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LETTER TO THE EDITOR

Relativistic response of a Fermi gas

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Abstract. The relativistic response of an assembly of non-interacting nucleons is calculated and compared with the non-relativistic limit.

In this Letter we compare the fully relativistic response of an assembly of non-interacting nucleons to an electromagnetic probe with the non-relativistic one. The finite size of the nucleons is also taken into account.

The motivation for this work is to point out that in most cases relativistic kinematics reduce the cross section for deep inelastic electron scattering (quasi-elastic peak). Our result contrasts some statements found in the literature (Moniz *et al* 1971, Brieva and Dellafiore 1977)†.

In the Born approximation the double differential cross section (with respect to the scattering angle θ and the final energy ε' of an electron scattered out of a nucleus) reads (Moniz 1969, Donnelly and Walecka 1975)

$$\frac{d^2\sigma}{d\Omega\, d\varepsilon'} = \sigma_M\{S_L(q,\omega) + [\tfrac{1}{2}(q_\lambda^2/q^2) + \tan^2 \tfrac{1}{2}\theta] S_T(q,\omega)\} \tag{1}$$

where σ_M is the Mott cross section, and $q_\lambda = \boldsymbol{q}, \omega/c$ is the four-momentum transfer. Keeping the full relativistic nucleon electromagnetic vertex one gets for the longitudinal and transverse structure functions (Moniz *et al* 1971)

$$S_L(q,\omega) = \frac{V}{(2\pi)^3}\int d\boldsymbol{k}\,\delta[\hbar\omega - (E(\boldsymbol{q}+\boldsymbol{k}) - E(\boldsymbol{k}))]\,\frac{\theta(k_F - k)\theta(|\boldsymbol{q}+\boldsymbol{k}| - k_F)}{E(\boldsymbol{k})E(\boldsymbol{q}+\boldsymbol{k})}$$
$$\times\left(\frac{T_2(q_\lambda^2)}{M^2c^4}[E(\boldsymbol{k}) - \hbar\omega(k/q)\cos\tau]^2 - (q_\lambda^2/q^2)T_1(q_\lambda^2)\right) \tag{2a}$$

$$S_T(q,\omega) = \frac{2V}{(2\pi)^3}\int d\boldsymbol{k}\,\delta[\hbar\omega - (E(\boldsymbol{q}+\boldsymbol{k}) - E(\boldsymbol{k}))]\,\frac{\theta(k_F - k)\theta(|\boldsymbol{q}+\boldsymbol{k}| - k_F)}{E(\boldsymbol{k})E(\boldsymbol{q}+\boldsymbol{k})}$$
$$\times\left(T_1(q_\lambda^2) + \frac{T_2(q_\lambda^2)}{2M^2c^4}(\hbar ck)^2\sin^2\tau\right) \tag{2b}$$

† Unfortunately there is a wrong sign in the coefficient of $T_2(q_\lambda^2)$ in formula (2*a*) of Brieva and Dellafiore (1977): this may be the reason for the incorrect conclusion about relativistic effects contained in that paper.

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where V is the volume enclosing the system, M the nucleon mass and τ the angle between $\boldsymbol{q}$ and $\boldsymbol{k}$. In (2a) and (2b)

$$T_1(q_\lambda^2)=\tfrac{1}{2}(\hbar c q_\lambda)^2 \sum_i (F_1^i(q_\lambda^2)+2Mc^2F_2^i(q_\lambda^2))^2$$

$$T_2(q_\lambda^2)=2M^2c^4 \sum_i [F_1^{i2}(q_\lambda^2)+(\hbar c q_\lambda)^2 F_2^{i2}(q_\lambda^2)]$$

where the sum goes over the proton and the neutron and

$$F_1^{\mathrm{p}}(0)=1 \qquad F_1^{\mathrm{n}}(0)=0$$

$$2Mc^2F_2^{\mathrm{p}}(0)=1.79 \qquad 2Mc^2F_2^{\mathrm{n}}(0)=-1.91.$$

The q_λ dependence of the form factors has been chosen as (Weber 1967)

$$F(q_\lambda^2)=\frac{1}{(1+q_\lambda^2/\Lambda^2)^2} \tag{3}$$

with $\Lambda^2=18.1\ \mathrm{fm}^{-2}$.

For an homogeneous, infinite system of non-interacting relativistic (both above and below the Fermi surface) nucleons one has, for the longitudinal cross section,

$$\begin{aligned}\left(\frac{1}{\sigma_{\mathrm{M}}}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega\,\mathrm{d}\varepsilon'}\right)_{\mathrm{L}} &= \frac{Vk_{\mathrm{F}}^2}{4\pi^2\hbar c q}\left(\frac{T_2(q_\lambda^2)}{2M^2c^4}\left(1-\frac{\alpha^2\nu^2}{q^2}\right)^2\left\{\frac{2}{3}\left[\left(\frac{1}{\alpha^2}+1\right)^{3/2}\right.\right.\right.\\ &\left.\left.-\left(\frac{1}{\alpha^2}+q_0^2\right)^{3/2}\right]+\alpha\nu(1-q_0^2)+\tfrac{1}{2}\alpha^2\nu^2\left[\left(\frac{1}{\alpha^2}+1\right)^{1/2}-\left(\frac{1}{\alpha^2}+q_0^2\right)^{1/2}\right]\right\}\\ &\left.-\left(1-\frac{\alpha^2\nu^2}{q^2}\right)\frac{T_1(q_\lambda^2)}{(\hbar c k_{\mathrm{F}})^2}\left[\left(\frac{1}{\alpha^2}+1\right)^{1/2}-\left(\frac{1}{\alpha^2}+q_0^2\right)^{1/2}\right]\right)\end{aligned} \tag{4}$$

and, for the transverse cross section,

$$\begin{aligned}\left(\frac{1}{\sigma_{\mathrm{M}}}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega\,\mathrm{d}\varepsilon'}\right)_{\mathrm{T}} &= \frac{Vk_{\mathrm{F}}^2}{2\pi^2\hbar c q}\left[\frac{1}{2}\left(1-\frac{\alpha^2\nu^2}{q^2}\right)+\tan^2\tfrac{1}{2}\theta\right]\\ &\times\left(\frac{T_1(q_\lambda^2)}{(\hbar c k_{\mathrm{F}})^2}\left[\left(\frac{1}{\alpha^2}+1\right)^{1/2}-\left(\frac{1}{\alpha^2}+q_0^2\right)^{1/2}\right]+\frac{T_2(q_\lambda^2)}{2M^2c^4}\right.\\ &\times\left\{\left(\frac{1}{\alpha^2}+1\right)^{1/2}-q_0^2\left(\frac{1}{\alpha^2}+q_0^2\right)^{1/2}-\frac{2}{3}\left(1+\frac{\alpha^2\nu^2}{2q^2}\right)\left[\left(\frac{1}{\alpha^2}+1\right)^{3/2}\right.\right.\\ &\left.-\left(\frac{1}{\alpha^2}+q_0^2\right)^{3/2}\right]+\frac{\alpha\nu}{2}\left(1-\frac{\alpha^2\nu^2}{q^2}\right)(1-q_0^2)-\frac{q^2}{4}\left(1-\frac{\alpha^2\nu^2}{q^2}\right)^2\\ &\left.\left.\times\left[\left(\frac{1}{\alpha^2}+1\right)^{1/2}-\left(\frac{1}{\alpha^2}+q_0^2\right)^{1/2}\right]\right\}\right)\end{aligned} \tag{5}$$

where q, ν are the dimensionless momentum and energy transfer (q in units of the Fermi momentum k_F and $\nu = M\omega/\hbar k_F^2$), $\alpha = \hbar k_F/Mc$ and

$$q_0 = \frac{\nu}{q}\left(\frac{1}{1-\alpha^2\nu^2/q^2} + \frac{\alpha^2 q^2}{4}\right)^{1/2} - \frac{q}{2}. \tag{6}$$

Formulae (4) and (5) hold in the regions of the (ν, q) plane defined by $q \geqslant 2$ and

$$\{[1+\alpha^2(1+q)^2]^{1/2} - (1+\alpha^2)^{1/2}\}/\alpha^2 \geqslant \nu \geqslant \{[1+\alpha^2(q-1)^2]^{1/2} - (1+\alpha^2)^{1/2}\}/\alpha^2 \tag{7'}$$

where relativistic effects are expected to be sizable, or $q < 2$ and

$$\{[1+\alpha^2(1+q)^2]^{1/2} - (1+\alpha^2)^{1/2}\}/\alpha^2 \geqslant \nu \geqslant \{(1+\alpha^2)^{1/2} - [1+\alpha^2(1-q)^2]^{1/2}\}/\alpha^2. \tag{7''}$$

To appreciate the role of relativity in the nuclear response we expand (4) and (5) in powers of α^2. Keeping only the first two terms of the expansion one gets

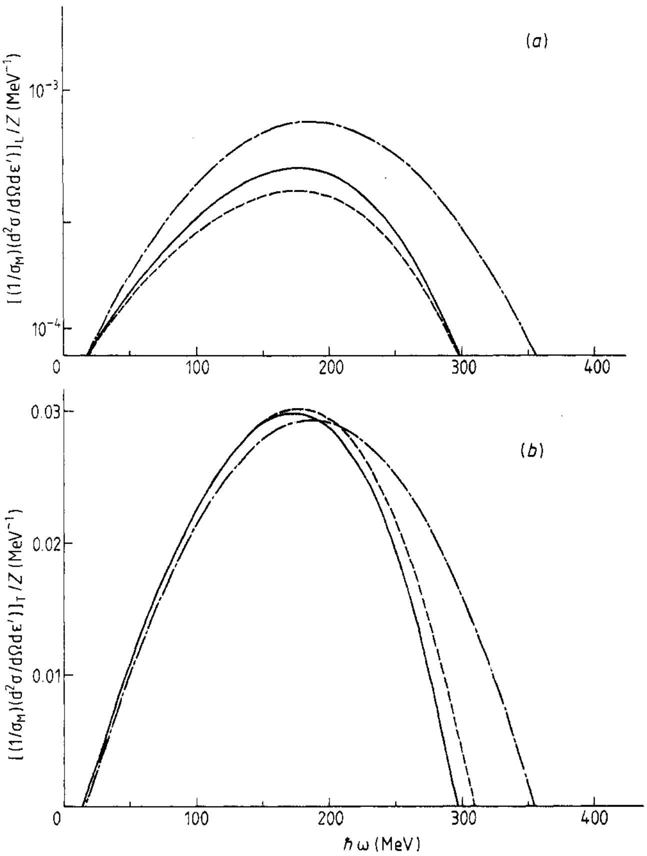

Figure 1. The longitudinal (a) and transverse at $\theta = 150°$ (b) double differential cross sections, in the Born approximation, as a function of the transferred energy at $q = 3\ \mathrm{fm}^{-1}$. The full curves correspond to the relativistic calculation, the chain curves to the non-relativistic one and the broken curves to formula (8). The finite size of nucleons is taken into account; k_F is $1.36\ \mathrm{fm}^{-1}$.

$$\left(\frac{1}{\sigma_{\mathrm{M}}}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega\,\mathrm{d}\varepsilon'}\right)_{\mathrm{L}}$$

$$\simeq\left\{-\frac{V}{\pi}\operatorname{Im}\Pi^0(q,\nu)\left[1+\tfrac{1}{2}\alpha^2\left(\nu+\tfrac{1}{2}(1+k_0^2)-2(\nu^2/q^2)\right.\right.\right.$$

$$\left.\left.\left.-2.29q^2+8\frac{k_{\mathrm{F}}^2\nu^2}{\Lambda^2+k_{\mathrm{F}}^2q^2}\right)\right]-\frac{VMk_{\mathrm{F}}}{(2\pi\hbar)^2}\alpha^2(1+\nu)\frac{\nu k_0}{q^2}\right\}\left(1+\frac{k_{\mathrm{F}}^2q^2}{\Lambda^2}\right)^{-4} \qquad (8)$$

$$\left(\frac{1}{\sigma_{\mathrm{M}}}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega\,\mathrm{d}\varepsilon'}\right)_{\mathrm{T}}$$

$$\simeq\left\{-\frac{V}{\pi}\operatorname{Im}\Pi^0(q,\nu)\alpha^2(\tfrac{1}{2}+\tan^2\tfrac{1}{2}\theta)\left[[11.43\times\tfrac{1}{2}q^2+\tfrac{1}{2}(1-k_0^2)]\right.\right.$$

$$\times\left(1+\frac{4\alpha^2k_{\mathrm{F}}^2\nu^2}{\Lambda^2+k_{\mathrm{F}}^2q^2}\right)-\tfrac{1}{2}\alpha^2\{10.43[\tfrac{1}{4}q^2(1+k_0^2)+\nu^2]-6.85\times\tfrac{1}{4}q^2(1-k_0^2)$$

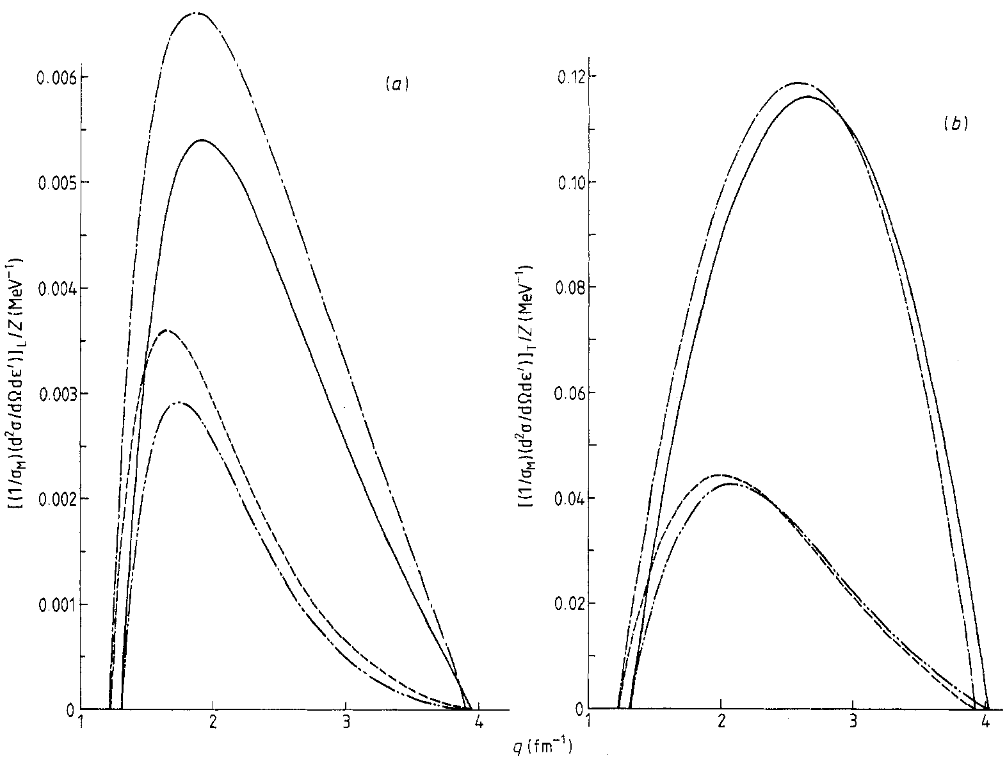

Figure 2. The longitudinal (a) and transverse at $\theta = 150°$ (b) double differential cross sections, in the Born approximation, as a function of the transferred momentum q at fixed energy $\hbar\omega = 100$ MeV. The full curves correspond to the relativistic calculation and the chain curves to the non-relativistic one for point-like nucleons; the double chain and broken curves are, respectively, the relativistic and non-relativistic response from extended nucleons; k_{F} is $1.36\ \mathrm{fm}^{-1}$.

$$
\begin{aligned}
&+2\frac{\nu^3}{q^2}+\frac{\nu^2}{2q^2}\left(1-\frac{\nu^2}{q^2}\right)+\tfrac{1}{8}\nu(q^2-\nu)+\tfrac{1}{3}(1+k_0^2+k_0^4)\}\Bigg]\\
&-\alpha^4\frac{VMk_{\mathrm{F}}}{(2\pi\hbar)^2}\frac{\nu}{2q^2}(11.43q^2+1-k_0^2)\left[\frac{\nu}{2q}(1-k_0^2)+(\tfrac{1}{2}+\tan^2\tfrac{1}{2}\theta)\right.\\
&\left.\times\left(\frac{\nu^2}{q^2}+\frac{q^2}{4}\right)k_0\right]\Bigg\}\left(1+\frac{k_{\mathrm{F}}^2q^2}{\Lambda^2}\right)^{-4} \qquad (9)
\end{aligned}
$$

where $k_0=\nu/q-\frac{1}{2}q$ and $\Pi^0(q,\nu)$ is the time-ordered density–density correlation function of a free Fermi gas:

$$
\operatorname{Im}\Pi^0(q,\nu)=-\frac{Mk_{\mathrm{F}}}{4\pi\hbar^2}\frac{1}{q}\left[1-\left(\frac{\nu}{q}-\frac{q}{2}\right)^2\right] \qquad (q\geqslant 2). \qquad (10)
$$

Thus, to leading order, one recovers the familiar result of linear response theory. Numerically (8) and (9) are very close to (4) and (5) as the expansion parameter α^2 is small.

In figure 1(*a*) the longitudinal cross section is plotted as a function of the energy for $q=3\ \mathrm{fm}^{-1}$. The shrinking effect associated with relativistic kinematics is clearly exhibited, particularly at large electron energy loss.

The transverse cross section is shown in figure 1(*b*), where the same effect is seen to occur for large ω. Note that at $\theta=150°$ the transverse cross section is about 20 times as large as the longitudinal one.

For smaller values of ω the reduction of the cross section due to relativistic kinematics takes place, essentially, only in the longitudinal case and for momentum transfers which are not too large.

The corresponding situation is illustrated in figures 2(*a*) and 2(*b*), where the cross sections are displayed as a function of q for $\hbar\omega=100$ MeV.

Perhaps it is worthwhile to recall that a much stronger reduction in the nuclear response is linked with the finite size of the nucleon which, furthermore, substantially reshapes the cross section: indeed for large momenta the exchanged virtual photon has a wavelength $\lambda\!\!\!^{-}$ significantly shorter than the root-mean-square proton radius (~0.8 fm). This result is, of course, known, but we felt it worthwhile to present it again in conjunction with the relativistic effects.

We believe that these features will qualitatively persist when the interaction among the nucleons is switched on.

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