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This is the author's manuscript

Original Citation:

Availability:

This version is available <http://hdl.handle.net/2318/146053> since 2017-07-17T23:04:42Z

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UNIVERSITÀ DEGLI STUDI DI TORINO

This is an author version of the contribution published on:

Questa è la versione dell'autore dell'opera:

Rend. Sem. Mat. Univ. Politec. Torino, 70, 4 (2012), 469—495.

The definitive version is available at:

La versione definitiva è disponibile alla URL:

<http://www.seminariomatematico.unito.it/rendiconti/>

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LINEAR ALGEBRA AND TORIC DATA OF WEIGHTED PROJECTIVE SPACES

Abstract. This paper is devoted to give characterizations of suitable matrices associated with fans and polytopes defining a weighted projective space and switching rules between them.

Introduction

The aim of the present paper is to give characterizations of fans and polytopes defining a weighted projective space (*wps* for short) and switching rules between them, from a linear algebraic point of view.

After recalling some notation and preliminaries on toric varieties, the starting point here is the usual definition of a (complex) *wps* as a geometric quotient (see Definition 2), directly checking its natural toric structure. The bridge with the classical presentation of toric varieties via fans is then given by the Cox Theorem [8] Thm. 2.1.

Section 2 is devoted to the characterization of a *wps*'s fan: up to permutations on generators it is possible to associate a $n \times (n + 1)$ integer matrix with the fan of a n -dimensional *wps* (so called *fan matrix*) whose entries turn out to verify an amount of relations (see equivalent conditions in the Theorem 3, giving a linear algebraic characterization of a *wps*'s fan). Until here almost nothing is new, since the equivalence of conditions (1) and (2) in Theorem 3 can be recovered from [4] and [2], while condition (3) can be deduced from [7] Thm. 3.6. Anyway, we were not able to find in the literature the relations between the fan generators \mathbf{v}_j 's of a *wps* $\mathbb{P}(Q)$, the associated primitive vectors \mathbf{n}_j 's and the reduction Q' of the weight vector Q , as explained in Lemma 1, although probably well-known to the experts. Notice that Lemma 1 can be used as a key step to get a completely combinatoric proof of the well-known Reduction Theorem 2 (see [18] Thm. 1.26 and its proof). Probably the most original result in Section 2 is the Proposition 5 where it is shown that the fan of a given *wps* $\mathbb{P}(Q)$ is *encoded* in the switching matrix giving the *Hermite normal form* (HNF for short) of the transposed weight vector Q^T . This section ends up with the Proposition 6 in which on the one hand (parts from (1) to (3)) we rewrite the Conrads's presentation of a *wps*'s fan matrix, but proved by directly starting from relations given in the Theorem 3, and on the other hand (part (4)) we present a *Q-canonical* form for the fan of $\mathbb{P}(Q)$, only depending on the weights order in Q (see Remark 4), which can be simply obtained by the HNF of *any* fan matrix of $\mathbb{P}(Q)$. In our opinion this Proposition describes a clean and easy method to get a fan of $\mathbb{P}(Q)$ even by hand (see Example 1).

Section 3 is dedicated to characterize polytopes associated with a polarized *wps*.

*The authors were partially supported by the Local Project "Computational Algebra and Applications"(2007) and the MIUR-PRIN "Geometria delle varietà algebriche"2010 Research Funds

As far as we know, results of this section were not known before. Let $O(1)$ be the *minimal* polarization given by a generator of the Picard group $\text{Pic}(\mathbb{P}(Q))$. Then we draw a *fan-polytope correspondence* between fans of $\mathbb{P}(Q)$ and polytopes of $(\mathbb{P}(Q), O(1))$: this is given, up to suitable weightings, in the one direction by taking the transposed inverse (so called *transverse*) of a maximal submatrix of a fan matrix, in the other direction by an obvious completion of the transposed adjoint matrix of the polytope matrix (see Definitions 5 and 7 and Remark 10). In our opinion this correspondence combined with the previous Proposition 6 provides a clean and easy method to get a polytope of $(\mathbb{P}(Q), O(m))$, even by hand, up to the elementary computation of the inverse of a (possibly big) matrix (see Example 2). Main results of this section are given by the Theorem 4, which is a direct consequence of Lemma 1, and the Proposition 9. This section ends up with the Theorem 5 giving a linear algebraic characterization of a polarized wps's polytope. This result has to be thought of as the polytopal counterpart of Theorem 3.

1. Preliminaries and notation

1.1. Toric varieties

A n -dimensional toric variety is an algebraic normal variety X containing the torus $T := (\mathbb{C}^*)^n$ as a Zariski open subset such that the natural multiplicative self-action of the torus can be extended to an action $T \times X \rightarrow X$.

Let us quickly recall the classical approach to toric varieties by means of *cones* and *fans*. For proofs and details the interested reader is referred to the extensive treatments [11], [15], [17] and the recent and quite comprehensive [10].

As usual M denotes the group of characters $\chi : T \rightarrow \mathbb{C}^*$ of T and N the group of 1-parameter subgroups $\lambda : \mathbb{C}^* \rightarrow T$. It follows that M and N are n -dimensional dual lattices via the pairing

$$\begin{aligned} M \times N &\longrightarrow \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{C}^* \\ (\chi, \lambda) &\longmapsto \chi \circ \lambda \end{aligned}$$

which translates into the standard pairing $\langle u, v \rangle = \sum u_i v_i$ under the identifications $M \cong \mathbb{Z}^n \cong N$ obtained by setting $\chi(\mathbf{t}) = \mathbf{t}^{\mathbf{u}} := \prod t_i^{u_i}$ and $\lambda(t) = t^{\mathbf{v}} := (t^{v_1}, \dots, t^{v_n})$.

Cones and affine toric varieties

Define $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \text{Hom}(N, \mathbb{Z}) \otimes \mathbb{R} \cong \text{Hom}(N_{\mathbb{R}}, \mathbb{R})$.

A convex polyhedral cone (or simply a cone) σ is the subset of $N_{\mathbb{R}}$ defined by

$$\sigma = \langle \mathbf{v}_1, \dots, \mathbf{v}_s \rangle := \{r_1 \mathbf{v}_1 + \dots + r_s \mathbf{v}_s \in N_{\mathbb{R}} \mid r_i \in \mathbb{R}_{\geq 0}\}$$

The s vectors $\mathbf{v}_1, \dots, \mathbf{v}_s \in N_{\mathbb{R}}$ are said to generate σ . A cone $\sigma = \langle \mathbf{v}_1, \dots, \mathbf{v}_s \rangle$ is called *rational* if $\mathbf{v}_1, \dots, \mathbf{v}_s \in N$, *simplicial* if $\mathbf{v}_1, \dots, \mathbf{v}_s$ are \mathbb{R} -linear independent and *non-singular* if $\mathbf{v}_1, \dots, \mathbf{v}_s$ can be extended by $n - s$ further elements of N to give a basis of the lattice N .

A cone σ is called *strictly convex* if it does not contain a linear subspace of positive dimension of $N_{\mathbb{R}}$.

The *dual cone* σ^{\vee} of σ is the subset of $M_{\mathbb{R}}$ defined by

$$\sigma^{\vee} = \{\mathbf{u} \in M_{\mathbb{R}} \mid \forall \mathbf{v} \in \sigma \quad \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}$$

A *face* τ of σ (denoted by $\tau < \sigma$) is the subset defined by

$$\tau = \sigma \cap \mathbf{u}^{\perp} = \{\mathbf{v} \in \sigma \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$$

for some $\mathbf{u} \in \sigma^{\vee}$. Observe that also τ is a cone.

Gordon's Lemma (see [15] §1.2, Proposition 1) ensures that the semigroup $S_{\sigma} := \sigma^{\vee} \cap M$ is *finitely generated*. Then also the associated \mathbb{C} -algebra $A_{\sigma} := \mathbb{C}[S_{\sigma}]$ is finitely generated. A choice of r generators gives a presentation of A_{σ}

$$A_{\sigma} \cong \mathbb{C}[X_1, \dots, X_r]/I_{\sigma}$$

where I_{σ} is the ideal generated by the relations between generators. Then

$$U_{\sigma} := \mathcal{V}(I_{\sigma}) \subset \mathbb{C}^r$$

turns out to be an *affine toric variety*. In other terms an affine toric variety is given by $U_{\sigma} := \text{Spec}(A_{\sigma})$. Since a closed point $x \in U_{\sigma}$ is an evaluation of elements in $\mathbb{C}[S_{\sigma}]$ satisfying the relations generating I_{σ} , then it can be identified with a semigroup morphism $x : S_{\sigma} \rightarrow \mathbb{C}$ assigned by thinking of \mathbb{C} as a multiplicative semigroup. In particular the *characteristic morphism*

$$(1) \quad \begin{array}{ccc} x_{\sigma} : \sigma^{\vee} \cap M & \longrightarrow & \mathbb{C} \\ \mathbf{u} & \longmapsto & \begin{cases} 1 & \text{if } \mathbf{u} \in \sigma^{\perp} \\ 0 & \text{otherwise} \end{cases} \end{array}$$

which is well defined since $\sigma^{\perp} < \sigma^{\vee}$, defines a *characteristic point* $x_{\sigma} \in U_{\sigma}$ whose torus orbit O_{σ} turns out to be a $(n - \dim(\sigma))$ -dimensional torus embedded in U_{σ} (see e.g. [15] §3).

Fans and toric varieties

A *fan* Σ is a finite set of cones $\sigma \subset N_{\mathbb{R}}$ such that

1. for any cone $\sigma \in \Sigma$ and for any face $\tau < \sigma$ then $\tau \in \Sigma$,
2. for any $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$.

For every i with $0 \leq i \leq n$ denote by $\Sigma(i) \subset \Sigma$ the subset of i -dimensional cones, called the *i -skeleton* of Σ . A fan Σ is called *simplicial* if every cone $\sigma \in \Sigma$ is simplicial and *non-singular* if every such cone is non-singular. The *support* of a fan Σ is the subset $|\Sigma| \subset N_{\mathbb{R}}$ obtained as the union of all of its cones i.e.

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}.$$

If $|\Sigma| = N_{\mathbb{R}}$ then Σ will be called *complete* or *compact*.

Since for any face $\tau < \sigma$ the semigroup S_{σ} turns out to be a sub-semigroup of S_{τ} , there is an induced immersion $U_{\tau} \hookrightarrow U_{\sigma}$ between the associated affine toric varieties which embeds U_{τ} as a principal open subset of U_{σ} . Given a fan Σ one can construct *an associated toric variety* $X(\Sigma)$ by patching all the affine toric varieties $\{U_{\sigma} \mid \sigma \in \Sigma\}$ along the principal open subsets associated with any common face. Moreover *for every toric variety X there exists a fan Σ such that $X \cong X(\Sigma)$* (see [17] Theorem 1.5). It turns out that ([17] Theorems 1.10 and 1.11; [15] §2):

- $X(\Sigma)$ is non-singular if and only if the fan Σ is non-singular,
- $X(\Sigma)$ is complete if and only if the fan Σ is complete.

In the following a *1-generated fan* Σ is a fan generated by a set of $n+1$ integral vectors i.e. a fan whose cones $\sigma \subset N \otimes \mathbb{R}$ are generated by any proper subset of a given finite subset $\{\mathbf{v}_0, \dots, \mathbf{v}_n\} \subset N$: we will write

$$(2) \quad \Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) .$$

Given a 1-generated fan $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$, the matrix $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ will be called *a fan matrix of Σ* . Notice that Σ determines V up to a permutations of columns, meaning that Σ admits $(n+1)!$ associated fan matrices.

If $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ is a fan matrix of $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$ then we will denote the maximal square sub-matrices of V and the associated n -minors as follows

$$(3) \quad \forall 0 \leq j \leq n \quad V^j := (\mathbf{v}_0, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n) , \quad V_j = \det(V^j) .$$

Polytopes and projective toric varieties

A *polytope* $\Delta \subset M_{\mathbb{R}}$ is the convex hull of a finite set of points. If this set is a subset of M then the polytope is called *integral*. Starting from an integral polytope one can construct a projective toric variety as follows. Here we will follow the approach of [1], which the interested reader is referred to for proofs and details (see also [9] §3.2.2).

For any $k \in \mathbb{N}$ one can define the dilated polytope $k\Delta := \{k\mathbf{u} \mid \mathbf{u} \in \Delta\}$. It is then possible to define a graded \mathbb{C} -algebra S_{Δ} , associated with the integral polytope Δ , as follows. For any $\mathbf{u} \in k\Delta \cap M$ consider the associated character $\chi^{\mathbf{u}} : \mathbf{t} \mapsto \mathbf{t}^{\mathbf{u}}$. Given $t \in \mathbb{C}^*$ consider the *monomial* $t^k \chi^{\mathbf{u}} : \mathbf{t} \mapsto t^k \mathbf{t}^{\mathbf{u}}$. It well defines a *monomial product* $t^{k_1} \chi^{\mathbf{u}_1} \cdot t^{k_2} \chi^{\mathbf{u}_2} := t^{k_1+k_2} \chi^{\mathbf{u}_1+\mathbf{u}_2}$ where $\mathbf{u}_1 + \mathbf{u}_2 \in (k_1 + k_2)\Delta$. Let S_{Δ} be the \mathbb{C} -algebra generated by all monomials $\{t^k \chi^{\mathbf{u}} \mid k \in \mathbb{N}, \mathbf{u} \in k\Delta\}$ which is a graded object by setting $\deg(t^k \chi^{\mathbf{u}}) = k$.

The *projective variety* $\mathbb{P}_{\Delta} := \text{Proj}(S_{\Delta})$ turns out to be naturally a *toric variety* whose fan Σ_{Δ} can be recovered as follows. For any nonempty face $F \prec \Delta$ consider the cone

$$\check{\sigma}_F := \{r(\mathbf{u} - \mathbf{u}') \mid \mathbf{u} \in \Delta, \mathbf{u}' \in F, r \in \mathbb{R}_{\geq 0}\} \subset M_{\mathbb{R}}$$

and define $\sigma_F := \check{\sigma}_F^{\vee} \subset N_{\mathbb{R}}$. Then $\Sigma_{\Delta} := \{\sigma_F \mid F \prec \Delta\}$ turns out to be a fan, called the *normal fan* of the polytope Δ , such that there exists a very ample divisor H of $X(\Sigma_{\Delta})$ for

which $(X(\Sigma_\Delta), H) \cong (\mathbb{P}_\Delta, O(1))$, where $O(1)$ is the natural polarization of $\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$ (see [1] Proposition 1.1.2).

Viceversa a *projective toric variety* is the couple $(X(\Sigma), H)$ of a toric variety $X(\Sigma)$ and a polarization given by (the linear equivalence class of) a hyperplane section H . For any 1-cone $\rho \in \Sigma(1)$, consider the torus stable divisor $D_\rho := \overline{O}_\rho$ defined as the closure of the torus orbit of the characteristic point x_ρ , defined in (1). Since those divisors generate the Chow group of Weil divisors $A_{n-1}(X(\Sigma))$ (see [15] §3.4), there exist integer coefficients $a_\rho \in \mathbb{Z}$ such that $H = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. It is then well defined the integral polytope

$$(4) \quad \Delta_H := \{ \mathbf{u} \in M_{\mathbb{R}} \mid \forall \rho \in \Sigma(1) \langle \mathbf{u}, \mathbf{n}_\rho \rangle \geq -a_\rho \}$$

where \mathbf{n}_ρ is the unique generator of the semigroup $\rho \cap N$. Then

$$(\mathbb{P}_{\Delta_H}, O(1)) \cong (X(\Sigma), H) .$$

1.2. Hermite normal form

It is well known that Hermite algorithm provides an effective way to determine a basis of a subgroup of \mathbb{Z}^n . We briefly recall the definition and the main properties. For details, see for example [6].

DEFINITION 1. *An $m \times n$ matrix $M = (m_{ij})$ with integral coefficients is in Hermite normal form (abbreviated HNF) if there exists $r \leq m$ and a strictly increasing map $f : \{1, \dots, r\} \rightarrow \{1, \dots, n\}$ satisfying the following properties:*

1. *For $1 \leq i \leq r$, $m_{i,f(i)} \geq 1$, $m_{ij} = 0$ if $j < f(i)$ and $0 \leq m_{i,f(k)} < m_{k,f(k)}$ if $i < k$.*
2. *The last $m - r$ rows of M are equal to 0.*

THEOREM 1 ([6] Theorem 2.4.3). *Let A be an $m \times n$ matrix with coefficients in \mathbb{Z} . Then there exists a unique $m \times n$ matrix $B = (b_{ij})$ in HNF of the form $B = U \cdot A$ where $U \in \text{GL}(m, \mathbb{Z})$.*

We will refer to matrix B as the HNF of matrix A . The construction of B and U is effective, see [6, Algorithm 2.4.4], based on Eulid's algorithm for greatest common divisor. In the following two applications of this algorithm will be considered: for computing a fan of a given wps (see Prop. 5) and the so-called Q -canonical fan of $\mathbb{P}(Q)$ (see Prop. 6). At this purpose, a key theoretical tool is the following (for the proof see [6, §2.4.3])

PROPOSITION 1.

1. *Let L be a subgroup of \mathbb{Z}^n , $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ a set of generators, and let A be the $m \times n$ matrix having $\mathbf{v}_1, \dots, \mathbf{v}_m$ as rows. Let B be the HNF of A . Then the nonzero rows of B are a basis of L .*

2. Let A be a $m \times n$ matrix, and let $B = U \cdot A^T$ be the HNF of the transposed of A , and let r be the number of nonzero rows of B . Then a \mathbb{Z} -basis for the kernel of A is given by the last $m - r$ rows of U .

1.3. Transversion of a matrix

In the following, given a matrix $A \in \text{GL}(n, \mathbb{Q})$, the matrix obtained by taking the transposed matrix of the inverse matrix

$$A^* := ((A)^{-1})^T$$

is called the *transverse matrix* of A . We will see in the following (see subsection 3.1, in particular Thm. 4) that *transversion* of a matrix describes, up to the multiplication by a diagonal matrix of weights, the passage from a fan to a polytope (and back) associated with the same weighted projective space $\mathbb{P}(Q)$.

Here are some elementary properties of transversion:

PROPOSITION 2. Let A and B be matrices of $\text{GL}(n, \mathbb{Q})$. Then:

1. $(A^*)^* = A$ i.e. transversion is an involution in $\text{GL}(n, \mathbb{Q})$,
2. $(A \cdot B)^* = A^* \cdot B^*$,
3. $\det(A^*) = 1/\det(A)$,
4. if A is a upper (lower) triangular matrix then A^* is an lower (upper) triangular matrix,
5. if $A \in \text{GL}(n, \mathbb{Z})$ then $A^* \in \text{GL}(n, \mathbb{Z})$ too.

1.4. Weighted projective spaces

In the present subsection we will briefly recall the definition and some well known fact about *weighted projective spaces* (wps in the following). Proofs and details can be recovered in the extensive treatments [12] [16], [13] and [3].

DEFINITION 2. Set $Q := (q_0, \dots, q_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$ and consider the multiplicative group $\mu_Q := \mu_{q_0} \oplus \dots \oplus \mu_{q_n}$ where μ_{q_i} is the group of q_i -th roots of unity. Consider the following action of μ_Q over the n -dimensional complex projective space \mathbb{P}^n

$$\mu_Q : \begin{array}{ccc} \mu_Q \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ ((\xi_j), [z_j]) & \longmapsto & [\xi_j z_j] . \end{array}$$

Let $\Delta_Q \subset \mu_Q$ be the diagonal subgroup and consider the quotient group $\mathbb{W}_Q := \mu_Q / \Delta_Q$. Then the induced quotient space

$$\mathbb{P}(Q) := \mathbb{P}^n / \mathbb{W}_Q$$

is called the Q -weighted projective space (Q -wps).

REMARK 1. If q is the greatest common divisor of (q_0, \dots, q_n) then

$$\Delta_Q \cong \mu_q$$

Therefore we get the canonical isomorphism

$$\mathbb{P}(Q) \cong \mathbb{P}\left(\frac{q_0}{q}, \dots, \frac{q_n}{q}\right)$$

For this reason in the following we will always assume that

$$q = \gcd(q_0, \dots, q_n) = 1.$$

Let us recall the following standard notation

$$(5) \quad \begin{aligned} d_j &:= \gcd(q_0, \dots, q_{j-1}, q_{j+1}, \dots, q_n), \\ a_j &:= \text{lcm}(d_0, \dots, d_{j-1}, d_{j+1}, \dots, d_n), \\ a &:= \text{lcm}(a_0, \dots, a_n). \end{aligned}$$

DEFINITION 3 (Weight vector). *In the following a weight vector $Q = (q_0, \dots, q_n)$ will denote a $n + 1$ -tuple of coprime positive integer numbers. Referring to notation defined in (5), a weight vector Q will be called reduced if $d_j = 1$, or equivalently $a_j = 1$, for any $j = 0, \dots, n$.*

REMARK 2. *Every weighted projective space is a toric variety.* In fact the natural torus action over \mathbb{P}^n passes through the quotient as follows

$$\begin{array}{ccc} (\mathbb{C}^*)^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ \tau_Q \times \pi_Q \downarrow & & \downarrow \pi_Q \\ (\mathbb{C}^*)^n \times \mathbb{P}^n(Q) & \longrightarrow & \mathbb{P}^n(Q) \end{array}$$

where π_Q is the natural quotient map and τ_Q is the quotient map associated with the action

$$\begin{aligned} \mu_Q \times (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ ((\zeta_j), (t_i)) &\longmapsto (\zeta_0^{-1} \zeta_i t_i) \end{aligned}$$

Then the torus $(\mathbb{C}^*)^n$ can be embedded in $\mathbb{P}(Q)$ via the following map

$$\begin{aligned} (\mathbb{C}^*)^n &\hookrightarrow \mathbb{P}(Q) \\ (t_1, \dots, t_n) &\longmapsto [1 : t_1 : \dots : t_n] \end{aligned}$$

whose image is the open subset $\mathbb{P}(Q) \setminus \mathcal{V}(\prod_j z_j)$.

PROPOSITION 3. *Since $\gcd(q_0, \dots, q_n) = 1$, the following facts are true:*

1. $\gcd(q_j, d_j) = 1$,

2. if $i \neq j$ then $\gcd(d_i, d_j) = 1$,
3. $a_j \mid q_j$,
4. $\gcd(a_j, d_j) = 1$,
5. $a_j d_j = a$,
6. setting $q'_j := q_j/a_j$, then $Q' = (q'_0, \dots, q'_n)$ is reduced; Q' is then called the reduction of Q .

The proofs of these well known properties (see [13] 1.3.1) are elementary.

Instead we will prove the following property, which does not appear in the main treatments of the subject.

PROPOSITION 4. Let $Q = (q_0, \dots, q_n)$ be a weight vector and $Q' = (q'_0, \dots, q'_n)$ be its reduction. Define

$$(6) \quad \delta := \text{lcm}(q_0, \dots, q_n) \quad \text{and} \quad \delta' := \text{lcm}(q'_0, \dots, q'_n).$$

Then $\delta = a\delta'$, where a is defined in (5).

REMARK 3. The Proposition 4 still holds when $q := \gcd(q_0, \dots, q_n) > 1$.

Proof of Proposition 4. We will prove that δ divides $a\delta'$ and, viceversa, $a\delta'$ divides δ . On the one hand, to show that δ divides $a\delta'$ it suffices to show that q_j divides $a\delta'$ for every j and this fact follows immediately by definitions since $q_j = a_j q'_j \mid a\delta'$. On the other hand, by definitions of a and δ' , to prove that $a\delta'$ divides δ it suffices to prove that $a_i q'_k \mid \delta$, which is $a_i \mid \frac{\delta}{q_k} a_k$, for every i, k . By the definition of a_i given in (5), the latter is obtained by showing that d_j divides $\frac{\delta}{q_k} a_k$ for every j, k .

If $j \neq k$ then d_j divides a_k and we are done.

Suppose now $j = k$. Let p be a prime dividing d_k and let p^t, p^r be the highest powers of p dividing d_k and q_k respectively. Then p^t divides q_i for every $i \neq k$, by the definition of d_k : in particular $p^t \mid \delta$. If $r \geq t$ then

$$\forall i \quad p^t \mid q_i \Rightarrow \forall j \quad p^t \mid d_j \Rightarrow \forall k \quad p^t \mid a_k.$$

If $r < t$ then p^{t-r} divides $\frac{\delta}{q_k}$; moreover p^r divides d_i for every $i \neq k$, since $p^r \mid q_k$ and $p^t \mid q_i$ for every $i \neq k$: then p^r divides a_k . Therefore p^t divides $\frac{\delta}{q_k} a_k$. Thus we proved that d_k divides $\frac{\delta}{q_k} a_k$. \square

Let us now recall the following well-known result to which we will refer below as to the *Reduction Theorem*.

THEOREM 2 (Reduction Theorem ([12] §1, [13] 1.3.1)). Let $Q' = (q'_0, \dots, q'_n)$ be the reduced weight vector of $Q = (q_0, \dots, q_n)$. Then

$$(7) \quad \mathbb{P}(Q) \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* = \mathbb{P}(Q')$$

where the quotient is realized by means of the (reduced) action

$$\begin{aligned} \mathbf{v}_{Q'} : \mathbb{C}^* \times \mathbb{C}^{n+1} &\longrightarrow \mathbb{C}^{n+1} \\ (t, (z_j)) &\longmapsto (t^{q'_0} z_0, \dots, t^{q'_n} z_n) . \end{aligned}$$

Let us end up this preliminary section with the following technical statement which will be useful below. Partial proofs of this result may recovered from [4] §2 and [14] Prop. 2.3. Moreover it is certainly well-know to experts. However for purposes of definiteness we include here a detailed proof.

LEMMA 1. *Let $Q = (q_0, \dots, q_n)$ be a weight vector; let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a set of vectors in \mathbb{Q}^n , generating \mathbb{Q}^n and such that $\sum_{j=0}^n q_j \mathbf{v}_j = 0$. Let L be the lattice generated in \mathbb{Q}^n by $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ and L' be the sublattice generated by $\{q_0 \mathbf{v}_0, \dots, q_n \mathbf{v}_n\}$. Then the following properties hold:*

- (a) $[L : L'] = \prod_{j=0}^n q_j$;
- (b) let $V := (v_{ij})$ be the $n \times (n+1)$ matrix whose columns are given by components of $\mathbf{v}_0, \dots, \mathbf{v}_n$ over a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of L i.e. $\mathbf{v}_j = \sum_{i=1}^n v_{ij} \mathbf{e}_i$, for every $j = 0, \dots, n$, and denote by V_j the n -minor of V obtained by deleting the j -th column as in (3). Then

$$\forall j = 0, \dots, n \quad V_j = (-1)^{\varepsilon+j} q_j, \text{ for a fixed } \varepsilon \in \{0, 1\},$$
- (c) $\forall j = 0, \dots, n \quad \mathbf{v}_j = d_j \mathbf{n}_j$, where \mathbf{n}_j is the generator of the semigroup $\langle \mathbf{v}_j \rangle \cap L$ and d_j is defined in (5); in particular L is the lattice generated by $\{\mathbf{n}_0, \dots, \mathbf{n}_n\}$; moreover $\{\mathbf{n}_0, \dots, \mathbf{n}_n\}$ satisfy the hypotheses of this Lemma with respect to the reduced weight vector Q' i.e. they generate \mathbb{Q}^n and $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$.

Proof. For (a), observe that L' has $q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n$ as a basis. Then L' has index $\prod_{j=1}^n q_j$ in the lattice L_0 generated by $\mathbf{v}_1, \dots, \mathbf{v}_n$. The quotient L/L_0 is cyclic generated by the image of \mathbf{v}_0 , so that $[L : L_0]$ divides q_0 . If $r \mathbf{v}_0 \in L_0$, with $r \in \mathbb{Z}$ then $r \mathbf{v}_0 = \sum_{j=1}^n s_j \mathbf{v}_j$ with $s_1, \dots, s_n \in \mathbb{Z}$. Since $\gcd(q_0, \dots, q_n) = 1$ then there exists $\lambda \in \mathbb{Z}$ such that $r = -\lambda q_0$, $s_i = \lambda q_i$ for $i = 1, \dots, n$; in particular q_0 divides r , so that $[L : L_0] = q_0$ and $[L : L'] = [L : L_0][L_0 : L'] = \prod_{i=0}^n q_i$.

(b): for $j = 0, \dots, n$, let L_j be the lattice generated by $\mathbf{v}_0, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n$. Then $|V_j| = [L : L_j] = q_j$, as we have shown in (a) for the case $j = 0$. Let $\varepsilon \in \{0, 1\}$ be such that $V_0 = (-1)^\varepsilon q_0$. Then

$$\forall j = 0, \dots, n \quad V_j = (-1)^j \frac{q_j}{q_0} V_0 = (-1)^{\varepsilon+j} q_j$$

since $\sum_{j=0}^n q_j \mathbf{v}_j = 0$.

(c): we have

$$\forall j = 0, \dots, n \quad q_j \mathbf{v}_j = - \sum_{k \neq j} q_k \mathbf{v}_k = -d_j \sum_{k \neq j} \tilde{q}_k \mathbf{v}_k$$

where $\tilde{q}_k := q_k/d_j \in \mathbb{N}$. By (1) in Proposition 3, $\gcd(q_j, d_j) = 1$ meaning that

$$\forall j = 0, \dots, n \quad \exists \mathbf{v}'_j \in L : \mathbf{v}_j = d_j \mathbf{v}'_j .$$

Then (5) in Proposition 3 allows to write

$$(8) \quad 0 = \sum_{j=0}^n q_j \mathbf{v}_j = \sum_{j=0}^n (q'_j a_j) (d_j \mathbf{v}'_j) = a \sum_{j=0}^n q'_j \mathbf{v}'_j \implies \sum_{j=0}^n q'_j \mathbf{v}'_j = 0 .$$

Moreover $\mathbf{v}'_0, \dots, \mathbf{v}'_n$ generate L and (a) ensures that the following index

$$(9) \quad [L : \langle q'_j \mathbf{v}'_j \mid j = 0, \dots, n \rangle] = \prod_{j=0}^n q'_j .$$

Then the proof ends up by showing that, for all j , $\mathbf{v}'_j = \mathbf{n}_j$. With this goal in mind, consider $h_j \in \mathbb{N}$ such that $\mathbf{v}'_j = h_j \mathbf{n}_j$. If $V' = (v'_{ij})$ is the matrix of components of $\mathbf{v}'_0, \dots, \mathbf{v}'_n$, over the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of L , then

$$\forall j = 0, \dots, n \quad |V'_j| = q'_j$$

On the other hand

$$\mathbf{v}'_0 \in h_0 L \implies \forall i = 1, \dots, n \quad h_0 \mid v'_{i0} \implies \forall k = 1, \dots, n \quad h_0 \mid |V'_k| = q'_k .$$

Therefore (6) in Proposition 3 implies that

$$h_0 \mid \gcd(q'_1, \dots, q'_n) = 1 \implies h_0 = 1$$

Analogously $h_j = 1$, for all $1 \leq j \leq n$. Hence $\mathbf{v}'_j = \mathbf{n}_j$. □

2. Characterization of fans giving $\mathbb{P}(Q)$

2.1. Characterizing the fan

Let us fix an n -dimensional lattice N and a subset of $n+1$ vectors $\{\mathbf{v}_0, \dots, \mathbf{v}_n\} \subset N$. The following theorem is an application of the previous Lemma 1 to known results such as e.g. Lemma 2.11 in [2], Prop. 5.4 in [5] and Thm. 3.6 in [7].

THEOREM 3. *Let $Q = (q_0, \dots, q_n)$ be a weight vector. Consider the fan $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$ and the associated matrix $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ with respect to a fixed basis of N . Then the following facts are equivalent:*

1. Σ is a fan of $\mathbb{P}(Q)$,
2. $\sum_{j=0}^n q_j \mathbf{v}_j = 0$ and the sub-lattice $N' := \langle q_0 \mathbf{v}_0, \dots, q_n \mathbf{v}_n \rangle \subset N$ has finite index

$$[N : N'] = \prod_{j=0}^n q_j ,$$

3. $\forall j = 0, \dots, n \quad V_j = (-1)^{\varepsilon+j} q_j$, for a fixed $\varepsilon \in \{0, 1\}$,

$$4. \quad q_0 \mathbf{v}_0 = -\sum_{i=1}^n q_i \mathbf{v}_i \text{ and } |V_0| := |\det(\mathbf{v}_1, \dots, \mathbf{v}_n)| = q_0.$$

Proof of Theorem 3. (1) \Rightarrow (2). A fan of the wps $\mathbb{P}(Q)$ with $Q = (q_0, \dots, q_n)$ is presented in [15] at the end of §2.3. Then, by Lemma 1(a) one may check that fan to satisfy conditions stated in (2).

(2) \Rightarrow (3). This is Lemma 1(b).

(3) \Rightarrow (4). For any $k = 1, \dots, n$ consider the $(n+1) \times (n+1)$ matrix

$$A_k := \begin{pmatrix} v_{k0} & \cdots & v_{kn} \\ \mathbf{v} \end{pmatrix}.$$

Since the first and the $(k+1)$ -st rows of A_k are equal we get

$$\forall k = 1, \dots, n \quad 0 = \det(A_k) = \sum_{j=0}^n (-1)^j v_{kj} v_j \stackrel{(3)}{=} (-1)^{\varepsilon} \sum_{j=0}^n q_j v_{kj} \Rightarrow \sum_{j=0}^n q_j v_j = 0.$$

(4) \Rightarrow (2). Since $|V_0| = q_0$ then $\{q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n\}$ is a basis of the sub-lattice N' . Hence

$$[N : N'] = |\det(q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n)| = \left(\prod_{i=1}^n q_i \right) |V_0| \stackrel{(4)}{=} \prod_{j=0}^n q_j.$$

(2) \Rightarrow (1). First note that (2) and Lemma 1(a) imply that $\mathbf{v}_0, \dots, \mathbf{v}_n$ generate N . It follows by Lemma 1(c) that $\mathbf{n}_0, \dots, \mathbf{n}_n$ generate N and $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$.

Let D_j be the torus invariant divisor associated with the 1-dimensional cone $\langle \mathbf{n}_j \rangle \in \Sigma(1)$ for the toric variety with fan Σ . Consider the sequence

$$(10) \quad 0 \longrightarrow M \xrightarrow{\text{div}} \bigoplus_{j=0}^n \mathbb{Z} \cdot D_j \xrightarrow{d} \mathbb{Z} \longrightarrow 0$$

where $\text{div}(\mathbf{m}) = \sum_{j=0}^n \langle \mathbf{m}, \mathbf{n}_j \rangle D_j$ and $d(\sum_{j=0}^n b_j D_j) = \sum_{j=0}^n b_j q'_j$. Then div is injective since the \mathbf{n}_j span $N = M^\vee$ and d is surjective since $\gcd(q'_0, \dots, q'_n) = 1$. Furthermore, $d \circ \text{div} = 0$ follows easily from $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$.

Hence, to prove that (10) is exact, it suffices to show that $\ker(d) \subset \text{Im}(\text{div})$. Take $\sum_{j=0}^n b_j D_j \in \ker(d)$. Since $\mathbf{n}_1, \dots, \mathbf{n}_n$ are linearly independent over \mathbb{Q} , one can find $\mathbf{m} \in M \otimes \mathbb{Q}$ such that $\langle \mathbf{m}, \mathbf{n}_i \rangle = b_i$ for $1 \leq i \leq n$. Since $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$ and $\sum_{j=0}^n b_j q'_j = 0$, we obtain

$$q'_0 \langle \mathbf{m}, \mathbf{n}_0 \rangle = -\sum_{i=1}^n \langle \mathbf{m}, q'_i \mathbf{n}_i \rangle = -\sum_{i=1}^n q'_i \langle \mathbf{m}, \mathbf{n}_i \rangle = -\sum_{i=1}^n q'_i b_i = q'_0 b_0.$$

It follows that $\langle \mathbf{m}, \mathbf{n}_0 \rangle = b_0$. Thus $\langle \mathbf{m}, \mathbf{n}_j \rangle = b_j \in \mathbb{Z}$ for all j . This implies $\mathbf{m} \in M$ since $\mathbf{n}_0, \dots, \mathbf{n}_n$ span $N = M^\vee$, and the desired exactness follows immediately.

We are now in a position to apply the Cox Theorem, [8] Thm. 2.1, to give a geometric quotient description of $X(\Sigma)$. In particular the exact sequence (10) suffices

to show that the *Chow group* of Weil divisors modulo rational equivalence for the toric variety $X(\Sigma)$ is given by $A_{n-1}(X) \cong \mathbb{Z}$ (see e.g. [15] §3.4). Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the polynomial ring obtained by associating the variable x_j with the 1-dimensional cone $\langle \mathbf{n}_j \rangle \in \Sigma(1)$. The grading on S is defined by setting

$$\deg(x_j) := \deg(d(D_j)) = q_j.$$

Since $\text{Hom}(A_{n-1}(X), \mathbb{C}^*) \cong \mathbb{C}^*$, it is then possible to exhibit $X(\Sigma)$ as a geometric quotient

$$X(\Sigma) \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$$

where the quotient is realized by means of the action $\mathbf{v}_{Q'}$ in the statement of the Reduction Theorem 2. Then $X(\Sigma) \cong \mathbb{P}(Q') \cong \mathbb{P}(Q)$. \square

The following definition will be useful in section 3, when speaking about the *fan-polytope correspondence*:

DEFINITION 4 (*F*-admissible matrices). *A matrix $V \in \text{Mat}(n, n+1, \mathbb{Z})$ will be called *F*-admissible if it satisfies the following conditions*

1. *the matrix $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ admits only nonvanishing coprime maximal minors i.e. $\forall j = 0, 1, \dots, n \quad V_j \neq 0$ and $\gcd(V_j \mid 0 \leq j \leq n) = 1$;*
2. *the columns \mathbf{v}_j of V satisfy one of the equivalent conditions (2), (3), (4) of Theorem 3 with respect to the weights $q_j := |V_j|$.*

*The subset of *F*-admissible matrices will be denoted by $\mathfrak{V}_n \subset \text{Mat}(n, n+1, \mathbb{Z})$.*

2.2. Hermite normal form of weights and fans of $\mathbb{P}(Q)$

The following result, which is a direct consequence of Theorem 3, exhibit a rather surprising method to get a fan of a given wps $\mathbb{P}(Q)$: in fact this fan turns out to be *encoded* in the switching matrix giving the HNF of the transposed weight vector Q^T . Since such a matrix is obtained by a well known algorithm, based on Euclid's algorithm for greatest common divisor, [6] Algorithm 2.4.4, this gives a constructive method to produce a fan of $\mathbb{P}(Q)$ which can be performed by any procedures computing elementary linear algebra operations.

PROPOSITION 5. *Let $Q = (q_0, \dots, q_n)$ be a weight vector, B the HNF of the transposed vector Q^T and $U \in \text{GL}(n+1, \mathbb{Z})$ be such that $U \cdot Q^T = B$. Let C be the matrix consisting of the last n rows of U and let \mathbf{v}_j be the j^{th} column vector of C , for $0 \leq j \leq n$. Let L, L' be the lattices generated in \mathbb{Z}^n by $\mathbf{v}_0, \dots, \mathbf{v}_n$ and $q_0\mathbf{v}_0, \dots, q_n\mathbf{v}_n$ respectively. Then*

$$1. B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

2. $L = \mathbb{Z}^n$;
3. $\sum_{j=0}^n q_j \mathbf{v}_j = \mathbf{0}$;
4. there exists $\varepsilon \in \{0, 1\}$ such that $C_j = (-1)^{\varepsilon+j} q_j$ for all $0 \leq j \leq n$.
5. $[L : L'] = \prod_{j=0}^n q_j$.

As a consequence of Theorem 3, $\text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$ is a fan of $\mathbb{P}(Q)$.

Proof. The rank of Q is 1, so by definition of HNF, $B = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $a \geq 1$. By the

equality $Q^T = U^{-1} \cdot B$ we see that a must divide q_0, \dots, q_n , so that $a = 1$; this proves (1). Then (3) follows immediately by $U \cdot Q^T = B$.

To prove part (2), let (c_0, \dots, c_n) be the 2nd column of U^{-1} . Then $U \cdot U^{-1} = I$ easily implies that $\sum_{j=0}^n c_j \mathbf{v}_j = \mathbf{e}_1$, the first standard basis vector of \mathbb{Z}^n . Columns $3, \dots, n+1$ of U^{-1} similarly show that $\mathbf{e}_2, \dots, \mathbf{e}_n$ are in the sublattice generated by $\mathbf{v}_0, \dots, \mathbf{v}_n$. Hence this sublattice must be \mathbb{Z}^n .

Finally parts (4) and (5) follow immediately from parts (2) and (3) of Lemma 1. \square

2.3. A Q -canonical fan of $\mathbb{P}(Q)$

In the present subsection we want to use the characterization (4) in Theorem 3 to get a Q -canonical fan of $\mathbb{P}(Q)$, in the sense that the associated fan matrix is in HNF, up to a permutation of columns (see the following Remark 4). This fact presents the fan in a triangular shape and generated by as many as possible of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ in a given basis of the lattice N . Moreover it turns out to be a convenient procedure to get a fan of $\mathbb{P}(Q)$ by hands (see Example 1 below).

PROPOSITION 6. *Let $Q = (q_0, \dots, q_n)$ be a weight vector. For any j with $1 \leq j \leq n$, define $k_j := \gcd(q_0, q_j, q_{j+1}, \dots, q_n)$. Then:*

1. $k_j \mid k_{j+1}$,
2. either $k_n = (q_0, q_n) = 1$ or there exists a positive integer i , with $1 \leq i \leq n-1$, such that $k_i = 1$ and $k_{i+1} > 1$,
3. consider a upper triangular matrix $V^0 = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \text{Mat}(n, n, \mathbb{Z})$ whose columns \mathbf{v}_j are such that:

$$\begin{aligned} \forall 1 \leq j \leq i-1 \quad \mathbf{v}_j &= \mathbf{e}_j \\ \forall i \leq j \leq n-1 \quad v_{jj} &= k_{j+1}/k_j \\ v_{nn} &= q_0/k_n \end{aligned}$$

where v_{kj} is the k -th entry of the column \mathbf{v}_j ; then there exists a choice for v_{kj} with $i \leq j$ and $k < j$ such that V^0 can be completed to a matrix $V = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n) \in \text{Mat}(n, n+1, \mathbb{Z})$ whose columns satisfy the following condition

$$\sum_{j=0}^n q_j \mathbf{v}_j = \mathbf{0};$$

in particular the columns of V satisfy condition (4) of Theorem 3, hence generate a fan of $\mathbb{P}(Q)$.

4. there exists a unique choice of the previous matrices V and V^0 such that V^0 is in HNF with only nonnegative entries; then the column \mathbf{v}_0 in V admits only negative entries. Moreover the matrix $V' = (\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_0)$ is in HNF.

Proof. (1) is obvious. (2) follows from (1) by recalling the hypothesis

$$k_1 = \gcd(q_0, q_1, \dots, q_n) = 1.$$

To prove (3) we have to show the existence of an integral vector $\mathbf{v}_0 = \begin{pmatrix} v_{10} \\ \vdots \\ v_{n0} \end{pmatrix}$ and integers v_{jk} satisfying the following equations

$$(11) \quad \forall 1 \leq j \leq i-1 \quad q_0 v_{j0} + q_j + \sum_{k=i}^n q_k v_{jk} = 0$$

$$(12) \quad \forall i \leq j \leq n-1 \quad q_0 v_{j0} + q_j \frac{k_{j+1}}{k_j} + \sum_{k=j+1}^n q_k v_{jk} = 0$$

$$(13) \quad q_0 v_{n0} + q_n \frac{q_0}{k_n} = 0.$$

The last equation (13) is clearly satisfied by putting $v_{n0} = -q_n/k_n = -q_n/(q_0, q_n)$. The j -th equation in (12) admits integer solutions for $(v_{j0}, v_{j,j+1}, \dots, v_{jn})$ if and only if

$$\gcd(q_0, q_{j+1}, \dots, q_n) = k_{j+1} \mid q_j \frac{k_{j+1}}{k_j}$$

which is clearly true since $k_j \mid q_j$, by definition. Finally the j -th equation in (11) admits integer solutions for $(v_{j0}, v_{ji}, v_{j,i+1}, \dots, v_{jn})$ if and only if

$$\forall 1 \leq j \leq i-1 \quad \gcd(q_0, q_i, \dots, q_n) = k_i \mid q_j$$

which is clearly true since $k_i = 1$, by the previous part (2). Recall now that V^0 is a triangular matrix, giving

$$\det(V^0) = \prod_{j=1}^n v_{jj} = k_{i+1} \cdot \frac{k_{i+2}}{k_{i+1}} \cdots \frac{k_n}{k_{n-1}} \cdot \frac{q_0}{k_n} = q_0$$

which is enough to get condition (4) of Theorem 3 for the columns of V .

To prove (4) let us first of all observe that, for any $1 \leq j \leq i-1$, $\mathbf{v}_j = \mathbf{e}_j$, meaning that the first $i-1$ columns of V^0 are composed of nonnegative entries satisfying the HNF conditions. Moreover V^0 is upper triangular. Then it remains to prove that there exists a unique choice for v_{jk} such that

$$\forall k : i \leq k \leq n, \forall j : j < k \quad 0 \leq v_{jk} < v_{kk} .$$

The j -th equation in (12) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk} .$$

Fixing variables v_{jk} , for $j+1 \leq k \leq n-1$, the previous diophantine equation admits solutions for v_{j0}, v_{jn} if and only if

$$(14) \quad k_n = \gcd(q_0, q_n) \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk} .$$

Moreover, given a particular solution $v_{jn}^{(0)}$, all the possible integer solutions for v_{jn} are given by

$$v_{jn} = v_{jn}^{(0)} - \frac{q_0}{k_n} \cdot h_{jn} = v_{jn}^{(0)} - v_{nn} \cdot h_{jn}, \quad \forall h_{jn} \in \mathbb{Z} .$$

Divide $v_{jn}^{(0)}$ by v_{nn} . Then the remainder of such a division gives a unique choice for v_{jn} such that

$$\forall i \leq j \leq n-1 \quad 0 \leq v_{jn} < v_{nn} .$$

Analogously the j -th equation in (11) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j - \sum_{k=i}^{n-1} q_k v_{jk}$$

and the same argument ensures the existence of a unique choice for v_{jn} such that

$$\forall 1 \leq j \leq i-1 \quad 0 \leq v_{jn} < v_{nn} .$$

Then the last column in V^0 can be uniquely chosen with non-negative entries satisfying the HNF condition. Iteratively, condition (14) is satisfied if and only if there exist integer solutions for x, v_{jk} in the diophantine equation

$$k_n x + q_{n-1} v_{j,n-1} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk}$$

which is if and only if

$$\gcd(k_n, q_{n-1}) = \gcd(q_0, q_{n-1}, q_n) =: k_{n-1} \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk} .$$

In particular, given a solution $v_{j,n-1}^{(0)}$, all the possible integer solutions for $v_{j,n-1}$ are given by

$$v_{j,n-1} = v_{j,n-1}^{(0)} - \frac{k_n}{k_{n-1}} \cdot h_{j,n-1} = v_{j,n-1}^{(0)} - v_{n-1,n-1} \cdot h_{j,n-1}, \quad \forall h_{j,n-1} \in \mathbb{Z}.$$

Therefore, the division algorithm ensures the existence of a unique choice for $v_{j,n-1}$ such that

$$\forall i \leq j \leq n-1 \quad 0 \leq v_{j,n-1} < v_{n-1,n-1}.$$

The same argument ensures the existence of a unique choice for $v_{j,n-1}$ such that

$$\forall 1 \leq j \leq i-1 \quad 0 \leq v_{j,n-1} < v_{n-1,n-1}.$$

Then the $(n-1)$ -st column in V^0 can be uniquely chosen with non-negative entries satisfying the HNF condition. By completing the iteration, V_0 can then be uniquely chosen in HNF. Consequently \mathbf{v}_0 has to necessarily admit only negative entries. To prove that V' is in HNF it suffices to observe that, for V' , the function $f: \{1, \dots, n\} \rightarrow \{1, \dots, n+1\}$, in Definition 1, is given by setting $f(i) = i$, for any $1 \leq i \leq n$. Then V' is in HNF if and only if V^0 is in HNF, since there are no condition for the entries of \mathbf{v}_0 which is the $(n+1)$ -st column of V' . \square

REMARK 4. When the weight vector Q is fixed, a significant consequence of Proposition 6 is that the fan of $\mathbb{P}(Q)$ presented in (4) is unique and is given by the HNF of a matrix V associated with any fan of $\mathbb{P}(Q)$. Clearly the uniqueness of the Q -canonical fan of $\mathbb{P}(Q)$ depends on the weights order in Q . Then we can't define a *canonical* fan of $\mathbb{P}(Q)$ but just a Q -canonical one.

EXAMPLE 1. Let us apply the Proposition 6 to produce by hand the Q -canonical fan (hence a fan) of $\mathbb{P}(Q)$ for $Q = (2, 3, 4, 15, 25)$. First of all observe that in this case

$$k_1 = \gcd(Q) = 1, \quad k_2 = d_1 = 1, \quad k_3 = \gcd(2, 15, 25) = 1, \quad k_4 = \gcd(2, 25) = 1.$$

The matrix V' in Proposition 6(4) is in HNF, then it looks as follows

$$V' = \begin{pmatrix} 1 & 0 & 0 & v_{1,3} & v_{1,0} \\ 0 & 1 & 0 & v_{2,3} & v_{2,0} \\ 0 & 0 & 1 & v_{3,3} & v_{3,0} \\ 0 & 0 & 0 & 2 & v_{4,0} \end{pmatrix}$$

with $0 \leq v_{k,3} \leq 1$, for $1 \leq k \leq 3$. Moreover we get the following conditions

$$\begin{aligned} 0 &= q_0 v_{4,0} + \frac{q_0 q_4}{k_4} = 2v_{4,0} + 50 \Rightarrow v_{4,0} = -25 \\ 0 &= q_0 v_{3,0} + \frac{k_3 q_3}{k_4} + q_4 v_{3,3} = 2v_{3,0} + 15 + 25v_{3,3} \Rightarrow v_{3,3} = 1 \text{ and } v_{3,0} = -20 \\ 0 &= q_0 v_{2,0} + \frac{k_2 q_2}{k_3} + q_4 v_{2,3} = 2v_{2,0} + 4 + 25v_{2,3} \Rightarrow v_{2,3} = 0 \text{ and } v_{2,0} = -2 \\ 0 &= q_0 v_{1,0} + \frac{k_1 q_1}{k_2} + q_4 v_{1,3} = 2v_{1,0} + 3 + 25v_{1,3} \Rightarrow v_{1,3} = 1 \text{ and } v_{1,0} = -14 \end{aligned}$$

giving the following Q -canonical fan for $\mathbb{P}(2, 3, 4, 15, 25)$:

$$\Sigma = \text{fan} \left(\left(\begin{pmatrix} -14 \\ -2 \\ -20 \\ -25 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right) .$$

REMARK 5. Proposition 6 above has to be compared with results in §3 and §4 of [7]. In particular parts from (1) to (3) give a rewrite of Proposition 3.2, Remark 3.3 and Theorem 3.6 in [7]. For what concerns part (4), although it apparently looks to be related with Remark 4.3 and Theorem 4.5 in [7], it seems to us to be a rather new result in the literature. In fact Conrads, in §4 of [7], discusses HNFs of square matrices in $\text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$ since he's interested in classifying isomorphism classes of simplices of a given type. Here the aim is quite different since we study HNF of the simplex itself, which turns out to be unique and then identifying the Q -canonical fan of $\mathbb{P}(Q)$.

3. Characterization of polytopes giving $\mathbb{P}(Q)$

3.1. From fans to polytopes and back

We shall use the following notation: given a $n \times (n + 1)$ matrix $V = (\mathbf{v}_0, \dots, \mathbf{v}_n) = (v_{ij})$ with $1 \leq i \leq n, 0 \leq j \leq n$, the $n \times n$ sub-matrix of V obtained by removing the first column is denoted by $V^0 = (\mathbf{v}_1, \dots, \mathbf{v}_n) = (v_{ik})$ with $1 \leq i \leq n$ and $1 \leq k \leq n$.

DEFINITION 5. Let $V \in \text{Mat}(n, n + 1, \mathbb{Z})$ be a matrix whose maximal minors do not vanish i.e., in the same notation given above, $V_l \neq 0$ for every $0 \leq l \leq n$. Consider the vector of absolute values of maximal minors $Q = (|V_0|, \dots, |V_n|)$. Recalling 1.3, the $(0, Q)$ -weighted transverse matrix of V (or simply weighted transverse) is defined to be the following $n \times n$ rational matrix

$$(V^0)_Q^* := (V^0)^* \cdot (\delta I_Q^0)$$

where $I_Q^0 := \text{diag}(1/|V_1|, \dots, 1/|V_n|)$ and $\delta := \text{lcm}(|V_0|, \dots, |V_n|)$.

REMARK 6. If $V \in \mathfrak{V}_n$, as defined in the Definition 4, then Theorem 4 below implicitly shows that the weighted transverse matrix $(V^0)_Q^*$ is a $n \times n$ integral matrix. In particular this fact is also proved explicitly in the following Proposition 7.

PROPOSITION 7. If $V = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$ is a fan matrix of $\mathbb{P}(Q)$, with $Q = (q_0, \dots, q_n)$, then the weighted transverse $(V^0)_Q^*$ has integral entries.

Proof. Recall that the adjoint matrix of an invertible square matrix A is defined by setting $\text{Adj}(A) := \det(A) A^{-1}$. Set $W = \text{Adj}(V^0)$ and let \mathbf{w}_i be the i -th row of W .

Observe that parts (3) and (4) in Theorem 3 give, for $i = 1, \dots, n$,

$$\begin{aligned} |\mathbf{w}_i \cdot \mathbf{v}_i| &= |\det(V^0)| = q_0 \\ \mathbf{w}_i \cdot \mathbf{v}_k &= 0 \quad \text{for } 1 \leq k \leq n \text{ and } k \neq i \\ |\mathbf{w}_i \cdot \mathbf{v}_0| &= |\det(V^0)| \frac{q_i}{q_0} = q_i, \end{aligned}$$

where the dot product is the usual matrix product. Therefore $\frac{\delta}{q_0 q_i} \mathbf{w}_i \cdot \mathbf{v}_j \in \mathbb{Z}$ for any $0 \leq j \leq n$. This means that

$$\forall 1 \leq i \leq n \quad \frac{\delta}{q_0 q_i} \mathbf{w}_i \in \mathbb{Z}^n$$

since $\mathcal{L}(\mathbf{v}_0, \dots, \mathbf{v}_n) = \mathbb{Z}^n$. The proof concludes by transposing W . \square

Let us denote by $\mathcal{O}_{\mathbb{P}(Q)}(1)$, or $\mathcal{O}(1)$ for short, the generator of the Picard group $\text{Pic}(\mathbb{P}(Q)) \cong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}(Q)}(1)$.

PROPOSITION 8. *If D_j is the torus invariant divisor associated with $\langle \mathbf{v}_j \rangle \in \Sigma(1)$ then $(\delta'/q'_j)D_j$ is an ample divisor in the linear system $|\mathcal{O}_{\mathbb{P}(Q)}(1)|$, where as usual $Q' = (q'_0, \dots, q'_n)$ is the reduced weight vector of Q and $\delta' = \text{lcm}(Q')$.*

Proof. Recall the exact sequence (10) showing that the Chow group of $\mathbb{P}(Q)$ is given by $A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$. By construction, the morphism $d : \bigoplus_{i=0}^n \mathbb{Z} \cdot D_j \rightarrow \mathbb{Z}$ sends a Weil divisor $\sum_{j=0}^n b_j D_j$ to the generator $1 \in \mathbb{Z}$ if and only if (b_0, \dots, b_n) is a solution of the diophantine equation $\sum_{j=0}^n q'_j x_j = 1$. It is a well known fact that the Picard group of a normal toric variety can be identified with the subgroup of $A_{n-1}(X)$ generated by the classes of torus invariant Cartier divisors (see e.g. [15] § 3.4, [10] § 4.2). In particular $\text{Pic}(\mathbb{P}(Q)) \subset A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$ is a free cyclic subgroup. Then a generator of $\text{Pic}(\mathbb{P}(Q))$ is given by a suitable multiple kD of a generator D of $A_{n-1}(\mathbb{P}(Q))$, where k is the least positive integer such that kD is Cartier. There is a Criterion to determine when a Weil divisor of a toric variety is a Cartier divisor ([17] Prop. 2.4) which applied to the case of $\mathbb{P}(Q)$ can be rewritten as follows:

$$(15) \quad \sum_{j=0}^n b_j D_j \text{ is a Cartier divisor} \iff \forall 0 \leq l \leq n \exists \mathbf{u}_l \in M : \forall j \neq l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = b_j$$

where \mathbf{n}_j is a generator of the monoid $\langle \mathbf{v}_j \rangle \cap N$. Recall the exact sequence (10) and let $D = \sum_{j=0}^n b_j D_j$ be a generator of $A_{n-1}(\mathbb{P}(Q))$, i.e. $d(D) = 1$, and consider the positive integer multiple kD . Then (15) gives that kD is a Cartier divisor if and only if, for every $l = 0, \dots, n$,

$$(16) \quad \exists \mathbf{u}_l \in M : \forall j \neq l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = kb_j.$$

Since $\text{div}(\mathbf{u}_l) = \sum_{j=0}^n \langle \mathbf{u}_l, \mathbf{n}_j \rangle D_j$, then (16) is equivalent to requiring that

$$(17) \quad \exists \mathbf{u}_l \in M : \text{div}(\mathbf{u}_l) = \sum_{j \neq l} kb_j D_j + \langle \mathbf{u}_l, \mathbf{n}_l \rangle D_l = kD + (\langle \mathbf{u}_l, \mathbf{n}_l \rangle - kb_l) D_l.$$

The exactness of (10) ensures that (17) is equivalent to asking that

$$(18) \quad \begin{aligned} \exists \mathbf{u}_l \in M & : d(kD + (\langle \mathbf{u}_l, \mathbf{n}_l \rangle - kb_l)D_l) = 0 \\ \Leftrightarrow k &= q'_l(kb_l - \langle \mathbf{u}_l, \mathbf{n}_l \rangle) . \end{aligned}$$

Then (18) gives that kD is Cartier if and only if $q'_l \mid k$ for every $0 \leq l \leq n$. Then the inclusion $\text{Pic}(\mathbb{P}(Q)) \hookrightarrow A_{n-1}(\mathbb{P}(Q))$ turns out to be the multiplication by δ' . To complete the proof, notice that D_j and $q'_j D$ give the same class in $A_{n-1}(\mathbb{P}(Q))$: in fact $d(D_j - q'_j D) = 0$. Then $(\delta'/q'_j)D_j$ and $\delta' D$ give the generator of $\text{Pic}(\mathbb{P}(Q)) = \delta' A_{n-1}(\mathbb{P}(Q))$. In particular $(\delta'/q'_j)D_j \in |\mathcal{O}_{\mathbb{P}(Q)}(1)|$. This also suffices to prove that $(\delta'/q'_j)D_j$ is ample. \square

Set Δ_j be the integral polytope associated with the divisor $H = (\delta'/q'_j)D_j$, as in (4). One can easily check that there exist n points $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$, depending on the choice of D_j , such that Δ_j is the convex hull $\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$: in particular *the ampleness of $(\delta'/q'_j)D_j$ implies that $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a set of n linearly independent integral vectors* ([17] Corollary 2.14).

Let \mathfrak{P}_n be the set of integral polytopes in $M_{\mathbb{R}}$ obtained as the convex hull of the origin and n linearly independent integral vectors and $\mathfrak{F}(Q)$ be the set of fans in $N_{\mathbb{R}}$ defining $\mathbb{P}(Q)$. Then we have established maps

$$(19) \quad \begin{aligned} \forall 0 \leq j \leq n, \quad \Delta_Q^j : \mathfrak{F}(Q) &\longrightarrow \mathfrak{P}_n \\ \Sigma &\longmapsto \Delta_Q^j(\Sigma) := \Delta_j \end{aligned}$$

Let $W = (w_{ik})$ be the $n \times n$ matrix of the components of vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$ over the dual basis: namely

$$\forall k = 1, \dots, n \quad \mathbf{w}_k = \sum_{i=1}^n w_{ik} \mathbf{e}_i^{\vee}$$

where $\{\mathbf{e}_1^{\vee}, \dots, \mathbf{e}_n^{\vee}\}$ is the dual basis of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we get the following representation of the map Δ_Q^0 :

THEOREM 4. *Given the fan $\Sigma := \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathfrak{F}(Q)$, the image $\Delta_Q^0(\Sigma)$ defined in (19) is the convex hull $\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$ of the origin with the n linearly independent integral vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$ giving the columns of the $(0, Q)$ -weighted transverse matrix of $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$, i.e.*

$$W = (V^0)_Q^*,$$

where $Q = (|V_0|, \dots, |V_n|)$. Namely the entries of W are given by

$$\forall 1 \leq i \leq n, 1 \leq k \leq n \quad w_{ik} = \frac{\delta V_{ik}^0}{q_k V_0}$$

where V_{ik}^0 is the cofactor of v_{ik} in V^0 and $V_0 = \det(V^0) = \pm q_0$ (by either (3) or (4) in Theorem 3).

Proof. Recalling (4), to define $\Delta_Q^0(\Sigma) = \Delta_0$ one has to write down the hyperplanes of $M_{\mathbb{R}}$

$$(20) \quad \forall \rho \in \Sigma(1) \quad \langle \mathbf{u}, \mathbf{n}_\rho \rangle = -a_\rho, \quad \text{where } \mathbf{n}_\rho \text{ generates } \rho \cap N,$$

for the divisor $H = (\delta'/q'_0)D_0$. Since $\Sigma(1) = \{\langle \mathbf{v}_j \rangle \subset N_{\mathbb{R}} \mid j = 0, \dots, n\}$ the hyperplanes (20) are then given by

$$(21) \quad \begin{aligned} \sum_{i=1}^n n_{i0} u_i &= -\delta'/q'_0 \\ \forall k = 1, \dots, n \quad \sum_{i=1}^n n_{ik} u_i &= 0 \end{aligned}$$

where $\mathbf{n}_j = \sum_{i=1}^n n_{ij} \mathbf{e}_i$ generates the 1-dimensional cone $\langle \mathbf{v}_j \rangle \cap N$. In the part (c) of Lemma 1 it has been observed that $q'_0 \mathbf{n}_0 = -\sum_{k=1}^n q'_k \mathbf{n}_k$. Then the first equation in (21) can be rewritten as follows

$$\sum_{i=1}^n \left(\sum_{k=1}^n q'_k n_{ik} \right) u_i = \delta'.$$

Let us represent equations in (21) by the following $(n+1) \times (n+1)$ -matrix

$$M = \left(\begin{array}{ccc|c} \sum_{k=1}^n q'_k n_{1k} & \cdots & \sum_{k=1}^n q'_k n_{nk} & \delta' \\ n_{11} & \cdots & n_{n1} & 0 \\ & \vdots & & \vdots \\ n_{1n} & \cdots & n_{nn} & 0 \end{array} \right).$$

For $j = 0, 1, \dots, n$, the vertex \mathbf{w}_j of $\Delta_Q^0(\Sigma)$ is then given by the (unique, for (3) in Theorem 3 and recalling that $v_{ij} = d_j n_{ij}$) solution of the linear system associated with the matrix M^{j+1} , obtained removing the $(j+1)$ -st row in M . Clearly $\mathbf{w}_0 = 0$. For $j = k = 1, \dots, n$ we get

$$w_{ik} = M_{k+1,i} / M_{k+1,n+1}$$

where $M_{a,b}$ is the (a,b) -cofactor in M . Observe that $M_{k+1,n+1} = (-1)^{k-1} q'_k V_0 / a_0$ and $M_{k+1,i} = (-1)^{k+1} \delta' d_k V_{ik}^0 / a_0$. Then

$$w_{ik} = \frac{\delta' d_k V_{ik}^0}{q'_k V_0} = \frac{\delta' a_k d_k}{q_k} v_{ik}^* = \frac{\delta}{q_k} v_{ik}^*$$

where $v_{ik}^* = V_{ik}^0 / V_0$ is the (i,k) -entry of $V^{0*} := ((V^0)^{-1})^T$. The last equality on the right is obtained by recalling Proposition 3(5) and Proposition 4. \square

REMARK 7. Clearly same conclusions as in Theorem 4 can be obtained by exchanging 0 with any other value j such that $0 \leq j \leq n$.

REMARK 8. Let Q be a weight vector whose reduction is given by Q' . Consider $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathfrak{F}(Q)$ and, for any $0 \leq j \leq n$, consider the generator \mathbf{n}_j of the semigroup $\langle \mathbf{v}_j \rangle \cap N$, where N is the lattice generated by $\mathbf{v}_0, \dots, \mathbf{v}_n$. Then Lemma 1(c) and Theorem 3 ensure that $\Sigma := \text{fan}(\mathbf{n}_0, \dots, \mathbf{n}_n) \in \mathfrak{F}(Q')$. Then the previous Theorem 4 gives that

$$\Delta_Q^0(\Sigma) = \Delta_{Q'}^0(\Sigma)$$

since, recalling once again Propositions 3 and 4,

$$w_{ik} = (\delta/q_k)(V_{ik}^0/V_0) = (\delta'a/q'_k a_k)(N_{ik}^0/d_k N_0) = (\delta'/q'_k)(N_{ik}^0/N_0)$$

(here N denotes the matrix $N = (\mathbf{n}_0, \dots, \mathbf{n}_n)$).

EXAMPLE 2. Let us still consider Example 1 to apply the weighted transversion and Theorem 4 for producing by hand a polytope of a given wps $\mathbb{P}(Q)$ with the minimal polarization.

Recall that $Q = (2, 3, 4, 15, 25)$ and the matrix fan obtained in the Example 1 is

$$V = \begin{pmatrix} -14 & 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & 0 \\ -20 & 0 & 0 & 1 & 1 \\ -25 & 0 & 0 & 0 & 2 \end{pmatrix} \implies (V^0)^* = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

Since $\delta = \text{lcm}(2, 3, 4, 15, 25) = 300$, we get

$$W = (V^0)^*_Q = (V^0)^* \cdot \delta I_Q = 150 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/15 & 0 \\ 0 & 0 & 0 & 1/25 \end{pmatrix}$$

$$\text{giving } W = \begin{pmatrix} 100 & 0 & 0 & 0 \\ 0 & 75 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ -50 & 0 & -10 & 6 \end{pmatrix}. \text{ Then the polytope we are looking for is}$$

$$\Delta = \text{Conv} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 0 \\ 0 \\ -50 \end{pmatrix}, \begin{pmatrix} 0 \\ 75 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 20 \\ -10 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right).$$

More precisely $(\mathbb{P}_\Delta, \mathcal{O}(1)) \cong (\mathbb{P}(Q), \delta/q_0 D_0) = (\mathbb{P}^4(2, 3, 4, 15, 25), 150 D_0)$.

DEFINITION 6 (P -admissible matrices). A square matrix $W \in \text{Mat}(n, n, \mathbb{Z})$ is called P -admissible if there exists an F -admissible matrix $V \in \mathfrak{V}_n$ such that W is the weighted transverse matrix of V , which is

$$W = (V^0)^*_Q \quad \text{with} \quad Q = (|V_0|, \dots, |V_n|).$$

In other words $W = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ is admissible if and only if the polytope

$$\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$$

belongs to the image of the map Δ_Q^0 , as defined in (19). In this case we say that Q, W, V are associated to each other.

Let us denote $\mathfrak{W}_n \subset \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$ the subset of P -admissible matrices: notice that any such matrix has integer entries by either Theorem 4 or the following Proposition 7.

REMARK 9. Remark 8 guarantees that weight vectors Q_1 and Q_2 admitting the same reduction Q' are associated with the same P -admissible matrix W , which is the P -admissible matrix associated with the reduced weight vector Q' . Conversely, there exists a unique reduced weight vector Q' to which W is associated. Proposition 9(c) will prove this fact in a purely algebraic setting; moreover Proposition 9(b) will exhibit a constructive method for finding Q' .

DEFINITION 7. Consider a matrix $W \in \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$. Let s_i be the gcd of entries in the i -th row of $\text{Adj}(W)$. Then we define the reduced adjoint of W as follows

$$\begin{aligned} \widehat{W} &:= \frac{|\det(W)|}{\det(W)} \text{diag}\left(\frac{1}{s_1}, \dots, \frac{1}{s_n}\right) \cdot \text{Adj}(W) \\ &= \text{diag}\left(\frac{|\det(W)|}{s_1}, \dots, \frac{|\det(W)|}{s_n}\right) \cdot W^{-1} \end{aligned}$$

Notice that if V is a square matrix in $\text{Mat}(n, n, \mathbb{Z})$ such that $V \cdot W$ is a diagonal matrix with positive entries then

$$(22) \quad V = \text{diag}(r_1, \dots, r_n) \cdot \widehat{W}$$

for some $r_1, \dots, r_n \in \mathbb{N}$.

PROPOSITION 9. Let W be a P -admissible matrix and let $Q = (q_0, \dots, q_n)$ be a reduced weight vector associated to W . Then

$$(a) \quad \left(\widehat{W}^T\right)_Q^* = W;$$

(b) if $s := \gcd(s_1, \dots, s_n)$ is the greatest common divisor of the terms in $\text{Adj}(W)$ then

$$q_0 = |\det(\widehat{W})| \quad , \quad \forall 1 \leq i \leq n \quad q_i = \frac{s_i}{s} \quad , \quad \text{lcm}(Q) = \frac{|\det(W)|}{s}$$

(c) if Q_1 and Q_2 are reduced weight vectors associated with the same P -admissible matrix W , then $Q_1 = Q_2$;

(d) there exists a unique F -admissible matrix V associated with W and Q i.e. such that $W = (V^0)_Q^*$ with $Q = (|V_0|, \dots, |V_n|)$.

Proof. (a). W is a P -admissible matrix. Then there exists a F -admissible matrix V such that $W = ((V^0)^T)^{-1} \delta I_Q$ and $Q = (|V_0|, \dots, |V_n|)$, meaning that $(V^0)^T W = \delta I_Q^0$ is diagonal with positive entries. Recalling (22) we get that $(V^0)^T = \text{diag}(r_1, \dots, r_n) \cdot \widehat{W}$ for some $r_1, \dots, r_n \in \mathbb{N}$. But Q is reduced, which implies that the columns of V^0 have coprime entries. Therefore $r_1 = \dots = r_n = 1$ and $(V^0)^T = \widehat{W}$. (a) follows immediately. (b) On the one hand $\widehat{W} \cdot W = \text{diag}(|\det(W)|/s_1, \dots, |\det(W)|/s_n)$. On the other hand, by (a), $\widehat{W} = (V^0)^T$ and $\widehat{W} \cdot W = \text{diag}(\delta/q_1, \dots, \delta/q_n)$, where $\delta := \text{lcm}(Q)$. Therefore

$$(23) \quad \forall 1 \leq i \leq n \quad \frac{\delta}{q_i} = \frac{|\det(W)|}{s_i}.$$

Observe now that

$$\begin{aligned} \text{lcm}\left(\frac{\delta}{q_1}, \dots, \frac{\delta}{q_n}\right) &= \frac{\delta}{\text{gcd}(q_1, \dots, q_n)} = \delta \\ \text{lcm}\left(\frac{|\det(W)|}{s_1}, \dots, \frac{|\det(W)|}{s_n}\right) &= \frac{|\det(W)|}{s} \end{aligned}$$

Then (23) gives that $\delta = |\det(W)|/s$ and, for any $1 \leq i \leq n$, $q_i = s_i/s$. Finally (a) gives that $q_0 = |V_0| = |\det(\widehat{W})|$.

(c) follows immediately by the previous part (b).

(d). If there exist two F -admissible matrix U, V such that they are both associated with W and Q , then

$$(\mathbf{v}_1, \dots, \mathbf{v}_n) = V^0 = U^0 = (\mathbf{u}_1, \dots, \mathbf{u}_n) \Rightarrow \mathbf{v}_0 = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{v}_i = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{u}_i = \mathbf{u}_0$$

implying that $V = U$. □

REMARK 10. In a sense the previous Proposition 9 states that, when restricted to wps fans associated with *reduced* weight vector, the weighted transversion process giving a polytope starting from a fan, can be inverted by considering the *transposed reduced adjoint* of the polytope matrix. Namely if W is a polytope matrix of $(\mathbb{P}(Q), \mathcal{O}(1))$, with Q reduced, then $V := \left(\begin{array}{c|c} \mathbf{v}_0 & \widehat{W}^T \end{array} \right)$ is a fan matrix of $\mathbb{P}(Q)$ when \mathbf{v}_0 is defined by setting $\mathbf{v}_0 = -(\sum_{i=1}^n q_i \mathbf{v}_i)/q_0$, where $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \widehat{W}^T$.

The following Proposition 10 shows criteria for a matrix W to be P -admissible.

PROPOSITION 10. *Let $W = (w_{ij}) \in \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$ be a matrix such that $\text{gcd}(w_{ij}) = 1$. Let s be the greatest common divisor of the entries in $\text{Adj}(W)$ and \mathbf{v} be the sum of the rows of $\text{Adj}(W)$. Define $q_0 = |\det(\widehat{W})|$, $\delta = \frac{|\det(W)|}{s}$. The following statements are equivalent:*

- (a) W is P -admissible;
- (b) the vector \mathbf{v} is divisible by $q_0 s$;

(c) q_0 divides δ and the vector $\frac{\delta}{q_0}(1, \dots, 1)$ is in the lattice generated by the rows of W .

Proof. (a) \Rightarrow (b): If W is P -admissible then there exist a unique reduced weight vector Q and a unique F -admissible matrix V , associated with W like in Definition 6. By Proposition 9, $\widehat{W} = (V^0)^T$ and $Q = (q_0, \dots, q_n)$ with $q_i = s_i/s$, for $i = 1, \dots, n$. Let \mathbf{v}_i be the i -th row of \widehat{W} ; then $\sum_{i=1}^n q_i \mathbf{v}_i$ is divisible by q_0 since $\widehat{W}^T = V^0$ is F -admissible, meaning that its columns satisfy the relation $\sum_{i=1}^n q_i \mathbf{v}_i = -q_0 \mathbf{v}_0$. Then $\sum_{i=1}^n s_i \mathbf{v}_i$ is divisible by $q_0 s$ and $s_i \mathbf{v}_i$ is the i -th row of $\text{Adj}(W)$.

(b) \Rightarrow (a): Assume that $q_0 s$ divides any entry in \mathbf{v} . For $1 \leq i \leq n$, let \mathbf{v}_i be the i -th row of \widehat{W} and $q_i = s_i/s$ be defined as in Proposition 9(b); then $\sum_{i=1}^n q_i \mathbf{v}_i$ is divisible by q_0 . Put $\mathbf{v}_0 = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{v}_i$. Then the matrix

$$V := \left(\mathbf{v}_0 \mid \widehat{W}^T \right) = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$$

turns out to be F -admissible with respect to Q by Theorem 3(4). Then $W = (V^0)_Q^*$ is P -admissible.

(b) \Leftrightarrow (c): the sum of the rows of $\text{Adj}(W)$ is the row vector $(1, \dots, 1) \cdot \text{Adj}(W) = (1, \dots, 1) \cdot \det(W) W^{-1}$. Thus it is divisible by $q_0 s$ if and only if there exists $(x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $(x_1, \dots, x_n) \cdot W = \frac{\delta}{q_0}(1, \dots, 1)$, that is if and only if (c) holds. \square

3.2. Characterizing the polytope of a polarized wps

Given an integral polytope $\Delta = \text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$, for a suitable subset $\{\mathbf{w}_1, \dots, \mathbf{w}_n\} \subset M$, let $W := (\mathbf{w}_1, \dots, \mathbf{w}_n)$ be the associated polytope matrix. Then the following result is a consequence of Propositions 9 and 10.

THEOREM 5. *Let $\Delta = \text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n) \subset M_{\mathbb{R}}$ be a n -dimensional integral polytope. Set $m := \gcd(w_{ij})$ and define $W' := \frac{1}{m}W$. Let $Q = (q_0, \dots, q_n)$ be the reduction of the weight vector defined as in Proposition 9(b). Then the following facts are equivalent:*

1. W' is a P -admissible matrix, hence it satisfies one of the equivalent conditions in Proposition 10,
2. $(\mathbb{P}_{\Delta}, \mathcal{O}(1)) \cong (\mathbb{P}(Q), \mathcal{O}(m))$.

Proof. (1) \Rightarrow (2): By definition if W' is P -admissible then there exists a F -admissible matrix V such that $W' = (V^0)_{\tilde{Q}}^*$, where $\tilde{Q} = (|V_0|, \dots, |V_n|)$. By Proposition 9(b) $\tilde{Q} = Q$. Then the polytope

$$\text{Conv}\left(\mathbf{0}, \frac{\mathbf{w}_1}{m}, \dots, \frac{\mathbf{w}_n}{m}\right)$$

belongs to the image of the map Δ_Q^0 , as defined in (19). Then W' is the polytope matrix of $\Delta_{D'}$ for some divisor $D' \in \mathcal{O}_{\mathbb{P}(Q)}(1)$ and $W = mW'$ is the polytope matrix of $\Delta := \Delta_{mD'}$ giving (2).

(2) \Rightarrow (1): There exists a divisor D of $\mathbb{P}(Q)$, belonging to the linear system $|O(m)|$, such that $\Delta = \Delta_D$. Moreover there exists a divisor $D' \in |O(1)|$ such that $D = mD'$ and in particular $\Delta = \Delta_D = m\Delta_{D'}$. This means that $\Delta_{D'} = \text{Conv}(\mathbf{0}, \mathbf{w}'_1, \dots, \mathbf{w}'_n)$ and $W' := (\mathbf{w}'_1, \dots, \mathbf{w}'_n) = \frac{1}{m}W$ is a P -admissible matrix associated with Q . \square

Acknowledgment. The authors would like to thank the referee for his careful reading of the manuscript and his pertinent remarks and corrections. In particular Theorem 5 and the proofs of Theorem 3 and Proposition 5 have been significantly improved by his observations.

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AMS Subject Classification: 14M25, 52B20

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Lavoro pervenuto in redazione il MM.GG.AAAA.