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THE FUNDAMENTAL GROUP OF THE OPEN SYMMETRIC PRODUCT OF A HYPERELLIPTIC CURVE

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ABSTRACT. On the second symmetric product $C^{(2)}$ of a hyperelliptic curve C of genus g let L be the line given by the divisors on the standard linear series g^1_2 and for a point $b \in C$ let C_b be the curve $\{(x+b) : x \in C\}$. It is proved that $\pi_1(C^{(2)} \setminus (L \cup C_b))$ is the integer-valued Heisenberg group, which is the central extension of $\mathbb{Z}^{2g}$ by $\mathbb{Z}$ determined by the symplectic form on $H_1(C,\mathbb{Z})$.

1. INTRODUCTION

If C is a smooth curve the second symmetric product $C^{(2)}$ is a non singular surface, which parametrizes the effective divisors of degree 2 on C. The choice of a point $b \in C$ determines a copy of C inside $C^{(2)}$, this is $C_b := \{(x+b) : x \in C\}$. When C is projective and hyperelliptic there is one line L, the locus of divisors in the linear series g^1_2.

According to Nori [9] the complement of the theta-divisor Θ in a general principally polarized abelian variety of dimension $g \geq 2$ has the integer-valued Heisenberg group $H(g)$ for its fundamental group. Recall that $H(g)$ is generated by $2g+1$ letters $\{a_i, b_i, \delta\}$, the commutators among generators are all trivial but for $[a_i, b_i]$, they are all identified with δ. This gives the central extension

$$0 \to \mathbb{Z}_\delta \to H(g) \to \mathbb{Z}^{2g} \to 0$$

When $g = 2$ the p.p. abelian variety is the Jacobian $J(C)$ of a curve C of genus 2 and then C is hyperelliptic. In this case the Abel-Jacobi map $C^{(2)} \to J(C)$ is the blow down of L and the theta-divisor is the image of C_b, up to translation. Restriction gives a homeomorphism $C^{(2)} \setminus (L \cup C_b) \to J(C) \setminus \Theta$, therefore $\pi_1(C^{(2)} \setminus (L \cup C_b))$ is $H(2)$. We extend this version of Nori's result to higher genus:

Theorem 1.1. *Let C be a hyperelliptic curve of genus $g \geq 2$ then*

$$\pi_1(C^{(2)} \setminus (L \cup C_b)) \simeq H(g)\ .$$

The generator δ is the class σ_L of a meridian loop around L, moreover $\delta^{(g-1)} = \sigma_{C_b}$, the class of a meridian around C_b .

2. PROOF OF THEOREM 1.1

2.1. **The semistable degeneration.** Let $\phi : \mathscr{F} \to \Delta$ be a morphism from a smooth scheme to the disk. This is called a semistable degeneration if the fibration ϕ has non singular fibres over over the punctured disk, while the central fibre $\phi^{-1}(0)$ is a reduced scheme with simple normal crossing singularities. We deal with the situation in which a nonsingular curve C of genus g

degenerates to a reduced nodal curve with two simple crossing components G and E meeting at one point p, E being an elliptic curve and G a non singular curve of genus $(g-1)$.

The corresponding semistable degeneration of $C^{(2)}$ has as central fibre a reducible surface with three components, cf. [11]. One component is the blow up B of the product $G\times E$ at the point (p_G, p_E), the other components are $G^{(2)}$ and $E^{(2)}$. It is $G^{(2)}\cap E^{(2)}=\emptyset$. The surface B intersects $G^{(2)}$ along a copy of G. On B this is the proper transform of $G\times\{p_E\}$, while it is G_{p_G} on $G^{(2)}$. In the same way B intersects $E^{(2)}$ along a copy of E, which is on B the proper transform of $\{p_G\}\times E$ and which is E_{p_E} on $E^{(2)}$. We summarize the situation by drawing a schematic diagram.

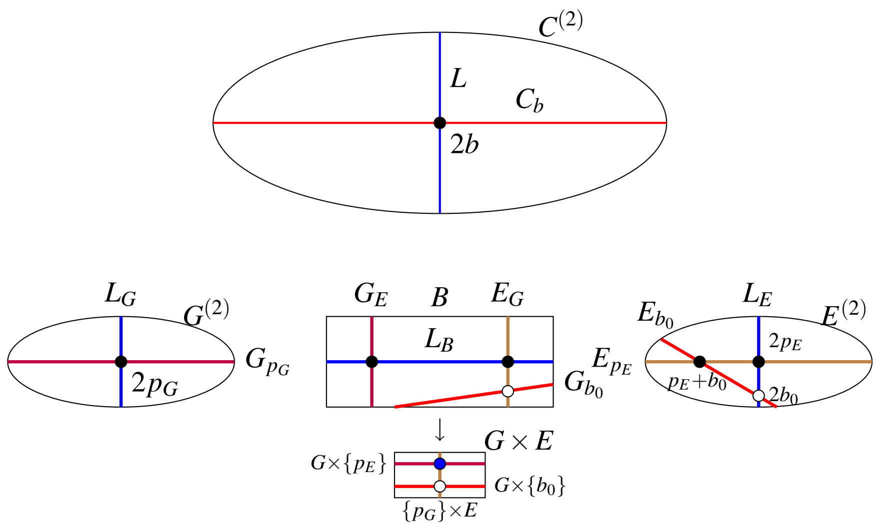

FIGURE 1. L_B is the exceptional divisor. The lines L, L_G and L_E appear only for the hyperelliptic degeneration.

As a topological space the smooth fibre $C^{(2)}$ is reconstructed from the central fibre of the degeneration using a surgery procedure, see [4], [10]. Remove on each component an open normal disc bundle around the double curves of intersection, then $C^{(2)}$ is obtained by gluing along the corresponding circle bundles.

We now make the assumption that the degeneration takes place inside the hyperelliptic locus, then the point p_G is a Weierstrass point on G, namely one of the ramification points of the hyperelliptic map $G\to\mathbb{P}^1$, cf. [7]. This fact is discussed also in [2], an explicit computation is given there in section 4.

Up to a finite base change over the disk, we can choose the semistable family of curves so that the section given by the base point $b_t\in C_t$ intersects the central curve $C_0=G+E$ is a point b_o in $E\setminus\{p_E\}$. Moreover, although this is not needed, we may take b_t to be a Weierstrass point, so on E the divisor $2b_o$ is linearly equivalent to $2p_E$. Looking at the corresponding degeneration of the symmetric products we find that the curve C_b degenerates to the reducible curve $G\times\{b_o\}+E_{b_o}$ inside the central fibre of the degeneration. The two components meet in one point, this is (p_G, b_o) on B and it is p_E+b_o on $E^{(2)}$. The degeneration of the line L is a reducible connected curve with three components. On B it is the exceptional divisor L_B, while the component L_E on $E^{(2)}$ is the linear system on E to which the divisor $2p_E$ belongs. Similarly the component L_G on $G^{(2)}$ is the linear system on G to which the divisor $2p_G$ belongs.

2.2. **Open second symmetric products.** Given a possibly reducible divisor $D = \sum D_i$ on a smooth variety V, we assign to each component of D an element σ_{D_i} in $\pi_1(V \setminus D)$ represented by some choice of a simple loop around D_i. This is well defined up to conjugacy. On the other hand if $\Gamma \to X$ is an oriented circle bundle we write σ_Γ to represent the class of the circle in $\pi_1(\Gamma)$.

The choice of the base point p_E determines the Abel-Jacobi map $\zeta : E^{(2)} \to E$, given by the rule $\zeta(x+y) = z$, where $(z+p_E) \sim (x+y)$. It is a $\mathbb{P}^1$-fibration and the curves E_b are sections. Let L_1 be the fibre containing $2p_E$. The open surface $E^{(2)o} := E^{(2)} \setminus (L_1 \cup E_p)$ is topologically a $\mathbb{C}$-fibration over $E \setminus \{p_E\}$, then its fundamental group is F_2, the free group on two letters. Consider $E^{(2)oo} := E^{(2)} \setminus (L_1 \cup E_b \cup E_p)$.

Lemma 2.3.

$$\pi_1(E^{(2)oo}) \simeq \mathbb{Z}\sigma_{E_p} \times F_2$$

Proof. The sections E_p and E_b intersect each of the fibers of ζ in two different points, but for the fiber L_2 through $(p+b)$, since $E_p \cap E_b = (p+b)$. By removing L_2 we have then that $E^{(2)oo} \setminus L_2$ is topologically an oriented $\mathbb{C}^*$-bundle over the twice punctured elliptic curve. The long exact sequence of homotopy yields $\pi_1(E^{(2)oo} \setminus L_2) \simeq \mathbb{Z}\sigma_{E_p} \times F_3$, because our bundle is oriented. We note for later use that $\sigma_{E_p} = \sigma_{E_b}^{-1}$. Lemma 4.18 from [8] and corollary 2.5 [9] give that a meridian σ_{L_2} normally generates the kernel $\pi_1(E^{(2)oo} \setminus L_2) \twoheadrightarrow \pi_1(E^{(2)oo})$. Since σ_{L_2} is part of a basis of F_3 then the result follows immediately. □ □

2.4. **Surgeries.** Let $X := G \setminus \Delta_{p_G}$ be the complement of an open disc around p_G and let $Y := E \setminus (\Delta_{p_E} \cup \Delta_{b_o})$, here we require that the two discs are disjoint. We choose the generators for $\pi_1(Y)$ to be $\{\alpha, \beta, \sigma_{b_o}\}$ so that $\sigma_{p_E}\sigma_{b_o} = [\alpha, \beta]$ and $\{\alpha, \beta\}$ freely generate $\pi_1(E \setminus \Delta_{p_E})$. The generators for $\pi_1(X)$ are $\{\alpha_i, \beta_i : i = 1, \dots, (g-1)\}$ with the condition $\sigma_{p_G} = \prod_{i=1}^{g-1}[\alpha_i, \beta_i]$.

We are interested in the topology of the open surface $C^{(2)o} := C^{(2)} \setminus (L \cup C_b)$. Consider each component of the normal crossing curve $L \cup C_b$, take an open tubular neighbourhood from each component and by plumbing them together construct a regular neighbourhood (r.n.) of $L \cup C_b$ inside $C^{(2)}$. By removing this last neighbourhood from $C^{(2)}$ we obtain $C^{(2)\partial}$, this is a surface with boundary which is a deformation retract of $C^{(2)o}$.

The construction of $C^{(2)}$ by surgery gives

$$C^{(2)\partial} = G^{(2)\partial} \cup (X \times Y) \cup E^{(2)\partial}$$

where the three surfaces are

(1) $E^{(2)\partial}$ the complement of a r.n. around $L_E \cup E_{b_o} \cup E_{p_E}$ on $E^{(2)}$, a deformation retract of $E^{(2)oo}$.
(2) $X \times Y$, this is the complement B^∂ of a r.n. for $L_B \cup (G \times \{b_o, p_E\}) \cup (\{p_G\} \times E)$ in B.
(3) $G^{(2)\partial}$ the complement of a r.n. for $L_G \cup G_{p_G}$ on $G^{(2)}$.

We define $M := (X \times Y) \cap G^{(2)\partial}$, recall that the gluing of $X \times Y$ with $G^{(2)\partial}$ is done by identification of the circle bundle around $X \times \{p_E\}$ with the normal circle bundle around $G_{p_G} \cap G^{(2)\partial}$

inside $G^{(2)}$, with the proviso that the orientation of corresponding circles are reversed. Note $\pi_1(M) = F_{2g-2} \times \mathbb{Z}\sigma_M$, where σ_M corresponds to the class of the circle around p_E on Y.

In the same way the gluing of $(X \times Y) \cup G^{(2)\partial}$ with $E^{(2)\partial}$ is done by identification of the circle bundle around $\{p_G\} \times Y$ with the circle bundle around $E_{p_E} \cap E^{(2)\partial}$. We write $N := ((X \times Y) \cup G^{(2)\partial} \cap E^{(2)\partial})$, so $\pi_1(N) = \mathbb{Z}\sigma_N \times F_3$, where σ_N represents the class of the circle around p_G on X.

2.5. **The computation.** The proof of thm1 is by induction on the genus g of C, we have therefore

- if $g(G) = 1$ then $\pi_1(G^{(2)\partial}) \simeq F_2$, $\sigma_{L_G} = [a_1, b_1]$ and $\sigma_{G_p} = 1$
- if $g(G) \geq 2$ then $\pi_1(G^{(2)\partial}) \simeq H(g-1)$, $\sigma_{L_G} = [a_i, b_i] = \delta$ and $\sigma_{G_p} = \delta^{(g(G)-1)}$

The morphism $\pi_1(M) \to \pi_1(G^{(2)\partial})$ sends σ_M to $\sigma_{G_p}^{-1}$ and it maps α_i and β_j to generators which we can take to be a_i and b_j without restriction. On the other hand $\pi_1(M) \to \pi_1(X \times Y)$ is the identity on F_{2g-2} and it maps $\sigma_M \to \sigma_{p_E} = [\alpha, \beta]\sigma_{b_o}^{-1}$. By van Kampen theorem a presentation for $\pi_1((X \times Y) \cup G^{(2)\partial})$ is obtained from the one for $\pi_1(G^{(2)\partial}) \times F_3$ by the addition of the further relation $\delta^{(g(G)-1)} = \sigma_{b_o}[\alpha, \beta]^{-1}$.

Similarly $\pi_1(N) \to \pi_1((X \times Y) \cup G^{(2)\partial})$ sends σ_N to $\sigma_{p_G} = \prod_{i=1}^{g-1}[a_i, b_i]$ and it is the identity on F_3. The epimorphism $\pi_1(N) \to \pi_1(E^{(2)\partial})$ is given by $\sigma_N \to \sigma_{E_p}^{-1}$, $\sigma_{b_o} \to \sigma_{E_b} = \sigma_{E_p}^{-1}$, while $\alpha \to a$ and $\beta \to b$. It follows that $\pi_1(C^{(2)\partial})$ is the quotient of $\pi_1(G^{(2)\partial}) \times F_2$ modulo the relations $\delta^{(g(G)-1)} = \sigma_{b_o}[a, b]^{-1}$ and $\prod_{i=1}^{g-1}[a_i, b_i] = \sigma_{b_o}$, this implies

Lemma 2.6. *(1) $[a, b] = [a_i, b_i]$ is a central element δ in $\pi_1(C^{(2)\partial})$.*
(2) $\pi_1(C^{(2)\partial}) \simeq H(g)$.
(3) $\delta = \sigma_{L_C}$.

Proof. Only the last item requires a proof. We know that the line L in $C^{(2)}$ is obtained by gluing the line L_G with the exceptional divisor L_B on B and then with L_E, therefore $\sigma_{L_C} = \sigma_{L_G} = \delta$. □ □

Lemma 2.7. $\sigma_{C_b} = \delta^{g-1}$.

Proof. By construction $\sigma_{C_b} = \sigma_{G \times \{b_0\}} = \sigma_{b_o} = \prod_{i=1}^{g-1}[a_i, b_i] = \delta^{g-1}$. □ □

The proof of Theorem 1.1 is completed.

Corollary 2.8. *$\pi_1(C^{(2)} \setminus L)$ is a central extension of $\mathbb{Z}^{2g}$.*

This fact was used without proof in my paper [5] where it was computed that the fundamental group of the Fano surface F is the Heisenberg like central extension of $H_1(F, \mathbb{Z})$, but with kernel $\mathbb{Z}/2$. The homotopy properties of F have been revisited recently, see [3] and [6].

Remark 2.9. The referee knows of a different way to prove the result. Let $w : C \to \mathbb{P}^1$ be the hyperelliptic projection, let $b_1, \ldots, b_{2g+2} \in C$ be the branching points. Then w induces $h : C^{(2)} \to (\mathbb{P}^1)^{(2)} \equiv \mathbb{P}^2$. It is a non-Galois 4-fold covering ramified at $D := Q \cup L_1 \cup \ldots L_{2g+2}$, where Q is a smooth conic and L_j are tangent lines to Q. We have $h^{-1}(L_j) = C_{b_j}$ and $h^{-1}(Q) = L \cup \Delta$, where $\Delta := \{2x | x \in C\}$. Using Zariski-van Kampen theorem $\pi_1(\mathbb{P}^2 \setminus D)$ is computed, cf. [1]. The fundamental group $\pi_1(C^2 \setminus h^{-1}(D))$ is determined next using the Reidemeister-Schreier method. Finally the referee uses Lemma 4.18 from [8], to the effect that the desired fundamental group can be computed by tracking and killing the meridians of $\Delta, C_1 \dots C_{2g+1}$. He finds that the quotient is $H(g)$ and that σ_C is the class which was written above.

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