2}}$$
(3.12)

for every  $z_0 \in S_T$ ,  $z, w \in S_{2T}$  such that  $||w^{-1} \circ z|| \le 1$ ;

$$|k(z_0; w^{-1} \circ z) - k(z_0; w^{-1} \circ \overline{z})| +$$

$$|k(z_0; z^{-1} \circ w) - k(z_0; \overline{z}^{-1} \circ w)| \le c \frac{\|z^{-1} \circ \overline{z}\|}{\|w^{-1} \circ z\|^{Q+3}}$$
(3.13)

for every  $z_0 \in S_T, z, \overline{z}, w \in S_{2T}$  such that  $||z^{-1} \circ \overline{z}|| \leq M ||w^{-1} \circ z||$  and  $||w^{-1} \circ z|| \leq 1$ ;

$$\left| \int_{r_1 \le \|\zeta^{-1} \circ z\| \le r_2} k(z_0; \zeta^{-1} \circ z) \, d\zeta \right| + \left| \int_{r_1 \le \|\zeta^{-1} \circ z\| \le r_2} k(z_0; z^{-1} \circ \zeta) \, d\zeta \right| \le c$$
(3.14)

for every  $r_1, r_2$  with  $0 < r_1 < r_2$  and for all  $z \in S_{2T}$  and  $z_0 \in S_T$ ;

$$h(z_0, \cdot) - h^*(z_0, \cdot) \in C^{\gamma}(B_r(z_0))$$
 (3.15)

for some positive  $\gamma$ , where

$$h(z_0, z) = \lim_{r \to 0} \int_{r \le \|\zeta^{-1} \circ z\|} k(z_0; \zeta^{-1} \circ z) \, d\zeta;$$
(3.16)

$$h^*(z_0, z) = \lim_{r \to 0} \int_{r \le \|\zeta^{-1} \circ z\|} k(z_0; z^{-1} \circ \zeta) \, d\zeta.$$
(3.17)

Now: estimates (3.12) and (3.13) follow from Theorem 4.1 and Remark 4.2 contained in the Appendix.

Let us prove (3.14). Actually, we will bound the first integral, the bound on the second being analogous. Moreover, we actually prove the following

$$\left| \int_{r_1 \le \|\zeta^{-1} \circ z\| \le r_2} k_0(z_0; \zeta^{-1} \circ z) \, d\zeta \right| \le c, \tag{3.18}$$

which implies the analogous bound on k by the same argument contained in [4, Prop. 18]. To show (3.18), we proceed as in [4], page 803. Without loss of generality we assume  $r_2 \leq \rho_0$ , where  $\rho_0$  is the positive constant introduced at the beginning of the proof of Proposition 3.2; in fact,  $k_0(z_0; w) = 0$  for  $||w|| > \rho_0$ .

We have:

.

$$\int_{r_1 \le \|\zeta^{-1} \circ z\| \le r_2} k_0(z_0; \zeta^{-1} \circ z) \, d\zeta = A(z_0; r_1, r_2) + B(z_0; r_1, r_2),$$

where

$$A(z_0; r_1, r_2) = \int_{r_1 \le \|w\| \le r_2} \eta(w) \partial_{x_i x_j}^2 \gamma(z_0; w) \, dw$$

and

$$B(z_0; r_1, r_2) = \int_{r_1 \le ||w|| \le r_2} \eta(w) \partial_{x_i x_j}^2 \gamma(z_0; w) \left( e^{\tau Tr(B)} - 1 \right) dw, \qquad w = (\xi, \tau).$$

Then, by (4.3)

$$B(z_0; r_1, r_2) \le c \int_{r_1 \le \|w\| \le r_2} \frac{1}{\|w\|^Q} \, dw \le c \int_{\|w\| \le \rho_0} \frac{1}{\|w\|^Q} \, dw$$

with c independent of  $z_0 \in S_T$ . Moreover, if  $r_2 \leq \frac{\rho_0}{2}$ , then integrating by parts

$$A(z_0; r_1, r_2) = \int_{\|w\| = r_2} \partial_{x_i} \gamma(z_0; w) \nu_j \, d\sigma(w) - \int_{\|w\| = r_1} \partial_{x_i} \gamma(z_0; w) \nu_j \, d\sigma(w)$$
  
=:  $I(z_0; r_2) - I(z_0; r_1).$ 

Now we estimate  $I(z_0; \rho)$  by proceeding as in [9], page 1280. We have

$$I(z_0; \rho) = \int_{\|\zeta\|=1} \partial_{x_i} \gamma_{\rho}(z_0; \zeta) \nu_j \, d\sigma(\zeta)$$
  
= 
$$\int_{\|\zeta\|=1} \left( \partial_{x_i} \gamma_{\rho}(z_0; \zeta) - \partial_{x_i} \gamma_0(z_0; \zeta) \right) \nu_j \, d\sigma(\zeta)$$
  
+ 
$$\int_{\|\zeta\|=1} \partial_{x_i} \gamma_0(z_0; \zeta) \nu_j \, d\sigma(\zeta)$$

where  $\gamma_{\rho}(z_0; \cdot)$  is defined as in [9], (2.24).

The last integrand can be estimated by a constant independent of  $z_0 \in S_T$ , thanks to (4.2) and Remark 4.2. On the other hand, from (2.45) in [9] we get, for a suitable c independent of  $z_0 \in S_T$ :

$$\left| \int_{\|\zeta\|=1} \left( \partial_{x_i} \gamma_{\rho}(z_0;\zeta) - \partial_{x_i} \gamma_0(z_0;\zeta) \right) \nu_j \, d\sigma(\zeta) \right| \le c \, \rho \int_{\|\zeta\|=1} \frac{1}{\sqrt{\tau}} \gamma(\zeta) \, d\sigma(\zeta),$$

 $\zeta = (x, \tau)$ , where  $\gamma$  is the fundamental solution with pole at the origin of

$$\mu \sum_{i=1}^{p_0} \partial_{x_i}^2 + \langle x, B_0 \nabla \rangle - \partial_t$$

for a suitable  $\mu > 0$  independent of  $z_0 \in S_T$ . Note that the last integral is an absolute constant.

Suppose now  $\frac{\rho_0}{2} \leq r_2 \leq \rho_0$ . Then we can write

$$A(z_0; r_1, r_2) \le \left| \int_{r_1 < \|w\| < \rho_0/2} k_0(z_0; w) \, dw \right| + \left| \int_{\rho_0/2 < \|w\| < r_2} k_0(z_0; w) \, dw \right|.$$

The first term can be bounded as above, while the second one is bounded by

$$\int_{\rho_0/2 \le \|w\| \le \rho_0} c \, \|w\|^{-(2+Q)} \, dw$$

with c independent of  $z_0$ , see (4.3). This completes the proof of (3.18).

Finally, let us prove the Hölder continuity of the function

$$h(z_0, \cdot) - h^*(z_0, \cdot)$$

defined in (3.16)-(3.17).<sup>1</sup>

$$\begin{split} h\left(z_{0},z\right) &= \lim_{r \to 0} a\left(z\right) \int_{r \leq \|\zeta^{-1} \circ z\|} k_{0}(z_{0};\zeta^{-1} \circ z) b\left(\zeta\right) \, d\zeta = \\ &= \lim_{r \to 0} a\left(z\right) \int_{r \leq \|w\|} k_{0}(z_{0};w) b\left(z \circ w^{-1}\right) \, e^{\tau Tr(B)} dw \\ &= a\left(z\right) \int_{\|w\| \leq \rho_{0}} k_{0}(z_{0};w) \left[b\left(z \circ w^{-1}\right) - b\left(z\right)\right] e^{\tau Tr(B)} dw + \\ &+ a\left(z\right) b\left(z\right) \int_{\|w\| \leq \rho_{0}} k_{0}(z_{0};w) \left[e^{\tau Tr(B)} - 1\right] dw \\ &+ \lim_{r \to 0} a\left(z\right) b\left(z\right) \int_{r \leq \|w\|} k_{0}(z_{0};w) dw \\ &= h_{1}\left(z_{0},z\right) + h_{2}\left(z_{0},z\right) + h_{3}\left(z_{0},z\right). \end{split}$$

 $<sup>^{1}</sup>$ We take this opportunity to notice that in [4] this check has not been explicitly done.

Now:

$$h_{3}(z_{0}, z) = a(z) b(z) c(z_{0})$$

with  $a(\cdot)$ ,  $b(\cdot)$  smooth and  $c(z_0)$  uniformly bounded in  $z_0$  by the previous bound (3.18). Also,

$$h_2(z_0, z) = a(z) b(z) c_1(z_0)$$

with  $c(z_0)$  uniformly bounded in  $z_0$  by the same argument used above to bound  $B(z_0; r_1, r_2)$ . Let us come to  $h_1(z_0, z)$ . If Z is any right-invariant differential operator, then

$$Zh_{1}(z_{0},z) = Za(z) \int_{\|w\| \le \rho_{0}} k_{0}(z_{0};w) \left[ b\left(z \circ w^{-1}\right) - b(z) \right] e^{\tau Tr(B)} dw$$
  
+  $a(z) \int_{\|w\| \le \rho_{0}} k_{0}(z_{0};w) \left[ Zb\left(z \circ w^{-1}\right) - Zb(z) \right] e^{\tau Tr(B)} dw,$ 

hence

$$|Zh_1(z_0, z)| \le c \int_{\|w\| \le \rho_0} |k_0(z_0; w)| \, |w| \, dw \le c.$$

Since this procedure can be iterated, we get an upper bound on any derivative of the kind  $|Z_1Z_2...Z_kh_1(z_0,z)|$ , hence (since the commutators of suitable right invariant vector fields span  $\mathbb{R}^{N+1}$ ) also on  $|\nabla h_1(z_0,z)|$ . Therefore the function  $h_1(z_0,\cdot)$  is Lipschitz continuous, uniformly with respect to  $z_0$ . The function  $h^*(z_0,\cdot)$  can be handled similarly. This completes the proof of the conditions which are sufficient to apply [1, Thm 3] and deduce (3.11), with a constant c independent of  $z_0$ .

In order to deduce Theorem 3.1 from Proposition 3.2, we now need to recall a covering lemma, see Lemma 21 in [4] (note that this result is not standard since our space is not globally doubling):

**Lemma 3.4** For every  $r_0 > 0$  and K > 1 there exist  $r \in (0, r_0)$ , a positive integer M and a sequence of points  $\{z_i\}_{i=1}^{\infty} \subset S_T$  such that:

$$S_T \subset \bigcup_{i=1}^{\infty} B_r(z_i); \qquad (3.19)$$

$$\sum_{i=1}^{\infty} \chi_{B_{Kr}(z_i)}(z) \le M \quad \forall z \in S_T.$$
(3.20)

**Proof of Theorem 3.1.** Let us apply the previous lemma with  $r_0$  as in Proposition 3.2; for a fixed  $r \in (0, r_0)$ , with r/2 satisfying (3.19), (3.20). Pick  $A \in C_0^{\infty}(B_r(0)), A = 1$  in  $B_{r/2}(0), 0 \le A \le 1$  and let  $a_k(z) = A(z_k^{-1} \circ z)$ .

Let now  $u \in C_0^{\infty}(S_T)$ . By (3.19) we can write

$$\begin{aligned} \left\| \partial_{x_{i}x_{j}}^{2} u \right\|_{L^{p}(S_{T})}^{p} &\leq \sum_{k=1}^{\infty} \left\| \partial_{x_{i}x_{j}}^{2} u \right\|_{L^{p}\left(B_{r/2}(z_{k})\right)}^{p} = \sum_{k=1}^{\infty} \left\| \partial_{x_{i}x_{j}}^{2} \left(a_{k}u\right) \right\|_{L^{p}\left(B_{r/2}(z_{k})\right)}^{p} \\ &\leq \sum_{k=1}^{\infty} \left\| \partial_{x_{i}x_{j}}^{2} \left(a_{k}u\right) \right\|_{L^{p}\left(B_{r}(z_{k})\right)}^{p}. \end{aligned}$$
(3.21)

On the other hand, by (3.2) we have

$$\begin{split} \left\| \partial_{x_{i}x_{j}}^{2}\left(a_{k}u\right) \right\|_{L^{p}\left(B_{r}\left(z_{k}\right)\right)} &\leq c \left\| L\left(a_{k}u\right) \right\|_{L^{p}\left(B_{r}\left(z_{k}\right)\right)} \\ &\leq c \left\{ \left\| a_{k}Lu \right\|_{L^{p}\left(B_{r}\left(z_{k}\right)\right)} + 2 \sum_{l,m=1}^{p_{0}} \left\| \partial_{x_{l}}a_{k}\partial_{x_{m}}u \right\|_{L^{p}\left(B_{r}\left(z_{k}\right)\right)} + \left\| uLa_{k} \right\|_{L^{p}\left(B_{r}\left(z_{k}\right)\right)} \right\}. \end{split}$$

$$\tag{3.22}$$

By recalling that the operators  $\partial_{x_l}$ ,  $l = 1, ..., p_0$ , and  $Y_0 := \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j}$  are left invariant with respect to the group law  $\circ$ , we have

$$\sup_{z \in B_{r}(z_{k})} |\partial_{x_{l}} a_{k}(z)| = \sup_{z \in B_{r}(0)} |\partial_{x_{l}} A(z)| \le c, \qquad l = 1, ..., p_{0},$$
$$\sup_{z \in B_{r}(z_{k})} |Y_{0} a_{k}(z)| = \sup_{z \in B_{r}(0)} |Y_{0} A(z)| \le c$$

and

$$\sup_{z \in B_{r}(z_{k})} \left| \partial_{x_{i}x_{j}}^{2} a_{k}(z) \right| = \sup_{z \in B_{r}(0)} \left| \partial_{x_{i}x_{j}}^{2} A(z) \right| \le c, \qquad i, j = 1, 2, ..., p_{0}.$$

As a consequence

$$\sup_{z \in B_r(z_k)} \left| La_k(z) \right| \le c$$

with c independent of k. Hence (3.22) gives

$$\left\|\partial_{x_i x_j}^2\left(a_k u\right)\right\|_{L^p(B_r(z_k))} \le c \left\{ \|L u\|_{L^p(B_r(z_k))} + 2\sum_{l,m=1}^{p_0} \|\partial_{x_m} u\|_{L^p(B_r(z_k))} + \|u\|_{L^p(B_r(z_k))} \right\},$$

c independent of k. Inserting the last inequality in (3.21) and recalling (3.20) we get

$$\begin{aligned} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(S_T)}^p &\leq c \sum_{k=1}^\infty \left\{ \| L u \|_{L^p(B_r(z_k))}^p + \sum_{m=1}^{p_0} \| \partial_{x_m} u \|_{L^p(B_r(z_k))}^p + \| u \|_{L^p(B_r(z_k))}^p \right\} \\ &\leq c M \left\{ \| L u \|_{L^p(S_T)}^p + \sum_{m=1}^{p_0} \| \partial_{x_m} u \|_{L^p(S_T)}^p + \| u \|_{L^p(S_T)}^p \right\}. \end{aligned}$$

This also gives

$$\sum_{i,j=1}^{p_0} \left\| \partial_{x_i x_j}^2 u \right\|_{L^p(S_T)} \le cM \left\{ \|Lu\|_{L^p(S_T)} + \sum_{m=1}^{p_0} \|\partial_{x_m} u\|_{L^p(S_T)} + \|u\|_{L^p(S_T)} \right\}$$

which, by the classical interpolation inequality

$$\|\partial_{x_m} u\|_{L^p(S_T)} \le \varepsilon \left\|\partial_{x_m x_m}^2 u\right\|_{L^p(S_T)} + \frac{c}{\varepsilon} \|u\|_{L^p(S_T)},$$

yields (3.1). So we are done.  $\blacksquare$ 

## 4 Appendix: uniform bounds on the fundamental solution of $L_{z_0}$

The aim of this section is to prove the following result, which has been exploited in the proof of Proposition 3.2 and Theorem 3.3:

**Theorem 4.1** There exists a positive constant c independent of  $z_0 \in S_T$  such that

$$|\gamma(z_0;\zeta)| \le \frac{c}{\|\zeta\|^Q},\tag{4.1}$$

$$\left|\partial_{x_j}\gamma(z_0;\zeta)\right| \le \frac{c}{\|\zeta\|^{Q+1}} \qquad j = 1,...,p_0,$$
(4.2)

$$\left|\partial_{x_i x_j}^2 \gamma(z_0; \zeta)\right| \le \frac{c}{\|\zeta\|^{Q+2}} \qquad i, j = 1, ..., p_0,$$
(4.3)

for every  $\zeta \in S_{2T}$ .

Morever, if  $H \subset \mathbb{R}^N$  is a compact set there exist constants c' and M, depending on H but not on  $z_0$ , such that

$$\begin{aligned} \left|\gamma(z_{0}; w^{-1} \circ z) - \gamma(z_{0}; w^{-1} \circ \bar{z})\right| &\leq c' \frac{\|z^{-1} \circ \bar{z}\|}{\|w^{-1} \circ z\|^{Q+1}}, \\ \left|\partial_{x_{j}}\gamma(z_{0}; w^{-1} \circ z) - \partial_{x_{j}}\gamma(z_{0}; w^{-1} \circ \bar{z})\right| &\leq c' \frac{\|z^{-1} \circ \bar{z}\|}{\|w^{-1} \circ z\|^{Q+2}} \qquad j = 1, ..., p_{0}, \\ \left|\partial_{x_{i}x_{j}}^{2}\gamma(z_{0}; w^{-1} \circ z) - \partial_{x_{i}x_{j}}^{2}\gamma(z_{0}; w^{-1} \circ \bar{z})\right| &\leq c' \frac{\|z^{-1} \circ \bar{z}\|}{\|w^{-1} \circ z\|^{Q+3}} \qquad i, j = 1, ..., p_{0} \end{aligned}$$

$$(4.4)$$

for every  $z, \overline{z}, w \in S_{2T}$  such that  $||z^{-1} \circ \overline{z}|| \leq M ||w^{-1} \circ z||$  and  $w^{-1} \circ z \in H \times [-2T, 2T]$ .

The previous estimates still hold replacing  $\gamma(z_0; z)$  with  $\gamma(z_0; z^{-1})$ .

**Remark 4.2** The estimates of Theorem 4.1 obviously hold if we replace  $\gamma(z_0; \cdot)$ with  $\gamma_0(z_0; \cdot)$ . In this case we can exploit the homogeneity of  $\gamma_0$  to obtain (4.1)– (4.3) for every  $\zeta$  in the strip  $\mathbb{R}^N \times [-1, 1]$ .

The above theorem will follow by a careful inspection of several arguments contained in [9] and [12]. We first need to establish several lemmas.

In the following,  $\mathcal{I}$  denotes the  $N \times N$  matrix

$$\mathcal{I} := \begin{bmatrix} I_{p_0} & 0\\ 0 & 0 \end{bmatrix}$$

where  $I_{p_0}$  is the  $p_0 \times p_0$  identity matrix. Moreover, for every t > 0,  $\tilde{C}(t)$  is the  $N \times N$  matrix defined as follows

$$\widetilde{C}(t) = \int_0^t E_0(s) \mathcal{I} E_0^T(s) \, ds \tag{4.5}$$

with  $E_0(s)$  as in (2.4). Notice that  $\widetilde{C}(t) > 0$  for every t > 0, or, equivalently, that the operator

$$\sum_{i=1}^{p_0} \partial_{x_i x_i}^2 u(z) + \langle x, B_0 \nabla u(z) \rangle - \partial_t u(z)$$

is hypoelliptic (see [12]).

The following preliminary lemma holds.

**Lemma 4.3** The inequalities below hold true for all  $z_0 \in \mathbb{R}^{N+1}$ :

$$\frac{1}{\Lambda} \langle \widetilde{C}(1)y, y \rangle \leq \langle C_0(z_0; 1)y, y \rangle \leq \Lambda \langle \widetilde{C}(1)y, y \rangle \qquad \forall y \in \mathbb{R}^N$$
(4.6)

and

$$\frac{1}{\Lambda^N} \det \widetilde{C}(1) \le \det C_0(z_0; 1) \le \Lambda^N \det \widetilde{C}(1).$$
(4.7)

**Proof.** We have that

$$\frac{1}{\Lambda} \mathcal{I} \le A(z_0) \le \Lambda \mathcal{I} \quad \text{for all } z_0 \in \mathbb{R}^{N+1}.$$

Thus, (4.6) holds. Inequalities (4.7) are an easy consequence of (4.6).  $\blacksquare$ 

**Lemma 4.4** There exist  $M \ge 1, T > 0$  such that for every  $x \in \mathbb{R}^N$ ,  $z_0 \in \mathbb{R}^{N+1}$ ,  $t \in [0, T]$ ,

$$\frac{1}{M} \langle \widetilde{C}(t)x, x \rangle \le \langle C(z_0; t)x, x \rangle \le M \langle \widetilde{C}(t)x, x \rangle$$
(4.8)

and

$$\frac{1}{M}\det\widetilde{C}(t) \le \det C(z_0;t) \le M \det\widetilde{C}(t).$$
(4.9)

**Proof.** It is a known fact (see [12, Proposition 2.3]) that

$$\begin{aligned} C_0 (z_0; t) &= D(\sqrt{t}) C_0 (z_0; 1) D(\sqrt{t}) \\ \tilde{C} (t) &= D(\sqrt{t}) \tilde{C} (1) D(\sqrt{t}), \quad \forall t > 0. \end{aligned}$$

Then (4.6) implies

$$\frac{1}{\Lambda} \langle \widetilde{C}(t)x, x \rangle \leq \langle C_0(z_0; t)x, x \rangle \leq \Lambda \langle \widetilde{C}(t)x, x \rangle.$$
(4.10)

Therefore, to prove (4.8) it is enough to look for positive  $c_1$ ,  $c_2$  such that

$$c_1 \langle C_0(z_0;t) x, x \rangle (1+O(t)) \leq \langle C(z_0;t) x, x \rangle$$
  
$$\leq c_2 \langle C_0(z_0;t) x, x \rangle (1+O(t)) \qquad \text{as } t \to 0$$
  
(4.11)

with O(t) uniform w.r.t.  $z_0$ .

This follows using the arguments in [12, p. 46]. Indeed, set  $x = D\left(\frac{1}{\sqrt{t}}\right)y$ we get

$$\begin{aligned} \frac{\langle C\left(z_{0};t\right)x,x\rangle}{\langle C_{0}\left(z_{0};t\right)x,x\rangle} &= 1 + \frac{\langle (C(z_{0};t) - C_{0}(z_{0};t))x,x\rangle}{\langle C_{0}(z_{0};t)x,x\rangle} \\ &= 1 + \frac{\langle D\left(\frac{1}{\sqrt{t}}\right)\left(C(z_{0};t) - C_{0}(z_{0};t)\right)D\left(\frac{1}{\sqrt{t}}\right)y,y\rangle}{\langle C_{0}(z_{0};1)y,y\rangle}.\end{aligned}$$

Now, by the proof of Lemma 3.2 in [12] and a careful check of the block decomposition of the matrices  $C(z_0;t)$  and  $C_0(z_0;t)$ , see Lemma 3.1 in [12], we get

$$\left\| D\left(\frac{1}{\sqrt{t}}\right) \left(C(z_0;t) - C_0(z_0;t)\right) D\left(\frac{1}{\sqrt{t}}\right) \right\| \le ct \quad \text{as } t \to 0^+, \tag{4.12}$$

uniformly w.r.t.  $z_0$ . Thus, we get (4.11).

Let us now prove (4.9). By (4.10), we get

$$\frac{1}{\Lambda^N} \det \widetilde{C}(t) \le \det C_0(z_0; t) \le \Lambda^N \det \widetilde{C}(t).$$

Moreover, by (4.11) there exist positive constants  $c_3$ ,  $c_4$  such that

$$c_3(1+O(t)) \det C_0(z_0;t) \le \det C(z_0;t) \le c_4(1+O(t)) \det C_0(z_0;t)$$

as t goes to  $0^+$ , uniformly w.r.t.  $z_0 \in \mathbb{R}^{N+1}$ . Thus, (4.9) follows.

Now, we turn to prove estimates for  $C^{-1}(z_0; \cdot)$ .

Lemma 4.5 The following inequalities hold:

(1) there exist  $M \ge 1, T > 0$  such that for every  $x \in \mathbb{R}^N$ ,  $z_0 \in \mathbb{R}^{N+1}$ ,  $t \in [0, T]$ ,

$$\frac{1}{M} \langle C_0^{-1}(z_0; t) x, x \rangle \le \langle C^{-1}(z_0; t) x, x \rangle \le M \langle C_0^{-1}(z_0; t) x, x \rangle$$
(4.13)

(2) let  $\lambda_{\widetilde{C}}$  and  $\Lambda_{\widetilde{C}}$  be the smallest and the largest eigenvalue of the symmetric positive definite matrix  $\widetilde{C}(1)$ , respectively. Then

$$\frac{1}{\Lambda\Lambda_{\widetilde{C}}} \left| D\left(\frac{1}{\sqrt{t}}\right) x \right|^2 \le \langle C_0^{-1}(z_0; t) x, x \rangle \le \frac{\Lambda}{\lambda_{\widetilde{C}}} \left| D\left(\frac{1}{\sqrt{t}}\right) x \right|^2, \quad (4.14)$$

for all  $x \in \mathbb{R}^N$  and for all  $z_0 \in \mathbb{R}^{N+1}$ .

**Proof.** The proof of (4.13) follows the lines of the proof of (3.10) in [12], using (4.12) in place of (3.8) in [12].

As far as (4.14) it is concerned, we begin noticing that, see [12, p. 42],

$$C_0^{-1}(z_0;t) = D\left(\frac{1}{\sqrt{t}}\right)C_0^{-1}(z_0;1)D\left(\frac{1}{\sqrt{t}}\right), \quad \forall t > 0.$$

Thus we have

$$\begin{aligned} \langle C_0^{-1}(z_0;t)x,x \rangle &= \langle C_0^{-1}(z_0;1)D\left(\frac{1}{\sqrt{t}}\right)x, D\left(\frac{1}{\sqrt{t}}\right)x \rangle \\ &\leq \max_{|y|=1} \langle C_0^{-1}(z_0;1)y,y \rangle \left| D\left(\frac{1}{\sqrt{t}}\right)x \right|^2 = \frac{\left| D\left(\frac{1}{\sqrt{t}}\right)x \right|^2}{\min_{|y|=1} \langle C_0(z_0;1)y,y \rangle}. \end{aligned}$$

By (4.6)

$$\min_{|y|=1} \langle C_0(z_0;1)y,y \rangle \ge \frac{1}{\Lambda} \min_{|y|=1} \langle \tilde{C}(1)y,y \rangle = \frac{\lambda_{\tilde{C}}}{\Lambda}$$

and the last inequality in (4.14) follows. Analogously the first one can be proved.

Collecting the results in Lemma 4.4 and Lemma 4.5 we easily get the following:

**Proposition 4.6** Let  $\widetilde{C}$  be defined as in (4.5). There exist positive constants T and m, depending only on the operator L, such that the following inequalities hold for every  $t \in [-2T, 2T]$ , every  $z_0 \in \mathbb{R}^{N+1}$  and every  $x \in \mathbb{R}^N$ :

(a) 
$$\frac{1}{m} \langle \widetilde{C}(t)x, x \rangle \leq \langle C(z_0; t)x, x \rangle \leq m \langle \widetilde{C}(t)x, x \rangle;$$
  
(b)  $\frac{1}{m} \det \widetilde{C}(t) \leq \det C(z_0; t) \leq m \det \widetilde{C}(t);$   
(c)  $\frac{1}{m} \left| D\left(\frac{1}{\sqrt{t}}\right)x \right|^2 \leq \langle C^{-1}(z_0; t)x, x \rangle \leq m \left| D\left(\frac{1}{\sqrt{t}}\right)x \right|^2.$ 

The above estimates, together with the procedure in [9, proof of Proposition 2.7], imply the uniform bounds in Theorem 4.1 for  $\gamma(z_0; z)$ . To prove analogous estimates for  $\gamma(z_0; z^{-1})$  and its derivatives, one can proceed in a similar way.

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