"Dig where you stand" 2

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Editors:
Kristín Bjarnadóttir
Fulvia Furinghetti
José Manuel Matos
Gert Schubring

uied
Unidade de Investigação Educação e Desenvolvimento
FCT/UNL, Portugal

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Faculdade de Ciência e Tecnologia da Universidade Nova de Lisboa
Campus da Caparica
2829-516 Caparica, Portugal
Tel: +351 212948383
e-mail: uied.secretariado@fct.unl.pt

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Tel: +351 262098008
Fax: +351 262098582

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The emergence of the idea of the mathematics laboratory at the turn of the twentieth century

Livia Giacardi
Dipartimento di Matematica dell'Università di Torino. Italy

Abstract

The idea of offering students spaces where they could carry out activities spontaneously and constructively, develop their own individuality, and socialise, frequently appears in the studies of psychologists and educators at the turn of the twentieth century. Examples of this are found in the works of the American John Dewey, whose vision on education is related to the pragmatism of Charles S. Peirce and William James; the German Georg Kerschensteiner, an advocate of the Arbeitsschule, or 'work-school'; the Belgian Ovide Decroly; the Swiss Edouard Claparède and Adolphe Ferrière; the French Alfred Binet, one of the principal promoters of the 'active school'; and the Italian physician and educator Maria Montessori, among others.

All these scholars were especially interested in the formation of children during the first years of their lives, and mathematics is not always mentioned in their reflections, but the idea of a school-laboratory spread also among mathematicians, who extended it to secondary schools.

In my paper, after briefly mentioning the points of view of some of the educators who were active at the turn of the twentieth century and either had an interest in mathematics or were in contact with scientific circles (Dewey, Kerschensteiner, Wells), I will discuss the contributions of the mathematicians John Perry, Eliakim Hastings Moore, Émile Borel and, Felix Klein, and then focus on Giovanni Vailati's 'school as laboratory'. By comparing the various models of mathematics laboratories proposed, I will try to make clear the most significant differences between them, and their innovative aspects.

The school-laboratory according to well-known educators interested in mathematics

John Dewey (1859-1952) can rightly be considered the father of the active school and a source of inspiration for a large number of educators of the first half of the 1900s. Believing that the school of his day was anachronistic, passive, anti-psychological and antisocial, he proposed instead an active school that was centred, not on teachers or on books, but on the activity of the students, organised in a social kind of work. Knowledge, therefore, was not to be provided ready-made, but rather presented in the form of problems, and was to spring from the personal research of the student. Because the traditional classroom was inadequate for this kind of teaching, he believed that it was necessary to transfer the educational process into laboratories, libraries, playgrounds, workshops and kitchens, where work itself would transform school into an in-embryo community. In 1896 Dewey founded an 'experimental school' in Chicago based on these educational ideals and attempted to interact with

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mathematicians as well, in particular with Eliakim Hastings Moore and George B. Halsted. In his article 'The psychological and the logical in teaching Geometry' (1903), he says that in the practise of teaching psychological aspects must be taken into account as well, and that it is thus necessary to begin with concrete reality and ordinary experience, and present the practical applications of mathematics in such a way as to arrive gradually at logical rigour.

Among the European educators influenced by Dewey was Georg Kerschensteiner (1854-1932). A teacher of mathematics and physics in gymnasia for many years, and later a school inspector in Munich, he was a social educator and promoter of the *Arbeitsschule*, or 'work-school' (Simons, 1966). He believed that in order to reform schools it was not so much necessary to broaden programs or increase the number of hours as it was to transform schools into laboratories for practical exercises, where the student could learn to use knowledge and acquire a sense of social duty. The importance he attributed to manual work and practical activity goes beyond the acquisition of abilities and skills; rather, it is connected to the capacity for carrying out an activity responsibly and autonomously: manual labour disjoined from intellectual effort is merely mechanical, and thus from the point of view of education its essential characteristics are autonomous planning and realisation combined with the possibility for self-analysis. According to Kerschensteiner, the main aim of education should be civic education (*staatsbürgerliche Erziehung*). Having completed his mathematical studies at university, he was particularly aware of the problems related to teaching of the sciences, and in 1914 dedicated the short book *Wesen und Wert des naturwissenschaftlichen Unterrichts* to these problems. Here he maintained that the study of sciences was valuable for its ability both to train the mind to follow a logical and precise process of thought, and to increase the students' power of observation, where observation meant the combination of perception with thought. For this reason he campaigned vigorously for the introduction of laboratories and practical works in science teaching (see also Wolff, 1937, p. 97 and Simons, 1966, pp. 79-81).

Herbert George Wells (1866-1946), although mainly known as an author of science fiction, had scientific training in zoology and biology and also wrote many articles of a pedagogical and social nature. In his book *Mankind in the Making* (1904), Wells criticises English schools, in particular the programs, which were redundant or lacking in what truly made it possible for students to understand the society they live in, and the textbooks, inadequate for an active teaching (p. 226). According to Wells, schools should ideally be connected to public libraries (p. 213) and the actual lessons should be alternated with sessions dedicated to individual activities such as reading, painting, and play, intended as a

'spontaneous activity that involves the imagination' (p. 235). In his book he also cites laboratory-style teaching of mathematics proposed at the time by the mathematician John Perry, but he is frank about the difficulties of putting it into practise: because this kind of teaching requires such great commitment and planning on the part of the teacher, as long as there were no adequate textbooks, it remained practically impossible (pp. 224-225). Wells's book was reviewed by the Italian mathematician Giovanni Vailati (1906) and probably contributed to directing him towards his conception of a laboratory-type teaching of mathematics.

The laboratory for mathematics teaching in the international context

The idea of a laboratory for mathematics was introduced by John Perry (1850-1920), a professor of mechanics and mathematics at the Royal College of Science in London from 1896. In fact he maintained that mathematics had to be taught 'as any other physical science is taught, … with experiment and common-sense reasoning' and proposed a new teaching method that he called 'Practical Mathematics': 'The most essential idea in the method of study called Practical Mathematics is that the student should become familiar with things before he is asked to reason about them' (Perry, 1913, p. 21). Before concentrating on theorems and proofs, the student should become acquainted with the concepts by means of experiments and measurements using squared paper, data gathering, drawing, graphic methods, and relationships with physics and other sciences.

In working out his method, Perry was inspired by the methods used in the kindergartens, which, under the influence of Pestalozzi and Froebel, were based on activity and on 'hand and eye training' (Price, 1986, p. 109-114) and he was stimulated by the discernment of the failure of traditional teaching with respect to the average student:

> Academic methods of teaching Mathematics succeed with about five per cent of all students, the small minority who are fond of abstract reasoning: they fail altogether with the average student (Perry 1913, p. VII).
>
> So we now teach all boys what is called mathematical philosophy, that we may catch in our net the one demigod, the one pure mathematician, and we do our best to ruin all the others (Perry 1902, p. 6).

According to Perry, the cause of this failure was the English system of separate examinations that induced teachers to teach the various subjects in 'water-tight compartments', as well as the tendency to place too much importance on the abstract aspects of mathematics and on the 'laboursaving rules', neglecting the fundamental principles and concepts (Perry, 1913, p. X). He believed that the aim of teaching was not that of 'producing finished products either at school or the university' but rather that teachers 'ought to try to produce learners' (Perry, 1909, p. 11). Perry himself first began to use this new type of approach in an English public school, later in Japan (1875-1879) and then he developed a syllabus of practical mathematics for engineers at the Finsbury Technical College, where he was appointed professor of Mechanical Engineering in 1882. In 1899 he was able to convince the Board of Education to adopt it for science classes and in 1901 at Glasgow he communicated the results at a meeting of the British Association for the Advancement of Science, giving rise to a lively debate (Brock & Price, 1980; Price 1986). He gathered the various talks in a small volume published the following year, *Discussion on the teaching of mathematics*, in which he first outlined what he considered to be the purposes and usefulness of mathematics:

> In producing the higher emotions and giving mental pleasure. ... In producing logical ways of thinking. ... In the aid given by mathematical weapons in the study of physical science. ... In passing examinations. The only form that has not been neglected. The only form really recognised by teachers. In giving men mental tools as easy to use as their legs or arms ... In making men in any profession of applied science feel that they know the principles on which it is founded and according to which it is being developed ... (Perry, 1902, pp. 4-5).

Then he reiterated his criticisms of the English methods of teaching, and illustrated the programs of his courses both in elementary and advanced mathematics (pp. 25-32).[1]

In arithmetic emphasis was placed on decimals rather than fractions and on approximations; in algebra, on the comprehension and manipulation of the formulas as well as on the variations in the value of certain expressions with the varying of the values of the variables that appear in them; in geometry the Euclidean method was completely abandoned, replaced by a treatment based on measurement and drawing with the freedom to use arithmetic and algebraic methods. In the advanced course elements of trigonometry, three-dimensional geometry, calculus as well as vector methods were introduced.

Among the numerous comments, I limit myself to citing the one by David E. Smith who, while basically in agreement with Perry's point of view, indicated the problems that must be faced in order to put the

[1] See also (Howson 1982, pp. 148-149). The appendix on pages 222-224 gives Perry's 1900 proposals for a mathematics syllabus.

proposed reform into practice: new textbooks, teacher training, and the modifications of examinations (Perry, 1902, p. 90-91).[2]

In 1913 Perry published his best known book, *Elementary Practical Mathematics*, which was intended as a guide for teachers, with many carefully chosen exercises to pose to students. Perry begins with topics in arithmetic before going on to topics and problems in algebra, geometry, physics and calculus. In fact, according to him, the method of Practical Mathematics can be used at all levels of teaching as long as the presentation of subjects remains tied to real phenomena and concrete problems. The treatment of the subjects reflects a laboratory-like approach: the starting point is generally a practical problem; numerical data is gathered and interpreted; squared paper is used to tabulate the observations, solve equations graphically, represent functions, find the slope of the tangent to a curve; instructions are given for the construction of a slide rule, and its use, and so forth. Above all care is taken to provide a unified vision of mathematics, linking algebra, geometry and trigonometry, and to show how useful mathematical instruments are in addressing problems of physics and engineering. In particular, with regard to geometry, Perry criticises the Euclidean method and suggests that: practical experimentation and measuring with squared paper be carried out before rational geometry; that the experimental geometry be flanked by some deductive reasoning; that greater emphasis be given to solid geometry; that trigonometric functions be used in the study of geometry; and that more attention be paid to applications.[3]

Here and there Perry also provides remarks on methodology and advice for teachers,[4] noting the difficulties and most frequent errors on the part of the students, and the reasons for them.

Many of the problems addressed by Perry are similar to those proposed today in teaching experiments involving the use of graphic-symbolic calculators, with a strong use of numerical data, but the final object is different. The method that he proposed is based on problem solving, and on a transversal approach to mathematics highly concentrated on procedures. His text presents a mathematics to be 'practised' and not to be formulated in a theory. This constitutes the originality of the method, but is at the same time also its limit.

It is noteworthy that pioneers in mathematical education in England such as Charles Godfrey, Benchara Branford, Percy Nunn and William D.

[2] See also (Smith, 1913), where he underlines how the American school is aimed at the masses (p. 3).

[3] See, for example, (Perry 1902, p. 102).

[4] See, for example, (Perry 1913, pp. 21, 25, 32, 51-52).

Eggar stressed a heuristic and experimental approach to mathematics and the importance of a close correlation with other sciences. For example Eggar in his book *Practical exercise in Geometry* writes:

> This book is an attempt to adapt the experimental method to the teaching of Geometry in schools. The main object of this method, sometime called "heuristic", is to make the student think for himself, to give him something to do with his hands for which the brain must be called in as a fellow-worker. The plan has been tried with success in the laboratory, and it seems to be equally well-suited to the Mathematical class-room (Eggar, 1903, p. V).

In the volume *The teaching of algebra*, Nunn – a mathematician with strong interests in philosophy and other sciences, and professor of education at London University from 1913 – underlines the twofold purpose in mathematics teaching: to enable pupil to understand the importance of mathematics as 'an instrument of material conquests and of social organization', and to accustom him to appreciate 'the value and significance of an ordered system of mathematical ideas' (Nunn, 1914, p. 17). Concerning the teaching of algebra he maintains that this subject should be introduced 'as a symbolic language specially adapted for making concise statements of a numerical kind about matters with which he is already more o less familiar' (p. 18). Moreover pupils should be made to perceive from the very beginning that formulae refer to realities beyond themselves. As Perry did, also Nunn invited teachers to attract pupils' attention to the connexion between variables so that they can be gradually arrive to the study of functions, and to highlight the links among the various sectors of mathematics and between mathematics and other sciences. The work of Perry was one of his point of reference (Nunn, 1914, p. VII, 24, 311, 556).

Independent of his actual influence on technical education in England, Perry's movement favoured the dissemination of the idea of a laboratory-style teaching of mathematics for students of all types, and more generally the statement of some fundamental principles: greater democracy in education, greater consideration for what is useful in real life, greater attention to pedagogical aspects. The influence of his ideas can be perceived above all in the teaching of geometry: more space was given to experimental work, laboratories were set up in many schools, and there appeared many textbooks oriented in this direction (Price, 1986, p. 124-130). In 1908 the London County inspector Benchara Branford underlined the increasing attention paid to the experimental and graphical side of mathematics and the emergence of 'mathematical laboratories, well stocked with clay, cardboard, wire, wooden, metal, and other models and material, and apparatus for the investigation of form, mensuration and

movement' (Branford, 1908, p. VIII). In his speech at the International Congress of Mathematicians held in Cambridge in 1912, the mathematics educator Charles Godfrey reported the results concerning a questionnaire on the use of intuitive and experimental methods in English schools: in particular graphical representation of functions, graphical study of statistic, graphical statics, estimation of area by means of squared paper were adopted by public schools (pupils from 12 to 18 years old) in a percentage which varied from 90 per cent to 98 per cent (Godfrey, 1913).

Perry's movement aroused interest not only in Europe, but in America as well. In a 1902 lecture, Eliakim Hastings Moore, then president of the American Mathematical Society, gave his celebrated talk *On the foundations of mathematics* (Moore, 1903) in which he invited teachers' associations to concern themselves with secondary education, underlined the defects of a compartmentalised teaching too focussed on aspects that were theoretical and abstract, and expressed his hopes for an integrated teaching of pure and applied mathematics, citing Perry's experimental teaching as an example.

An entire section of his talk was dedicated to the 'laboratory method' in mathematics, which he compared to the physics laboratory, highlighting its advantages for teaching. According to Moore, this method is the only one capable of making young people understand that 'mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions' (Moore, 1903, pp. 417-420). He believed that laboratory teaching, which must be characterised by a practical approach – that is, one that is 'computational, or graphical or experimental' (p. 419) – has the following advantages: it allows the student to understand the importance of a theorem and creates in him the desire for a formal proof (p. 419); it stimulates his personal research; it permits individual work as well as work in groups, where the teacher is at once a member of the group and the leader. One of Moore's recommendations for making the method work is to present only interesting experiments: for example, in the laboratory for physics, the mere explanation of how the instruments are used is not interesting; it is better to pose problems whose solutions involve the use of those instruments, so that the student learns 'the use of the instruments as a matter of course, and not as a matter of difficulty' (p. 418).

In his conclusion he states that in his opinion the laboratory method for secondary teaching of mathematics and physics 'is the best method of instruction for students in general, and for students expecting to specialize in pure mathematics, in pure physics, in mathematical physics or astronomy, or in any branch of engineering' (p. 420).

Moore's program was taken up by several mathematicians who specialised in pedagogy and didactics, including Jacob William Albert

Young, a professor of mathematical pedagogy at the University of Chicago. He dedicated an entire chapter of his book *The teaching of mathematics in the elementary and the secondary school* (1906) to Perry's movement, devoting ample space to experimental method and to the mathematics laboratory. Young also went into detail about the equipment that a good mathematics laboratory should have: geometrical models, surveying instruments, scales, pendulums, levels, barometers and thermometers. Besides he believed that there should be a well-stocked library with a good collection of textbooks, workbooks and various kinds of tables, as well as books on the history of mathematics, recreational mathematics and journals about teaching. He thus underlined the importance of the laboratory as a physical place where students could work under the guidance of the teacher or assistant.

In 1902 in France, the reform of secondary teaching known as *humanités scientifiques*, introduced infinitesimal analysis in secondary schools, and also emphasized the importance of a concrete teaching method that takes account of relationships to the real world (Belhoste, Gispert, & Hulin 1996; Gispert 2009). Presiding over the commission to oversee the revision of the programs was the mathematician Gaston Darboux, but many other illustrious scholars also contributed. In particular, Émile Borel encouraged teachers to introduce 'more of life and a sense of reality into our mathematics teaching' and suggested creating an *atelier mathématique*, a 'mathematical workshop', where students could personally build models, take measurements, and so forth with the aim of 'bringing not only students but also teachers, but above all the mind of the public, to a more exact idea of what mathematics is and the role it actually plays in modern life' (Borel, 1904 (1967), p. 14). Borel's view is made clear in his 1905 handbook for geometry (Borel, 1905), in which the practical and intuitive aspects are amply emphasised. His aim was to 'write a more concrete geometry, where considerations of symmetry, of displacement are invoked as often as possible' and 'substitute more and more the dynamic study of the phenomena in place of their static study' (Borel, 1905, p. V, VII). The book opens with an introduction to the use of straightedge and compass, in which applications are skilfully coordinated with theory; among the exercises proposed there are some of a practical nature that involve symmetries, the use of instruments, etc. There is no rigid division between plane and solid geometry; the topics introduced include, for example, tiling the plane (pp. 111-113), approximations (pp. 280-281) and, in the complements notions regarding the conic sections and other curves, the approximate calculation of areas and land surveying (pp. 353-375). However, the idea of the mathematics laboratory he proposed was rather limited. He writes:

> … in my opinion, the ideal mathematics laboratory would be, for example, a carpentry workshop; the laboratory assistant would be a carpenter who in small institutions would come only a few hours per week, while in the large schools he would almost always be present. Under the guidance of the professor of mathematics and following his instructions, the students, aided and advised by the laboratory assistant, would work in small groups to build models or simple devises (Borel, 1904 (1967), pp. 15-16).

It was this kind of mathematics laboratory, the *Laboratoire d'enseignement mathématique*, that Borel created together with Jules Tannery at the École Normale Supérieure, aimed at training future teachers for a laboratory-type teaching. Here models in either wood or cardboard, wire and cork were conceived and built for teaching geometry and mechanics. The didactic uses of other instruments such as mechanical linkages, pantographs, inversors, calculating machines, and instruments for geodesy and land surveying were also taught. Furthermore, the laboratory was to be equipped with a library where future teachers could find the principal French publications on mathematics teaching, the most important pedagogical journals, and the scholastic handbooks of various countries (Châtelet, 1909).

In Germany, beginning in the 1890s, Felix Klein had begun to formulate his famous program for the reform of mathematics teaching which redefined the relationship between secondary schools and universities. It was first given formal expression, although with some compromises, in the *Meraner Lehrplan* of 1905 (Bericht, 1905; Klein, 1907, pp. 208-219), developed by the *Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte*. It addressed not only mathematics teaching but also that of physics and the other natural sciences. Klein's principal innovation was the introduction of 'functional thinking' (*funktionales Denken*) in secondary teaching, but other aspects, such as the importance of applications, the use of geometric models, the relationship to real problems, the connections with physics teaching, and the value of experiments were also underlined (pp. 550-553). For physics teaching, the *Meraner Lehrplan* highlighted the necessity of outfitting suitable work spaces (*Arbeitsräume*) where adequately trained teachers and their assistants could work and experiment alongside the students.

With regard to the teaching of geometry, and mathematics in general, the following aspects were emphasised: the strengthening of spatial intuition (p. 543); the use of straightedge and compass, drawing, measuring (p. 547); the consideration of geometrical configurations as dynamic objects (p. 548); the strengthening of the use of graphic representations; providing room for applications (p. 549); making use of models; the coordination of planimetry and stereometry; making mention of the historical and philosophical point of view (p. 550). The importance of using instruments and models in mathematics teaching also emerges from the books on elementary mathematics from an advanced standpoint, which Klein intended expressly for the training of mathematics teachers. The volume on geometry (Klein, 1925-1933 II; Klein, 2004) includes an introduction to various instruments such as Peaucellier's inversor, an instrument for creating affine transformations, and others. Klein observed:

> Instead of mentioning further details, I should like here to sound a general warning against neglecting the *actual practical demonstration* when such instruments are considered in illustration of a theory. The pure mathematician is often too prone to do so. Such neglect is just as unjustifiably one-sided as is the opposite extreme of the mechanician who, without taking an interest in the theory, loses himself in details of construction. Applied mathematics should supply here a bond of union (1925-1933, II; 2004, p. 15).[5]

Klein was also involved in the reorganisation and modernisation of the *Modellkammer* in Göttingen for educational purposes, in particular to aid in *Raumanschauung* (spatial intuition) (Schubring 2010) and, together with P. Treutlein, at the congress of the International Commission on the Teaching of Mathematics (later ICMI) held in Brussels in 1910 he presented the use of models in secondary and university teaching to develop geometrical intuition (Giacardi, 2008).

An important occasion for international comparison and contrast of teaching experiences in the field of mathematics in various countries was the fourth International Congress of Mathematicians held in Rome in 1908, which led to the founding of ICMI, with Klein as the first president. The session dedicated to teaching was quite rich: it was organised by Vailati, as archival documents show. Some of the numerous reports contain explicit or implicit references to the idea of laboratory: the importance of measuring, graphical representations, and the use of instruments for practical teaching is highlighted by Smith and Vailati, and the term 'laboratory work in mathematics' is introduced by Godfrey.

A picture of the international situation is found in the report prepared by Smith (1912) on the enquiry promoted by ICMI in 1911 on 'Intuition and experiment in mathematical teaching in the secondary schools', which concerned students from 10 to 19 years of age.[6] He presented a general

[5] For more on this, see (Bartolini Bussi et al., 2010) and (Schubring, 2010).

[6] Some of the topics discussed in this report had already appeared in an unpublished 1906 letter by Smith to Gino Loria. See the website http://www.icmihistory.unito.it/19081910.php.

outline of the situation in the various countries, from which it emerged that, in the teaching of mathematics in secondary schools, recourse to intuition and to practical experiments was generally more greatly prized in Austria, Germany and Switzerland, than in Great Britain, France and the United States (Smith, 1912, p. 512), and that the most debated subjects were above all the teaching of geometry, and whether or not to introduce the concept of function. Continuing on, he focused his attention on particular aspects of an 'active' teaching of mathematics: 'Measuring and Estimating' (geodetic, astronomical measurements, triangulations, etc.), 'Geometric Drawing and Graphic Representation' (teaching of descriptive geometry), 'Graphic Methods' (representation of functions on graph paper, graphic statics, evaluation of surfaces with the aid of graph paper, etc.), 'Numerical Computation' (use of tables, of the slide rule, methods of approximate calculation, etc.) (Giacardi, 2008). The aim of this report, as Smith himself made clear, was to illustrate data concerning the various countries and the efforts made to apply mathematics to real life, and not to offer solutions to problems or recommendations for the future. Nevertheless, it suggested questions which indicated a line of research and study for people concerned with education, such as: What simple, inexpensive instruments could be used to increase the interest in the early stages of mathematics teaching? What can be done to make the inductive phase of geometry teaching more real and interesting without weakening the deductive side? Could drawing and geometric constructions be used more adequately in the teaching of geometry? What should schools do to take advantage of the decreasing prices of the calculating machine?

The laboratory for mathematics according to Vailati

In Italy it was Giovanni Vailati who proposed the idea of 'school as a laboratory'. A mathematician, philosopher and educator, Vailati was a member of the Peano School. He earned his degree in engineering and then in mathematics in Torino. After having taught for several years at the University of Torino as an assistant or in open courses, in 1899 he left Torino and began teaching in secondary schools of various Italian towns and cities. His commitment to education found various expressions, but his most important contribution concerned the design of the programs of mathematics in the context of work performed for the Royal Commission for the reform of secondary schools, which he carried about from 1905 until his death in 1909. Elsewhere I have illustrated Vailati's contributions to the work of the Royal Commission (Giacardi, 2010), so here I will concentrate on his vision of the mathematics laboratory.

In the past it has been underlined that the idea of the 'school as laboratory' proposed by Vailati was analogous to the teaching experiments undertaken by the experts in pedagogy (Arzarello, 1987, p. 35), but on the basis of the scant evidence found in the published works and correspondence (*Epistolario*) it is difficult to say with certainty at what extent they were an actual reference.[7] To be sure, Vailati was very familiar with the work of Charles Sanders Peirce and William James and knew some of their papers that were expressly dedicated to questions of education,[8] while Dewey is mentioned only in passing in his writings.[9] He was also in contact with Claparède,[10] and owned a copy of Binet's *La psychologie du raisonnement* (Paris: Alcan, 1886). In his correspondence and writings he makes no references to Decroly nor even to Montessori, whose work in education began after her participation in the Italian national congress on pedagogy, which took place in Torino in 1898.[11] Even though at that time Vailati still lived in Torino, he is not listed among those who attended the congress, although the names of other members of the Peano School – Rodolfo Bettazzi, Alessandro Padoa and Giovanni Vacca – appear.

The idea of a school-laboratory proposed by Vailati appears explicitly in his review of the book of the educator Maria Begey, *Del lavoro manuale educativo* (1901), and in his brief paper 'Idee pedagogiche di H. G. Wells' (1906), and afterwards emerges in the mathematics programs that he devised for the Royal Commission and in the related instructions on methodology. However, Vailati never presented a systematic and complete exposition of his ideas. His thoughts regarding the field of pedagogy and education are in large measure found in marginal, fragmentary observations in the vast and heterogeneous collection of his writings, and are mostly contained in the innumerable reviews.

In order to understand the originality of Vailati's thinking in education, it is important to frame it within his particular vision of the function of

7 Only a detailed, complete examination of the vast quantity of material conserved in the Vailati archive (*Fondo Vailati*, Biblioteca di Filosofia, Università di Milano), in particular in the *Notes*, which I have only explored in part, could provide a definitive answer. See (Ronchetti, 1998).

8 See for example (Vailati, 1905c), where Vailati shows that he knows W. JAMES, *Talks to teachers on psychology and to students on some of life's ideals* (1899).

9 See *Scritti*, I, p. 202, 210.

10 See *Epistolario,* p. 231 and (Busino, 1972) where Vailati's letters to Claparède are transcribed.

11 See (Molineri & Alesio, 1899). Montessori's talk on the education of 'degenerate youths' appears on pp. 122-124. See also Maria Montessori, *Il Metodo della pedagogia scientifica applicato all'educazione infantile nelle Case dei Bambini* (1909), where there appears the idea of an active school and the use of specially prepared materials; of particular interest for the teaching of mathematics are the much later works *Psico-Aritmética* (Barcelona, 1934, trad. it. 1971) and *Psico-Geometría* (Barcelona, 1934).

mathematics and its teaching, a vision in which different motives and needs converge. The relationship with Peano and his School led to his solid mastery of mathematical logic, his ideas about deductive and systematic rigour, and to his reflections on language combined with a deep interest in education and the history of mathematics, and a genuine desire to democratise knowledge. The pragmatism of Peirce also influenced Vailati, who saw pragmatism as an instrument in the struggle against senseless problems and against metaphysics; in particular, he made his own the operative and functional criteria for giving meaning to the propositions, that is, he believed that their meaning depends on the consequences that can be drawn from these propositions. Intertwined with pragmatism are positivistic requirements: the idea of a scientific *humanitas*, an appreciation of the applied knowledge, the founding teaching on a positive knowledge of man (biology, psychology), in the constant awareness that the cognitive process proceeds from facts to abstraction. Underlying all of this is the Herbartian assumption that the aim of teaching is the formation of character.

The fact that Vailati was interested in psychology leads us to think that this influenced his vision of mathematics teaching. He took part in three international congresses in psychology (Munich 1896, Paris 1900, and Rome 1905) and from his writings it emerges that he was above all interested in questions of method in psychology, and in the applications of psychology in the studies of art, literature and anthropology. In his talk given at the congress in Rome he dealt with the 'psychology of intellectual operations' (today we would say cognitive psychology), a topic already present in *The will to believe* by the pragmatist philosopher James, who rehabilitated the constructive and anticipatory activities of the human mind against those that are purely receptive and classificatory.[12] However, there was no explicit reflection on education, although, as we will see later, the influence of modern psychology can be found in his considering general concepts (including those of science) as mere instruments that make it possible to order, classify, use the raw material of experiences (Vailati, 1905b, p. 280).

In order to address problems connected to mathematics teaching, Vailati took into consideration the programs and educational organisation of other European countries, as well as the movements for school reform of Klein in Germany, Perry in England, and Darboux and Borel in France.[13] The organisation of the session dedicated to teaching of the fourth International Congress of Mathematicians (Rome, 1908) helped

12 See (Vailati, 1905d); for more on this see (Sava 2006).

13 See (Vailati, 1910) and (*Fondo Vailati*, Cartella 41, fasc. 346; Cartella 31, fasc. 272).

him to build his international contacts. Moreover, his teaching experience in various secondary schools in northern and southern Italy had allowed Vailati to see first-hand the shortcomings and defects of Italian schools. It was precisely the desire to remedy these that guided him in formulating his proposals for reform.

In his opinion, an improvement of the teaching of certain subjects, both scientific and not, could have been possible if teaching were organised in the form of a *laboratory*, thus eliminating the *frontal character* and *verbalism* of the traditional lesson. For Vailati, the 'school as laboratory' must not, however, be conceived in the reductive sense of a laboratory for scientific experiments, but 'as a place where the student is given the means to train himself, under the guidance and advice of the teacher, to experiment and resolve questions, to … test himself in the face of obstacles and difficulties aimed at provoking his perspicacity and cultivating his initiative' (Vailati 1906, p. 292). Maieutic lessons, hands-on work and games as suitable aids for learning, the operative experimental method, the unitary vision of mathematics, the right balance between rigour and intuition, the use of the history of mathematics, are the salient aspects of Vailati's vision of mathematics teaching, and implement what he calls the 'school as laboratory'.

Maieutic lessons, hands-on work and games

According to Vailati, one of the major causes of the ill functioning of secondary schools is the deplorable habit of conceiving teaching as a lecture where the student can't do anything but listen, to then be interrogated 'for purposes of diagnosis' (Vailati 1905c, p. 287), that is, to make sure that he has understood and memorised all that he has heard. In contrast, the kind of method more suitable for educational purposes is the maieutic or Socratic, which allows teachers to guide their students towards the discovery of mathematical truths, while at the same time stimulating enquiry and reflection. Further, to arouse the student's attention, it may be useful to exploit moments of play during the process of learning, which far from 'diminishing the dignity of the science of mathematics' (Vailati, 1899, p. 261), instead increases its attraction. Manual activity, appropriately directed and not aimed at learning a trade, can serve to 'practise the various skills of observation, discernment, attention, and judgment' (Vailati, 1901, p. 265) and constitutes an excellent antidote to the common misconception that one knows something simply because one has learned certain words.

It is clear that in this kind of teaching, the mode of examination must change as well. It is not by asking the student to define the concepts

verbally that the teacher can grasp his level of comprehension, but rather by verifying that he is capable of applying them:

> In fact, there is no other point on which there is such a jarring contrast between the educational procedures ordinarily followed and the fundamental tendency of modern psychology to regard general concepts as simple instruments (*Denkmittel*), having no other role than of making it possible for us to order, classify, fashion for determined purposes, the raw material of particular experiences. In accordance with this view, not knowing how to apply a concept … is equivalent to not possessing the concept itself at all, regardless of the ability one has on the other hand to repeat the words that presume to define it or explain it. (Vailati, 1905b, p. 280).

The operative experimental method

One of the cardinal points on which Vailati's proposals were based was the conviction that since the process of learning moves from the concrete to the abstract, pupils should never be forced to 'learn theories before knowing the facts to which they refer' (Vailati, 1899, p. 261). On the contrary, they should show that they know *how to do things*, not merely *how to repeat things*. Therefore a mathematics teaching which takes these premises into account should adopt an approach that is experimental and active. The usefulness of this method can be perceived particularly in the teaching of geometry where drawing, the construction of the figures, the recourse to squared paper, to scales, etc., can aid the student in the process of learning. Vailati wrote:

> Guiding and pushing the student to procure for himself, by means of experiment and, in particular, with recourse to the instruments of drawing, the greatest possible number of real cognitions about how to construct the figures and about their properties – above all not 'intuitive' –, is on the other hand the best way to create in him the desire and need to understand 'how' and 'why' such properties exist, and to predispose him to think of learning and the search for the deductive connections between [these properties] as interesting, as well as the arguments that lead him to recognise each of them as a consequence of the other. (Vailati, 1907, p. 305).

Thus we see that in Vailati's vision of the 'school as laboratory', the 'operative' moment must be followed in a second phase of learning, by the search for 'deductive connections' and the arrangement of the knowledge acquired into a theory. The passage between the two types of teaching, experimental-operative and rational, has to be done gradually, 'applying first of all deductive reasoning, not to demonstrate propositions that students already find quite obvious … but rather to use these propositions to arrive at others which they do not yet know' (Vailati, 1909,

p. 485). In this way the deductive procedure will also appear as an instrument for discovery.

For each area of mathematics, Vailati then encourages teachers to stimulate the students' creativity by providing them with several proofs of the most significant propositions to show them how they can arrive at the same conclusion by different routes, or even by using different mathematical instruments. For example, in the notebook relative to the classes held in 1901-1904 at the Technical Institute in Como, Vailati addresses the problem of finding the sum of the first *n* odd natural numbers, of the squares of the first *n* natural numbers, and then of the cubes, presenting various kinds of proofs: direct, with the aid of graphic visualisations, and by induction.[14]

Further, Vailati maintains that as far as possible, the statements of theorems should be presented as problems,

> 'thus, for example, the Pythagorean theorem could be advantageously presented as the solution to the problem of finding a square whose area is equal to the sum of the areas of two given squares, or as an answer to the problem of how to construct, on the ground, a right angle when the only instrument available is a cord that can be divided for example, into twelve equal parts'. (Vailati, 1910, p. 38).

A unified vision of mathematics and knowledge

In Vailati's view, teachers should lead pupils to perceive the unity of the various branches of mathematics as soon as possible, making evident the close connections between arithmetic, algebra and geometry, in order to accustom them to addressing a single problem with various methods, choosing each time the most appropriate one. For instance, in the lecture given in 1908 in Rome during the International Congress of Mathematicians, Vailati provides the following example of connecting arithmetic and geometry:

> Think, for example, how much easier it would be for the student to recognise the meaning and the significance of a proposition like this one: that 'the geometric mean of two numbers can never exceed their arithmetic mean', when they are made to see that, in a circle whose diameter is the sum of two segments, the second is represented by the radius, and the other instead by half of a chord. (Vailati 1909, p. 487).

The connection that can be established between arithmetic and algebra is of particular use from the point of view of didactics, inasmuch as the

[14] Giovanni Vailati, *Appunti per Lezioni, Istituto Tecnico*, Como 1901-1904 (*Fondo Vailati*, Cartella 38, fasc. 340).

teaching of arithmetic is suitable, from the very beginning, to pave the way for the teaching of algebra. The aim is to lead him to view algebra as simply a new form of language that is much more precise than ordinary language and capable of reducing questions or problems that seem at first complicated to forms that are so simple that they require almost no mental effort to solve.[15].

Similarly, according to Vailati one of the most efficient means for preparing students to understand the significance and usefulness of formulas is to accustom them from the very beginning 'to recognise the necessary and sufficient conditions so that a given algebraic expression, a given equation, a given identity, can be interpreted as expressing, respectively, a construction, a problem, a theorem of geometry'. (Vailati, 1910, p. 57).

Vailati was not only convinced that the students ought to be offered a unified vision of mathematics, but he believed that it was fundamental for teaching to transmit a unified vision of knowledge, establishing a dialogue between humanistic culture and scientific culture. This objective could be reached by means of the historical method. Applied as much to the sciences as to the study of Latin and Greek, according to Vailati, this method can also take on a didactic function because it is particularly suited to 'avoiding pedantry'[16] and 'to rendering the teaching more fruitful ... more efficacious, and altogether more attractive' (Vailati 1897, p. 10). It also constitutes a good antidote to all forms of dogmatism. The teacher can help young people approach the history of science by means of commentated readings of passages from the classics of science. Vailati himself read and commented on passages from Euclid's Elements to his students, as can be seen from his class notes.[17] To make this kind of teaching possible, he also deemed it desirable that schools be equipped with well-organised libraries, containing not only textbooks, but clear and concise works of popular science, books providing an orientation to study, editions of the works of great authors, encyclopaedias, etc. (Vailati, 1906, pp. 293-294).

The dialectic between rigour and intuition

In his writings Vailati appears to want to avoid any clear contrast between 'intuition' and 'rigour', and in particular in the article (Vailati, 1907), in which he illustrates the new programs for mathematics, he

[15] See (Vailati 1910, p. 40) and G. Vailati to G. Vacca, Crema, 20 July 1902, in *Epistolario*, p. 207.
[16] See G. Vailati to G. Vacca, 25 May 1901, in *Epistolario*, p. 187.
[17] See Vailati, *Lib. V* (*Fondo Vailati*, Cartella 38, fasc. 340).

observes that the application of new research on the foundations of elementary geometry in education had made it evident that 'rigour or the logical correctness of a proof is not something that depends on the number or the quality of the assumptions or hypotheses which are used in it, but rather depends on the way in which these are applied'. What is important is that 'each hypothesis, or assumption which … is used be clearly recognised, and formulated explicitly' (Vailati, 1907, pp. 305-306), and the only indispensable requirement for the rigour of the proof is that the postulates be compatible among themselves. Only when the students have acquired a greater degree of maturity, will they be shown if and by how many the number of postulates can be reduced.

Far from discouraging geometric intuition, Vailati believed that the good teacher should 'discipline and refine intuition' in order to avoid the errors that can arise from a 'rash and instinctive trust in it' (Vailati, 1904, p. 268). Instead, in his review of Halsted's textbook on rational geometry, based on Hilbert's work on the foundations, Vailati pointed out the educational hazards which can derive from 'the concern for guaranteeing the absolute rigour and perfect logical coherence of the proofs, purging them of any intuitive suggestion' (Vailati, 1905a, p. 289). Therefore in the practice of teaching it is necessary to reach a balance between intuition and rigour.

Furthermore, Vailati maintained that deduction had to be assigned a role more extensive than that generally attributed to it. The metaphors that represent deduction as a process aimed at 'extracting' from the premises what is already contained in them tend to diminish its importance with respect to the other process of reasoning and of research (see Vailati, 1898, p. 25). According to Vailati, deduction can also have heuristic value and efficaciousness; in fact beginning with premises that are only hypothetical can serve to develop ideal constructions to be compared with reality, so that the premises and consequences can provide confirmation of each other in a 'reciprocal check' and 'mutual support' (Vailati, 1898, p. 42).[18]

Vailati's reflections, together with the notes of the lessons he taught in secondary schools, are illustrative of a mathematics laboratory intended in a broader sense that the various meanings we mentioned earlier: it is a laboratory that involves people (students and teachers), structures (classrooms, equipment, instruments), working methods, experimentation and commented readings, but it also involves the search for new deductions and proofs, and for different ways of interpreting a single

[18] See also the letter from Vailati to F. Brentano, Como 16 April 1904 in *Epistolario*, p. 305.

result. It is a place where the work is done with both the hands and the mind, beginning with the problems; a place where the student becomes accustomed to using concrete objects and instruments to measure, but also to 'take his own measure', to communicate his own hypotheses, propose new solutions, new proofs, in close alliance between the experimental and theoretical aspects. In the meantime, all of this contributes to the formation of character.

Conclusions

In 1909 the reform project of the Royal Commission on which Vailati had served was published, and in May of that same year Vailati passed away. A few months later, during the congress of the Associazione Mathesis, an Italian association for mathematics teachers, held in Padua in 1909, the distinguished scholar of algebraic geometry Guido Castelnuovo, speaking on the work of the newly created International Commission on the Teaching of Mathematics, spoke in praise of the reform proposals developed by Vailati and suggested that teachers put the general lines into practice at once in their classes (Castelnuovo, 1909, p. 3). Outside of Italy Vailati's project was seen as innovative and following the footsteps of the reforming proposals of Klein. In 1910 Florian Cajori wrote:

> Under the leadership of Loria and Vailati there is a movement afoot that favours greater emphasis upon intuition, the introduction of some modern geometrical notions, the fusion of geometry with arithmetic, and the concession to the demands for practical applications made by this age of industrial development. In fact, Italy is entering upon a reform much like that of Germany and France (Cajori, 1910, p. 192).[19]

Various factors prevented the mathematics laboratory proposed by Vailati from becoming widespread in practice, or from being realised in textbooks. First of all, the reform set forth by the Royal Commission was never carried through (Giacardi, 2009). Second, Vailati, unlike Perry, never wrote a systematic exposition, his ideas are scattered throughout his writings, and his premature death prevented any further developments. Third, laboratory method-inspired textbooks were never published in Italy even though some authors paid attention to the geometrical constructions and use of the instruments, to experiments with folded or cut paper, sand, or small models in geometry, or made use of squared paper to introduce the concept of function.[20] Fourth, the discussions of the various sections of the Mathesis Association show, that not all mathematicians in Italy

[19] See also (Lietzmann, 1908, p. 181).
[20] See for example the textbook by Federigo Enriques and Ugo Amaldi, *Nozioni di matematica ad uso dei licei moderni* (Bologna: Zanichelli, 1914-1915).

shared Vailati's approach to teaching of mathematics (Giacardi 2009, pp. 17-20). Ultimately, his efforts would have been in any case nullified by the Gentile Reform of 1923, which made the humanities the cultural axis of national life in Italy, and especially of education.

Nevertheless, his portrayal of the 'school as laboratory' remains significant even today, and in recent times has been taken up again in the mathematics curricula proposed by the Italian Commission for Mathematics Teaching, in which we read:

> The mathematics *laboratory* is not a physical space outside the classroom, but is rather a structured set of activities aimed at constructing the *meanings* of mathematical objects. Thus, the laboratory involves people (students and teachers), structures (classrooms, instruments, organisation of spaces and times), and ideas (projects, plans for educational activities, experimentation). (Matematica 2003, p. 28).

The mathematics laboratory therefore has a significance that extends beyond both the carpentry laboratory proposed by Borel, and Perry's Practical Mathematics, based on problem solving and a transversal approach to mathematics that was highly concentrated on procedures, and beyond even the laboratory method proposed by Moore, which concentrated above all on 'computational, or graphical or experimental' aspects or, as Klein suggested, on the use of models and mathematical machines, or as we would say, technological tools for visualisation.

It is, as Vailati had hoped, a methodology based on problem solving, conjecture and argumentation, but whose ultimate goal is that of arriving to the construction of meanings and to a theoretical systemisation of mathematics.

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