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C. Dagnino, S. Remogna, P. Sablonniere
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On the solution of Fredholm integral equations based on spline quasi-interpolating projectors

Catterina Dagnino · **Sara Remogna** · **Paul Sablonnière**

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Abstract We use spline quasi-interpolating projectors on a bounded interval for the numerical solution of linear Fredholm integral equations of the second kind by Galerkin, Kantorovich, Sloan and Kulkarni schemes. We get theoretical results related to the convergence order of the methods, in case of quadratic and cubic spline projectors, and we describe computational aspects for the construction of the approximate solutions. Finally, we give several numerical examples, that confirm the theoretical results and show that higher orders of convergence can be obtained by Kulkarni's scheme.

Keywords Integral equation · Spline quasi-interpolation

Mathematics Subject Classification (2000) 65R20 · 65D07

1 Introduction

Consider the linear equation

$$u - Tu = f, \tag{1.1}$$

where $T : \mathscr{X} \to \mathscr{X}$ is a compact linear operator on the Banach space $\mathscr{X}$. The operator $I - T$ is assumed to be invertible, so that the equation has a unique solution $u \in \mathscr{X}$ for any given $f \in \mathscr{X}$. Let $\pi_n : \mathscr{X} \to \mathscr{X}_n \subset \mathscr{X}$ be a sequence of linear projectors onto finite dimensional subspaces $\mathscr{X}_n$ of $\mathscr{X}$, converging to the identity operator pointwise.

C. Dagnino · S. Remogna
Dipartimento di Matematica, Università di Torino, via C. Alberto, 10, 10123 Torino, Italy
E-mail: catterina.dagnino@unito.it
E-mail: sara.remogna@unito.it

P. Sablonnière
INSA de Rennes, 20 av. des Buttes de Coesmes, CS 14315, 35043 Rennes CEDEX 7, France E-mail: Paul.Sablonniere@insa-rennes.fr

In this paper, we consider more specifically the integral operator

$$Tx(s) := \int_a^b k(s,t)x(t)dt, \quad s \in I := [a,b], \tag{1.2}$$

where $\mathscr{X} = C(I)$ and the kernel $k \in C(I^2)$. Then T is a compact linear operator defined on $\mathscr{X}$.

Let $\mathscr{X}_n := \mathscr{S}_d^{d-1}(I, \mathscr{T}_n)$ be the space of splines of degree d on the uniform knot sequence $\mathscr{T}_n := \{t_i = a + ih, 0 \le i \le n\}$, with $h = (b-a)/n$, and C^{d-1} smoothness. In particular we consider quadratic ($d = 2$) and cubic ($d = 3$) splines, because such a choice lies on our experience of using such functions which has proved to be efficient in many integration problems (see e.g. [10–13]).

Let π_n be a *quasi-interpolating projector* (abbr. QIP) on $\mathscr{X}_n$ (i.e. $\pi_n f = f, \forall f \in \mathscr{X}_n$) described in Section 2 below. For $u \in \mathscr{X}$, we can write $\pi_n u$ as

$$\pi_n u = \sum_{i=1}^{N} \lambda_i(u) B_i, \tag{1.3}$$

where $N = \dim(\mathscr{X}_n) = n + d$, the $B_i's$ are B-splines and the coefficients $\lambda_i(u)$ are local functionals using discrete values of u in some neighbourhood of $\mathrm{supp}(B_i)$.

We use such spline QIPs (1.3) for the numerical solution of (1.1)-(1.2) by Galerkin, Kantorovich, Sloan schemes (see e.g. [3,14]) and by the more recent Kulkarni scheme (see [7–9]).

We remark that, recently, the use of the spline quasi-interpolation has been proved to work well for the approximation of solution of integral equations (see e.g. [1,2]). In particular, in [1] a degenerate kernel method based on (left and right) partial approximation of the kernel by a discrete quartic spline quasi-interpolant is provided. In [2], the authors propose and analyse a collocation method and a modified Kulkarni's scheme based on spline quasi-interpolating operators, which are not projectors, but reproduce polynomial spaces, while the original Kulkarni's scheme requires the use of projection operators.

Here is an outline of the paper. In Section 2 we introduce the quadratic and cubic spline QIPs and present their convergence properties. In Section 3 we consider the four projection methods based on the spline QIPs (1.3):

1. *Galerkin's method*, where T, in (1.1), is approximated by $T_n^g := \pi_n T \pi_n$, and the right hand side f by $\pi_n f$. The approximate equation is then

$$u_n^g - \pi_n T \pi_n u_n^g = \pi_n f, \tag{1.4}$$

2. *Kantorovich method*, where T is approximated by $T_n^k := \pi_n T$. The approximate equation is then

$$u_n^k - \pi_n T u_n^k = f, \tag{1.5}$$

3. *Sloan's iterated version*, where T is approximated by $T_n^s := T \pi_n$. The approximate equation is then

$$u_n^s - T \pi_n u_n^s = f, \tag{1.6}$$

4. *Kulkarni's method*, where T is approximated by

$$T_n^m := \pi_n T\pi_n + \pi_n T(I-\pi_n) + (I-\pi_n)T\pi_n = \pi_n T + T\pi_n - \pi_n T\pi_n = T_n^k + T_n^s - T_n^g$$

The approximate equation is then

$$u_n^m - T_n^m u_n^m = f. \tag{1.7}$$

Moreover, in such a section, we construct the corresponding approximate solutions by solving linear systems.

In Section 4 the convergence of the above methods is analysed and the obtained results show that the Kulkarni's method has the highest convergence order with respect to the other three ones. Moreover, in case $d = 2$, superconvergence properties at specific points occur for Galerkin, Kantorovich and Kulkarni methods.

In Section 5 we describe the computational aspects for the construction of approximate solutions.

In Section 6, we present some *quadrature formulas of product type* with B-spline weight functions (details will be given in [4]) used in the computation of the approximate solutions.

In Section 7, we give numerical results on examples of integral equations with more or less smooth kernels, comparing the four above methods. The numerical comparisons among Galerkin, Kantorovich, Sloan and Kulkarni methods based on our quadratic or cubic spline QIPs, confirm the theoretical results of Section 4.

Finally, Section 8 contains the proofs of some theorems and technical lemmas, presented in Sections 2 and 4.

2 Spline quasi-interpolating projectors

2.1 A quadratic spline quasi-interpolating projector

Setting $J := \{0,1,\ldots n+1\}$, the $n+2$ quadratic B-splines $\{B_i, i \in J\}$, with support $[t_{i-2}, t_{i+1}]$, on the usual extended knot sequence $\mathscr{T}_n \cup \{t_{-2} = t_{-1} = t_0 = a; b = t_n = t_{n+1} = t_{n+2}\}$ form a basis of the space $\mathscr{S}_2^1(I, \mathscr{T}_n)$ of C^1 quadratic splines on the partition $\mathscr{T}_n$. We set $s_i := \frac{1}{2}(t_{i-1} + t_i)$, for $1 \le i \le n$, $f_{2i} := f(t_i)$ for all $0 \le i \le n$ and $f_{2i-1} := f(s_i)$ for $1 \le i \le n$. We also introduce the set $\mathscr{S}_n := \{s_j, 1 \le j \le n\}$.

We choose the quasi-interpolating projector P_2 defined as

$$P_2 f := \sum_{i\in J} \lambda_i(f) B_i, \tag{2.1}$$

where the linear coefficient functionals have the following expressions

$$\begin{aligned} &\lambda_0(f) := f_0, \qquad \lambda_1(f) := 2f_1 - \frac{1}{2}(f_0 + f_2), \\ &\lambda_n(f) := 2f_{2n-1} - \frac{1}{2}(f_{2n-2} + f_{2n}), \qquad \lambda_{n+1}(f) := f_{2n}, \end{aligned} \tag{2.2}$$

and, for $2 \le i \le n-1$,

$$\lambda_i(f) = \frac{1}{14} f_{2i-4} - \frac{2}{7} f_{2i-3} + \frac{10}{7} f_{2i-1} - \frac{2}{7} f_{2i+1} + \frac{1}{14} f_{2i+2}. \tag{2.3}$$

These coefficients are computed in order to make P_2 a *projector*, i.e. to make functionals a dual basis to B-splines: $\lambda_i(B_j) = \delta_{ij}$ for all pairs (i, j). For instance, in order to obtain (2.3), starting from the following expression

$$\lambda_i(f) = c_1 f_{2i-4} + c_2 f_{2i-3} + c_3 f_{2i-1} + c_2 f_{2i+1} + c_1 f_{2i+2},$$

we see that $\lambda_i(B_j) = 0$ for $j < i-2$ and $j > i+2$. Then writing the conditions $\lambda_i(B_j) = \delta_{i,j}$ for $j = i-2, i-1, i$ respectively, we obtain the equations

$$\begin{aligned} &4c_1 + c_2 = 0 \\ &4c_1 + 6c_2 + c_3 = 0 \\ &c_2 + 3c_3 = 4 \end{aligned}$$

whose solution is $c_1 = \frac{1}{14}$, $c_2 = -\frac{2}{7}$, $c_3 = \frac{10}{7}$. Similarly, we get the coefficient functionals (2.2).

This projector can be written in the *quasi-Lagrange form*

$$P_2 f = \sum_{i=0}^{2n} f_i L_i,$$

where the quasi-Lagrange functions are linear combinations of a finite number of B-splines. For the sake of completeness, we give their expressions in terms of B-splines:

$$L_0 = B_0 - \frac{1}{2}B_1 + \frac{1}{14}B_2, \quad L_1 = 2B_1 - \frac{2}{7}B_2,$$

$$L_2 = -\frac{1}{2}B_1 + \frac{1}{14}B_3, \quad L_3 = \frac{10}{7}B_2 - \frac{2}{7}B_3, \quad L_4 = \frac{1}{14}B_4,$$

$$L_{2i-1} = -\frac{2}{7}B_{i-1} + \frac{10}{7}B_i - \frac{2}{7}B_{i+1}, \quad 3 \le i \le n-2,$$

$$L_{2i} = \frac{1}{14}(B_{i-1} + B_{i+2}), \quad 3 \le i \le n-3, \tag{2.4}$$

$$L_{2n-4} = \frac{1}{14}B_{n-3}, \quad L_{2n-3} = \frac{10}{7}B_{n-1} - \frac{2}{7}B_{n-2},$$

$$L_{2n-2} = -\frac{1}{2}B_n + \frac{1}{14}B_{n-2}, \quad L_{2n-1} = 2B_n - \frac{2}{7}B_{n-1},$$

$$L_{2n} = B_{n+1} - \frac{1}{2}B_n + \frac{1}{14}B_{n-1}.$$

As, for $\|f\|_\infty \le 1$, $|\lambda_i(f)| \le 3$ for all $i \in J$, one deduces that the infinity norm of P_2 is bounded above by 3, independently of the partition.

The exact value is obtained in the following theorem, whose proof is given in Section 8, by considering the corresponding Lebesgue function $\Lambda := \sum_{i=0}^{2n} |L_i|$.

Theorem 2.1 *The infinite norm of the quadratic spline projector P_2 is equal to*

$$\|P_2\|_\infty = \frac{157}{67} \approx 2.34.$$

2.2 A cubic spline quasi-interpolating projector

Setting $J := \{0, 1, \ldots n+2\}$, the $n+3$ cubic B-splines $\{B_i,\, i \in J\}$, with support $[t_{i-3}, t_{i+1}]$, on the usual extended knot sequence $\mathscr{T}_n \cup \{t_{-3} = t_{-2} = t_{-1} = t_0 = a;\, b = t_n = t_{n+1} = t_{n+2} = t_{n+3}\}$ form a basis of the space $\mathscr{S}_3^2(I, \mathscr{T}_n)$ of C^2 cubic splines on the partition $\mathscr{T}_n$.

We consider a projector whose general coefficient functional is based on 7 values of f. There is a simpler one, whose general coefficient functional is based on 5 values of f. However, as its norm is rather high, we prefer to use the former, which is slightly more complicated, but has a smaller norm.

The projector is defined by

$$P_3 f := \sum_{i \in J} \lambda_i(f) B_i, \tag{2.5}$$

where the linear coefficient functionals have the following expressions

$$\begin{aligned}
&\lambda_0(f) := f_0, \qquad \lambda_1(f) := -\frac{5}{18} f_0 + \frac{20}{9} f_1 - \frac{4}{3} f_2 + \frac{4}{9} f_3 - \frac{1}{18} f_4,\\
&\lambda_2(f) := \frac{1}{8} f_0 - f_1 + \frac{19}{8} f_2 - \frac{19}{24} f_4 + \frac{1}{3} f_5 - \frac{1}{24} f_6,\\
&\lambda_n(f) := \frac{1}{8} f_{2n} - f_{2n-1} + \frac{19}{8} f_{2n-2} - \frac{19}{24} f_{2n-4} + \frac{1}{3} f_{2n-5} - \frac{1}{24} f_{2n-6},\\
&\lambda_{n+1}(f) := -\frac{5}{18} f_{2n} + \frac{20}{9} f_{2n-1} - \frac{4}{3} f_{2n-2} + \frac{4}{9} f_{2n-3} - \frac{1}{18} f_{2n-4},\\
&\lambda_{n+2}(f) := f_{2n},
\end{aligned} \tag{2.6}$$

and, for $3 \le i \le n-1$,

$$\lambda_i(f) := -\frac{1}{30} f_{2i-6} + \frac{4}{15} f_{2i-5} - \frac{19}{30} f_{2i-4} + \frac{9}{5} f_{2i-2} - \frac{19}{30} f_{2i} + \frac{4}{15} f_{2i+1} - \frac{1}{30} f_{2i+2}. \tag{2.7}$$

Also in this case, the coefficients are computed in order to make P_3 a *projector*, i.e. to make functionals a dual basis to B-splines: $\lambda_i(B_j) = \delta_{ij}$ for all pairs (i,j). For instance, in order to obtain (2.7), starting from the following expression

$$\lambda_i(f) = c_1 f_{2i-6} + c_2 f_{2i-5} + c_3 f_{2i-4} + c_4 f_{2i-2} + c_3 f_{2i} + c_2 f_{2i+1} + c_1 f_{2i+2}.$$

It is easy to see that $\lambda_i(B_j)=0$ for $j<i-3$ and $j>i+3$. Then, writing the conditions $\lambda_i(B_j)=\delta_{i,j}$ for $j=i-3,i-2,i-1,i$ respectively, we obtain the equations

$$\begin{array}{l} 8c_1+c_2=0 \\ 32c_1+23c_2+8c_3=0 \\ 8c_1+23c_2+32c_3+8c_4=0 \\ c_2+8c_3+16c_4=24 \end{array}$$

whose unique solution is $c_1=-\frac{1}{30}$, $c_2=\frac{4}{15}$, $c_3=-\frac{19}{30}$, $c_4=\frac{9}{5}$. Similarly, we get the coefficient functionals (2.6).

This projector can be written in the *quasi-Lagrange form*

$$P_3 f=\sum_{i=0}^{2n} f_i \overline{L}_i,$$

where the quasi-Lagrange functions are linear combinations of a finite number of B-splines. For the sake of completeness, we give their expressions in terms of B-splines:

$$\begin{gathered} \overline{L}_0=B_0-\frac{5}{18}B_1+\frac{1}{8}B_2-\frac{1}{30}B_3, \quad \overline{L}_1=\frac{20}{9}B_1-B_2+\frac{4}{15}B_3, \\ \overline{L}_2=-\frac{4}{3}B_1+\frac{19}{8}B_2-\frac{19}{30}B_3-\frac{1}{30}B_4, \quad \overline{L}_3=\frac{4}{9}B_1+\frac{4}{15}B_4, \\ \overline{L}_4=-\frac{1}{18}B_1-\frac{19}{24}B_2+\frac{9}{5}B_3-\frac{19}{30}B_4-\frac{1}{30}B_5, \\ \overline{L}_5=-\frac{1}{3}B_2+\frac{4}{15}B_5, \quad \overline{L}_6=-\frac{1}{24}B_2-\frac{19}{30}B_3+\frac{9}{5}B_4-\frac{19}{30}B_5-\frac{1}{30}B_6, \\ \overline{L}_{2i-1}=\frac{4}{15}\left(B_{i-1}+B_{i+2}\right), \quad 4\le i\le n-3. \\ \overline{L}_{2i}=-\frac{1}{30}B_{i-1}-\frac{19}{30}B_i+\frac{9}{5}B_{i+1}-\frac{19}{30}B_{i+2}-\frac{1}{30}B_{i+3}, \quad 4\le i\le n-4. \end{gathered} \tag{2.8}$$

The quasi-Lagrange functions $\overline{L}_{2n-j}$, $j=0,\dots,6$, have symmetric expressions with respect to $\overline{L}_j$, $j=0,\dots,6$.

For $\|f\|_\infty\le 1$, we get $|\lambda_1(f)|$ and $|\lambda_{n+1}(f)|\le 17/3$, $|\lambda_2(f)|$ and $|\lambda_n(f)|\le 14/3$ and, for $3\le i\le n-1$, $|\lambda_i(f)|\le 11/3$. Therefore we deduce that $\|P_3\|_\infty\le 17/3\approx 5.33$, for any uniform partition.

The exact value is obtained in the following theorem, whose proof is given in Section 8, by considering the corresponding Lebesgue function $\overline{\Lambda}:=\sum_{i=0}^{2n}|\overline{L}_i|$.

Theorem 2.2 *The infinite norm of the cubic spline projector P_3 is equal to*

$$\|P_3\|_\infty=\frac{292460}{390963}+\frac{222277}{3518667}\sqrt{501}\approx 2.16.$$

2.3 Convergence properties of the spline QIPs

Since the operators $\pi_n = P_2$ or P_3 are projectors that are uniformly bounded independently of the uniform partition $\mathscr{T}_n$, classical results in approximation theory (see e.g. [6], chapter 5) provide

$$\|f-\pi_n f\|_\infty \le C\,\mathrm{dist}(f,\mathscr{X}_n),$$

where

$$C = 1+\|\pi_n\|_\infty \le \begin{cases} 3.35 & \text{for } \pi_n = P_2 \\ 3.17 & \text{for } \pi_n = P_3 \end{cases}$$

Therefore, using the fact that $\Pi_d \subset \mathscr{S}_d^{d-1}(I,\mathscr{T}_n)$ (for $d=2,3$), where Π_d is the space of polynomials of degree d, and a Jackson type theorem for splines ([5], chapter XII), we can conclude that there exist constants $\overline{C}_j$, depending on C and j, such that for all $f \in C^j[a,b]$

$$\|f-\pi_n f\|_\infty \le \overline{C}_j h^j \omega(f^{(j)},h), \quad \text{with} \quad \begin{cases} 0\le j\le 2 & \text{for } \pi_n = P_2 \\ 0\le j\le 3 & \text{for } \pi_n = P_3 \end{cases}$$

where ω is the modulus of continuity of $f^{(j)}$.

In particular for $j=2$ (resp. $j=3$) and when f has a third (resp. fourth) order continuous derivative, we obtain

$$\|f-P_2 f\|_\infty = O(h^3), \quad (\text{resp.}\ \|f-P_3 f\|_\infty = O(h^4)).$$

Moreover, using some majorations and a graphical study, one can get the following error bounds for smooth functions.

Theorem 2.3 *1. For the quadratic projector P_2 and $f^{(3)}$ bounded, there holds*

$$\|f-P_2 f\|_\infty \le \overline{C}_2 h^3 \|f^{(3)}\|_\infty, \ \text{with } \overline{C}_2 = \frac{7}{24}.$$

2. For the cubic projector P_3 and $f^{(4)}$ bounded, there holds

$$\|f-P_3 f\|_\infty \le \overline{C}_3 h^4 \|f^{(4)}\|_\infty, \ \text{with } \overline{C}_3 = \frac{4}{9}.$$

Proof In the first case, using Taylor's formulas

$$f(t_i) = f(x)+(t_i-x)f'(x)+\frac{1}{2}(t_i-x)^2 f''(x)+\frac{1}{2}\int_x^{t_i}(t_i-u)^2 f^{(3)}(u)du,$$

$$f(s_i) = f(x)+(s_i-x)f'(x)+\frac{1}{2}(s_i-x)^2 f''(x)+\frac{1}{2}\int_x^{s_i}(s_i-u)^2 f^{(3)}(u)du$$

and the fact that P_2 is exact on Π_2, we get

$$P_2 f(x) = f(x)+\frac{1}{2}\sum_{i=0}^{n} L_{2i}(x)\int_x^{t_i}(t_i-u)^2 f^{(3)}(u)du+\frac{1}{2}\sum_{i=1}^{n} L_{2i-1}(x)\int_x^{s_i}(s_i-u)^2 f^{(3)}(u)du,$$

Then, from the majorations

$$\left|\int_x^{t_i}(t_i-u)^2 f^{(3)}(u)du\right| \le \frac{1}{3}\|f^{(3)}\|_\infty |x-t_i|^3,$$

$$\left|\int_x^{s_i}(s_i-u)^2 f^{(3)}(u)du\right| \le \frac{1}{3}\|f^{(3)}\|_\infty |x-s_i|^3,$$

one gets

$$|P_2 f(x)-f(x)| \le \frac{1}{6}\|f^{(3)}\|_\infty L(x),$$

where

$$L(x) := \left(\sum_{i=0}^{n}|x-t_i|^3|L_{2i}(x)| + \sum_{i=1}^{n}|x-s_i|^3|L_{2i-1}(x)|\right).$$

Taking into account that the quasi-Lagrange functions have local support and the knot sequence is uniform, the graphical study of this function, by using a computer algebra system, provides $L(x) \le (7/4)h^3$ and finally

$$|P_2 f(x)-f(x)| \le \overline{C}_2 h^3\|f^{(3)}\|_\infty, \quad \text{with } \overline{C}_2 = \frac{7}{24}.$$

A similar method is used for the cubic projector P_3. Using a Taylor expansion of order 3, we first obtain

$$P_3 f(x) = f(x) + \frac{1}{6}\sum_{i=0}^{n}\overline{L}_{2i}(x)\int_x^{t_i}(t_i-u)^3 f^{(4)}(u)du + \frac{1}{6}\sum_{i=1}^{n}\overline{L}_{2i-1}(x)\int_x^{s_i}(s_i-u)^3 f^{(4)}(u)du.$$

Then, from the two upper bounds

$$\left|\int_x^{t_i}(t_i-u)^3 f^{(4)}(u)du\right| \le \frac{1}{4}\|f^{(4)}\|_\infty |x-t_i|^4$$

$$\left|\int_x^{s_i}(s_i-u)^3 f^{(4)}(u)du\right| \le \frac{1}{4}\|f^{(4)}\|_\infty |x-s_i|^4$$

we deduce

$$|P_3 f(x)-f(x)| \le \frac{1}{24}\|f^{(4)}\|_\infty \overline{L}(x)$$

where

$$\overline{L}(x) := \left(\sum_{i=0}^{n}|x-t_i|^4|\overline{L}_{2i}(x)| + \sum_{i=1}^{n}|x-s_i|^4|\overline{L}_{2i-1}(x)|\right).$$

The graphical study of this function provides $\overline{L}(x) \le (32/3)h^4$ and finally

$$|P_3 f(x)-f(x)| \le \overline{C}_3 h^4\|f^{(4)}\|_\infty, \quad \text{with } \overline{C}_3 = \frac{4}{9},$$

which completes the proof. □

The quadratic spline projector P_2 has the particularly interesting property to be superconvergent on the sets of evaluation points $\mathscr{T}_n$ and $\mathscr{S}_n$, as shown in Lemma 4.1 given in Section 4.2. It seems that there is no similar result for cubic splines.

3 The four projection methods

Considering the approximate equations (1.4), (1.5), (1.6) and (1.7), where π_n is P_2 or P_3, defined in (2.1) and (2.5), respectively, here we propose the construction of the corresponding approximate solutions.

3.1 Galerkin method

The approximate solution of (1.4) can be written in the form

$$u_n^g = \pi_n f + \sum_{j\in J} X_j B_j,$$

where the X_j's are obtained as follows.

Substituting in the equation (1.4), as $\pi_n u_n = u_n$, we get

$$\pi_n f + \sum_{j\in J} X_j B_j = \pi_n f + \pi_n(T\pi_n f + \sum_{j\in J} X_j T B_j).$$

On the other hand, we have

$$\pi_n T \pi_n f = \sum_{i\in J} \lambda_i(T\pi_n f) B_i \quad \text{and} \quad \pi_n T B_j = \sum_{i\in J} \lambda_i(TB_j) B_i,$$

therefore, by identifying the coefficients of B_i, we obtain the linear equations

$$X_i = \lambda_i(T\pi_n f) + \sum_{j\in J} \lambda_i(TB_j) X_j, \quad i\in J.$$

Introducing, respectively, the vector $\mathbf{g}$ and the matrix $\mathbf{B}$ defined by

$$g_i := \lambda_i(T\pi_n f) \quad \text{and} \quad B_{i,j} := \lambda_i(TB_j),$$

the linear system to solve is then

$$(\mathbf{I}-\mathbf{B})\mathbf{X} = \mathbf{g}, \tag{3.1}$$

with $\mathbf{X}$ the vector whose components are the unknown X_j.

3.2 Kantorovich method

The approximate solution of (1.5) can be written in the form

$$u_n^k = f + \sum_{j\in J} X_j B_j,$$

where the X_j's are obtained as follows.

Substituting in the equation (1.5), we get

$$f + \sum_{j\in J} X_j B_j = f + \pi_n(Tf + \sum_{j\in J} X_j T B_j).$$

As we have

$$\pi_n T f = \sum_{i\in J} \lambda_i(Tf) B_i \quad \text{and} \quad \pi_n T B_j = \sum_{i\in J} \lambda_i(TB_j) B_i,$$

therefore, by identifying the coefficients of B_i, we obtain the equations

$$X_i = \lambda_i(Tf) + \sum_{j\in J} \lambda_i(TB_j) X_j, \quad i \in J.$$

Let $\mathbf{c}$ be the vector with components

$$c_i := \lambda_i(Tf),$$

and let $\mathbf{B}$ be the matrix defined in Section 3.1, then the linear system to solve is

$$(\mathbf{I} - \mathbf{B})\mathbf{X} = \mathbf{c}. \tag{3.2}$$

3.3 Sloan method

The approximate solution of (1.6) is obtained as an iterate of Galerkin's solution

$$u_n^s := f + T u_n^g.$$

Therefore, we have first to compute $u_n^g = \pi_n f + \sum_{i\in J} X_i B_i$ (Section 3.1), then

$$T u_n^g = T \pi_n f + \sum_{i\in J} X_i T B_i,$$

where

$$T \pi_n f = \sum_{i\in J} \lambda_i(f) T B_i.$$

So, we finally get

$$u_n^s := f + \sum_{i\in J} (\lambda_i(f) + X_i) T B_i, \tag{3.3}$$

for which we need the computation of integrals with B-spline weight functions

$$T B_i(s) := \int_a^b B_i(t) k(s,t) dt.$$

3.4 Kulkarni method

We recall that the equation to solve is the following (the upper index m is deleted for the sake of clearness)

$$u_n - T_n u_n = f,$$

where the operator T_n is defined by

$$T_n := \pi_n T + T\pi_n - \pi_n T\pi_n.$$

We can deduce the expressions:

- $\pi_n Tu = \sum_{i\in J} \lambda_i(Tu)B_i$;
- $T\pi_n u = \sum_{i\in J} \lambda_i(u)\tilde{B}_i$, with $\tilde{B}_i := TB_i$;
- $\pi_n T\pi_n u = \sum_{(i,j)\in J\times J} \lambda_j(u)\lambda_i(\tilde{B}_j)B_i$.

Therefore, we obtain the following expression for u_n:

$$u_n = f + \sum_{i\in J}\lambda_i(Tu_n)B_i + \sum_{i\in J}\lambda_i(u_n)\tilde{B}_i - \sum_{(i,j)\in J\times J}\lambda_j(u_n)\lambda_i(\tilde{B}_j)B_i, \tag{3.4}$$

which has also the following form, with two vectors $\mathbf{X}$ and $\mathbf{Y}$ of unknown coefficients:

$$u_n = f + \sum_{k\in J} X_k B_k + \sum_{\ell\in J} Y_\ell \tilde{B}_\ell. \tag{3.5}$$

Thus, the problem has $2N$ unknowns.

Substituting (3.5) in (3.4) and setting $B_i^* := T\tilde{B}_i$, we get

$$\begin{aligned}\sum_{i\in J} X_i B_i + \sum_{j\in J} Y_j \tilde{B}_j &= \sum_{i\in J}\lambda_i(Tu_n)B_i + \sum_{j\in J}\lambda_j(u_n)\tilde{B}_j - \sum_{(i,j)\in J\times J}\lambda_j(u_n)\lambda_i(\tilde{B}_j)B_i.\\
&= \sum_{i\in J}\left(\lambda_i(Tf) + \sum_{k\in J} X_k\lambda_i(\tilde{B}_k) + \sum_{\ell\in J} Y_\ell\lambda_i(B_\ell^*)\right)B_i\\
&+ \sum_{j\in J}\left(\lambda_j(f) + \sum_{k\in J} X_k\lambda_j(B_k) + \sum_{\ell\in J} Y_\ell\lambda_j(\tilde{B}_\ell)\right)\tilde{B}_j\\
&- \sum_{(i,j)\in J\times J}\left(\lambda_j(f) + \sum_{k\in J} X_k\lambda_j(B_k) + \sum_{\ell\in J} Y_\ell\lambda_j(\tilde{B}_\ell)\right)\lambda_i(\tilde{B}_j)B_i.\end{aligned}$$

Consider the vectors $\mathbf{b},\mathbf{c}$ and the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ with components :

$$b_i := \lambda_i(f), \quad c_i := \lambda_i(Tf), \quad A_{i,j} := \lambda_i(B_j), \quad B_{i,j} := \lambda_i(\tilde{B}_j), \quad C_{i,j} := \lambda_i(B_j^*).$$

We notice that $A_{i,j} := \lambda_i(B_j) = \delta_{i,j}$, since the functionals are a dual basis to B-splines, therefore $\mathbf{A} = \mathbf{I}$. Thus, identifying the coefficients of B_i and $\tilde{B}_j$ (we assume that they are linearly independent), we obtain the double system of linear equations

$$\begin{aligned}\mathbf{X} &= \mathbf{c} + \mathbf{B}\mathbf{X} + \mathbf{C}\mathbf{Y} - (\mathbf{Bb} + \mathbf{B}\mathbf{X} + \mathbf{B}^2\mathbf{Y}),\\
\mathbf{Y} &= \mathbf{b} + \mathbf{X} + \mathbf{B}\mathbf{Y}.\end{aligned}$$

It can be written in a simpler form, since the second equation can be substituted in the first:

$$\mathbf{X} = \mathbf{c} + \mathbf{B}\,\mathbf{X} + (\mathbf{C} - \mathbf{B})\,\mathbf{Y} \tag{3.6}$$

$$\mathbf{Y} = \quad \mathbf{b} + \mathbf{X} + \mathbf{B}\,\mathbf{Y} \tag{3.7}$$

Introducing the block vectors and matrices, of size $2N$,

$$\mathbf{Z} := \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}, \quad \mathbf{D} := \begin{bmatrix} \mathbf{B} & \mathbf{C} - \mathbf{B} \\ \mathbf{I} & \mathbf{B} \end{bmatrix},$$

finally we are led to solve the system of $2N$ linear equations:

$$(\mathbf{I} - \mathbf{D})\,\mathbf{Z} = \mathbf{d}.$$

This system can be reduced to the solution of **one system** of N algebraic equations.

Indeed, substituting (3.6) for (3.6)+(3.7), we get

$$\mathbf{Y} = \mathbf{b} + \mathbf{c} + \mathbf{BX} + \mathbf{CY} \tag{3.8}$$

From equation (3.7), we now take

$$\mathbf{X} = (\mathbf{I} - \mathbf{B})\mathbf{Y} - \mathbf{b}, \tag{3.9}$$

that we substitute in (3.8) to get

$$((\mathbf{I} - \mathbf{B})^2 + \mathbf{B} - \mathbf{C})\mathbf{Y} = \mathbf{c} + (\mathbf{I} - \mathbf{B})\mathbf{b}. \tag{3.10}$$

Solving this equation gives $\mathbf{Y}$, then $\mathbf{X}$ is computed by (3.9).

4 Convergence of the methods

4.1 Error bounds

For the four methods, since $(I - T)$ is invertible, then $(I - \pi_n T)$, $(I - T\pi_n)$ and $(I - T_n^m)$ are invertible for n large enough and we have

$$\left\|(I - \pi_n T)^{-1}\right\|_\infty \leq \Gamma_1, \quad \left\|(I - T\pi_n)^{-1}\right\|_\infty \leq \Gamma_2, \quad \|(I - T_n^m)^{-1}\|_\infty \leq \Gamma_3$$

where Γ_1, Γ_2 and Γ_3 are constants independent of n ([3,7]). Hence for n large enough, the equations have unique solutions and we get respectively

$$\|u - u_n^g\|_\infty \leq \Gamma_1\,\|u - \pi_n u\|_\infty\,, \tag{4.1}$$

$$\|u - u_n^k\|_\infty \leq \Gamma_1\,\|(I - \pi_n)Tu\|_\infty\,, \tag{4.2}$$

$$\|u - u_n^s\|_\infty \leq \Gamma_2\,\|T(I - \pi_n)u\|_\infty\,, \tag{4.3}$$

$$\|u - u_n^m\|_\infty \leq \Gamma_3\,\|(I - \pi_n)T(I - \pi_n)u\|_\infty\,. \tag{4.4}$$

4.2 Convergence orders of the solution

From the error bounds (4.1)÷(4.4) on the solution of the integral equation, we deduce the convergence order of the methods.

In case of $\pi_n = P_2$, we need specific results on the projector. We present them in Lemma 4.1, 4.2 and 4.3, whose proofs are given in Section 8, where we denote respectively $e_3(x) = \pi_n m_3(x) - m_3(x)$, with $m_3(x) := x^3$, and $v_n = \pi_n u - u$.

Lemma 4.1 *(Superconvergence of π_n on $\mathscr{T}_n$ and $\mathscr{S}_n$). If $\|u^{(4)}\|_\infty$ is bounded, for $0 \le i \le n$ and $1 \le j \le n$,*

- $e_3(t_i) = e_3(s_j) = 0$.
- $v_n(t_i) = O(h^4)$ and $v_n(s_j) = O(h^4)$.

Lemma 4.2 *There holds*

$$\int_a^b (\pi_n m_3(x) - m_3(x))dx = 0.$$

More precisely, for all $i = 1 \dots n$, there holds

$$\int_{t_{i-1}}^{t_i} (\pi_n m_3(x) - m_3(x))dx = 0.$$

Lemma 4.3 *For any function $g \in W^{1,1}$ (i.e. with $\|g'\|_1$ bounded), there holds*

$$\int_a^b g(t)(\pi_n m_3(t) - m_3(t))dt = O(h^4).$$

More generally, if $\left\|u^{(4)}\right\|_\infty$ is bounded, then

$$\int_a^b g(t)(\pi_n u(t) - u(t))dt = O(h^4).$$

Theorem 4.1 *Assume that the solution u has a bounded fourth derivative, then, for $\pi_n = P_2$, there holds*

(i) for the three first methods

$$\|u - u_n^g\|_\infty = O(h^3), \quad \left\|u - u_n^k\right\|_\infty = O(h^3), \quad \|u - u_n^s\|_\infty = O(h^4);$$

(ii) for the Kulkarni's method

$$\|u - u_n^m\|_\infty = O(h^7).$$

Proof The first results are straightforward consequences of inequalities (4.1), (4.2), (4.3) and of the above lemmas.

The last one comes from inequality (4.4), Lemma 4.3 and Theorem 2.3. □

Remark 4.1 Let $\pi_n = P_2$ and let z be equal to $t_i \in \mathscr{T}_n$ or $s_i \in \mathscr{S}_n$, then, from Lemma 4.1 and (4.1), (4.2), (4.4), it results

$$u(z) - u_n^g(z) = O(h^4),$$
$$u(z) - u_n^k(z) = O(h^4),$$
$$u(z) - u_n^m(z) = O(h^8),$$

i.e. a superconvergence phenomenon occurs at the sets of evaluation points $\mathscr{T}_n$ and $\mathscr{S}_n$, in case of Galerkin, Kantorovich and Kulkarni methods.

Theorem 4.2 *Assume that the solution u has a bounded fourth derivative, then, for $\pi_n = P_3$, there holds*

(i) for the three first methods

$$\|u - u_n^g\|_\infty = O(h^4), \quad \left\|u - u_n^k\right\|_\infty = O(h^4), \quad \|u - u_n^s\|_\infty = O(h^4 \varepsilon(h)), \; \lim_{h\to 0} \varepsilon(h) = 0;$$

(ii) for the Kulkarni's method

$$\|u - u_n^m\|_\infty = O(h^8).$$

Proof These results are straightforward consequences of inequalities (4.1), (4.2), (4.3), (4.4) and Theorem 2.3. □

We remark that the Kulkarni's scheme, based on quadratic and cubic spline QIPs, has a convergence order higher than the other three ones based on the same QIPs. We also notice that, in [7], Kulkarni proposes a scheme for the solution of (1.1)-(1.2), based on orthogonal projections in the space of (discontinuous) piecewise polynomials of degree d and she shows that the corresponding error bound is $O(h^{3(d+1)})$. The proof of such a superconvergence result is based on the orthogonality of the projections. Since our Kulkarni's scheme is based on spline operators that are projectors but are not orthogonal, we can not get the superconvergence result obtained in [7]. However, we have shown that our method has a good convergence order (i.e. seven in case $d = 2$ and eight in case $d = 3$) and a superconvergence property at the evaluation points in case of $\pi_n = P_2$.

5 Computation of the solutions

In this section, we briefly describe the computational aspects needed for the computation of approximate solutions in the four projection methods.

5.1 Vectors and matrices for the Galerkin, Kantorovich and Sloan methods

The components of the right-hand side $\mathbf{g}$ in (3.1) are

$$g_i := \lambda_i(T\pi_n f) = \lambda_i\left(\sum_{k\in J} \lambda_k(f) T B_k\right) = \sum_{k\in J} \lambda_i(TB_k)\lambda_k(f) = \sum_{k\in J} B_{i,k}\lambda_k(f),$$

therefore we have $\mathbf{g} = \mathbf{B}\mathbf{b}$, where $\mathbf{b}$ denotes the vector with components $b_k := \lambda_k(f)$, $k \in J$.

For the computation of the vector $\mathbf{b}$, we need the band matrix $\mathbf{L}$ of size $N \times (2n+1)$ associated with the linear forms λ_i of the projector π_n. So, we have $\mathbf{b} = \mathbf{L}\tilde{\mathbf{f}}$, where $\tilde{\mathbf{f}} \in \mathbb{R}^{2n+1}$ is the vector of discrete values of f at the points of sets $\mathscr{T}_n$ and $\mathscr{S}_n$.

The coefficients of the matrix $\mathbf{B}$ in (3.1) are $B_{i,j} := \lambda_i(TB_j)$, with

$$TB_j(s) = \int_a^b B_j(t)k(s,t)dt. \tag{5.1}$$

In order to evaluate these integrals, we need the values $TB_j(t_k)$ and $TB_j(s_\ell)$, i.e. the values of this function at the points of $\mathscr{T}_n$ and $\mathscr{S}_n$, so we have to construct a $(2n+1) \times N$ matrix that we denote by $\mathbf{V}$. Then, we use suitable product quadrature formulas with B-spline weight functions B_j, presented in Section 6. Finally, one gets $\mathbf{B} = \mathbf{L}\mathbf{V}$.

For the construction of $\mathbf{c}$ in (3.2), we need the intermediate vector with components

$$\int_a^b k(t_k,t)f(t)dt \quad \text{or} \quad \int_a^b k(s_\ell,t)f(t)dt, \tag{5.2}$$

that can be evaluated by using a suitable Romberg's quadrature formula.

The vectors and matrices in (3.3) are known by Galerkin method implementation.

5.2 Vectors and matrices for Kulkarni's method

For the computation of the solution $\mathbf{Y}$ of (3.10) and of the vector $\mathbf{X}$ in (3.9), we need the vectors $\mathbf{b}$, $\mathbf{c}$ and the matrices $\mathbf{B}$, $\mathbf{C}$.

- The vector $\mathbf{b}$ of components $b_i = \lambda_i(f)$ is the same used for the Galerkin method and defined in Section 5.1.
- The vector $\mathbf{c}$ of components $c_i = \lambda_i(Tf)$ is the same used for the Kantorovich method and defined in Section 5.1.
- The matrix $\mathbf{B}$ of components $B_{i,j} = \lambda_i(\tilde{B}_j)$, $\tilde{B}_j = TB_j$, is the same used for the Galerkin method and defined in Section 5.1.
- The elements of the matrix $\mathbf{C}$ are $C_{i,j} = \lambda_i(B_j^*)$, where $B_j^* = T\tilde{B}_j$. As $T\tilde{B}_j(x) = \int_a^b k(x,s)\tilde{B}_j(s)ds$, we compute the matrix $\mathbf{B}^*$ with elements

  $$B_j^*(\tau_i) = \int_a^b k(\tau_i,s)\tilde{B}_j(s)ds, \quad \tau_i = t_i \text{ or } s_i, \tag{5.3}$$

  by using a suitable Romberg's quadrature formula. Finally, one gets $\mathbf{C} = \mathbf{L}\mathbf{B}^*$.

6 Quadrature formulas with B-spline weight functions

In numerical experiments, we use *product type quadrature formulas* (abbr. PQF) with B-spline weight functions and classical quadrature formulas. As there are many possibilities for the construction of such PQF, we have done several tests on various rules and selected those that appeared to be the best in numerical examples, in particular those having the least number of negative weights. The latter formulas are listed below, where we write that a formula is of order ℓ if it is exact on $\mathbb{P}_{\ell-1}$.

6.1 PQF for quadratic B-splines

Formula of order 4 for inner B-splines

$$\int_{t_{i-2}}^{t_{i+1}} B_i(t)f(t)dt \approx \frac{h}{8}(f(s_{i-1})+6f(s_i)+f(s_{i+1})), \quad 2 \le i \le n-1.$$

Formulas of order 3 for boundary B-splines

There are two specific boundary B-splines: B_0, with support $[a,t_1]$ and B_1, with support $[a,t_2]$.

- $\displaystyle\int_a^{t_1} B_0(t)f(t)\,dt \approx \frac{h}{60}(9f(a)+12f(s_1)-f(t_1))$;
- $\displaystyle\int_a^{t_2} B_1(t)f(t)\,dt \approx \frac{h}{30}(13f(s_1)+4f(t_1)+3f(s_2))$;

and similar formulas for B_n and B_{n+1}.

Formula of order 9 for inner B-splines

$$\begin{aligned}\int_{t_{i-2}}^{t_{i+1}} B_i(t)f(t)dt \approx h\Bigg(&\frac{41}{51975}f(t_{i-2})+\frac{1024}{779625}f\left(\frac{t_{i-2}+s_{i-1}}{2}\right)+\frac{827}{14175}f(s_{i-1})+\frac{95}{378}f(t_{i-1})\\ &+\frac{8894}{23625}f(s_i)+\frac{95}{378}f(t_i)+\frac{827}{14175}f(s_{i+1})+\frac{1024}{779625}f\left(\frac{t_{i+1}+s_{i+1}}{2}\right)\\ &+\frac{41}{51975}f(t_{i+1})\Bigg), \quad 2 \le i \le n-1.\end{aligned}$$

Formulas of order 9 for boundary B-splines

- $\displaystyle\int_a^{t_1} B_0(t)f(t)dt \approx \sum_{j=0}^{8} a_j f(r_j), \quad r_j = a+jh/8$, with

$$a_0=\frac{3029}{89100},\quad a_1=\frac{25904}{155925},\quad a_2=-\frac{2252}{51975},\quad a_3=\frac{6064}{31185},$$

$$a_4=-\frac{3191}{31185},\quad a_5=\frac{5296}{51975},\quad a_6=-\frac{4204}{155925},\quad a_7=\frac{1616}{155925},\quad a_8=-\frac{37}{41580};$$

– $\int_a^{t_2} B_1(t)f(t)dt \approx \sum_{j=0}^{8} a_j f(r_j), \quad r_j = a + jh/4$, with

$$a_0 = \frac{4519}{623700}, \quad a_1 = \frac{17912}{155925}, \quad a_2 = \frac{2858}{22275}, \quad a_3 = \frac{31576}{155925},$$

$$a_4 = \frac{2776}{31185}, \quad a_5 = \frac{2072}{22275}, \quad a_6 = \frac{3418}{155925}, \quad a_7 = \frac{1544}{155925}, \quad a_8 = -\frac{79}{623700};$$

and similar formulas for B_n and B_{n+1}.

6.2 PQF for cubic B-splines

Formula of order 4 for inner B-splines

$$\int_{t_{i-3}}^{t_{i+1}} B_i(t)f(t)dt \approx \frac{h}{6}\left(f(t_{i-2}) + 4f(t_{i-1}) + f(t_i)\right), \quad 3 \le i \le n-1.$$

Formulas of order 4 for boundary B-splines

There are three specific boundary cubic B-splines: B_0, with support $[a,t_1]$, B_1, with support $[a,t_2]$ and B_2, with support $[a,t_3]$.

– $\int_a^{t_1} B_0(t)f(t)dt \approx h(a_0 f(t_0) + a_1 f(s_1) + a_2 f(t_1) + a_3 f(s_2)))$, with coefficients

$$a_0 = \frac{13}{105}, \quad a_1 = \frac{17}{105}, \quad a_2 = -\frac{19}{420}, \quad a_3 = \frac{1}{105};$$

– $\int_a^{t_2} B_1(t)f(t)dt \approx h(a_0 f(t_0) + a_1 f(s_1) + a_2 f(t_1) + a_3 f(s_2)))$, with coefficients

$$a_0 := \frac{1}{21}, \quad a_1 = \frac{34}{105}, \quad a_2 = \frac{23}{210}, \quad a_3 = \frac{2}{105};$$

– $\int_a^{t_3} B_2(t)f(t)dt \approx h(a_0 f(t_0) + a_1 f(t_1) + a_2 f(t_2) + a_3 f(t_3)))$, with coefficients

$$a_0 = \frac{1}{35}, \quad a_1 = \frac{151}{280}, \quad a_2 = \frac{13}{70}, \quad a_3 = -\frac{1}{280}.$$

Symmetric formulas hold for the three last boundary B-splines.

Formula of order 8 for inner B-splines

$$\int_{t_{i-3}}^{t_{i+1}} B_i(t)f(t)dt \approx \frac{h}{1890}\left(19f(s_{i-2}) + 159f(t_{i-2}) + 453f(s_{i-1}) + 628f(t_{i-1}) \right.$$
$$\left. + 453f(s_i) + 159f(t_i) + 19f(s_{i+1})\right), \quad 3 \le i \le n-1.$$

Formulas of order 8 for boundary B-splines

– $\int_a^{t_1} B_0(t)f(t)dt \approx h\sum_{j=0}^{7} a_j f(r_j)$, $r_j := a + jh/8$, with coefficients

$$a_0 = \frac{3029}{89100}, \quad a_1 = \frac{3238}{22275}, \quad a_2 = -\frac{563}{17325}, \quad a_3 = \frac{758}{6237},$$
$$a_4 = -\frac{3191}{62370}, \quad a_5 = \frac{662}{17325}, \quad a_6 = -\frac{1051}{155925}, \quad a_7 = \frac{202}{155925};$$

– $\int_a^{t_2} B_1(t)f(t)dt \approx h\sum_{j=0}^{7} a_j f(r_j)$, $r_j := a + jh/4$, with coefficients

$$a_0 = \frac{37}{3465}, \quad a_1 = \frac{2456}{17325}, \quad a_2 = \frac{2026}{17325}, \quad a_3 = \frac{104}{693},$$
$$a_4 = \frac{247}{6930}, \quad a_5 = \frac{136}{3465}, \quad a_6 = \frac{74}{17325}, \quad a_7 = \frac{8}{5775};$$

– $\int_a^{t_3} B_2(t)f(t)dt \approx h\sum_{j=0}^{7} a_j f(r_j)$, $r_j := a + jh/2$, with coefficients

$$a_0 = \frac{53}{17325}, \quad a_1 = \frac{701}{5544}, \quad a_2 = \frac{2699}{9240}, \quad a_3 = \frac{929}{3960},$$
$$a_4 = \frac{577}{6930}, \quad a_5 = \frac{71}{6600}, \quad a_6 := -\frac{1}{3960}, \quad a_7 = \frac{1}{27720}.$$

Symmetric formulas hold for the three last boundary B-splines.

7 Numerical results

In this section, we compare the numerical results obtained by the Galerkin, Kantorovich, Sloan and Kulkarni's methods on integral equations of kind (1.1)-(1.2), whose exact solution u is known (see Table 7.1, with $x_1 := 1 - s$, $c_1 := \cos(1)$, $s_1 := \sin(1)$ in the function f of Test 2).

For the evaluation of integrals (5.1) we use the PQF proposed in Section 6 of order at least equal to the expected convergence order of the method, given in Theorems 4.1 and 4.2. For the evaluation of integrals (5.2) and (5.3) we apply the quadrature formulas generated by the 4th column of Romberg's algorithm (the first being the trapezoidal formula), that is written as follows:

$$\int_0^1 g(s)ds \approx \sum_{r=0}^{8} A_r g(rh), \quad h = 1/8,$$

where the coefficients A_r are given by

$$A_0 = A_8 = \frac{31}{810}, \quad A_1 = A_7 = \frac{512}{2835}, \quad A_2 = A_6 = \frac{176}{2835}, \quad A_3 = A_5 = \frac{512}{2835}, \quad A_4 = \frac{218}{2835}.$$

Firstly, we compute the maximum absolute error

$$e_n^\beta = \max_{z\in G} |u(z) - u_n^\beta(z)|,$$

where G is a set of 1500 equally spaced points in $I=[a,b]$ and $\beta = g, k, s, m$, in case of methods based both on the spline operator P_2 and on P_3, for increasing values of n. The results are reported in Tables 7.2÷7.6, where the quantities O_g, O_k, O_s and O_m are the numerical convergence orders, obtained by the logarithm to base 2 of the ratio between two consecutive errors.

Table 7.1 Numerical tests

Test	Interval $I=[a,b]$	Kernel k	Function f	Solution u
1	$[0,1]$	$\frac{1}{2}(s+1)\exp(-st)$	$\exp(-s)+\frac{1}{2}(\exp(-(s+1))-1)$	$\exp(-s)$
2	$[0,1]$	$\exp(st)$	$\exp(-s)\cos(s)+\frac{\exp(-x_1)(c_1x_1-s_1)-x_1}{s^2-2s+2}$	$\exp(-s)\cos(s)$
3	$[0,\pi]$	$\sin(s-t)$	$\cos(s)$	$\frac{2}{4+\pi^2}(2\cos(s)+\pi\sin(s))$
4	$[0,1]$	$s^{5/2}t^5$	$\sqrt{s}$	$\sqrt{s}+\frac{34}{195}s^{5/2}$
5	$[0,1]$	$s^5t^{5/2}$	$\sqrt{s}$	$\sqrt{s}+\frac{17}{60}s^5$

Concerning the smoothness of the test functions, in the first three tests, the kernel k, the function f and the solution u are sufficiently smooth so we expect and get the optimal convergence orders stated in Theorems 4.1 and 4.2. In Test 4, the kernel k is $C^2(I)$, but not $C^3(I)$, with respect to the variable s, $f\in C^0(I)$, but $f \notin C^1(I)$ and consequently we expect and get reduced convergence orders in case of Galerkin, Kantorovich and Kulkarni methods (as noticed in [7] in case of other projector choices). Similarly, in Test 5, the kernel k is $C^2(I)$, but not $C^3(I)$, with respect to the variable t, $f\in C^0(I)$, but $f\notin C^1(I)$ and so we expect and get a reduced convergence order in case of Galerkin and Sloan method.

For the Tests 1, 2 and 3, we also compute the maximum absolute error at the evaluation points belonging to $\mathscr{T}_n$ and $\mathscr{S}_n$

$$es_n^\beta = \max_{z\in \mathscr{T}_n\cup\mathscr{S}_n} |u(z) - u_n^\beta(z)|$$

where $\beta = g, k, m$, in case of methods based on the spline projector P_2, for increasing values of n. The results, reported in Table 7.7, confirm the theoretical superconvergence properties at the evaluation points given in Remark 4.1.

Table 7.2 Maximum absolute errors for Test 1

	Methods based on P_2							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	1.4(-04)		2.6(-05)		8.1(-06)		2.5(-08)	
8	1.7(-05)	3.1	3.0(-06)	3.1	3.3(-07)	4.6	1.1(-10)	7.9
16	2.0(-06)	3.0	3.7(-07)	3.0	1.5(-08)	4.5	4.2-(13)	8.0
32	2.5(-07)	3.0	4.6(-08)	3.0	7.3(-10)	4.3	1.7(-15)	7.9
64	3.1(-08)	3.0	5.6(-09)	3.0	3.9(-11)	4.2	6.7(-16)	–
128	3.8(-09)	3.0	7.0(-10)	3.0	2.3(-12)	4.1		–
256	4.7(-10)	3.0	8.7(-11)	3.0	1.4(-13)	4.1		–
	Methods based on P_3							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	7.4(-06)		1.5(-06)		3.3(-07)		2.5(-08)	
8	5.2(-07)	3.8	1.1(-07)	3.8	9.0(-09)	5.2	1.1(-10)	7.9
16	3.5(-08)	3.9	7.6(-09)	3.9	2.5(-10)	5.2	4.2(-13)	8.0
32	2.2(-09)	4.0	4.9(-10)	4.0	7.1(-12)	5.1	1.7(-15)	7.9
64	1.4(-10)	4.0	3.2(-11)	4.0	2.1(-13)	5.1	6.7(-16)	–
128	8.7(-12)	4.0	2.1(-12)	3.9	6.8(-15)	5.0		–
256	5.4(-13)	4.0	1.3(-13)	3.9	1.1(-15)	–		–

Table 7.3 Maximum absolute errors for Test 2

	Methods based on P_2							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	2.7(-04)		2.4(-05)		1.3(-04)		2.1(-09)	
8	3.1(-05)	3.1	2.8(-06)	3.1	5.7(-06)	4.5	1.0(-11)	7.7
16	3.9(-06)	3.0	3.4(-07)	3.0	2.8(-07)	4.3	5.4(-14)	7.6
32	4.9(-07)	3.0	4.2(-08)	3.0	1.5(-08)	4.2	1.0(-15)	–
64	6.1(-08)	3.0	5.3(-09)	3.0	8.8(-10)	4.1		–
128	7.6(-09)	3.0	6.5(-10)	3.0	5.3(-11)	4.0		–
256	1.1(-10)	3.0	8.1(-11)	3.0	3.2(-12)	4.0		–
	Methods based on P_3							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	3.1(-05)		3.3(-06)		2.3(-06)		1.2(-09)	
8	2.1(-06)	3.9	2.5(-07)	3.7	7.3(-08)	4.9	5.0(-12)	7.9
16	1.4(-07)	4.0	1.8(-08)	3.8	2.1(-09)	5.1	2.0(-14)	8.0
32	8.7(-09)	4.0	1.2(-09)	3.9	6.0(-11)	5.1	8.0(-16)	–
64	5.5(-10)	4.0	8.1(-11)	3.9	1.8(-12)	5.1		–
128	3.4(-11)	4.0	5.1(-12)	4.0	5.5(-14)	5.0		–
256	2.1(-12)	4.0	3.2(-13)	4.0	2.4(-15)	–		–

Table 7.4 Maximum absolute errors for Test 3

	Methods based on P_2							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
8	2.8(-04)		4.3(-04)		3.7(-05)		3.7(-08)	
16	3.3(-05)	3.1	5.2(-05)	3.1	1.9(-06)	4.2	2.3(-10)	7.3
32	4.1(-06)	3.0	6.4(-06)	3.0	1.1(-07)	4.1	1.6(-12)	7.1
64	5.1(-07)	3.0	8.0(-07)	3.0	6.8(-09)	4.0	1.3(-14)	7.0
128	6.4(-08)	3.0	1.0(-07)	3.0	4.2(-10)	4.0	1.6(-15)	–
256	8.0(-09)	3.0	1.2(-08)	3.0	2.6(-11)	4.0	1.5(-16)	–
	Methods based on P_3							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
8	3.8(-05)		6.4(-05)		2.0(-06)		2.4(-10)	
16	2.1(-06)	4.1	3.6(-06)	4.2	3.6(-08)	5.8	2.5(-13)	9.8
32	1.3(-07)	4.1	2.2(-07)	4.1	5.8(-10)	6.0	1.0(-15)	8.0
64	7.8(-09)	4.0	1.3(-08)	4.0	9.4(-12)	5.9		–
128	4.7(-10)	4.0	8.0(-10)	4.0	2.1(-13)	5.5		–
256	2.9(-11)	4.0	4.9(-11)	4.0	7.7(-15)	4.8		–

Table 7.5 Maximum absolute errors for Test 4

	Methods based on P_2							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
8	5.0(-02)		2.9(-05)		1.4(-07)		4.6(-09)	
16	3.5(-02)	0.5	5.2(-06)	2.5	3.2(-08)	2.1	3.8(-11)	6.9
32	2.5(-02)	0.5	9.1(-07)	2.5	2.2(-09)	3.8	3.6(-13)	6.7
64	1.8(-02)	0.5	1.6(-07)	2.5	1.4(-10)	3.9	4.0(-15)	6.5
128	1.3(-02)	0.5	2.8(-08)	2.5	8.6(-12)	4.0	8.9(-16)	–
256	7.7(-03)	0.7	4.9(-09)	2.5	5.4(-13)	4.0		–
	Methods based on P_3							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
8	4.4(-02)		5.3(-06)		4.1(-07)		4.6(-09)	
16	3.1(-02)	0.5	9.4(-07)	2.5	3.1(-09)	7.0	3.8(-11)	6.9
32	2.2(-02)	0.5	1.7(-07)	2.5	1.1(-11)	8.1	3.6(-13)	6.7
64	1.5(-02)	0.5	2.9(-08)	2.5	2.6(-13)	5.4	3.8(-15)	6.6
128	1.1(-02)	0.5	5.2(-09)	2.5	1.0(-14)	4.7	8.9(-16)	–
256	6.2(-03)	0.8	9.2(-10)	2.5	4.4(-16)	4.5		–

Table 7.6 Maximum absolute errors for Test 5

	Methods based on P_2							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	7.1(-02)		2.3(-03)		4.9(-05)		8.6(-07)	
8	5.0(-02)	0.5	2.8(-04)	3.0	2.4(-06)	4.3	4.6(-09)	7.5
16	3.5(-02)	0.5	3.4(-05)	3.0	6.3(-08)	5.3	1.9(-11)	7.9
32	2.5(-02)	0.5	4.2(-06)	3.0	7.8(-10)	6.3	5.9(-14)	8.3
64	1.8(-02)	0.5	5.2(-07)	3.0	2.7(-10)	1.5	1.0(-15)	–
128	1.3(-02)	0.5	6.5(-08)	3.0	2.8(-11)	3.3		–
256	7.7(-03)	0.7	8.0(-09)	3.0	2.3(-12)	3.6		–
	Methods based on P_3							
n	e_n^g	O_g	e_n^k	O_k	e_n^s	O_s	e_n^m	O_m
4	6.2(-02)		2.7(-04)		9.3(-05)		7.4(-07)	
8	4.4(-02)	0.5	1.9(-05)	3.8	5.2(-06)	4.2	3.2(-09)	7.9
16	3.1(-02)	0.5	1.2(-06)	3.9	3.2(-07)	4.1	1.3(-11)	7.9
32	2.2(-02)	0.5	7.9(-08)	4.0	1.9(-08)	4.0	5.5(-14)	7.9
64	1.5(-02)	0.5	5.0(-09)	4.0	1.2(-09)	4.0	5.5(-16)	–
128	1.1(-02)	0.5	3.1(-10)	4.0	7.5(-11)	4.0		–
256	6.2(-03)	0.8	1.9(-11)	4.0	4.7(-12)	4.0		–

Table 7.7 Maximum absolute errors at the sets of points $\mathscr{T}_n$ and $\mathscr{S}_n$, for methods based on P_2

	Test 1					
n	es_n^g	O_g	es_n^k	O_k	es_n^m	O_m
4	4.4(-05)		8.2(-06)		2.5(-08)	
8	2.8(-06)	4.0	5.3(-07)	4.0	1.1(-10)	7.9
16	1.8(-07)	4.0	3.4(-08)	4.0	4.2(-13)	7.9
32	1.1(-08)	4.0	2.1(-09)	4.0	1.7(-15)	7.9
64	7.1(-10)	4.0	1.3(-10)	4.0	5.6(-16)	–
128	4.4(-11)	4.0	8.4(-12)	4.0		–
256	2.8(-12)	4.0	5.3(-13)	4.0		–
	Test 2					
n	es_n^g	O_g	es_n^k	O_k	es_n^m	O_m
4	1.3(-04)		4.8(-06)		1.3(-09)	
8	5.9(-06)	4.5	2.3(-07)	4.4	4.9(-12)	8.0
16	4.4(-07)	3.7	1.2(-08)	4.3	2.0(-14)	7.9
32	3.0(-08)	3.9	8.3(-10)	3.9	4.4(-16)	–
64	1.9(-09)	3.9	5.6(-11)	3.9		–
128	1.2(-10)	4.0	3.6(-12)	4.0		–
256	7.8(-12)	4.0	2.3(-13)	4.0		–
	Test 3					
n	es_n^g	O_g	es_n^k	O_k	es_n^m	O_m
8	1.1(-04)		2.0(-04)		1.3(-08)	
16	6.1(-06)	4.1	1.3(-05)	4.0	4.3(-11)	8.2
32	3.7(-07)	4.1	7.9(-07)	4.0	1.6(-13)	8.1
64	2.3(-08)	4.0	4.9(-08)	4.0	1.5(-15)	–
128	1.5(-09)	4.0	3.0(-09)	4.0	1.5(-15)	–
256	9.1(-11)	4.0	1.9(-10)	4.0	1.4(-15)	–

8 Proofs of Theorems 2.1, 2.2 and Lemmas 4.1, 4.2, 4.3

In this section we report the proofs of Theorems 2.1, 2.2 and Lemmas 4.1, 4.2, 4.3. We recall that, in Lemmas 4.1, 4.2 and 4.3, $m_3(x) = x^3$, $e_3(x) = \pi_n m_3(x) - m_3(x)$, $v_n = \pi_n u - u$ and $\pi_n = P_2$.

Proof of Theorem 2.1

For the sake of simplicity, we take $I = [a,b] = [0,n]$ with $h = 1$. By using a computer algebra system, a first graphical study shows that the maximum of the Lebesgue function $\Lambda = \sum_{i=0}^{2n} |L_i|$ is attained in the first and last intervals. The first interval $[0,1]$ is covered by the supports of the seven first quasi-Lagrange functions (except L_4) and, using the local BB(=Bernstein-Bézier)-coefficients of B-splines B_0, B_1, B_2 and the definition of the quasi-Lagrange functions given in (2.4), we get the BB-coefficients of the latter, given in Table 8.1.

Table 8.1 BB-coefficients of $L_i(x)$, $i = 0,\dots 6$, $i \neq 4$, $x \in [0,1]$

	BB-coefficients of $L_i(x)$ $i = 0,\dots 6$, $i \neq 4$, $x \in [0,1]$
$L_0(x)$	[1,-1/2,-3/14]
$L_1(x)$	[0,2,6/7]
$L_2(x)$	[0,-1/2,-1/4]
$L_3(x)$	[0,0,5/7]
$L_5(x)$	[0,0,-1/7]
$L_6(x)$	[0,0,1/28]

From the BB-coefficients $[0,5/2,2]$ of $\Lambda_0 := \sum_{i=1,\, i\neq 4}^{6} |L_i| = L_1 - L_2 + L_3 - L_5 + L_6$, we deduce its equation

$$\Lambda_0(x) = 5x(1-x) + 2x^2 = 5x - 3x^2.$$

On the other hand, from the BB-coefficients of L_0, we deduce

$$L_0 = (1-x)^2 - x(1-x) - \frac{3}{14}x^2 = 1 - 3x + \frac{25}{14}x^2.$$

We have $L_0(x) \geq 0$ for $0 \leq x \leq x^*$ and $L_0(x) \leq 0$ for $x^* \leq x \leq 1$, where the unique zero of Λ_0 is

$$x^* = (21 - \sqrt{91})/25 \sim 0.46.$$

Hence the equations of Λ are respectively

$$\Lambda^-(x) = L_0(x) + \Lambda_0(x) = 1 + 2x - \frac{17}{14}x^2, \qquad x \in [0, x^*]$$

$$\Lambda^+(x) = -L_0(x) + \Lambda_0(x) = -1 + 8x - \frac{67}{14}x^2, \qquad x \in [x^*, 1]$$

It is easy to see that $\max_{x\in[0,x^*]} \Lambda^-(x) = \Lambda^-(x^*)$ and $\max_{x\in[x^*,1]} \Lambda^+(x) = \Lambda^+(\overline{x})$ with $\overline{x} = \frac{56}{67}$. We then deduce

$$\max_{x\in[0,1]} \Lambda(x) = \Lambda^+(\overline{x}) = \frac{157}{67},$$

which completes the proof. □

Proof of Theorem 2.2

For the sake of simplicity, we take $I = [a,b] = [0,n]$ with $h = 1$. By using a computer algebra system, a first graphical study shows that the maximum of the Lebesgue function $\overline{\Lambda} = \sum_{i=0}^{2n} |\overline{L}_i|$ is attained in the first and last intervals. The first interval $[0,1]$ is covered by the supports of the eight first quasi-Lagrange functions and, using the polynomial expressions of B-splines B_0, B_1, B_2, B_3 in $[0,1]$

$$B_0(x) = (1-x)^3, \quad B_1(x) = \frac{1}{4}x(7x^2 - 18x + 12), \quad B_2(x) = \frac{1}{12}x^2(18 - 11x), \quad B_3(x) = \frac{x^3}{6},$$

and the definition of the quasi-Lagrange functions given in (2.8), we get respectively

$$\overline{L}_0(x) = -\frac{257}{160}x^3 + \frac{71}{16}x^2 - \frac{23}{6}x + 1, \quad \overline{L}_1(x) = \frac{97}{20}x^3 - \frac{23}{2}x^2 + \frac{20}{3}x,$$

$$\overline{L}_2(x) = -\frac{6647}{1440}x^3 + \frac{153}{16}x^2 - 4x, \quad \overline{L}_3(x) = \frac{7}{9}x^3 - 2x^2 + \frac{4}{3}x,$$

$$\overline{L}_4(x) = \frac{1337}{1440}x^3 - \frac{15}{16}x^2 - \frac{1}{6}x, \quad \overline{L}_5(x) = -\frac{11}{36}x^3 + \frac{1}{2}x^2,$$

$$\overline{L}_6(x) = -\frac{97}{1440}x^3 - \frac{1}{16}x^2 \quad \overline{L}_7(x) = \frac{2}{45}x^3, \quad \overline{L}_8(x) = -\frac{1}{180}x^3.$$

In the interval $[0,1]$, the functions $\overline{L}_1, \overline{L}_3, \overline{L}_4, \overline{L}_5, \overline{L}_6, \overline{L}_7, \overline{L}_8$ have one sign while the other functions $\overline{L}_0$ and $\overline{L}_2$ have one zero, denoted by x_0^* and x_2^*, respectively

$$x_0^* = .4871225680, \quad x_2^* = .5815623494.$$

The maximum of $\overline{\Lambda}$ occurs in the subinterval $[0, x_0^*]$, where the equation of this function and its derivative are respectively

$$\overline{\Lambda}(x) = \frac{361}{48}x^3 - \frac{137}{8}x^2 + \frac{25}{3}x + 1, \quad \overline{\Lambda}'(x) = \frac{361}{16}x^2 - \frac{137}{4}x + \frac{25}{3}.$$

The function $\overline{\Lambda}'$ has one zero $x^* = \frac{274}{361} - \frac{22}{1083}\sqrt{501} \sim .304$ and the value of the maximum is equal to

$$\Lambda(x^*) = \frac{292460}{390963} + \frac{222277}{3518667}\sqrt{501}.$$

which concludes the proof. □

Proof of Lemma 4.1

One first prove that $e_3(t_i) = e_3(s_j) = 0$ for all i, j. This result is purely technical and it can be obtained by using the exact values of the B-splines on $\mathscr{T}_n$ and $\mathscr{S}_n$ and the definitions of coefficient functionals. We only give the proof for $e_3(t_i) = 0$, $2 \le i \le n$, the other cases being similar, but a little bit more specific near the endpoints of the interval.

Denoting $p_3(x) = (x - t_i)^3$, we have $m_3(x) = p_3(x) + p_2(x)$ where $p_2 \in \Pi_2$. Therefore

$$\pi_n m_3(x) = \pi_n p_3(x) + p_2(x) \Rightarrow e_3(x) = \pi_n p_3(x) - p_3(x) \Rightarrow e_3(t_i) = \pi_n p_3(t_i)$$

and

$$\pi_n p_3(t_i) = \frac{1}{2}(\lambda_i(p_3) + \lambda_{i+1}(p_3)) =$$
$$\frac{1}{14}(t_{i-2} - t_i)^3 - \frac{2}{7}(s_{i-1} - t_i)^3 + \frac{1}{14}(t_{i-1} - t_i)^3 + \frac{8}{7}(s_i - t_i)^3$$
$$+\frac{8}{7}(s_{i+1} - t_i)^3 + \frac{1}{14}(t_{i+1} - t_i)^3 - \frac{2}{7}(s_{i+2} - t_i)^3 + \frac{1}{14}(t_{i+2} - t_i)^3.$$

It is clear that this sum is equal to zero, as quantities with the same coefficients have opposite signs. For example, $t_{i-2} - t_i = -2h = -(t_{i+2} - t_i)$. This proves the first result.

Writing $\pi_n u$ in the quasi-Lagrange form

$$\pi_n u(x) = \sum_{i=0}^{n} u(t_i) L_{2i}(x) + \sum_{i=1}^{n} u(s_i) L_{2i-1}(x),$$

we observe that

$$e_3(x) = \sum_{i=0}^{n} (t_i^3 - x^3) L_{2i}(x) + \sum_{i=1}^{n} (s_i^3 - x^3) L_{2i-1}(x).$$

Now, starting from Taylor's formulas

$$u(t_i) = \sum_{k=0}^{2} u^{(k)}(x)(t_i - x)^k/k! + u^{(3)}(x)(t_i - x)^3/6 + R_i(x),$$

$$u(s_i) = \sum_{k=0}^{2} u^{(k)}(x)(s_i - x)^k/k! + u^{(3)}(x)(s_i - x)^3/6 + \tilde{R}_i(x),$$

with

$$R_i(x) = \frac{1}{6}\int_x^{t_i} (t_i - s)^3 u^{(4)}(s)ds, \quad \tilde{R}_i(x) = \frac{1}{6}\int_x^{s_i} (s_i - s)^3 u^{(4)}(s)ds,$$

we use the exactness of π_n on Π_2, the above expression of $e_3(x)$ and we define

$$R(x) = \sum_{i=0}^{n} R_i(x) L_{2i}(x) + \sum_{i=1}^{n} \tilde{R}_i(x) L_{2i-1}(x),$$

to get the following representation

$$\pi_n u(x) = u(x) + \frac{1}{6} u^{(3)}(x) e_3(x) + R(x). \tag{8.1}$$

Since

$$R(x) = \frac{1}{6}\left(\sum_{i=0}^{n} L_{2i}(x) \int_x^{t_i} (t_i - s)^3 u^{(4)}(s) ds + \sum_{i=1}^{n} L_{2i-1}(x) \int_x^{s_1} (s_1 - s)^3 u^{(4)}(s) ds \right),$$

therefore

$$\begin{aligned} |R(x)| &\le \frac{1}{6} \|u^{(4)}\|_\infty \left(\sum_{i=0}^{n} |L_{2i}(x)| \int_x^{t_i} (t_i - s)^3 ds + \sum_{i=1}^{n} |L_{2i-1}(x)| \int_x^{s_i} (s_i - s)^3 ds \right) \\ &= \frac{1}{24} \|u^{(4)}\|_\infty \left(\sum_{i=0}^{n} |L_{2i}(x)|(t_i - x)^4 + \sum_{i=1}^{n} |L_{2i-1}(x)|(s_i - x)^4 \right). \end{aligned}$$

Assume $x \in [t_{i-1}, t_i]$ which is covered by $\text{supp}(L_{2k-1}) = [t_{k-3}, t_{k+2}]$ for $k = i-2, \dots i+2$ and by $\text{supp}(L_{2\ell}) = [t_{\ell-3}, t_{\ell+3}]$ for $\ell = i-3, \dots i+2$. Since both $|s_k - x|$ and $|t_\ell - x|$ are $\le 3h$, we deduce

$$|R(x)| \le \frac{27}{8} h^4 \|u^{(4)}\|_\infty \left(\sum_{i=0}^{n} |L_{2i}(x)| + \sum_{i=1}^{n} |L_{2i-1}(x)| \right).$$

As the Lebesgue function $\Lambda(x) = \sum_{i=0}^{n} |L_{2i}(x)| + \sum_{i=1}^{n} |L_{2i-1}(x)|$ is bounded independently of n, we see that $R(x)| \le Ch^4 \|u^{(4)}\|_\infty$ for some constant C.

Now, from (8.1), we have

$$\pi_n u(x) - u(x) = \frac{1}{6} u^{(3)}(x) e_3(x) + O(h^4).$$

As $e_3(x) = 0$ on $\mathscr{S}_n \cup \mathscr{T}_n$, we see that $\pi_n u - u = O(h^4)$ at those points and we get the superconvergence of the quadratic projector. This ends the proof of Lemma 4.1. $\square$

Proof of Lemma 4.2

For the first equality, we use the symmetry of the abscissae with respect to the midpoint of the interval and the exactness of π_n on Π_2. Thus, the quadrature formula associated with the QIP is exact on Π_3.

For the second one, observe that, setting $m_3(x) = (x - s_i)^3 + q_2(x) = q_3(x) + q_2(x)$, where $q_2 \in \Pi_2$, as $\pi_n q_2 - q_2 = 0$, we get

$$\int_{t_{i-1}}^{t_i} e_3(x) dx = \int_{t_{i-1}}^{t_i} (\pi_n q_3(x) - q_3(x)) dx.$$

Now, as $\int_{t_{i-1}}^{t_i} q_3(x)dx = 0$, it is enough to prove that $\int_{t_{i-1}}^{t_i} \pi_n q_3(x)dx = 0$. Then, at least for interior subintervals, we have $\int_{t_{i-1}}^{t_i} \pi_n q_3(x)dx = \frac{h}{6}(\lambda_{i-1}(q_3)+4\lambda_i(q_3)+\lambda_{i+1}(q_3))$. Then, one computes

$$\begin{aligned}\lambda_{i-1}(q_3)+4\lambda_i(q_3)+\lambda_{i+1}(q_3) = {} & \frac{1}{14}(q_3(t_{i-3})+4q_3(t_{i-2})+q_3(t_{i-1})+q_3(t_i) \\ & +4q_3(t_{i+1})+q_3(t_{i+2}))-\frac{2}{7}(q_3(s_{i-2})+q_3(s_{i+2})) \\ & +\frac{2}{7}(q_3(s_{i-1})+q_3(s_{i+1}))+\frac{36}{7}q_3(s_i)\end{aligned}$$

and this quantity is equal to zero in view of the symmetry of data points with respect to s_i and the fact that q_3 satisfies $q_3(s_i+w) = -q_3(s_i-w)$. This is also the case for the first and last intervals. □

Proof of Lemma 4.3

For the first equality, setting $\gamma_j = \frac{1}{h}\int_{t_{j-1}}^{t_j} g(t)dt$, for all $j = 1\ldots n$, and defining the piecewise constant function γ by $\gamma(x) = \gamma_j$ for $x \in I_j = (t_{j-1}, t_j)$, then

$$\|g-\gamma\|_1 = \int_a^b |g(x)-\gamma(x)|dx = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |g(x)-\gamma_j|dx.$$

Since

$$|g(x)-\gamma_j| \leq \frac{1}{h}\int_{t_{j-1}}^{t_j} |g(x)-g(t)|dt \leq \frac{1}{h}\int_{t_{j-1}}^{t_j}\int_t^x |g'(s)|dsdt \leq \int_{t_{j-1}}^{t_j} |g'(s)|ds,$$

then

$$\|g-\gamma\|_1 \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t_j} |g'(s)|dsdx = h\sum_{j=1}^n \int_{t_{j-1}}^{t_j} [g'(s)|ds = h\|g'\|_1.$$

Now, we can write

$$\int_a^b g(t)e_3(t)dt = \int_a^b (g(t)-\gamma(t))e_3(t)dt + \int_a^b \gamma(t)e_3(t)dt.$$

On the one hand, we have the majoration

$$\left|\int_a^b (g(t)-\gamma(t))e_3(t)dt\right| \leq \|g-\gamma\|_1\|e_3\|_\infty = O(h^4).$$

On the other hand, for all $j = 1\ldots n$, we can write

$$\int_a^b \gamma(t)e_3(t)dt = \sum_{j=1}^n \gamma_j \int_{t_{j-1}}^{t_j} e_3(t)dt = 0,$$

in view of Lemma 4.2.

For the second equality, using the same technique, we obtain

$$\left|\int_a^b g(t)(\pi_n u(t)-u(t))dt\right| \le \|g-\gamma\|_1\|u-\pi_n u\|_\infty + \|\gamma\|_\infty \int_a^b |\pi_n u(t)-u(t)|dt.$$

As $\|\pi_n u - u\|_\infty = O(h^3)$ and $\|g-\gamma\|_1 = O(h)$, the first term is a $O(h^4)$. For the second one, we use Lemma 4.1 to deduce

$$\int_a^b (\pi_n u(x)-u(x))dx = \frac{1}{6}\int_a^b u^{(3)}(x)e_3(x)dx + \int_a^b R(x)dx.$$

As the second term of the right-hand side is a $O(h^4)$, taking $g(x)=u^{(3)}(x)$ in the first term, the first equality of the present lemma leads to the desired result. □

References

1. Allouch, Ch., Sablonnière, P., Sbibih, D.: Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants. Numer. Algorithms **56**, 437-453 (2011)
2. Allouch, Ch., Sablonnière, P., Sbibih, D.: A modified Kulkarni's method based on a discrete spline quasi-interpolant. Math. Comput. Simul. **81** 1991-2000 (2011)
3. Atkinson, K.E.: The numerical solution of integral equations of the second kind. Cambridge University Press (1997)
4. Barrera, D., Sablonnière, P.: Numerical computation of univariate integrals with B-spline weights. I. Gauss type quadrature formulas, II. Product type quadrature formulas. University of Rennes, in preparation, (2013).
5. de Boor, C.: A practical guide to splines (Revised edition). Springer Verlag, Berlin (2001)
6. DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Springer Verlag, Berlin (1993)
7. Kulkarni, R.: A superconvergence result for solutions of compact operator equations. Bull. Austral. Math. Soc. **68**, 517-528 (2003)
8. Kulkarni, R.: On improvement of the iterated Galerkin solution of the second kind integral equations. J. Numer. Math. **13**, 205-218 (2005)
9. Kulkarni, R., Grammont, L.: Extrapolation using a modified projection method. Numer. Funct. Anal. & Optimiz. **30**, 1339-1359 (2009)
10. Sablonnière, P.: Univariate spline quasi-interpolants and applications to numerical analysis. Rend. Sem. Mat. Univ. Pol. Torino **63**, 211–222 (2005)
11. Sablonnière, P.: A quadrature formula associated with a univariate quadratic spline quasi-interpolant. BIT Numer. Math. **47**, 825-837 (2007)
12. Sbibih, D., Allouch, Ch., Sablonnière, P., Tahrichi, M.: Product integration methods based on discrete spline quasi-interpolants and applications to weakly singular integral equations. J. Comput. Appl. Math. **233**, 2855-2866 (2010)
13. Sablonnière, P., Sbibih, D., Tahrichi, M.: High-order quadrature rules based on spline quasi-interpolants and application to integral equations. Appl. Numer. Math. **62**, 507-520 (2012)
14. Sloan, I.: Improvement by iteration for compact operator equations. Math. Comp. **30**, 758-764 (1976)