ON STATIONARY MARKOVIAN MODELS: A POISSON-DRIVEN APPROACH

Consuelo Nava[1], Ramsés H. Mena[2] and Igor Prünster[3]

[1] Department of Economics and Statistics and Collegio Carlo Alberto
Università degli Studi di Torino
C.so Unione Sovietica 218/bis, 1014 Torino, Italy
(e-mail: consuelo.nava@carloalberto.com)

[2] Departamento de Probabilidad y Estadstica, IIMAS, UNAM
Universidad Nacional Autónoma de México
Apartado Postal 20-726 01000 México, D.F. México
(e-mail: ramses@sigma.iimas.unam.mx)

[3] Department of Economics and Statistics and Collegio Carlo Alberto
Università degli Studi di Torino
C.so Unione Sovietica 218/bis, 1014 Torino, Italy
(e-mail: igor.pruenster@unito.it)

ABSTRACT Here we summarize some results that are further developed in Nava et al. (2013). We propose a method to construct strictly stationary Markovian models with fixed invariant distributions. Of particular interest are those models with invariant distributions belonging to the class of Generalized Inverse Gaussian (GIG) distributions family. The construction we propose is based on a Poisson transform which controls the dependence structure in the model. In particular, it allows to fully control the underlying transition probabilities that, an appealing feature, is then incorporated within standard estimation methods. A Bayesian estimate via a Gibbs sampler algorithm, based on the slice method, is proposed and implemented.

1 INTRODUCTION

When interested in random phenomena evolving in time, a natural starting point is to consider Markovian processes. If at the outset, we look to retain the marginal distributional features invariant over time, then the immediate question is how to construct the transition mechanism accomplishing this. This is the approach pursued by many of the constructions available in the literature, in both discrete and continuous time. Most of these approaches start from a stochastic equation describing the dynamics in time and thus not necessarily leading to analytic expressions for the corresponding transition probabilities. This then leads to the consideration of specific, sometimes too specific and not always computationally efficient, estimation techniques. On the counterpart, full control of the transition probabilities driving a Markovian process is desirable, especially for the conveyed advantages in estimation and prediction procedures.

By appropriately relaxing the distributional assumptions of the underlying stationary and transition distributions, data features typically associated to non-stationary sequences can also be captured through stationary models, e.g. widen the definition of stationary measure accordingly to the observed phenomena.

With similar targets in mind, Pitt e al. (2002) made use of the reversibility property characterizing Gibbs sampler Markov chains to propose strictly stationary AR(1)-type models with virtually any choice of marginal distribution. In particular, they demonstrate that various general existing approaches, e.g. the one by Joe (1996), can be seen as particular cases of their construction. However, being such a general constructive approach, concrete choices of dependence should be made to accommodate specific modeling needs. Indeed, examples of this construction, meeting some specific dependency or distributional features, can be found in Mena (2003), Mena and Walker (2005, 2007, 2007b, 2009) and Contreras-Cristán et al. (2009). The idea presented in Nava et al. (2013), and also explored in this note, aims at constructing stationary Markovian models using a Poisson transform. The resulting specific dependence structure is general enough to accommodate models with invariant distributions on R_+.

2 f-STATIONARY POISSON-DRIVEN MARKOV PROCESS

As mentioned, our target is the construction of stationary Markov process with arbitrary but given marginal distribution. For the sake of simplicity, we assume that our choice of invariant distribution has an absolutely continuous density function f supported on R_+. Let $y \in \{0,1,2,\ldots\}$ and $\phi > 0$. We define the *Poisson weighted density* as

$$\hat{f}(x;y,\phi) := \frac{x^y e^{-x\phi} f(x)}{\xi(y,\phi)} \quad \text{where } \xi(y,\phi) := \int_{R_+} z^y e^{-z\phi} f(z)\,\mathrm{d}z \tag{1}$$

Notice that if the above integral exists, then (1) is a well defined probability density with positive support. The interesting role of the Poisson weighted density is enhanced by the following observations:

- for $\phi \downarrow 0$, it reduces to the size-biased density of f;
- when $y = 0$, $\hat{f}$ reduces to the Esscher transform of f;
- eq. (1) can be seen as the posterior density of a $\text{Po}(\phi x)$ distribution with prior $x \sim f$.

The Poisson weighted density is crucial for our construction of stationary Markovian processes, $(X_n)_{n\in Z_+}$, with invariant distribution having density f. The construction we propose is to define the following time-homogeneous one-step ahead Markovian density

$$p(x_{n-1},x_n) = \exp\{-\phi(x_n + x_{n-1})\} f(x_n) \sum_{y=0}^{\infty} \frac{(x_n x_{n-1}\phi)^y}{y!\xi(y,\phi)} \tag{2}$$

which clearly satisfies the balance condition $p(x_{n-1},x_n)f(x_{n-1}) = p(x_n,x_{n-1})f(x_n)$ for all $x_{n-1},x_n \in R_+$ and hence leading to a time-reversible Markovian process.

Definition 1. We term the stationary Markovian process, driven by the transition density (2) and stationary density f, an *f-stationary Poisson-driven Markov process* (f-sPDMP).

It can be easily seen that n-step ahead transition probabilities based on (2), modulating a discrete-time *f-stationary Poisson-driven Markovian process*, $(X_n)_{n\in Z_+}$, satisfies the Chapman-Kolmogorov (CK) equation. An interesting extension of this construction is its continuous time counterpart. The crucial point here is that we need to add some further conditions on the transition density (2) to preserve the CK equation. The approach followed in Nava et al. (2013) is based on an underlying time-homogeneous effect that enters through one of the parameters in (2) in a way that CK equation is satisfied. Following this approach we will consider time homogeneous transition densities

$$p_t(x_0, x_t) = \exp\{-\phi_t(x_0 + x_t)\} f(x_t) \sum_{y=0}^{\infty} \frac{(x_t x_0 \phi_t)^y}{y! \xi(y, \phi_t)} \tag{3}$$

with $t \mapsto \phi_t$ a C_0-function chosen such that the CK equation is satisfied. Stating the problem in terms of the Laplace transform, we need to find ϕ_t such that

$$\mathcal{L}_{X_{t+s}|X_0}(\lambda) = E_{X_0}[\mathcal{L}_{X_{t+s}|X_s}(\lambda)] \tag{4}$$

where the Laplace transform of the corresponding transition distribution is given by

$$\mathcal{L}_{X_t|X_0}(\lambda) = \sum_{y=0}^{\infty} \mathrm{Po}(y; x_0 \phi_t) \frac{\xi(y, \phi_t + \lambda)}{\xi(y, \phi_t)}$$

Within the above setting it can be seen that the above condition is sufficient for constructing a Feller semigroup ensuring the existence of a right continuous modification.

Within the framework of this construction, various appealing features are at hand. In particular, using the conditional representation in (2), we see that an *f-stationary Poisson-driven Markov process* has conditional moments $E_{X_{n-1}}[X_n^r] := E[X_n^r \mid X_{n-1} = x_{n-1}]$ given by

$$E_{x_{n-1}}[X_n^r] = \sum_{y=0}^{\infty} \left[\frac{\xi(y+r, \phi)}{\xi(y, \phi)} \right] \mathrm{Po}(y; x_{n-1}\phi)$$

Therefore the autocorrelation can be expressed as

$$\mathrm{Corr}(X_n, X_{n-1}) = \frac{\sum_{y=0}^{\infty} \left[\frac{\xi(y+1,\phi)^2}{\xi(y,\phi)} \right] \frac{\phi^y}{y!} - \xi(1,0)^2}{\xi(2,0) - \xi(1,0)^2} \tag{5}$$

provided that f leads to the existence of second order moments. The continuous time case follows analogously.

2.1 GIG-STATIONARY DENSITY IN DISCRETE TIME

Let us concentrate our choice of f on the general family of Generalized Inverse Gaussian (GIG) distributions. Constructing Markovian models with this invariant distributions is appealing in several areas due to its ability to capture diverse empirical features. The GIG distribution has density given by

$$\mathrm{GIG}(x; \alpha, \delta, \gamma) = \frac{(\gamma/\delta)^\alpha}{2K_\alpha(\delta\gamma)} x^{\alpha-1} \exp\left\{ -\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x) \right\} 1_{\{x>0\}}$$

where K_ν denotes the modified Bessel function of the third type with index ν. Importantly, the GIG distribution encompasses several well known distributions: for instance, the gamma distribution ($\alpha > 0, \delta = 0, \gamma > 0$), the inverse gamma distribution ($\alpha > 0, \delta > 0, \gamma = 0$) and the inverse Gaussian distribution ($\alpha = -\frac{1}{2}, \gamma > 0, \delta > 0$). See Eberlein and v. Hammerstein (2002) for further details on GIG distributions.

Following the construction of an *f-stationary Poisson-driven Markov process*, we obtain

$$\hat{f}(x;y,\phi) = \mathrm{GIG}(x;\alpha+y,\delta,\sqrt{\gamma^2+2\phi}) \quad \text{with} \quad \xi(y,\phi) = \frac{A(\alpha+y,\delta,\sqrt{\gamma^2+2\phi})}{A(\alpha,\delta,\gamma)}$$

where $A(\alpha,\delta,\gamma) := (\delta/\gamma)^{\alpha}\, 2K_\alpha(\delta\gamma)$. Namely, the corresponding Poisson weighted distribution is also GIG. Given these results, the transition density (2) simplifies to

$$p(x_{n-1},x_n) = x_n^{\alpha-1} \exp\left\{-\phi(x_n+x_{n-1}) - \frac{1}{2}\left[\frac{\delta^2}{x_n} + \gamma^2 x_n\right]\right\} \sum_{y=0}^{\infty} \frac{(x_{n-1}x_n\,\phi)^y}{y!\,A(\alpha+y,\delta,\sqrt{\gamma^2+2\phi})}$$

2.2 Bayesian Estimation for GIG-sPDMP

The availability of a tractable expression for the transition density is a desirable feature in the analysis and estimation of Markov processes. In particular if the choice of f leads to a manageable analytic (or computable) expression in (2), the maximum likelihood estimator (MLE) can be typically determined exactly. If not, one could make use of such a representation for the transition density to obtain a MLE via the expectation-maximization (EM) algorithm based on the augmented likelihood or a Gibbs sampler algorithm for Bayesian estimation. Regarding, this latter approach, we could define a Gibbs sampler algorithm borrowing some slice sampling ideas from Mena et al. (2011). An augmentation of the transition density for the GIG-sPDMP can be introduced as follows

$$\begin{aligned} p(x_{n-1},x_n,u,s) =& \psi_s^{-1} 1(u \le \psi_s) \exp\left\{-\phi(x_n+x_{n-1}) - \frac{1}{2}(\delta^2 x_n^{-1} + \gamma^2 x_n)\right\} \\ & \frac{x_n^{\alpha+s-1}\,(x_{n-1}\,\phi)^s}{s!\,A(\alpha+s,\delta,\sqrt{\gamma^2+2\phi})} \end{aligned} \tag{6}$$

where $s \mapsto \psi_s$ is a N-valued function with known inverse ψ^*, e.g. $e^{-\eta s}$, for $0 \le \eta \le 1$. The use of these latent variables allows us to construct the augmented likelihood for a set of observations $\mathbf{x} = (x_1,\ldots,x_N)$ for $\theta = (\alpha,\delta,\gamma,\phi)$. With the assumption of independent priors π, the full conditionals can be obtained from the log-posterior distribution:

$$\begin{aligned} \log(\theta|\cdots) \propto& \log(\pi(\theta)) + \sum_{n=1}^{N} \log(\mathrm{GIG}(x_n;\theta)) + \sum_{n=2}^{N} [\log(1(u_n < \psi_{s_n})) - \log(\psi_{s_n})] \\ & - \sum_{n=2}^{N} \phi(x_n + x_{n-1}) + \sum_{n=2}^{N} \{s_n \log(x_n x_{n-1}\phi) - \log(s_n!\,\xi(s_n,\phi;\theta))\} \end{aligned}$$

where $\mathbf{u} = (u_2, \ldots, u_N)$ and $\mathbf{s} = (s_2, \ldots, s_N)$. Simulation from this posterior distribution can be done via the adaptive rejection Metropolis sampling (ARMS) algorithm. The full conditional distributions for the latent variables can be obtained via

$$\pi(u_n \mid \cdots) = \mathrm{U}(u_n; 0, \psi_{s_n})$$
$$\pi(s_n \mid \cdots) \propto \frac{[x_n\, x_{n-1}\, \phi]^{s_n}}{s_n!\, \xi(s_n, \phi)\, \psi_{s_n}} 1(u_n \leq \psi_{s_n})$$

for $n = 2, \ldots, N$. Note that the above distribution has support $s_n = 0, \ldots, \lfloor \psi^*(u_n) \rfloor$. This is the advantage of the slice method at issue, i.e. that we only need to sample from a finite distribution instead of performing the infinite summation involved in the transition.

Figure 1 shows the posterior estimates of the above MCMC scheme for a simulated GIG-stationary Poisson driven Markov process. In Figure 2 we illustrate the application of the same model to real data. See Nava et al. (2013) for further details.

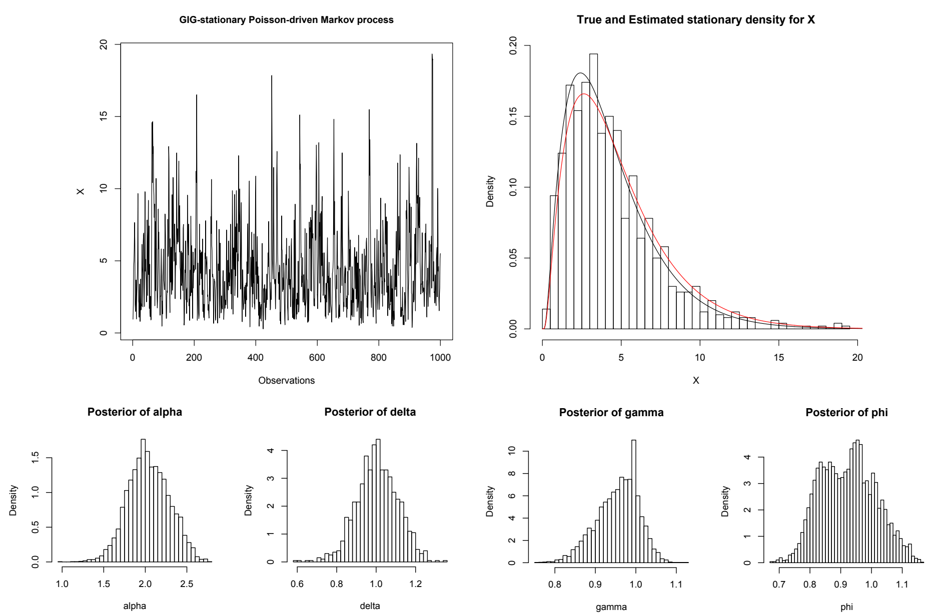

Figure1. Simulated data from a GIG-stationary Poisson driven Markov process. The top plots show the simulated time series (left) and the data histogram together with stationary GIG distribution (right). The Gibbs sampler algorithm is based on 5000 iterations, after 500 burn in sweeps. The posterior estimates for the model parameters, $\theta = (\alpha, \delta, \sigma, \phi)$, have modes located at $(2.029, 1.005, 0.963, 0.926)$ respectively as plotted in the bottom plots. The data were generated from the model using parameters $(2, 1, 1, 0.8)$, $\eta = 0.4$ and assuming independent priors $\pi(\theta) =$Exp(1).

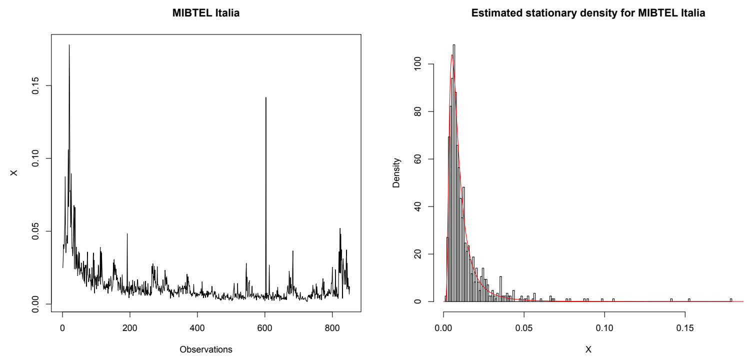

Figure2. Mibtel Italia volatility data modeled with a GIG-sPDMP. The plots show the time series from February 18, 2003 to June 27, 2006 (left) and the data histogram together with estimated stationary GIG distribution (right). The Gibbs sampler algorithm is based on 8000 iterations, after 1000 burn in sweeps. The posterior estimates for (α, δ, σ) have modes located at $(-2.710, 0.198, 0.649)$.

REFERENCES

BIBBY, M., SKOVGAARD, M. and SØRENSEN, M. (2005): Diffusion-type models with given marginal distribution and autocorrelation function. *Bernoulli*, *11*, 191–220.

BROCKWELL, P. J. and DAVIS, R. A. (1987): Time Series: Theory and Methods. *Springer*.

CONTRERAS-CRISTÁN, A., MENA, R. H. and WALKER, S. G. (2009): On the construction of stationary AR(1) models via random distributions. *Statistics*, *43*, 227–240.

EBERLEIN E. and V. HAMMERSTEIN E. A. (2002): Generalized Hyperbolic and Inverse Gaussian Distributions: Limiting Cases and Approximation of Processes. In Seminar on Stochastic Analysis, Random Fields and Applications IV, Progress in Probability 58, R.C. Dalang, M. Dozzi, F. Russo (Eds.), pages 221–264.

JOE, H. (1996): Time series models with univariate margins in the convolution-closed infinitely divisible class. *J. Appl. Probab.*. *33*, 664–677.

MENA, R. H. (2003): Stationary models using latent structures. PhD. Dissertation, University of Bath.

MENA, R. H., RUGGIERO, M. and WALKER, S. G. (2011): Geometric stick-breaking processes for continuous time Bayesian nonparametric modeling. *Journal of Statistical Planning and Inference*, *141*, 3217–3230.

MENA, R. H. and WALKER, S. G. (2005): Stationary autoregressive models via a Bayesian nonparametric approach. *Journal of Time Series Analysis*, *26*, 789–805.

MENA, R. H. and WALKER, S. G. (2007a): On the stationary version of the generalized hyperbolic ARCH model. *Annals of the Institute of Statistical Mathematics*, *59*, 325–348.

MENA, R. H. and WALKER, S. G. (2007b): Stationary Mixture Transition Distribution (MTD) models via predictive distributions. *Journal of Statistical Planning and Inference*, *137*, 3103–3112.

MENA, R. H. and WALKER, S. G. (2009): On a construction of Markov models in continuous time. *METRON - International Journal of Statistics*, *LXVII*, 303–323.

NAVA, C., MENA, R. H. and PRÜNSTER, I. (2013): Poisson Driven Stationary Markovian Models. Technical Report.

PITT, M. K., CHATFIELD, C., and WALKER, S. G. (2002): Constructing first order autoregressive models via latent processes. *Scand. J. Statist.*, *29*, 657–663.