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Extended Theories of Gravitation

Observation Protocols and Experimental Tests

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Abstract. Within the framework of extended theories of gravitation we shall discuss physical equivalences among different formalisms and classical tests. As suggested by the Ehlers-Pirani-Schild framework, the conformal invariance will be preserved and its effect on observational protocols discussed. Accordingly, we shall review standard tests showing how Palatini $f(\mathcal{R})$-theories naturally passes solar system tests. Observation protocols will be discussed in this wider framework.

1 Introduction

In [1] we defined and discussed extended theories of gravitation (ETG) and their EPS interpretation; see [2], [3]. In particular we introduced Palatini $f(\mathcal{R})$-theories in which one considers as fundamental fields a metric g, a (torsionless) connection $\tilde{\Gamma}$ together with a collection of matter fields ψ; see [4], [5], [6]. The dynamics is described by a Lagrangian in the form

$$L = \sqrt{g} f(\mathcal{R}) + L_m(\psi, g) \tag{1}$$

where $\mathcal{R}(g, \tilde{\Gamma}) := g^{\mu\nu}\tilde{R}_{\mu\nu}$, where $\tilde{R}_{\mu\nu}$ is the Ricci tensor of the independent connection $\tilde{\Gamma}$, and where f is a generic (analytic or *sufficiently regular*) function. Field equations can be recast as

$$\begin{cases} \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \kappa\left(\frac{1}{\varphi(T)}\left(T_{\mu\nu} - \frac{1}{4}T g_{\mu\nu}\right) - \frac{1}{4}\hat{r}(T) g_{\mu\nu}\right) =: \kappa\tilde{T}_{\mu\nu} \\ f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T := \kappa g^{\mu\nu}T_{\mu\nu} \end{cases} \tag{2}$$

where we introduced a conformal metric $\tilde{g} := \varphi(T)\cdot g$ and we set $T := g^{\mu\nu}T_{\mu\nu}$. The original connection which is associated to free fall is given by $\tilde{\Gamma} = \{\tilde{g}\}$. Here the conformal factor $\varphi(T)$ is obtained as $\varphi(T) = f'(r(T))$ and $r(T)$ is a solution of the second equation in (2), which is called the *master equation*. The triple $(M, [g], \tilde{\Gamma} = \{\tilde{g}\})$ is an integrable Weyl geometry on spacetime.

The EPS framework provides an interpretation of the objects involved. In particular the metric g (or better its conformal structure $\mathfrak{C} = [g] \equiv [\tilde{g}]$) is related to light cones; the metric g is a representative of the conformal structure and any choice of a conformal representative can be considered

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as a definition of geometrical distances on spacetime. We can thence conjecture that g is related to operational definitions of length and time lapses.

The connection $\tilde{\Gamma} = \{\tilde{g}\}$ is related to free fall of massive particles and in view of the EPS-compatibility condition

$$\tilde{\Gamma}^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} - \tfrac{1}{2}(g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{\mu)})\partial_{\epsilon}\ln\varphi \tag{3}$$

also to the free fall of light rays.

By observing field equations (2) one can see how Palatini $f(\mathcal{R})$-theories behave like standard GR, but with a different effective source term $\tilde{T}_{\mu\nu}$. The effective source term $\tilde{T}_{\mu\nu}$ is conserved because of Bianchi identities; however, not being fundamental, the usual energy conditions are less "stringent". For example, for some f they can correspond to effective exotic matter sources which are usually forbidden in standard GR and allow solutions like wormholes or singularities which potentially violate causality.

Hereafter we shall discuss in detail Palatini $f(\mathcal{R})$-theories. In particular we shall show that although Palatini $f(\mathcal{R})$-theories are mathematically equivalent to Brans-Dicke theories (which are ruled out by classical tests in the Solar System), they are physically inequivalent being free fall different in the two theories. We refer to the appendix for a discussion in a simple mechanical example about how observables can violate mathematical equivalence.

2 Equivalence with Brans-Dicke theories

One can consider field equations (2) back in the original metric g obtaining

$$\begin{cases}\varphi R_{\mu\nu} = \nabla_{\mu\nu}\varphi + \frac{1}{2}\Box\varphi g_{\mu\nu} - \frac{3}{2\varphi}\nabla_{\mu}\varphi\nabla_{\nu}\varphi + \frac{1}{4}\varphi\hat{r}(T)g_{\mu\nu} + \kappa\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right)\\ \varphi R = 3\Box\varphi - \frac{3}{2\varphi}\nabla_{\alpha}\varphi\nabla^{\alpha}\varphi + \kappa T + 2f\end{cases} \tag{4}$$

Within the framework for $f(\mathcal{R})$-theory, one can generically solve the definition of the conformal factor $\varphi = f'(\mathcal{R})$ to obtain $\mathcal{R} = \sigma(\varphi)$ and define a potential function

$$U(\varphi) = -\varphi\sigma(\varphi) + f(\sigma(\varphi)) \qquad (\Rightarrow\ U'(\varphi) = -\sigma'\varphi - \sigma + f'\sigma' = -\sigma) \tag{5}$$

Equations (4) can be recognized as field equations of a *Brans-Dicke* theory for a metric $g_{\mu\nu}$ with dynamics described by a Lagrangian in the following form

$$L_{BD} = \sqrt{g}\left[\varphi R - \frac{\omega}{\varphi}\nabla_{\mu}\varphi\nabla^{\mu}\varphi + U(\varphi)\right] + L_m(g,\psi) \tag{6}$$

where we set $\omega = -\frac{3}{2}$. Thus we can summarize the situation by saying that any Palatini $f(\mathcal{R})$-theory is equivalent to a Brans-Dicke theory (with $\omega = -\frac{3}{2}$ and a suitable potential). Let us remark that Brans-Dicke theories (without potential) are considered in testing standard GR; see [7]. In fact standard GR corresponds to the limit $\omega \to \infty$ and classical tests within Solar System rule out small values of ω (among which $\omega = -3/2$ as for $f(\mathcal{R})$-theories). Thus we have two theories which are mathematically equivalent (one can map one action –field equations and solutions– into the other by a 'conformal transformation') one of which (Brans-Dicke) is ruled out by observations. Can we conclude that the other theory (Palatini $f(\mathcal{R})$-theory) is ruled out as well?

In Appendix A we discuss the issue in a simple mechanical example. To answer let us remark that, first of all, what is ruled out by observations is Brans-Dicke *without a potential*. One should then discuss whether the potential has some influence on observation. Let us also remark that the potential (5) is singluar exactly on standard GR where the conformal factor $\varphi = f'(\mathcal{R}) \equiv 1$ cannot be solved for $\mathcal{R}$. For the sake of argument let us assume that the potential does not effect observation and Brans-Dicke model (6) *is* ruled out.

Secondly, there is a number of features in a field theory which are assumed independently of the action principle as independent assumptions. One is the interpretation. In Brans-Dicke theory the free fall is dictated by the metric g, in Palatini $f(\mathcal{R})$-theories with EPS interpetation it is dictated by $\tilde{g}$. In the two mathematically *"equivalent"* models bodies fall along different worldlines. For example Mercury will go along different orbits so that the perihelia precession test failed by Brans-Dicke theories will not apply to Palatini $f(\mathcal{R})$-theories. We shall be back on this test below to discuss it in further detail.

Moreover, also an independent assumption is: which metric should be used to define distances on spacetime? In Palatini $f(\mathcal{R})$-theories one has two natural conformal metrics (and in fact a whole conformal class). Each representative of the conformal class defines a different notion of distance and it is quite hard to see which metric is selected by the *usual* observational protocols. This issue has been noticed by EPS as well; they stop to discuss how firmly we know that *gravitational time* (which is what we call the proper time) is identical to *atomic time* (which is what we use); they concluded that the issue cannot be easily addressed; see also [8].

Finally, let us stress that we are using *conformal transformation* with two meanings. In EPS framework a conformal transformation is changing the metric, leaving the connection (as well as the spacetime point) unchanged. In view of EPS analysis these transformations are expected to be gauge transformations since one cannot observe representatives of the conformal structure. When discussing the equivalence with Brans-Dicke theories (see [9]) we made a *"conformal transformation"* to go back to the original metric g. However, at that point we already eliminated the connection $\tilde{\Gamma} = \{\tilde{g}\}$ so that by acting on the metric $\tilde{g}$ we *also* act on the connection. This is not a gauge transformation as the one found in EPS and in fact this affects the model.

The equivalent Brans-Dicke action (6) can be further recast (up to a divergence) as

$$L^* = \sqrt{\tilde{g}}\left[\tilde{R} - \left(\omega + \frac{3}{2}\right)\frac{1}{\varphi^2}\tilde{\nabla}_\mu\varphi\tilde{\nabla}^\mu\varphi + U(\varphi)\right] + L_m(\varphi^{-1}\tilde{g}, \psi) \tag{7}$$

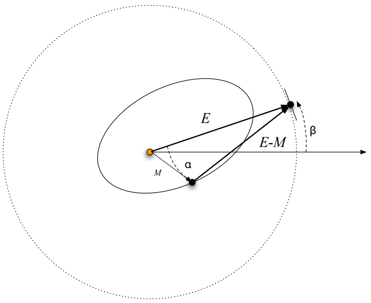
Notice how in Palatini $f(\mathcal{R})$-theories ($\omega = -3/2$) the conformal factor looses its dynamics. It enters the action as a Lagrange multiplier and it inherits its dynamics from the dynamics of $\tilde{g}$ and ψ.

3 Standard Test for Palatini $f(\mathcal{R})$-theories

Let us consider the test for precession of perihelia of Mercury for $f(\mathcal{R})$-theories. First we need a model for the gravitational field around the Sun; that is well approximated by a static, spherically symmetric vacuum solution. In view of universality theorem (see [10]), we *know* that the metric $\tilde{g}$ must be some sort of Schwarzschild-(A)dS solution and the conformal factor is a constant related to the cosmological constant. We know by experience that cosmological constant effects within the Solar System are hard to be detected so that we expect the solution to be well approximated by Schwarzschild solution. In view of the EPS interpretation Mercury, unlike in Brans-Dicke theories, goes along the geodesics of the Scharwschild $\tilde{g}$. Thence one expects $f(\mathcal{R})$-theories to be almost identical to standard GR and quite different from Brans-Dicke, at least in this situation. It is then natural to expect that a whole family of $f(\mathcal{R})$-theories to pass the precession test as well as standard GR. Unfortunately, by the same argument it is natural to expect to be difficult to test Palatini $f(\mathcal{R})$-theories against standard GR.

It is instead interesting to go through the Mercury test in the context of Palatini $f(\mathcal{R})$-theories, i.e. in view of EPS interpretation, tracing the influence of conformal transformations. In particular, let us go through it at first by relying only on conformally invariant quantities. This will provide insights about the meaning of the conformal factor.

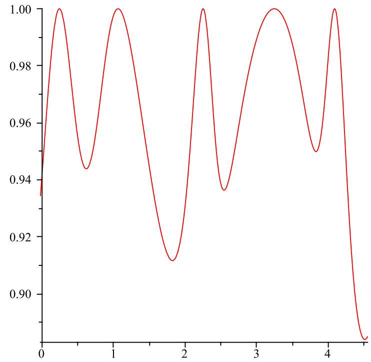
Let us simplify the situation by assuming that the Earth moves along a circular orbit coplanar to the one of Mercury and that it always faces the Sun.



Orbit of Mercury and Earth around the Sun

Let us start by using an apparatus apt to measure angles in the sky and no clock or ruler, so that we measure conformally invariant quantities, only. We can measure the angle β between a fixed star on the ecliptic plane and Sun (or equivalently to measure which fixed star is at right angle with Sun so staying away from possible deflections of light rays) and to measure the angle α between Mercury and the Sun.

Then one can obtain a dataset $(\beta, \cos(\alpha))$. We can easily obtain a prediction of the graph for the function $\cos(\alpha) = \chi(\beta)$ within Kepler approximation. There are two important features to be remarked about this prediction. First of all one can check that the function $\cos(\alpha) = \chi(\beta)$ is conformally invariant; in particular, the parameters depending on the conformal gauge (such as the dimension of the orbits or their periods) do not appear.



Prediction of $\cos(\alpha) = \chi(\beta)$ *using Kepler laws*

Secondly, the observed data collected during few orbits do fit well the Kepler prediction (otherwise one would be able to spot GR effects during few orbits and would not need observations along few centuries).

Then by fitting the dataset against the curve one can obtain an estimate of the best fit orbit parameters and in principle having a lot of redundancy to check *a posteriori* the hypothesis that Mercury motion is well described by Kepler approximation.

Let us remark that we are not using clocks. We are instead using the orbital motion of Earth as a clock, measuring only angles. This can be regarded as a typical implementation of Leibniz's (or

Einstein's) *relational time* according to which phenomena are not parametrized against an absolute Newtonian time, but they evolve one against the others; see [11].

Hence observed dataset by best fitting provides a prediction for the conformally invariant orbital parameters $\ell \simeq 0.380$, $\epsilon \simeq 0.205$ where ℓ is the ratio of the Earth's and Mercury's orbital radii in our solar system (which is also a conformal invariant parameter being the ratio of two distances) and ϵ is the eccentricity of Mercury orbit. One also obtains some value $\theta \simeq \theta_0$ for the orientation of Mercury orbit. These agree with the observed orbital parameters of Mercury.

Now we can repeat the measurement over and over and find that ϵ and ℓ are constant over *'time'* within the measurement errors while θ appears to be increasing linearly. This shows that we are able to check over time (or β) the evolution of the perihelia of Mercury.

Let us remark that standard GR predicts

$$\Delta\theta = \frac{6\pi}{1-\epsilon^2}\frac{GM_{\text{Sun}}}{a} \qquad \text{(rad per Mercury revolution)} \tag{8}$$

i.e. $\Delta\theta \simeq 43.03$" per century (per 100 revolutions of the Earth). Let us also stress that the mass of the Sun M_{Sun} is *defined* in Astronomy as $GM = 4\pi^2 a^3 T^{-2}$ where a and T are the orbital semiaxis and period, respectively, for any planet (e.g. Mercury). Accordingly, the quantity GM_{Sun}/a is conformally invariant, as of course is the factor $6\pi/(1-\epsilon^2)$. Let us also notice that the result follows only assuming that planets are moving along Schwarzschild (timelike) geodesics and thence extends to Palatini $f(\mathcal{R})$-theories, i.e. to a metric (conformal to) a Schwarzschild metric.

Within Brans-Dicke (with no potential) theory, Mercury moves along different orbit and the prediction would be different by a factor

$$\Delta\theta = \frac{6\pi}{1-\epsilon^2}\frac{GM_{\text{Sun}}}{a}\left(\frac{3\omega+4}{3\omega+6}\right) \qquad \text{(rad per Mercury revolution)} \tag{9}$$

which is sensibly different from the prediction of both standard GR and for Palatini $f(\mathcal{R})$-theories (by a factor $-\frac{1}{3}$ for the value $\omega = -\frac{3}{2}$). Thus to summarize both standard GR and Palatini $f(\mathcal{R})$-theories (unlike Brans-Dicke theories) predict a shift of perihelia of mercury of

$$\Delta\theta = 43.03\text{"} \qquad \text{(rad per 100 revolutions of the Earth)} \tag{10}$$

Such a prediction is conformally invariant (while of course it would not be so if stated in *rad per century* which would depend on a clock fixing). The conformal invariance tells us that until we fix a clock (or a ruler) we are not able to distinguish on observational stance between our Solar System and a star system which happens to be bigger, slower and with a star which is more massive than the Sun by a single factor which rescaled distances, times, and masses. If we misjudged distances and times for some reasons this implies extra gravitational mass of the star.

It is natural to ask for evidences that dark matter (see [12], [13], [14]) does not arise by such a mechanism (especially in view that at cosmological scales where one may expect a conformal factor depending on the matter density of visible matter, hence depending on time but at time scales comparable with the age of universe). In any event, at scales of the Solar System we would not see and difference; only when observations are concerned with dimensions or time lapses in which one can appreciate the variation of the conformal factor. Whenever the conformal factor can be approximated by a constant its effect naturally hides in the values of the "universal constants" (namely, G) which are measured here and now. Of course, here we are claiming that Palatini $f(\mathcal{R})$-theories (unlike the mathematically "equivalent" Brans-Dicke theory) passes standard tests in the Solar System; we are not claiming that other effects could not rule them out and spot the difference. The issue at bigger scales must be analyzed and discussed in details in view of EPS interpretation.

4 Extended Cosmologies

Let us now consider a cosmological situation in view of the discussion above for a general Palatini $f(\mathcal{R})$-model. As a first step one has to impose cosmological principle on the metric. However, in Palatini $f(\mathcal{R})$-theories one has at least two different metrics and it is not clear on which one should impose homogeneity and isotropy. Luckily enough, if one imposes cosmological principle on g then automatically $\tilde{g}$ is also homogeneous and isotropic. If one requires that $\tilde{g}$ is in the form

$$\tilde{g} = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)$$

then also g is homogeneous and isotropic, i.e. in the form

$$g = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)$$

provided one rescales the cosmological time with the conformal factor (which depends only on time) $d\tilde{t} = \sqrt{f'}\,dt$ (i.e. $\tilde{t}(t) = \int \sqrt{f'}\,dt$) and rescales the Friedmann-Lemaître-Robertson-Walker (FLRW) scale factor accordingly $\tilde{a}(\tilde{t}) = \sqrt{f'}\,a(t)$.

Then one should assume that the matter stress tensor $T_{\mu\nu}$ is the energy momentum tensor of a (perfect) fluid

$$T_{\mu\nu} := p g_{\mu\nu} + (p+\rho) u_\mu u_\nu \tag{11}$$

where $u^\alpha u^\beta g_{\alpha\beta} = -1$ and we set ρ for the fluid density and p for its pressure. Matter field equations are assumed to provide a relation between pressure and density under the form $p = w\rho$ for some (constant) w. Then one has $\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}_\mu \tilde{u}_\nu + \tilde{p}\tilde{g}_{\mu\nu}$ where we set $\tilde{u}_\mu = (f')^{\frac{1}{2}}\, u_\mu$ and

$$\begin{cases} \tilde{\rho} = \frac{3}{4}(p+\rho)(f')^{-2} + \frac{1}{4}\hat{r}(f')^{-\frac{1}{2}} \\ \tilde{p} = \frac{p+\rho}{4}(f')^{-2} - \frac{1}{4}\hat{r}(f')^{-1} \end{cases} \tag{12}$$

Thus the effect of a Palatini $f(\mathcal{R})$-dynamics is to modify the fluid tensor representing sources into another stress tensor which is again in the form of a (perfect) fluid, with modified pressure and density. This can be split quite naturally (though of course non-uniquely) into three fluids with

$$\begin{cases} \tilde{\rho}_1 = \rho \\ \tilde{p}_1 = p \end{cases} \qquad \begin{cases} \tilde{\rho}_2 = \frac{3}{4}(p+\rho)(f')^{-2} - \rho \\ \tilde{p}_2 = \frac{p+\rho}{4}(f')^{-2} - p \end{cases} \qquad \begin{cases} \tilde{\rho}_3 = \frac{1}{4}\hat{r}(f')^{-1} \\ \tilde{p}_3 = -\frac{1}{4}\hat{r}(f')^{-1} = -\tilde{\rho}_3 \end{cases} \tag{13}$$

The first fluid accounts for what we see as visible matter and it has standard equation of states $p_1 = w_1\rho_1$ with $w_1 = w$, i.e. the same state equation chosen for the visible matter. The third fluid has equation of states in the form $p_3 = w_3\rho_3$ with $w_3 = -1$, i.e. it is a quintessence field.

This is probably the main reason to consider Palatini $f(\mathcal{R})$-theories as an interesting model also for Cosmology and dark sources: although we assumed only dust at fundamental level, from the gravitational viewpoint that behaves effectively as a more general fluid the characteristics of which depend on the extended gravitational theories chosen, i.e. on f.

Morever, let us also remark that this simple toy model can be easily tested and falsified by current data and it makes predictions about near future surveys. In the standard ΛCDM one assumes a cosmological constant Λ which is here modeled by the third fluid. Thus in order to fit data one has to fix the current value for f', which in turn fixes the current equation of state for the CDM dark matter which is also observed. Of course one can consider other reasonable models considering more realistic and finer descriptions of visible matter. Near future surveys will provide data about the evolution of the cosmological constant in time allowing in principle to observe $f'(t)$ directly.

5 Conclusions and Perspectives

We considered Palatini $f(\mathcal{R})$-theories as alternative theories for gravity. We reviewed the equivalence with Brans-Dicke models though we argued that the mathematical equivalence does not extend to a physical equivalence between the two models. This is due to different assumptions done in the two models about free fall. As a consequence, for example, Palatini $f(\mathcal{R})$-theories (at least a whole family of them) make the same predictions about precession of Mercury perihelia and thence they pass the standard test in Solar System.

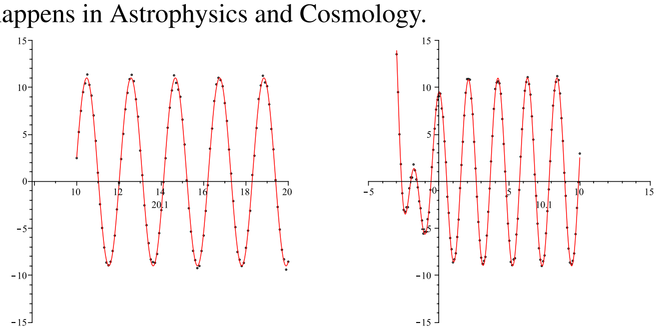
By analyzing in detail the classical test by restricting to measure only conformally invariant theory we pointed out that there is a strict relation between conformal gauge, gravitational masses as defined in Astronomy, and the standard operative definitions of distances on spacetime. This relation has been frozen in standard GR by assuming from the very beginning that all these physical quantities (which have quite independent origins) are associated by a single metric. Of course once the assumption is encoded into the theory it is then difficult to test it in a model independent fashion. Under this viewpoint Palatini $f(\mathcal{R})$-theories provide a more general framework in which projecting tests to decide the characteristics of gravitational phenomenology on observations.

In Palatini $f(\mathcal{R})$-theories one has essentially two types of effects: on one hand the modified dynamics produces extra effective dark sources. On the other hand, the mutual relation between different fixings of the conformal gauge (i.e. the conformal factor) provides an extra degree of freedom to discuss and adjust a possibly non–standard relation between gravitational field and our usual operative definition of distances on spacetime.

Both the effects may turn out to be physically relevant or not. In any case, this is a issue to be addressed experimentally and Palatini $f(\mathcal{R})$-theories provides a suitable framowork for this adjudication.

Appendix A: Equivalence between models

Here we shall show in a simple mechanical example how observational protocols can prevent a mathematical equivalence between models to be recognized as a complete physical equivalence. Here the point is that even in view of a mathematical equivalence which in principle allows to map one description into the other, observational protocols may break the equivalence. This may happen especially when observations are difficult and one is unable to measure everything but just some quantities can be measured, as it happens in Astrophysics and Cosmology.



$x(t)$ in the time range
a) $10 \le t \le 20$. *b)* $-3 \le t \le 10$.

Imagine we are given a material point constrained on a straight line free to move under unknown forces to be studied. We know that one can measure the position x and the the momentum π. We can easily plot x (or π) in their time evolution.

One question which is easily asked is whether the system can be described as a Hamiltonian system. For it, one should identify two quantities q and p, give a Hamiltonian function $H(q, p)$ and relate the evolution of the system and the solutions of the Hamilton equations

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \tag{14}$$

By observing the motion of the point in the range (a) $10 \leq t \leq 20$ one could make an educated guess for a harmonic oscillator. By a closer look one could fit the data very well by a function

$$x = A\cos(\omega(t + t_0)) + \lambda \tag{15}$$

with, say, $A = 10, \omega = 3, \lambda = 1, t_0 = 0$. The system is thence described by the Hamiltonian

$$\bar{H} = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 - \lambda\omega^2 q \tag{16}$$

Accordingly, one can guess the force to be a harmonic force plus a constant force.

After that one obtains a bigger dataset (b) which includes the time range $-3 \leq t \leq 10$. The new dataset clearly disagrees with previous guesses. It seems that something awkward happened at time $t \sim 0$. One could either suppose some extra temporary force acted and it is then switched off, or look for as single time dependent force explaining both datasets. In fact a very good fit with the dataset can be obtained by the function

$$x = \alpha(t)(A\cos(\omega(t - t_0)) + \lambda) \tag{17}$$

for a parameter function $\alpha(t) = 1 - e^{-(t+2)}$ and again with, e.g., $A = 10, \omega = 3, \lambda = 1, t_0 = 0$.

This is a solution of Hamilton equations of the following (time-dependent) Hamiltonian

$$H = \frac{\alpha^2}{2}p^2 + \frac{\omega^2}{2\alpha^2}q^2 - \lambda\frac{\omega^2}{\alpha}q + \frac{\dot{\alpha}}{\alpha}pq \tag{18}$$

Of course this model is a big step forward since it describes the whole dataset. On the other hand term in pq has not a direct mechanical interpretation.

Then one can try to look for a canonical transformation to simplify the model. In particular one can define a (time-dependent) canonical transformation

$$\begin{cases} Q = \frac{q}{\alpha(t)} - \lambda \\ P = \alpha(t)p \end{cases} \tag{19}$$

and check that the new Hamiltonian is simply

$$K = \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 \tag{20}$$

Obviously, the canonical transformation establishes a very well founded mathematical equivalence between the two Hamiltonian systems described by H and K (which is in fact *one* Hamiltonian system with two different local representations).

To what extent though the system is a pure harmonic oscillator on the physical stance? To answer the question maybe it is worth considering some remarks. First, the dynamics K is much simpler than the dynamics of H. Second, in the K framework there is an observable $x = \alpha(t)(Q + \lambda)$ which fits the dataset as perfectly as q does in the H framework. Third, in the H framework the observable used for fitting is simply $x = q$.

Accordingly, we have two mathematically equivalent frameworks, one in which the dynamics is particularly simple, the other in which what we observe (i.e. our observational protocols) is particularly simple. Because of the particular situation either we find a way of observing directly the quantity Q (which would make the K framework superior under all viewpoints) or we have to resign to have two frameworks each one being simpler under a different viewpoint.

Let us also finally remark that the transformation between the two frameworks, namely $q = \alpha(t)(Q + \lambda)$, can be directly related to a (time-dependent) mismatch of the protocol of measuring the position. It is as if, besides changing the origin λ of the position reference frame, we did change the unit of distances by a (time-dependent) factor $\alpha(t)$ (imagine for example we are using ultrasounds to define position and the speed of sound in changing with time, as it may happen on a space lab if pressurization is for some reason not preserved). The mismatch does not emerge in datasets covering intervals in which the function α can be considered constant (as it happens in the dataset a) while it become manifest once the dependence of α on time can be appreciated.

This rather trivial example in any event shows how one should not use mathematical equivalence to dismiss a framework without carefully reviewing the observational protocols and verifying their compatibility with the equivalence transformations, especially in Cosmology where we know from the very beginning that most of the time we are measuring quantities that are not gauge covariant; see [11].

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