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A note on the existence of multiple solutions for a class of systems of second order ODEs

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Abstract

We prove the existence of multiple solutions for some systems of second order ODEs with Dirichlet boundary conditions. Such systems are obtained by coupling scalar ODEs with different growth conditions. The proof relies on a global continuation technique.

Keywords: Systems of ODEs, Multiplicity results, Continuation theorems
2000 MSC: 34B15

1. Introduction

This paper deals with the existence of multiple solutions to some classes of systems of second order ordinary differential equations of the type

$$x'' + \gamma(t, x) = 0, \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d,\ t \in [0, T], \tag{1.1}$$

together with Dirichlet boundary conditions

$$x(0) = x(T) = 0. \tag{1.2}$$

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In (1.1), $\gamma = (\gamma_1, \dots, \gamma_d) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function. Incidentally notice that, since we are not assuming $\gamma(t,x) = \nabla_x \Gamma(t,x)$, system (1.1) is not in general Hamiltonian and cannot be treated in a variational framework.

We are interested in situations in which the components $\gamma_i(t,x)$ of the vector field $\gamma(t,x)$ have a different behavior. For instance, systems which will be covered by our results are

$$\begin{cases} x_1'' + x_1^3 = p(t, x_1, x_2) \\ x_2'' - x_2^3 = r(t, x_1, x_2), \end{cases} \tag{1.3}$$

with p, r bounded, and

$$\begin{cases} x_1'' + q(t, x_1, x_2)(\beta x_1 + (\alpha - \beta)\arctan(x_1)) = 0 \\ x_2'' + \mu x_2 = r(t, x_1, x_2), \end{cases} \tag{1.4}$$

with $\mu \neq \left(\frac{j\pi}{T}\right)^2$ for every $j = 1, 2, \dots$ (that is, μ is not an eigenvalue of the differential operator $x_2 \mapsto -x_2''$ with Dirichlet boundary conditions on $[0,T]$) and r bounded, q positive, bounded and bounded away from zero and $\alpha, \beta > 0$ with $|\beta - \alpha|$ large enough.

The common feature of systems (1.3) and (1.4) is that the equation for $x_1(t)$ has (for any fixed continuous function $x_2(t)$) a large number of solutions, which can be distinguished through their nodal behavior. More precisely, the first equation in (1.3) is superlinear in x_1 and possesses infinitely many solutions, with an arbitrarily large number of zeros in $[0,T)$, while the first equation in (1.4) is asymptotically linear for x_1 near zero and near infinity, thus having a finite number (larger and larger as the quantity $\beta - \alpha$ increases) of solutions. These results are nowadays well known (see, among many others, [9, 18] for the superlinear case and [8, 16] for the asymptotically linear one). On the other hand, the equation for $x_2(t)$ (both in (1.3) and in (1.4), and for any fixed continuous function $x_1(t)$) is also solvable, but no multiplicity is in general available and the nodal properties of the solution found cannot be described. Such existence results can be established via topological degree theory, proving suitable a priori-bounds and showing that the associated global Leray-Schauder degree is equal to ± 1 (see, for instance, [13, 17]).

The main aim of the present paper is to show that corresponding results can be obtained for systems like (1.3) and (1.4), which couple (in suitable weak ways, via the terms p, q, r) such different scenarios. Multiple solutions (infinitely many for (1.3) and a finite number for (1.4)) will be detected and distinguished via the nodal properties of the component $x_1(t)$ (more in general, in case of system (1.1), of the components giving rise to multiplicity). To this aim, a global continuation technique in the framework of Leray-Schauder degree theory, introduced in [4] and developed in some subsequent papers [6, 14], will be used.

We point out that the plan of extending multiplicity results valid for scalar second order ODEs to weakly coupled second order systems has been already initiated in previous papers (see [3, 7, 15] for superlinear systems and [2] for asymptotically linear ones). However, to the best of our knowledge, this natural idea of coupling equations with different growth assumptions has not been developed yet. In particular, we think that the main novelty of our result is the coupling of equations with a large number of solutions with equations for which multiplicity is not available. From the point of view of the proof, this requires a slight variant in the continuation technique, matching the evaluation of some local degrees (for the components x_i giving rise to the multiplicity of solutions) with global ones (for the components x_j for which only existence can be proved).

The plan of the paper is the following. In Section 2, we state the main result, Theorem 2.1, together with some comments. For simplicity, we have chosen to deal with a system in $\mathbb{R}^3$, with a superlinear behavior in its first component, an asymptotically linear behavior (at zero and at infinity) in its second component, and global a priori-bounds for its third one. This case should show the main idea of the paper, keeping the notation at a reasonable level. In Remark 2.2, we briefly discuss how to extend the result to systems with more degrees of freedom. The final part of the Section is devoted to a concise description of the global continuation technique which is used in the proof. In Section 3, we prove the technical estimates from which Theorem 2.1 follows.

Notation. We denote by $C^1_0([0,T];\mathbb{R}^d)$ ($C^1_0([0,T])$ if $d=1$) the Banach space of all functions $x:[0,T]\to\mathbb{R}^d$ of class C^1 such that $x(0)=x(T)=0$, endowed with the norm

$$\|x\| = \sup_{t\in[0,T]} \sqrt{|x(t)|^2+|x'(t)|^2}.$$

Here $|\cdot|$ stands for the Euclidean norm of an n-dimensional vector.

2. Statement of the main result

We consider the following system of ODEs

$$\begin{cases} u'' + f(u) = p(t,u,v,w) \\ v'' + q(t,u,v,w)g(v) = 0 \qquad (u,v,w) \in \mathbb{R}^3,\ t \in [0,T], \\ w'' + h(t,u,v,w) = 0, \end{cases} \tag{2.1}$$

where all the functions considered are continuous on their variables and real-valued. We are interested in the existence of solutions to (2.1) satisfying Dirichlet boundary conditions

$$u(0) = u(T) = 0, \quad v(0) = v(T) = 0, \quad w(0) = w(T) = 0. \tag{2.2}$$

The set of assumptions which we are going to consider on system (2.1) is the following:

(H_u) the function $f : \mathbb{R} \to \mathbb{R}$ is superlinear at infinity, i.e.

$$\lim_{|u|\to+\infty} \frac{f(u)}{u} = +\infty; \tag{2.3}$$

the function $p : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is bounded as a function of (u,v), precisely, for every $M > 0$ there exists $p_{\max}(M) > 0$ such that

$$|p(t,u,v,w)| \le p_{\max}(M), \qquad \text{for every } t \in [0,T],\ (u,v) \in \mathbb{R}^2,\ |w| \le M;$$

(H_v) the function $g : \mathbb{R} \to \mathbb{R}$ is asymptotically linear at zero and at infinity, i.e. $g(0) = 0$ and there exist $g_0, g_\infty > 0$ such that

$$\lim_{v\to 0} \frac{g(v)}{v} = g_0, \qquad \lim_{|v|\to+\infty} \frac{g(v)}{v} = g_\infty;$$

the function $q : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is positive, bounded and bounded away from zero, that is to say, there exist $q_{\min}, q_{\max} > 0$ such that

$$q_{\min} \le q(t,u,v,w) \le q_{\max}, \qquad \text{for every } t \in [0,T],\ (u,v,w) \in \mathbb{R}^3;$$

(H_w) there exist $R^* > 0$ and $\tilde{h} : [0,T] \times \mathbb{R}^3 \times [0,1] \to \mathbb{R}$ with $\tilde{h}(t,u,v,w,1) = h(t,u,v,w)$ and

$$\tilde{h}(t,u,v,w,0) = \mu w, \qquad \text{for } \mu \neq \left(\tfrac{j\pi}{T}\right)^2 \ \forall j \in \mathbb{N}_0,$$

such that for every $\lambda \in [0,1]$, for every $u, v \in C_0^1([0,T])$ and for every $w \in C_0^1([0,T])$ solving $w'' + \tilde{h}(t,u,v,w,\lambda) = 0$, it holds that $\|w\| \leq R^*$.

We recall that typical situations in which (H_w) is satisfied are the sublinear case

$$h(t,u,v,w) = k(w) + r(t,u,v,w),$$

with $k(w)w < 0$ for $|w|$ large and r bounded (with homotopy given by $\tilde{h}(t,u,v,w,\lambda) = \lambda h(t,u,v,w)$), and the non-resonant case

$$h(t,u,v,w) = \mu w + r(t,u,v,w),$$

with $\mu \neq \left(\tfrac{j\pi}{T}\right)^2$ for any $j \in \mathbb{N}_0$ and r bounded (with homotopy given by $\tilde{h}(t,u,v,w,\lambda) = \mu w + \lambda r(t,u,v,\lambda)$). It is also worth noticing that whenever one is able to find explicitly the constant R^* bounding $\|w\|$, then it is enough to verify the assumptions in (H_u) and (H_v) for $|w| \leq R^*$. This will be clear from the proof.

We are now in position to state the main result of the paper.

Theorem 2.1. *Assume $(H_u), (H_v), (H_w)$. Then there exists $n^* \in \mathbb{N}_0$ such that, for every $(n_u, n_v) \in \mathbb{N}^2$ with*

$$n_u \geq n^*, \qquad n_v \in \left(\frac{T}{\pi}\sqrt{q_{\max}\, g_0}, \frac{T}{\pi}\sqrt{q_{\min}\, g_\infty}\right),$$

(if any), there exist 4 solutions (u,v,w) to the boundary value problem (2.1)-(2.2) such that $u(t)$ has exactly n_u zeros on and $v(t)$ has exactly n_v zeros on $[0,T)$. Precisely, such solutions can be distinguished via the signs of the initial derivatives $u'(0), v'(0)$, according to the four possibilities $u'(0), v'(0) > 0$, $u'(0), v'(0) < 0$, $v'(0) < 0 < u'(0)$ and $u'(0) < 0 < v'(0)$.

Notice that solutions (u,v,w) are distinguished by means of the number of zeros of the components u and v; in general, due to the very mild assumptions on the function h in the equation satisfied by w, we cannot expect to be able to describe the oscillating properties of w.

Remark 2.1. We can rewrite the condition for the integer n_v in term of the spectrum of the linear differential operator $v \mapsto -v''$ with Dirichlet boundary conditions on $[0,T]$. Precisely, denoting by $\lambda_j = \left(\frac{j\pi}{T}\right)^2$ $(j \in \mathbb{N}_0)$ the eigenvalues and by $\sigma = \{\lambda_j\}_{j\in\mathbb{N}_0}$ the spectrum, we are assuming

$$(q_{\max}\, g_0, q_{\min}\, g_\infty) \cap \sigma \neq \emptyset. \tag{2.4}$$

Then, if j_1^*, j_2^* are, respectively, the smallest and the largest integer number such that

$$q_{\max}\, g_0 < \lambda_{j_1^*} \leq \lambda_{j_2^*} < q_{\min}\, g_\infty,$$

the conditions for n_v writes as $j_1^* \leq n_v \leq j_2^*$. The symmetric condition $(q_{\max}\, g_\infty, q_{\min}\, g_0) \cap \sigma \neq \emptyset$ could be considered, as well.

It is worth noticing that, if (2.4) is not satisfied, one can construct solutions to (2.1)-(2.2) of the type $(u, 0, w)$, with the same nodal information for u as in Theorem 2.1.

Remark 2.2. Variants of Theorem 2.1 can be obtained. For instance, one can consider a slightly different system, where the first equation in (2.1) is replaced by

$$u'' + q_1(t,u,v,w)f(u) = 0,$$

with f satisfying (2.3) and $f(0) = 0$, and q_1 fulfilling the same assumptions as q in hypothesis (H_v). In this case, suitable conditions on the behavior of f near zero can lead to precise estimates for the number n^* appearing in the statement of the theorem. For instance, if $f(u)/u \to 0$ for $u \to 0$, then it is possible to show (argue as in Proposition 3.3, but for the u-components of solutions) that $n^* = 1$, that is, we can obtain solutions (u, v, w) with $u(t) > 0$ for $t \in]0, T[$.

Another variant can be obtained by dealing with classes of systems in which only two of the three equations of (2.1) are present, namely, systems in (u, w) (compare with (1.3)), in (v, w) (see (1.4)) or in (u, v) (with, of course, the corresponding assumptions (H_u), (H_v) and (H_w) satisfied). In each case, the statement of the result has to be modified accordingly, giving respectively: two families of solutions (u, w), with nodal information on the component u; two families of solutions (v, w), with nodal information on v; four families of solutions (u, v) with nodal characterization for both u and v.

Finally, one could consider systems in higher dimension, like

$$\begin{cases} u_i'' + f_i(u_i) = p_i(t,u,v,w) \\ v_j'' + q_j(t,u,v,w)g_j(v) = 0 \\ w_k'' + h_k(t,u,v,w) = 0, \end{cases}$$

with $u = (u_1, \dots, u_{d_u}) \in \mathbb{R}^{d_u}$, $v = (v_1, \dots, v_{d_v}) \in \mathbb{R}^{d_v}$, $w = (w_1, \dots, w_{d_w}) \in \mathbb{R}^{d_w}$ and the functions f_i, p_i, q_j, g_j and h_k satisfying assumptions like the ones in (H_u), (H_v) and (H_w) of Theorem 2.1 (which corresponds to the case $d_u = d_v = d_w = 1$). Indeed, the equations for u_i give rise to a weakly-coupled superlinear system [3, 15] and the ones for v_j to a weakly-coupled asymptotically linear (at zero and at infinity) system [2]. In this case, $2^{(d_u+d_v)}$ solutions, with nodal information on the components u_i, v_j, can be provided. The proof of this result follows the same line of the one for Theorem 2.1.

Remark 2.3. We point out that, in a standard manner, multiple periodic solutions of (2.1) can be provided when the system exhibits suitable symmetry conditions. Precisely, if we assume that all the functions involved are defined for $t \in \mathbb{R}$, with $2T$-periodicity in time, and satisfy, for every $(t,u,v,w) \in \mathbb{R}^4$,

$$\begin{aligned} &f(u) = -f(-u), \quad g(v) = -g(-v), \\ &p(t,u,v,w) = -p(-t,-u,-v,-w), \quad h(t,u,v,w) = -h(-t,-u,-v,-w), \\ &q(t,u,v,w) = q(-t,-u,-v,-w), \end{aligned}$$

then it is easy to see that each solution (u,v,w) of (2.1)-(2.2) can be extended to an odd $2T$-periodic solution of the system.

In the absence of symmetry conditions, one could likely obtain (by arguing as in [5]) an *existence* result for T-periodic solutions to (2.1), while *multiplicity* cannot be in general obtained if (2.1) is not of Hamiltonian type. We remark that results proving the existence of multiple periodic solutions (with nodal characterization) to weakly coupled (Hamiltonian) systems of second order ODEs have appeared only very recently (see [1, 12]), but the arguments therein do not seem to be well suited to deal with systems like (2.1).

The proof of Theorem 2.1 follows from the application of a continuation theorem given in [3] (on the lines of [4, 6, 14]) for an abstract equation of the form

$$x = \mathcal{N}(x,\lambda), \tag{2.5}$$

where X is a Banach space and $\mathcal{N} : X \times [0,1] \longrightarrow X$ is a completely continuous operator. It is standard to prove that (2.12) can be written in the form (2.5), for a suitable choice of $\mathcal{N}$, with $X = \{x = (u,v,w) \in C^1([0,T],\mathbb{R}^3) : x(0) = x(T) = 0\}$.

For the statement of the continuation theorem, we shall consider two open sets A and B such that $A \subset \bar{A} \subset B \subset \bar{B}$ and $(\bar{B} \setminus A) \subset X$. Let Σ be the set of the solutions of (2.5), i.e.

$$\Sigma = \{(x,\lambda) \in X \times [0,1] : \ x = \mathcal{N}(x,\lambda)\}$$

and, for any subset $D \subset X \times [0,1]$, let us denote the section of D at $\lambda \in [0,1]$ by $D_\lambda = \{x \in X : (x,\lambda) \in D\}$; we also set $\mathcal{N}_\lambda = \mathcal{N}(\cdot,\lambda)$. We have the following:

Theorem 2.2. *(Th. 3.4 in [3]) Let $\mathbf{k} : \Sigma \cap (\bar{B} \setminus A) \longrightarrow \mathbb{N}^2$ be a continuous function; suppose that there exists $n \in \mathbb{N}^2$ satisfying the following conditions:*

$$n \notin \mathbf{k}(\partial(\bar{B} \setminus A)) \tag{2.6}$$

and

$$\mathbf{k}^{-1}(n) \quad \text{is bounded.} \tag{2.7}$$

Then, for an open set U_0^n such that $(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset (\bar{B} \setminus A)_0$ and $\Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0$, the Leray-Schauder degree $\deg(I - \mathcal{N}_0, U_0^n)$ is defined. If

$$\deg(I - \mathcal{N}_0, U_0^n) \neq 0, \tag{2.8}$$

then there is a continuum $C_n \subset \Sigma \cap (B \setminus \bar{A})$ whose projection on the λ-component covers $[0,1]$ and such that $\mathbf{k}(x,\lambda) = n$ for every $(x,\lambda) \in C_n$. In particular there exists at least one $\tilde{x} \in (B \setminus \bar{A})_1$ such that

$$\tilde{x} = \mathcal{N}(\tilde{x},1) \quad \text{and} \quad \mathbf{k}(\tilde{x},1) = n.$$

We point out that [3, Th. 3.4] actually dealt with the case $\mathbf{k} : \Sigma \cap (\bar{B} \setminus A) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{N}^3$ (having in mind the application to a weakly coupled superlinear system, with $\mathbf{k}$ taking into account the number of zeros of each component of a solution). However, the proof remains the same here, since only the discreteness of the codomain of the functional $\mathbf{k}$ matters. In our case, $\mathbf{k}(x) \in \mathbb{N}^2$ will take into account the number of zeros of u and v only, for $x = (u,v,w)$. For a proof of Theorem 2.2, we refer to [14].

In order to apply Theorem 2.2 we need to define a suitable homotopy; to this aim, let $\tilde{h}$ be the function given in (H_w) and let us define, for $\lambda \in [0,1]$,

$$\tilde{f}(u,\lambda) = \lambda f(u) + (1-\lambda)u^3, \tag{2.9}$$

(for technical reasons, we will assume henceforth that $f(u)u > 0$ for every $u \neq 0$; this is not restrictive, since it can by achieved by modifying f in a compact neighborhood of $u = 0$ and adding a corresponding bounded term to the function p) and

$$\tilde{g}(t,u,v,w,\lambda) = \lambda q(t,u,v,w)g(v) + (1-\lambda)\hat{g}(v), \tag{2.10}$$

where $\hat{g} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\hat{g}(v)v > 0$ for $v \neq 0$ and

$$\lim_{v\to 0} \frac{\hat{g}(v)}{v} = q_{\max}\, g_0, \qquad \lim_{|v|\to+\infty} \frac{\hat{g}(v)}{v} = q_{\min}\, g_\infty. \tag{2.11}$$

For every $\lambda \in [0,1]$ we then consider the system

$$\begin{cases} u'' + \tilde{f}(u,\lambda) = \lambda p(t,u,v,w) \\ v'' + \tilde{g}(t,u,v,w,\lambda) = 0 \\ w'' + \tilde{h}(t,u,v,w,\lambda) = 0, \end{cases} \tag{2.12}$$

Of course, system (2.12) for $\lambda = 1$ just coincides with system (2.1), while for $\lambda = 0$ it reduces to the autonomous uncoupled system

$$\begin{cases} u'' + u^3 = 0 \\ v'' + \hat{g}(v) = 0 \\ w'' + \mu w = 0, \end{cases} \tag{2.13}$$

For this system, it is possible to construct suitable open sets U_0^n such that the degree condition (2.8) is fulfilled (the precise definition will be given in Section 3 - see (3.14) - along the proof of the main result).

3. Qualitative properties of the solutions and proof of the result

We start this Section with the introduction of the so-called elastic properties for the components u and v of the solutions to system (2.12) (see Lemma 3.1 and Lemma 3.2).

The proof of these properties is based on the following version of Gronwall's lemma: if $e : I \to \mathbb{R}$ is a function of class C^1 (where $I \subset \mathbb{R}$ is an interval) and $L > 0$ a constant such that $|e'(t)| \le Le(t)$, then $e(t) \le e(t_0)\exp(L|t-t_0|)$ for every $t, t_0 \in I$. This fact is an easy consequence of the monotonicity of the function $e_1(t) = e(t)/\exp(L|t-t_0|))$; indeed, e_1 is non-increasing for $t > t_0$ and non-decreasing for $t < t_0$.

Lemma 3.1. *For every $R > 0$ and $\mu > 0$, there exists $\rho(R,\mu) \ge R$ such that, for every $(u,v,w,\lambda) \in \Sigma$,*

$$\min_{t\in[0,T]}(\mu^2 u(t)^2 + u'(t)^2) \le R^2 \Longrightarrow \max_{t\in[0,T]}(\mu^2 u(t)^2 + u'(t)^2) \le \rho(R,\mu)^2. \quad (3.1)$$

Proof. Let us set $\tilde F(x,\lambda) = \int_0^x \tilde f(u,\lambda)\,du$. In view of (H_u), $\tilde F(x,\lambda) \to +\infty$ for $|x| \to +\infty$, uniformly in $\lambda \in [0,1]$; moreover, $\tilde F(x,\lambda) > 0$ for $x \ne 0$. We define the function

$$E(x,y,\lambda) = \frac{1}{2}y^2 + \tilde F(x,\lambda) + \frac{1}{2}p_{\max}(R^*)^2;$$

we have $E(x,y,\lambda) \to +\infty$ for $x^2+y^2 \to +\infty$, uniformly in $\lambda \in [0,1]$. For $(u,v,w,\lambda) \in \Sigma$, let us take $t_0, t_1 \in [0,T]$ such that

$$\mu^2 u(t_0)^2 + u'(t_0)^2 = R^2, \quad \mu^2 u(t_1)^2 + u'(t_1)^2 = \max_{t\in[0,T]}(\mu^2 u(t)^2 + u'(t)^2) > R^2$$

(otherwise, one could take $\rho(R,\mu) = R$). For $e(t) = E(u(t),u'(t),\lambda)$ (it is not necessary to emphasize the dependence on λ), one has, in view of (H_u) and (H_w)

$$\begin{aligned} |e'(t)| &= |u'(t)(u''(t) + \tilde f(u(t),\lambda))| = |\lambda u'(t) p(t,u(t),v(t),w(t))| \\ &\le \frac{1}{2}u'(t)^2 + \frac{1}{2}p_{\max}(R^*)^2 \le e(t). \end{aligned}$$

Hence Gronwall's lemma yields

$$e(t_1) \le L(R,\mu)\exp(T), \qquad \text{for } L(R,\mu) = \max\{E(x,y,\lambda) :\ \lambda \in [0,1],\ \mu^2x^2+y^2 = R^2\}.$$

The thesis now follows choosing $\rho(R,\mu) > 0$ such that $E(\lambda,x,y) > L(R,\mu)\exp(T)$ for $\lambda \in [0,1]$ and $\mu^2x^2 + y^2 > \rho(R,\mu)^2$. □

Lemma 3.2. *For every $R > 0$ and $\mu > 0$, there exist $\sigma(R,\mu), \tau(R,\mu)$ with $0 < \tau(R,\mu) \le R \le \sigma(R,\mu)$ such that, for every $(u,v,w,\lambda) \in \Sigma$,*

$$\min_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) \le R^2 \Longrightarrow \max_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) \le \sigma(R,\mu)^2 \quad (3.2)$$

and

$$\max_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) > R^2 \Longrightarrow \min_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) > \tau(R,\mu)^2. \quad (3.3)$$

Proof. The proof of the first assertion is similar to the one of Lemma 3.1, but even simpler. Indeed, consider the function

$$E(x,y) = \frac{1}{2}\left(y^2 + \mu^2 x^2\right)$$

and, as before, take $t_0, t_1 \in [0,T]$ such that

$$\mu^2 v(t_0)^2 + v'(t_0)^2 = R^2, \quad \mu^2 v(t_1)^2 + v'(t_1)^2 = \max_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) > R^2.$$

For $e(t) = E(v(t), v'(t))$, one has, in view of (H_v)

$$\begin{aligned} |e'(t)| &= |v'(t)(v''(t) + \mu^2 v(t))| = |v'(t)(\mu^2 v(t) - \tilde{g}(\lambda, u(t), v(t), w(t)))| \\ &\le L_\mu |v'(t)v(t)| \le \frac{L_\mu}{2}\left(v'(t)^2 + v(t)^2\right) = L_\mu \max(1,\mu^2) e(t) \end{aligned}$$

where $L_\mu > \mu^2$ is a constant such that $|\tilde{g}(\lambda,u,v,w)| \le (L_\mu - \mu^2)|v|$. Hence Gronwall's lemma yields

$$e(t_1) \le \frac{R^2}{2} \exp(L_\mu \max(1,\mu^2) T),$$

giving the explicit estimate $\sigma(R,\mu) = R\exp(L_\mu \max(1,\mu^2)T/2)$.

At this point, the proof of the second assertion follows with the choice $\tau(R,\mu) = R\exp(-L_\mu \max(1,\mu^2)T/2)$. Indeed, if by contradiction

$$\min_{t\in[0,T]} (\mu^2 v(t)^2 + v'(t)^2) \le R\exp(-L_\mu \max(1,\mu^2)T/2),$$

then the first part of the proof shows that $\max_{t\in[0,T]}(\mu^2 v(t)^2 + v'(t)^2) \le R$. □

Now, let us concentrate on the oscillating behaviour of the solutions of (2.12); to this aim, for every $\beta \in C^1_0([0,T])$, we denote by $n(\beta)$ the number of zeros of β in $[0,T)$. We recall that if $\beta \in C^2$ and has only simple zeros, than $n(\beta)$ (is finite and) can be evaluated via the integral formula (given in [10], see also [11])

$$n(\beta) = \frac{\nu}{\pi} \int_0^T \frac{\beta'(t)^2 - \beta(t)\beta''(t)}{\nu^2\beta(t)^2 + \beta'(t)^2}\, dt, \qquad \forall\, \nu > 0. \tag{3.4}$$

We will estimate the number of zeros of the components u and v of solutions (u, v, w, λ) of (2.12).

3.1. The oscillating properties of the u-component of solutions

Let us first observe that we cannot ensure that for every $(u, v, w, \lambda) \in \Sigma$ the function u has finitely many zeros in $[0,T]$. However, this is certainly true for solutions such that u has sufficiently large initial values. Indeed, an application of Lemma 3.1 with $\mu = 1$ proves that there exists $u_0^* \geq 1$ such that for every $(u, v, w, \lambda) \in \Sigma$ we have

$$|u'(0)| \geq u_0^* \quad \Longrightarrow \quad u(t)^2 + u'(t)^2 \geq 1, \quad \forall\, t \in [0,T];$$

as a consequence, if $|u'(0)| \geq u_0^*$ the number $n(u)$ is well defined. Moreover, we are able to prove some bounds on this number; the first one is an upper estimate:

Proposition 3.1. *There exists $n^* \in \mathbb{N}$ such that for every $(u, v, w, \lambda) \in \Sigma$ we have*

$$|u'(0)| = u_0^* \quad \Longrightarrow \quad n(u) \leq n^* - 1. \tag{3.5}$$

Proof. Let us consider a nontrivial solution $(u, v, w, \lambda) \in \Sigma$ such that $|u'(0)| = u_0^*$; from Lemma 3.1 we deduce that there exists $\rho_0 := \rho(u_0^*, 1)$ such that

$$u(t)^2 + u'(t)^2 \leq \rho_0^2, \quad \forall\, t \in [0,T].$$

On the other hand, by the choice of u_0^* we also have

$$u(t)^2 + u'(t)^2 \geq 1, \quad \forall\, t \in [0,T].$$

Using (3.4) with $\nu = 1$ and recalling (2.9), we obtain

$$\begin{aligned} n(u) &\le \frac{1}{\pi}\int_0^T \frac{u(t)[\tilde{f}(u(t),\lambda) - \lambda p(t,u(t),v(t),w(t))] + u'(t)^2}{u(t)^2 + u'(t)^2}\,dt \\ &\le \frac{T}{\pi}\left(\rho_0(C_{\rho_0} + \rho_0^3 + p_{\max}(R^*)) + \rho_0^2\right), \end{aligned}$$

where

$$C_{\rho_0} = \max_{|\xi|\le\rho_0} |f(\xi)|$$

and $p_{\max}(R^*)$ is as in assumption (H_u). □

Now, let us prove that the u-component of solutions of (2.12) has an arbitrarily large number of zeros for sufficiently large initial values:

Proposition 3.2. *For every $n \ge n^*$ there exists $u^*_{\infty,n} > 0$ such that for every $(u,v,w,\lambda) \in \Sigma$ we have*

$$|u'(0)| \ge u^*_{\infty,n} \quad \Longrightarrow \quad n(u) > n. \tag{3.6}$$

Proof. Let us observe that assumption (H_u) and (2.9) imply that

$$\lim_{|u|\to+\infty} \frac{\tilde{f}(u,\lambda) - \lambda p(t,u,v,w)}{u} = +\infty,$$

uniformly in $(t,v,w,\lambda) \in [0,T]\times\mathbb{R}\times[-R^*,R^*]\times[0,1]$. As a consequence, for every $n \ge n^*$ there exists $K_n > 0$ such that

$$u(\tilde{f}(u,\lambda) - \lambda p(t,u,v,w)) > 4\pi^2 n^2 u^2 - K_n, \tag{3.7}$$

for every $(t,u,v,w,\lambda) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times[-R^*,R^*]\times[0,1]$. Now, from an application of Lemma 3.1 with $\mu = 2\pi n$ we deduce that there exists $u^*_{\infty,n} > 0$ such that for every $(u,v,w,\lambda) \in \Sigma$ with $|u'(0)| \ge u^*_{\infty,n}$ we have

$$4\pi^2 n^2 u(t)^2 + u'(t)^2 \ge 2K_n, \quad \forall\, t \in [0,T]. \tag{3.8}$$

Assume now that $(u,v,w,\lambda) \in \Sigma$ is such that $|u'(0)| \ge u^*_{\infty,n}$; hence, from

(3.4) with $\nu = 2\pi n/T$ and from (3.7) and (3.8) we obtain

$$\begin{aligned} n(u) &\geq \frac{2\pi n}{T\pi} \int_0^T \frac{u(t)[\tilde{f}(u(t),\lambda) - \lambda p(t,u(t),v(t),w(t))] + u'(t)^2}{4\pi^2 n^2 u(t)^2 + u'(t)^2}\, dt \\ &> \frac{2n}{T} \int_0^T \frac{4\pi^2 n^2 u(t)^2 + u'(t)^2 - K_n}{4\pi^2 n^2 u(t)^2 + u'(t)^2}\, dt \\ &= \frac{2n}{T} \left(T - \int_0^T \frac{K_n}{4\pi^2 n^2 u(t)^2 + u'(t)^2}\, dt \right) \\ &\geq \frac{2n}{T} \left(T - \int_0^T \frac{1}{2}\, dt \right) = n. \end{aligned}$$

□

3.2. The oscillating properties of the v-component of solutions

In this Subsection we study the rotating behaviour of the v-component of nontrivial solutions of (2.12); let us first observe that (3.3) of Lemma 3.2 shows that for every solution (u, v, w, λ) of (2.12) with $v \not\equiv 0$ it holds

$$v(t)^2 + v'(t)^2 > 0, \quad \forall\, t \in [0, T].$$

Hence, for every solution (u, v, w, λ) of (2.12) with $v \not\equiv 0$ the number $n(v)$ is finite. We are able to estimate this number, using the asymptotic assumptions on g given in (H_v):

Proposition 3.3. *For every $\epsilon_1 > 0$ there exists $v_0^* > 0$ such that for every $(u, v, w, \lambda) \in \Sigma$ we have*

$$|v'(0)| \leq v_0^* \quad \Longrightarrow \quad n(v) \leq \frac{T}{\pi} \sqrt{q_{\max}\, g_0 + \epsilon_1}. \tag{3.9}$$

Proof. Let us observe that assumption (H_v) and (2.10) and (2.11) imply that for every $\epsilon_1 > 0$ there exists $v_0^* > 0$ such that

$$v\tilde{g}(t, u, v, w, \lambda) \leq (q_{\max}\, g_0 + \epsilon_1) v^2, \tag{3.10}$$

for every $(t, u, v, w, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times [-R^*, R^*] \times [0, 1]$ with $|v| \leq v_0^*$. Assume now that $(u, v, w, \lambda) \in \Sigma$ is such that $|v'(0)| \leq v_0^*$; hence, from (3.4) with $\nu = \sqrt{q_{\max}\, g_0 + \epsilon_1}$ and from (3.10) we obtain

$$\begin{aligned} n(v) &\leq \frac{\sqrt{q_{\max}\, g_0 + \epsilon_1}}{\pi} \int_0^T \frac{v(t)\tilde{g}(t, u(t), v(t), w(t), \lambda) + v'(t)^2}{(q_{\max}\, g_0 + \epsilon_1) v(t)^2 + v'(t)^2}\, dt \\ &\leq \frac{\sqrt{q_{\max}\, g_0 + \epsilon_1}}{\pi} \int_0^T dt = \frac{T}{\pi} \sqrt{q_{\max}\, g_0 + \epsilon_1}. \end{aligned}$$

□

Proposition 3.4. *For every $\epsilon_2 > 0$ and $\epsilon_3 > 0$ there exists $v^*_\infty > 0$ such that for every $(u, v, w, \lambda) \in \Sigma$ we have*

$$|v'(0)| \geq v^*_\infty \quad \Longrightarrow \quad n(v) \geq (1 - \epsilon_3)\frac{T}{\pi}\sqrt{q_{\min}\, g_\infty - \epsilon_2}. \tag{3.11}$$

Proof. We first observe that assumption (H_v) and (2.10) and (2.11) imply that for every $\epsilon_2 > 0$ there exists $M > 0$ such that

$$v\tilde{g}(t, u, v, w, \lambda) \geq (q_{\min}\, g_\infty - \epsilon_2)v^2 - M, \tag{3.12}$$

for every $(t, u, v, w, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times [-R^*, R^*] \times [0, 1]$. Now, for every $\epsilon_3 > 0$, from an application of Lemma 3.2 with $\mu = \sqrt{q_{\min}\, g_\infty - \epsilon_2}$ we deduce that there exists $v^*_\infty > 0$ such that for every $(u, v, w, \lambda) \in \Sigma$ with $|v'(0)| \geq u^*_\infty$ we have

$$(q_{\min}\, g_\infty - \epsilon_2)v(t)^2 + v'(t)^2 \geq \frac{M}{\epsilon_3}, \quad \forall\ t \in [0, T]. \tag{3.13}$$

The proof continues now as the one of Proposition 3.6 using formula (3.4) with $\nu = \sqrt{q_{\min}\, g_\infty - \epsilon_2}$. □

3.3. Proof of the result

Let us fix $(n_u, n_v) \in \mathbb{N}^2$ such that $n_u \geq n^*$, with n^* as in Proposition 3.1, and

$$n_v \in \left(\frac{T}{\pi}\sqrt{q_{\max}\, g_0}, \frac{T}{\pi}\sqrt{q_{\min}\, g_\infty}\right).$$

Let us fix $\epsilon_i > 0$ $(i = 1, 2, 3)$ such that

$$\frac{T}{\pi}\sqrt{q_{\max} g_0 + \epsilon_1} < n_v < \frac{T}{\pi}(1 - \epsilon_3)\sqrt{q_{\min} g_\infty - \epsilon_2}.$$

Let us consider $u^*_0, u^*_{\infty,n_u}, v^*_0, v^*_\infty$ as in Propositions 3.1, 3.2, 3.3 and 3.4 and let $R^* > 0$ be as in assumption (H_w); we apply Theorem 2.2 with $n = (n_u, n_v)$,

$$B = \{(u, v, w, \lambda) \in X \times [0, 1] :\ u'(0) < u^*_{\infty,n_u},\ v'(0) < v^*_\infty,\ ||w|| < R^* + 1\}$$

and

$$A = \{(u, v, w, \lambda) \in X \times [0, 1] :\ u'(0) > u^*_0,\ v'(0) > v^*_0\}.$$

We also set $C = \overline{B} \setminus A$ and define

$$\mathbf{k}(u, v, w, \lambda) = (n(u), n(v)), \quad \forall\ (u, v, w, \lambda) \in \Sigma \cap C.$$

The continuity of $\mathbf{k}$ follows from the integral formula (3.4) (indeed, when used for u, v with $(u, v, w, \lambda) \in \Sigma$, the term involving the second derivative can be always expressed in terms of continuous functions of (u, v, w, λ)). From assumption (H_w) and Propositions 3.1, 3.2, 3.3 and 3.4 is is easy to see that the set

$$\{(u, v, w, \lambda) \in \Sigma \cap (\partial C) :\ \mathbf{k}(u, v, w, \lambda) = (n_u, n_v)\}$$

is empty; as a consequence, (2.6) is satisfied. Now, an application of Lemma 3.1 and Lemma 3.2 proves that

$$||u|| \leq \rho(u^*_{\infty,n_u}, 1), \quad ||v|| \leq \sigma(v^*_\infty, 1)$$

when $(u, v, w, \lambda) \in \Sigma \cap C$; on the other hand, by definition we obviously have $||w|| \leq R^* + 1$ when $(u, v, w, \lambda) \in \Sigma \cap C$. This is sufficient to conclude that (2.7) is fulfilled.

Finally, we have to check the validity of (2.8). For $\lambda = 0$ the problem is uncoupled, so that we can write $\mathcal{N}_0(u, v, w) = (\mathcal{N}^u_0(u), \mathcal{N}^v_0, \mathcal{N}^w_0(w))$. Accordingly, we define the open set U^n_0 as a product

$$U^n_0 = U^{n_u}_0 \times U^{n_v}_0 \times B_{R^*+1}, \tag{3.14}$$

where B_{R^*+1} denotes the open ball of radius R^*+1 in $C^1_0([0, T])$ and $U^{n_u}_0, U^{n_v}_0 \subset C^1_0([0, T])$ are open sets constructed as in [4, 6, 14] (using well-known arguments based on the use of time-maps associated with autonomous second order equations), such that the "local" degrees $\deg(I - \mathcal{N}^u_0, U^{n_u}_0), \deg(I - \mathcal{N}^v_0, U^{n_v}_0)$ are different from zero. An elementary property of the Leray-Schauder degree gives

$$\deg(I - \mathcal{N}_0, U^n_0) = \deg(I - \mathcal{N}^u_0, U^{n_u}_0) \deg(I - \mathcal{N}^v_0, U^{n_v}_0) \deg(I - \mathcal{N}^w_0, B_{R^*+1})$$

which is different from zero as well, since the "global" degree $\deg(I - \mathcal{N}^w_0, B_{R^*+1})$ equals 1 or -1 (see [13, 17]).

Hence, all the assumptions of Theorem 2.2 are fulfilled and we deduce the existence of a solution (u, v, w) of (2.1) such that $n(u) = n_u$, $n(v) = n_v$ and $u'(0) > 0$, $v'(0) > 0$.

A straightforward modification of the definition of the sets A and B leads to the proof of the existence of the solutions (u, v, w) of (2.1) with $n(u) = n_u$, $n(v) = n_v$ and the different signs of the initial derivates of u and v.

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