Dipartimento Economia e Statistica Cognetti de Martiis

Working Paper Series

15/14

A COMPETITIVE IDEA-BASED GROWTH MODEL WITH SHRINKING WORKERS' INCOME

CARLA MARCHESE and FABIO PRIVILEGGI

Department of Economics and Statistics "Cognetti de Martiis"
Campus Luigi Einaudi, Lungo Dora Siena 100/A, 10153 Torino (Italy)
www.est.unito.it

UNIVERSITÀ DEGLI STUDI DI TORINO

The Department of Economics and Statistics "Cognetti de Martiis" publishes research papers authored by members and guests of the Department and of its research centers. ISSN: 2039-4004

A Competitive Idea-Based Growth Model with Shrinking Workers' Income

Carla Marchese* Fabio Privileggi†

May 17, 2014

Abstract

In this paper we present a model in which endogenous growth arises in competitive markets. Knowledge is described as a labor-augmenting factor used directly in the final goods' production. Firms demand both basic non-rival knowledge contents, which are supplied jointly and inelastically with raw labor, and further contents supplied by patent holders. This fact, together with Lindahl prices for knowledge, allows competition to work, while it also implies that workers' income share declines overtime. In a first version of the model with constant cost of knowledge production the first best is attained. In further versions of the model, in which the cost of knowledge production is allowed to change over time and externalities arise, in a decentralized economy a second best equilibrium occurs in the transitional period, while in the long run there is convergence to efficiency.

JEL Classification Numbers: C61, E10, O31, O41.
Keywords: Endogenous growth, Competitive markets, Lindahl prices, Scale effects

1 Introduction

The mainstream literature on growth follows Romer (1990) in maintaining that only market power can provide the economic incentives for research activity and thus for economic growth. Hence it departs from the idea that competition is a viable reference model, which is otherwise standard in economics. The arguments forcefully put forth by Romer are the following.

i) Technological change, which essentially consists in improvements of knowledge about the way in which we should combine raw materials, is the main engine of economic growth.

ii) Knowledge is a non rival good, only partially excludable.

iii) The non-rivalry of knowledge implies increasing returns to scale in the production of final goods, since it is possible to replicate every production process based on a given blueprint without having to run again the research effort that led to design that blueprint. Knowledge is thus a factor clearly different from human capital, which is rival, since, *e.g.*, if an engineer is needed in a production process, to replicate it a further engineer is needed.

*Institute POLIS - DiGSPES, Università del Piemonte Orientale "Amedeo Avogadro", Via Cavour 84, 15121 Alessandria (Italy). Phone: +39-0131-283718; fax: +39-0131-283704; e-mail carla.marchese@unipmn.it

†Dept. of Economics and Statistics "Cognetti de Martiis", Università di Torino, Lungo Dora Siena 100 A, 10153 Torino (Italy). Phone: +39-011-6702635; fax: +39-011-6703895; e-mail fabio.privileggi@unito.it

iv) While under constant returns to scale and in a competitive market the value of production is fully distributed by paying the production factors the value of their marginal product, this is not the case under decreasing or increasing returns. Under decreasing returns rents would arise, and losses would be incurred under increasing returns after paying factors this way. In other words, under increasing returns, after the payment to the other inputs, nothing is left for compensating the contribution of knowledge, which thus would lack any market financing.

On the basis of these premises, Romer (1990) concludes that a non-competitive market must occur somewhere, in order to provide private economic incentives for research. He presents a model in which each inventor, by patenting her blueprint, can become the sole producer of a differentiated capital or intermediate good, thus enjoying market power and receiving a monopoly profit that covers her research costs. Moreover, in his model research activity produces positive externalities, since the number of blueprint varieties operates as a multiplier of final goods' production. Hence, markets are incomplete.

In this paper we present a very simple model that maintains many of the Romer premises, but is characterized by assumptions that render competition viable. While our assumptions are demanding, they capture some features that might prevail in the economy thanks to the characteristics that technological change assumed in the last decades. The first feature we consider is that technical progress leads more and more to the provision of immaterial goods such as computer programs, internet applications, business models, etc., patentable[1] and directly usable in the final goods production (Chantrel et al., 2012). It thus seems appropriate to assume that knowledge enters directly into the final goods production function, without having to be incorporated into capital or intermediate goods. This direct penetration of immaterial contents into final goods production also implies that much more information that in the past is available to asses the value of the marginal product of knowledge.

The second feature that we consider and whose implications we explore, is the progressive commoditization of education, specifically with reference to training for work. This increasingly occurs through on-line courses, distance-learning, learning through games etc., with a frequent updating of competencies according to changes in technology, markets etc. An example of this process is provided by taxi drivers. Regulations often constrain them to follow traditional forms of human capital accumulation, that is they must show of knowing all the roads in the city area. Learning all the roads in a big city requires long time (estimated in 50 weeks for London nowadays), but if we judge only on the basis of the available technology, a GPS can enable an untrained driver to make a performance quite alike that of a taxi driver. In the foreseeable future firms cannot dispense with low-skilled workers, endowed with the basic levels of human capital, which we assume as being supplied tied with raw labor, and which are still needed to exploit patented intellectual contents. In the example about taxis, a driver able at reading a GPS navigator is needed[2]. However, also these basic skills, which render raw labor useful, can be traced back to the sharing in families or in social institutions of basic cultural contents, habits, and social attitudes as part of the rearing process. In this case too doubling the population would not imply doubling the effort that built the cultural contents

[1] This trend for the U.S. can be dated back to 1998, when in the so called State Street Bank case a business method was declared patentable. Many other similar rulings followed with respect to software. The Supreme Court should reconsider in 2014 the patentability of software with respect to the requisite of producing useful and concrete results.

[2] The development of driverless car might render also the driver redundant, but in building the model we do not consider processes of robotization with the potential for substituting labor altogether, since their perspective diffusion and economic impact is still questioned (Gordon, 2014).

that are routinely passed on from one generation to the other, while the private expenditures that might occur in this process – which can be huge in advanced societies – can be considered as consumption expenditures that a dynastic agent bears to continue to be alive.

The main technical ingredients that we use to incorporate these features into the model, which also render competition viable, are 1) Lindahl prices paid by the final goods' sector to compensate knowledge suppliers, and 2) the virtual disappearance of the labor income share – which happens to be subsumed under that of knowledge – due to the fact that knowledge supply (the inherited basic social knowledge endowment) is provided jointly with raw labor. Under these assumptions, as will become clear in the following, it is possible to overcome the problems arising from the non-convexity (*i.e.*, from increasing returns to scale). In the basic version of the model, knowledge is non-rival but fully excludable, so that the first best is reached. In a further version of the model, knowledge is only partially excludable. Hence externalities arise and a second best result is attained in the transitional period, while in the long run the decentralized solution converges to the optimal Asymptotic Balanced Growth Path (ABGP).

The model accords with some stylized facts, such as the decline of the income share of labor (Karabarbounis and Neiman, 2014), at the advantage of the share going to intangibles and particularly to holders of patents (Corrado et al., 2009). The novel contribution of the paper is actually that of providing a rationale for a declining workers' income share in a growing competitive economy. Hence, while we do not address the problem of public policies, our results suggest that governments should focus primarily on redistributive interventions to improve social welfare. While efficient competitive markets are viable in the long run, policies aimed at correcting market failures should be considered in specific cases only in the transitional period.

The paper is organized as follows: after presenting the basic model in sections 2 and 3, in section 4 we consider the case in which the cost of knowledge production is not constant overtime, establishing in Section 5 that the decentralized equilibrium attains Pareto optimality only in the long-run. Section 6 provides a parameterized example, while a version of the model with population growth and weak scale effects is presented in section 7. Conclusions follow in section 8.

2 A Simple "Immaterial" Model

Our model is a Ramsey-type model of growth with endogenous creation of knowledge. The economy is composed of households, firms and the government. Households receive compensations for supplying inputs to the production sector, purchase a composite consumption good, which also represents the numeraire, and choose how much to save in order to accumulate new knowledge. There are two types of firms: one producing the final consumption good (F–firms) and one performing knowledge creation activities ($R\&D$–firms).

2.1 Households

For now we shed population growth and assume that the size of the economy – *i.e.*, the total number of households – is constant. We adopt the standard assumption that all households have the same rate of time preference, $\rho > 0$, and an identical increasing and concave instantaneous utility function.

A. 1 *The aggregate representative consumer is endowed with an instantaneous objective* $u(C)$*, where* C *is aggregate consumption, with* $u'(C) > 0$ *and* $u''(C) < 0$*.*

Households' goal consists of choosing consumption in order to maximize their own lifetime discounted utility subject to the usual asset accumulation constraint, and their initial common knowledge endowment $A(0)$. Knowledge is a non rival but excludable good. The initial knowledge endowment $A(0)$ belongs to the households in the sense that they share it as a part of their cultural heritage, so that doubling the population would not imply doubling the effort that led to built it. Households supply $A(0)$ to the final good producers at all instants $t \geq 0$ jointly with constant labor $L(t) \equiv L$, and firms can exploit $A(0)$ only by hiring workers. Thanks to such joint supply and the fact that labor supply is inelastic, workers are compensated just as owners of their initial knowledge endowment. No specific compensation for labor is paid. However, the royalty that workers receive for $A(0)$ can alternatively be interpreted as a wage, as it clears the labor market.

As, under these assumptions the representative household earns only royalties from renting knowledge to F–firms while no wage is being earned for labor supplied, she faces the following maximization problem,

$$\max_{[C(t)]_{t=0}^{\infty}} \int_0^{+\infty} u\left[C(t)\right] e^{-\rho t}\,\mathrm{dt} \quad (1)$$
$$\text{subject to } \dot{B}(t) = r(t)\,B(t) - C(t)\,,$$

where $B(t)$ denotes an asset that will be specified later on and $r(t)$ is the market rate of returns on assets, with the additional constraint $0 \leq C(t) \leq r(t)\,B(t)$, for a given initial asset level $B(0) = B_0 > 0$. Standard analysis of the (concave) current-value Hamiltonian associated to (1) yields the following necessary condition of optimality, which is the well known Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u\left[C(t)\right]}\left[r(t) - \rho\right], \quad (2)$$

where $1/\varepsilon_u(C) = -u'(C)/\left[u''(C)\,C\right]$ is the intertemporal elasticity of substitution.

2.2 Producing Sector

In the final good sector F–firms are competitive and operate in a standard neoclassical framework: at each instant t F–firm i employs a composite intermediate good and knowledge-augmented labor to produce a composite consumption good according to a neoclassical production function, $Y_i = F(X_i, AL_i)$.

A. 2 $F(\cdot,\cdot)$ *is concave and linearly homogenous, with* $F_1 > 0$, $F_2 > 0$, $F_{11} < 0$ *and* $F_{22} < 0$, *where* F_j *and* F_{jj} *denote the first-order and second-order partial derivatives with respect to arguments* $j = 1, 2$ *respectively. Moreover, the standard Inada conditions hold for both arguments.*

The intermediate good X_i is made up of final goods destined to production, so that its price is the numeraire, $p_X = 1$, while L_i denotes the number of workers employed by firm i, who are picked from a large and constant labor population of size L. We assume that knowledge A represents a homogeneous good that can be directly used in production to augment the effectiveness of labor. The transfer of knowledge from the research sector to the final good's production, however, involves a rescaling in order to take into account its decreasing effectiveness[3] in terms of output augmentation, which, according to Assumption A.2, translates

[3] Due, *e.g.*, to partial substitution by new knowledge for previously used results.

in decreasing returns to knowledge of the production function F. Let us reformulate output as

$$Y_i = AL_i f\left(\frac{x}{A}\right), \qquad \text{with } f(\cdot) = F(\cdot, 1), \tag{3}$$

where[4] $x = X_i/L_i$ is the per capita intermediate good. At each instant t we assume that there is a large number of firms, say $M(t)$, and several, say $N(t) < M(t)$, output levels $Y_i(t)$, for $i = 1 \dots N(t)$, each corresponding to a different amount L_i of labor employed, that are being produced by $m_i(t)$ identical firms operating at the same level, with $m_i(t) \geq \underline{m} > 0$ where $\underline{m}$ is a number sufficiently large to sustain a competitive market. Therefore, at each instant t the economy is populated by $M(t) = \sum_{i=1}^{N(t)} m_i(t)$ firms producing a total amount $Y(t) = \sum_{i=1}^{N(t)} m_i(t) Y_i(t)$ of final consumption good.[5] If $N(t) = 1$ then $M(t) = m_1(t)$ and $Y(t) = m_1(t) Y_1(t)$.

Because labor is supplied inelastically and jointly with the initial knowledge $A(0)$, F–firms just demand the intermediate good and knowledge. Each firm is willing to pay for intermediate goods and for knowledge the respective marginal product,

$$\frac{\partial Y_i}{\partial X_i} = f'\left(\frac{x}{A}\right) \qquad \text{and} \qquad \frac{\partial Y_i}{\partial A} = L_i\left[f\left(\frac{x}{A}\right) - \frac{x}{A} f'\left(\frac{x}{A}\right)\right].$$

Firms are eager to pay for knowledge because it is a perfectly excludable good[6] and they can rent it on the market. As there are many firms operating at each output level, one can identify $N(t)$ sub-markets for knowledge, in which the demand price is scaled by the labor amount L_i used by each sub-market firm. As knowledge is a collective good when considered from the supply side, all the F–firms can be supplied at the same time with the whole lot available, which is the given stock A. In other words, knowledge supply is inelastic and at each instant the equilibrium price depends on demand. In order to the whole available A be employed by all the buyers, all the sub-markets must clear; however, because F–firms demand prices depend on the labor amount they use, the knowledge equilibrium price in each sub-market will differ accordingly.

Hence, the FOC for each F–firm's profit maximization are:

$$\frac{\partial Y_i}{\partial X_i} = f'\left(\frac{x}{A}\right) = 1 \tag{4}$$

$$\frac{\partial Y_i}{\partial A} = L_i\left[f\left(\frac{x}{A}\right) - \frac{x}{A} f'\left(\frac{x}{A}\right)\right] = L_i \gamma\left(\frac{x}{A}\right). \tag{5}$$

Condition (4) holds because the intermediate goods are priced at the numeraire, while in (5) the term $\gamma(x/A)$ denotes the *equilibrium royalty per augmented worker*, which depends on the ratio x/A. As long as the royalties $L_i \gamma(x/A)$ clear all the the sub-markets according to their labor usage, for all i, the whole knowledge market is cleared. The labor market is cleared as well because, as labor is supplied jointly with $A(0)$, $\sum_i L_i = L$ must hold. Households who invested in new patented knowledge receive payments from all the sub-markets and in total

[4] All firms employ the same intermediate good/labor ratio, $x \equiv x_i = X_i/L_i$, as will become clear in the sequel.

[5] Alternatively, one can assume that there is a continuum of output levels $Y(i,t) \geq 0$, $i \in [0, N(t)]$, each produced by a density $m(i,t) \geq 0$ of identical F-firms, so that the economy is populated by an absolutely continuous distribution of firms over the compact support $[0, N(t)]$, and total output is given by $Y(t) = \int_0^{N(t)} m(i,t) Y(i,t) \,\mathrm{d}i$. Note that when $m(i,t) > 0$ there is a continuum of firms each producing $Y(i,t)$, thus assuring that such sub-market is competitive, while if $m(i,t) = 0$ there are no firms producing the $Y(i,t)$ level.

[6] F–firms cannot re-rent knowledge or share it with other firms; that is, no arbitrage is allowed.

earn $L\gamma(x/A)[A - A(0)]$, while workers in total receive $L\gamma(x/A)A(0)$. The rental price of knowledge $L\gamma(x/A)$ is allocated according to each firm's willingness to pay; in other words, the market equilibrium occurs at *Lindahl prices*.[7]

While in general the resort to Lindahl prices for public goods is deemed unattainable due to the lack of proper information, in this case the tie between knowledge and labor allows to overcome the problem of demand revelation. The observable labor input used by each firm renders it possible to ascribe it to a specific sub-market, and all the suppliers of knowledge can easily identify such sub-markets and compete within each one. The demand revelation problem is in general less severe when the public good is an input than when it is a consumption good, as demand in the former case derives from the profit function while in the latter it derives from the utility function which has a lower degree of measurability (Dasgupta, 2001). In the specific case of patents it is in fact common that licences for, *e.g.*, software or access to repositories of papers, data, etc., are granted upon payments that depend on the number of people habilitated by the licence within an organization.

According to (4) and (5) the per capita product $y_i = Y_i/L_i = Af(x/A)$ is fully distributed to the per capita intermediate good, $x = X_i/L_i$, and knowledge, A, that is,

$$y_i - x - \gamma(x/A)A = Af(x/A) - f'(x/A)x - Af(x/A) + xf'(x/A) = 0,$$

so that each firm (and thus the whole industry) makes no profit. Both workers as suppliers of $A(0)$ and households as holders of the patents covering $[A(t) - A(0)]$ are paid according to their entitlements.[8] Because in the market for final goods all firms face the same rental price for intermediate goods and face Lindahl prices for knowledge, all use the same combination of capital and knowledge-augmented labor. We thus refer in the following to a representative firm and drop the suffix i.

The absence of a specific price for labor does not imply efficiency losses, since labor is inelastically supplied jointly with $A(0)$. One implication of the model is that the benefits of technical progress are funneled to the owners of the intangible capital protected by patents, leaving labor as a residual input factor whose compensation – which can be identified with a wage amounting to $\gamma(t)A(0)$, with $\gamma(t) = \gamma[x(t)/A(t)]$ – falls short of marginal product of labor for given A. The negative implication of the model with respect to labor income accords, at any rate, with the stylized facts pertaining to the long term evolution of the income shares. Corrado et al. (2009), *e.g.*, show that labor's income share decreased significantly over the last 50 years in the US, provided that investments in intangibles and their income share are properly accounted for. These negative effects on salaries might be mitigated if the role of worker and that of licensee would overlap. As long as worker's abilities can increase thanks to the access to non rival but excludable immaterial inputs which they pay for, labor demand can be rationalized as a way through which firms resort to intermediaries in the access to the (growing) pool of non-rival ideas, so that firms can pay Lindahl prices to these workers-intermediaries. In such a case the option for either a direct access (through patent renting) or an indirect one (through trained workers) would imply the consideration of the respective transaction costs, an extension of the model that we leave for future research.

[7] A similar approach has been pursued in Chantrel et al. (2012) with reference to knowledge demanded by producers of differentiated goods, each one with its specific demand (and thus Lindahl) price.

[8] Households actually play both roles of workers and patent holders; we distinguish between them to clarify the economic mechanisms governing each one.

2.3 R&D Sector

Consistently with (6) and (14) below, throughout the whole paper we will resort to the standard simplified approach that describes aggregate knowledge creation as a deterministic process. That is, we assume that there is no aggregate uncertainty in the innovation process while, of course, there may be idiosyncratic uncertainty.[9] In a first version of the model we simplify things further by considering a constant cost of new knowledge production.

A. 3 *Each new idea can be produced at a constant unit cost,* $\eta > 0$.

Under Assumption A.3 the innovation possibilities frontier is given by

$$\dot{A} = \frac{J}{\eta}, \tag{6}$$

where J represents investment in new knowledge production. Every *R&D*–firm produces new knowledge and aims at profit maximization. In order to obtain a patent, knowledge must be new and differentiated, while when knowledge is used by the F–firms, it represents a homogeneous good. Because the number of *R&D*–firms is large, the market for each invention is competitive and each *R&D*–firm makes zero profits. Hence, the value of the patent associated to each (differentiated) unit of new knowledge purchased by households at instant t corresponds to the same constant:[10] η.

3 Market Equilibrium

As, under Assumption A.2, the derivative of function f defined in (3) is decreasing, f' is invertible and from (4) we get the demand for the intermediate good x by the F–firms, which turns out to be linear in A:

$$x = \delta A, \qquad \text{with } \delta = (f')^{-1}(1). \tag{7}$$

Thus, δ is a constant uniquely defined by the choice of the production function $F(\cdot,\cdot)$. As the ratio $\delta \equiv x/A$ is constant, from (5) we obtain the per capita willingness to pay for knowledge, which turns out to be constant as well:

$$\gamma\left(\frac{x}{A}\right) \equiv \gamma = f(\delta) - \delta f'(\delta) = f(\delta) - \delta, \tag{8}$$

where in the third equality we used the definition of δ in (7).

In order to incentivate all households to invest in new knowledge production, under Assumption A.3 the following free-entry condition, postulating that the present value of future royalties, $\Gamma(t) = \gamma(t) L$, be equal to the (constant) production cost η of a unit of new knowledge, must hold:

$$V(t) = \int_t^{+\infty} \gamma(v) L e^{-\int_t^v r(s)\,\mathrm{ds}} \,\mathrm{dv} = \eta. \tag{9}$$

Differentiating with respect to t and recalling from (8) that γ is constant, (9) boils down to

$$\dot{V}(t) = r(t) \int_t^{+\infty} \gamma L e^{-\int_t^v r(s)\,\mathrm{ds}} \,\mathrm{dv} - \gamma L = 0,$$

[9] See, *e.g.*, pp. 428–429 in Acemoglu (2009).

[10] Let ψ denote the patent's price. Then, according to (6), the representative *R&D*–firm maximizes its profit, $\psi \dot{A} - J = \psi(J/\eta) - J$ with respect to J, which immediately yields $\psi = \eta$.

which implies that the present value of future royalties does not change in time and, after substituting the integral with (9), the interest rate regulating the transfer of wealth through time (via the only state variable, which is the knowledge A) is constant as well, $r(t) \equiv r$, and given by

$$r = \frac{\gamma L}{\eta}. \tag{10}$$

In order to use (2) to look for a balanced growth path (BGP) type of equilibrium, we must assume that $1/\varepsilon_u(C) = 1/\sigma$ is a constant,[11] with $\sigma > 0$. Hence, using (10) in (2) we obtain the following constant rate of growth of consumption, C, knowledge, A, and output, Y, along the BGP:

$$g = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left(\frac{\gamma L}{\eta} - \rho\right). \tag{11}$$

Proposition 1 *Suppose that Assumptions A.1–A.3 hold and the intertemporal elasticity of substitution is constant – i.e.,* $\varepsilon_u(C) = -[u''(C)C]/u'(C) \equiv \sigma > 0$. *Then, if*

$$\gamma L > \rho\eta \qquad \textit{and} \qquad (1-\sigma)\gamma L < \rho\eta, \tag{12}$$

the economy admits a unique BGP along which knowledge, output, and consumption all grow at the same rate $g > 0$ *given by (11). Moreover, there are no transition dynamics: the economy immediately jumps on the BGP starting from* $t = 0$.

Proof. See the Appendix. ■

The novelty introduced by assuming that F–firms pay for the use of knowledge through Lindahl prices both to compensate patent holders and suppliers of basic knowledge joint with labor, together with Assumption A.3 of a constant cost for new knowledge production, allows for the solution characterized in Proposition 1 to be Pareto optimal even in a totally decentralized setting.

Proposition 2 *The BGP equilibrium characterized in Proposition 1 is Pareto optimal.*

Proof. See the Appendix. ■

Three features of our model are crucial to explain Proposition 2: 1) as knowledge is paid its Lindahl price there is no room for monopolistic power exploitation in the economy as in the standard literature, that is, all prices turn out to be competitive and nowhere mark-ups are being applied; 2) as labor supply is fixed and joint with initial knowledge supply, workers are paid less than their marginal product and thus the problem of the non-convexity of the final goods production function does not arise; 3) Assumption A.3, by postulating a constant unit cost η for the production of new ideas, rules out knowledge spillovers or other types of externalities. Our next step is to relax the last assumption.

[11] If we are interested in an asymptotic balanced growth path (ABGP) it is sufficient to assume that $\lim_{t\to+\infty}[1/\varepsilon_u(C)] = \lim_{t\to+\infty}\{-u'(C)/[u''(C)C]\} = 1/\sigma$. This approach wil be pursued in Section 4.

4 Non Constant Cost of Knowledge Production

An example of a well known model in which the cost of knowledge production turns out to be constant like in our simple setting discussed in the previous sections is the celebrated original contribution by Romer (1990). However, as most of the major contributions that followed Romer's seminal paper confirm, it is widely accepted that Assumption A.3 introduces a definitely unrealistic restriction. For instance, Tsur and Zemel (2007) consider a continuous-time version of the original model by Weitzman (1998) in which knowledge evolves according to a recombinant technology characterized by a variable unit cost of knowledge production,

$$\eta(t) = \varphi[A(t)], \tag{13}$$

in which the unit cost function $\varphi(\cdot)$ indirectly depends on time through the knowledge stock $A(t)$ evolution. The recombinant knowledge production function by Weitzman is well suited for our approach because it assumes that its main input is expressed in terms of financial resources,[12] $J(t)$. Hence, in this subsection we allow the unit cost of knowledge production, η, to vary over time.

A. 4 *Each new idea can be produced at a time-dependent unit cost, $\eta(t) > 0$.*

According to Assumption A.4, define the innovation possibilities frontier as

$$\dot{A}(t) = \frac{J(t)}{\eta(t)}, \tag{14}$$

in which time-dependence has been emphasized.[13]

From the representative household optimization problem (1) we get the same necessary Euler condition as in (2). Also, nothing changes from the point of view of knowledge demand: the per augmented worker royalty is still constant and, according to (8), given by $\gamma(x/A) \equiv \gamma = f(\delta) - \delta$. Only the free-entry condition (9) changes, as now the RHS depends on time:

$$V(t) = \int_t^{+\infty} \gamma(v)\, L e^{-\int_t^v r(s)\,\mathrm{ds}}\,\mathrm{dv} = \eta(t). \tag{15}$$

Differentiating it with respect to time leads to

$$\dot{V}(t) = r(t) \int_t^{+\infty} \gamma L e^{-\int_t^v r(s)\,\mathrm{ds}}\,\mathrm{dv} - \gamma L = \dot{\eta}(t),$$

which, after substituting the integral with (15), yields the interest rate

$$r(t) = \frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)}. \tag{16}$$

[12] As a matter of fact, all mainstream extensions of Romer's model implicitly assume time-dependent costs of producing new ideas. However, like Romer's one, all these models are based on knowledge production functions that use labor as a main input factor, so that the cost of new knowledge depends on the equilibrium wage. Because, according to (6) and (14), in our setting we assume that knowledge is produced through financial investment rather than labor, these contributions are not directly comparable with our model. We will return to this issue in Section 7, where scale effects will be tackled.

[13] Note that (14) encompasses also the case in which new knowledge is being produced by decentralized *R&D*–firms for a price $\eta(t) = \psi[A(t)]$ that includes a mark-up over the Tsur and Zemel (2007) first-best cost $\varphi[A(t)]$, as in Marchese et al. (2014).

Note that, although the total royalty $\Gamma = \gamma L$ remains constant, under Assumption A.4 the interest rate varies in time according to the law of motion of $\eta(t)$.

Because from (15) $V(t) = \eta(t)$, equation (16) can be rewritten in the familiar form of a Hamilton-Jacobi-Bellman equation,

$$r(t) V(t) = \gamma L + \dot{V}(t), \tag{17}$$

in which the evolution through time of knowledge's value takes into account the assets' gains/ losses $\dot{V}(t) = \dot{\eta}(t)$ due to variations in the new knowledge's cost $\eta(t)$.

Using (16) in (2) we obtain the following time-dependent growth rate of consumption:

$$g(t) = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u [C(t)]} \left[\frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \rho \right]. \tag{18}$$

As, by construction, this version of the model exhibits transition dynamics, we look for an Asymptotic Balanced Growth Path (ABGP) type of equilibrium. To this aim, we assume that asymptotically the intertemporal elasticity of substitution becomes constant, while the growth rate of the unit cost of new knowledge is required to vanish in the long-run; that is, we set $\lim_{t\to+\infty} [1/\varepsilon_u(C)] = 1/\sigma$, $\sigma > 0$, and $\lim_{t\to+\infty} [\dot{\eta}(t)/\eta(t)] = 0$. Note that this setting is sufficiently general to encompass any type of new knowledge production function, envisaging either increasing or decreasing $\eta(t)$ along transition dynamics, while asymptotically the unit cost of new knowledge must converge to some positive constant, $\lim_{t\to+\infty} \eta(t) = \eta^* > 0$, in order to satisfy the transversality condition for a solution of problem (1), as will be explained in Remark 1.

Proposition 3 *Under Assumptions A.1, A.2 and A.4, assume that* $\lim_{t\to+\infty} [1/\varepsilon_u(C)] = 1/\sigma$, $\sigma > 0$, *and* $\lim_{t\to+\infty} \eta(t) = \eta^* > 0$. *Then, if*

$$\gamma L > \rho\eta^* \qquad \textit{and} \qquad (1-\sigma)\gamma L < \rho\eta^*, \tag{19}$$

then the economy admits a unique ABGP along which knowledge, output, and consumption all grow at the same asymptotic growth rate given by

$$g^* = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left(\frac{\gamma L}{\eta^*} - \rho\right). \tag{20}$$

Proof. See the Appendix. ■

Conditions (19) and the growth rate (20) are the same as those in (12) and (11) respectively, only with the asymptotic value η^* in place of the constant unit cost η. Note that along the transition dynamics the consumption growth rate g in (18) may be either larger or smaller than its asymptotic value g^* in (20), depending on the sign of $\dot{\eta}/\eta$, that is, on whether building new knowledge involves increasing or decreasing costs as time elapses.

Remark 1 *If we relax the assumption that* $\lim_{t\to+\infty} \eta(t) = \eta^* > 0$ *and allow either that* $\eta(t)$ *grows asymptotically,* $\lim_{t\to+\infty} [\dot{\eta}(t)/\eta(t)] > 0$ *with* $\eta(t) \to +\infty$ *as* $t \to +\infty$, *or that* $\eta(t)$ *keeps decreasing,* $\lim_{t\to+\infty} [\dot{\eta}(t)/\eta(t)] < 0$ *with* $\eta(t) \to 0$ *as* $t \to +\infty$ *(asymptotically vanishing cost of producing new ideas), an ABGP type of equilibrium cannot exist. To see this, suppose, on the contrary, that* $\lim_{t\to+\infty} [\dot{\eta}(t)/\eta(t)] = g_\eta > 0$. *Then* $\gamma L/\eta(t) \to 0$ *as* $t \to +\infty$ *and, from (16) and (18),* $\lim_{t\to+\infty} r(t) = r^* \equiv g_\eta$ *and* $\lim_{t\to+\infty} g(t) = (g_\eta - \rho)/\sigma \equiv g^*$ *respectively, so that,* *assuming $g_\eta > \rho$ to have $g^* > 0$ (positive asymptotic growth), one has $g_\eta + g^* = r^* + g^* > r^*$, which violates the transversality condition $\lim_{t\to+\infty} \eta(t) A(t) e^{-r^* t} = 0$.*

The interpretation is that, according to (15), an increasing cost of knowledge production must be compensated by an increasing market value of knowledge, $V(t)$, a too heavy burden for the economy to sustain. On the other hand, $\lim_{t\to+\infty} [\dot{\eta}(t)/\eta(t)] = g_\eta < 0$ implies $\gamma L/\eta(t) \to +\infty$ as $t \to +\infty$, which would generate explosive growth and is thus incompatible with an ABGP of the sort defined by (20). In this case, vanishing costs of new knowledge production leads an immaterial economy like ours to outburst.

5 The Social Optimum

When the cost of knowledge varies through time depending on the evolution of the knowledge stock, externalities arise, but the households are not able to keep them into account in their decision process, as they can only observe the ensuing price changes, $\dot{\eta}(t)$, embedded in the interest rate according to (16). Clearly, under these circumstances the equilibrium described in Proposition 3 turns out to be not Pareto optimal, as we now show by solving the social planner problem associated to (1).

Now we assume that the unit cost of new knowledge production $\eta(t)$ has the form in (13) and, as in Proposition 3, $\lim_{t\to+\infty} \eta(t) = \lim_{t\to+\infty} \varphi[A(t)] = \eta^* > 0$ when there is knowledge growth, $\dot{A}(t)/A(t) > 0$. Following the same argument as in the proof of Proposition 2 in the Appendix, the social planner first maximizes net output, $Y(t) - X(t) = A(t) Lf[x(t)/A(t)] - x(t) L$, with respect to $x(t)$, obtaining the optimal net output $Y^S(t) - X^S(t) = \gamma LA(t)$. Then, she maximizes the representative household's lifetime discounted utility as in (1) subject to the resource constraint $\gamma LA(t) = C(t) + J(t)$, which, according to (13) and (14), can be written as

$$\dot{A}(t) = \frac{\gamma LA(t) - C(t)}{\varphi[A(t)]}. \tag{21}$$

Denoting by $\lambda(t)$ the costate variable associated to the unique dynamic constraint (21) and dropping the time argument for simplicity, the current-value Hamiltonian of the social planner problem is

$$H(A, C, \lambda) = u(C) + \lambda \frac{\gamma LA - C}{\varphi(A)}.$$

Necessary conditions are

$$u'(C) = \frac{\lambda}{\varphi(A)} \tag{22}$$

$$\dot{\lambda} = \rho\lambda - \lambda \frac{\gamma L \varphi(A) - (\gamma LA - C)\varphi'(A)}{[\varphi(A)]^2} \tag{23}$$

$$\lim_{t\to+\infty} \lambda(t) A(t) e^{-\rho t} = 0, \tag{24}$$

where (24) is the transversality condition. Differentiating with respect to time (22) one gets

$$\frac{\dot{\lambda}}{\lambda} = \frac{\varphi'(A)\dot{A}}{\varphi(A)} - \varepsilon_u(C)\frac{\dot{C}}{C}, \tag{25}$$

where $\varepsilon_u(C)$, as usual, denotes the inverse of the intertemporal elasticity of substitution. Coupling (25) with (23), using (21) and rearranging terms we obtain the following transitory consumption growth rate:

$$g^S(t) = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u[C(t)]}\left\{\frac{\gamma L}{\varphi[A(t)]} - \rho\right\}. \tag{26}$$

Also this version of the model exhibits transition dynamics; thus, we again look for an ABGP type of equilibrium which, because as $t \to +\infty$ g^S in (26) converges to the same consumption growth rate g^* defined in (20), turns out to be the same as that characterized in Proposition 3.

Proposition 4 *Suppose that Assumptions A.1, A.2 and A.4 with the specification in (13) hold. Moreover assume that* $\lim_{t\to+\infty}\{1/\varepsilon_u[C(t)]\} = 1/\sigma$, $\sigma > 0$, *and* $\lim_{t\to+\infty}\varphi[A(t)] = \eta^* > 0$ *whenever there is positive knowledge growth,* $\dot{A}/A > 0$. *Then, if conditions (19) hold, the social planner economy admits a unique ABGP along which knowledge, output, and consumption all grow at the same asymptotic growth rate characterized by the common constant growth rate as in (20) of Proposition 3 for the decentralized economy.*

Furthermore, along the transition dynamics the consumption growth rate in (26) can be either larger or smaller than that in (18), specifically, $g^S > g$ *when* $\varphi'[A(t)] < 0$, *while* $g^S < g$ *when* $\varphi'[A(t)] > 0$.

Proof. See the Appendix. ■

From Proposition 4 we conclude that, asymptotically, the equilibrium in the decentralized model of Section 4 converges to the Pareto optimal solution. However, along the transition dynamics the consumption growth rate under social planner supervision in (26) can be either larger or smaller than that in the decentralized market economy in (18), depending on the sign of $\varphi'[A(t)]$ which, in turn, determines the sign of the term $\dot{\eta}(t)/\eta(t) = \varphi'[A(t)]\dot{A}(t)/\varphi[A(t)]$ in the square bracket of the RHS in (18). Indeed, a social planner controls the whole evolution of the knowledge stock $A(t)$ so that, with a unit cost of new knowledge production as in (13), the externalities of investments in knowledge leading to changes in $\varphi[A(t)]$ through time are now taken into account.

Specifically, with costs that decrease in time, $\varphi'[A(t)] < 0$, the growth rate in the transitional period is larger under central control than in the decentralized model of Section 4: $g^S > g$. This is due to the presence of a positive externality, that is, when knowledge costs are decreasing in time, it becomes possible to produce subsequent inventions by subtracting less and less resources from other uses. This external effect, however, is not accounted for by private investors, while it is being considered by the central planner, who, accordingly, chooses a larger growth rate g^S in the transitional period.

Conversely, if $\varphi'[A(t)] > 0$, the growth rate in the transitional period is smaller under the social planner supervision than in the decentralized model: $g^S < g$.

As inefficiencies arise in the transitional period under decentralization, corrective policy interventions might be designed. That is, a subsidy to R&D investments along the transition when $\varphi'[A(t)] < 0$ (a tax when $\varphi'[A(t)] > 0$) could be introduced to align the behavior of decentralized agents to the path envisaged by the social planner. We do not pursue here the detailed specification of such tools.

6 A Parameterized Example

Assume households have an instantaneous CIES utility, $u(C) = (C^{1-\sigma} - 1)/(1-\sigma)$, $\sigma > 0$, and F–firms production function has the Cobb-Douglas form, $Y = F(X, AL) = \theta X^\alpha (AL)^{1-\alpha} =$

$\theta AL\,(x/A)^{\alpha}$, $\theta > 0$ and $0 < \alpha < 1$. Thus, $f\,(x/A) = \theta\,(x/A)^{\alpha}$ and parameters in (7) and (8) are given by $\delta = (\theta\alpha)^{1/(1-\alpha)}$ and $\gamma = \theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)$ respectively. Under Assumption A.3 the unit cost of new knowledge production, η, is constant, and the interest rate in (10) is given by

$$r = \frac{\gamma L}{\eta} = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L}{\eta},$$

while, according to Proposition 1, the economy growth rate common to all variables is

$$g = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left[\frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L}{\eta} - \rho\right],$$

whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L > \rho\eta$ and $(1-\sigma)\,\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L < \rho\eta$ hold.

To consider a time-dependent unit cost of new knowledge production, $\eta\,(t)$, as foreseen by Assumption A.4 with the (13) specification, let

$$\eta\,(t) = \varphi\,[A\,(t)] = \frac{\beta}{A\,(t)} + \eta^*, \qquad \beta, \eta^* > 0.$$

As the function $\varphi\,(A)$ is decreasing in A, under positive knowledge growth, $\dot{A}/A > 0$, $\eta\,(t)$ is decreasing in time and $\lim_{t\to+\infty}\eta\,(t) = \lim_{A\to+\infty}\varphi\,(A) = \eta^* > 0$, so that the assumptions of Proposition 3 hold. According to (14), the new knowledge production function is

$$\dot{A}\,(t) = \frac{J\,(t)}{\varphi\,[A\,(t)]} = \frac{A\,(t)}{\beta + \eta^* A\,(t)}J\,(t)\,,$$

envisaging increasing knowledge spillovers as $(\partial/\partial A)\,[A/\,(\beta+\eta^*A)] = \beta/\,(\beta+\eta^*A)^2 > 0$, with decreasing returns as $(\partial^2/\partial A^2)\,[A/\,(\beta+\eta^*A)] = -2\beta\eta^*/\,(\beta+\eta^*A)^3 < 0$. From (16) we get the transition interest rate as

$$r\,(t) = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,LA\,(t)}{\beta+\eta^*A\,(t)} - \frac{\beta}{\beta+\eta^*A\,(t)}\frac{\dot{A}\,(t)}{A\,(t)},$$

and Proposition 3 predicts an ABGP characterized by the following growth rate common to all variables,

$$g^* = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left[\frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L}{\eta^*} - \rho\right] \tag{27}$$

whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L > \rho\eta^*$ and $(1-\sigma)\,\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\,(1-\alpha)\,L < \rho\eta^*$ hold.

If we let

$$\eta\,(t) = \varphi\,[A\,(t)] = \eta^*\left[1 - \frac{1}{A\,(t)+\beta}\right], \qquad \beta \geq 1 \text{ and } \eta^* > 0,$$

we have a function $\varphi\,(A)$ which is increasing in A so that, under positive knowledge growth, $\dot{A}/A > 0$, $\eta\,(t)$ is increasing in time and again $\lim_{t\to+\infty}\eta\,(t) = \lim_{A\to+\infty}\varphi\,(A) = \eta^* > 0$; moreover, with $\beta \geq 1$ and assuming $A\,(0) > 0$, $\eta\,(0) = \eta^*\,\{1 - 1/\,[A\,(0)+\beta]\} > 0$. Hence, the assumptions of Proposition 3 still hold, but now the unit cost of new knowledge increases in time. According to (14), the new knowledge production function is

$$\dot{A}\,(t) = \frac{J\,(t)}{\varphi\,[A\,(t)]} = \frac{A\,(t)+\beta}{\eta^*\,[A\,(t)+\beta-1]}J\,(t)\,,$$

characterized by decreasing knowledge spillovers because $(\partial/\partial A)\,\{(A+\beta)\,/\,[\eta^*\,(A+\beta-1)]\} = -1/\left[\eta^*\,(\beta+\eta^*A)^2\right] < 0$; however, now such spillovers occur at a decreasing negative rate as

$(\partial^2/\partial A^2)\{(A+\beta)/[\eta^*(A+\beta-1)]\} = 2/\left[\eta^*(\beta+\eta^* A)^3\right] > 0$. From (16) we get the transition interest rate as

$$r(t) = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L[A(t)+\beta]}{\eta^*[A(t)+\beta-1]} + \frac{\dot{A}(t)}{[A(t)+\beta-1][A(t)+\beta]},$$

and Proposition 3 again predicts the same ABGP constant growth rate common to all variables given by (27) whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L > \rho\eta^*$ and $(1-\sigma)\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L < \rho\eta^*$ hold.

7 Scale Effects and Population Growth

It is evident from (11) and (20) that, as one expects, both versions of our simplified economy exhibit the *strong scale effect* – the growth rate of the economy increases in population size – as postulated by Jones (1995, 1999 and 2005) for knowledge-based endogenous growth models. Scale effects are in a sense built-in in idea-based growth models: as ideas are collective goods, their returns increase as a larger number of agents are affected by their spillovers. While in the original Romer (1990) model the strong scale effect is determined by full knowledge spillovers among researchers, in our setting this effect is present because labor plays the role of the "carrier" of the knowledge input factor in the final good production, so that the economy's growth rate turns out to be directly affected by the total royalty $\Gamma = \gamma L$, corresponding to the Lindahl pricing of knowledge.

Also in our scenario, however, a more realistic weak scale effect can hold if there are increasing technical difficulties in combining labor and knowledge. That is, besides the decreasing marginal returns to augmented labor that characterize the instantaneous production function in the final good production sector, the "labor augmentation" process may become more difficult through time. To take this feature into account, we add a "damping" term, $\Omega(t)$, to the second argument in the neoclassical production function of Assumption A.2 by defining aggregate output at instant t as[14]

$$Y(t) = F\left[X(t), \frac{A(t)L(t)}{\Omega(t)}\right], \tag{28}$$

where $X(t)$ still denotes the aggregate intermediate good amount used in final production. Because our aim is to tackle scale effects, in the sequel it will be assumed that population – and thus workers – grows exogenously at a constant rate $n > 0$, *i.e.*, $\dot{L}(t) = nL(t)$. For now let us only assume that the term $\Omega(t)$ depends on time; a more precise characterization of $\Omega(t)$ will be specified later on. Note the role of the term $\Omega(t)$ in (28): it mirrors the same effect as in the new knowledge production functions of Segerstrom (1998) and Kuwahara (2013), except that here such term affects final rather than knowledge production.

7.1 Market equilibrium

As in Subsection 2.2 we rewrite (28) as

$$Y(t) = \frac{A(t)}{\Omega(t)}L(t)f\left[\frac{\Omega(t)x(t)}{A(t)}\right], \qquad \text{with } f(\cdot) = F(\cdot,1), \tag{29}$$

[14] As explained in Subsection 2.2, all produncing firms in the economy use the same production function; hence, there is no reason to keep the index i and we can consider the representative F-firm.

where, as usual, $x(t) = X(t)/L(t)$ denotes per capita intermediate good. Dropping the time index for simplicity, FOC for the representative F–firm now become:

$$\frac{\partial Y}{\partial X} = f'\left(\frac{\Omega x}{A}\right) = 1 \tag{30}$$

$$\frac{\partial Y}{\partial A} = \frac{L}{\Omega}\left[f\left(\frac{\Omega x}{A}\right) - \left(\frac{\Omega x}{A}\right) f'\left(\frac{\Omega x}{A}\right)\right] = \frac{L}{\Omega}\gamma\left(\frac{\Omega x}{A}\right), \tag{31}$$

where in (31) the term $\gamma(\Omega x/A)$ still denotes the equilibrium royalty per augmented worker, which here depends on the ratio $(\Omega x/A)$ and is being further 'diminished' by the term Ω. Following the same arguments as in Section 3 it is readily seen that (30) implies the demand for the intermediate good x be still linear in A, but now it depends also on Ω, according to $x = \left[(f')^{-1}(1)/\Omega\right]A = (\delta/\Omega)A$. Replacing the constant $\delta = (f')^{-1}(1) = (\Omega x/A)$ in (31) we obtain the per capita willingness to pay for knowledge, which is still the constant given by (8): $\gamma(\Omega x/A) \equiv \gamma = f(\delta) - \delta f'(\delta) = f(\delta) - \delta$. Hence, according to (31), the total royalty at instant t is now given by

$$\Gamma(t) = \gamma\frac{L(t)}{\Omega(t)}, \tag{32}$$

which is no more a constant because both $L(t)$ and $\Omega(t)$ depend on time.

Under Assumption A.4 the free-entry condition in (15) can be rewritten as:

$$V(t) = \int_t^{+\infty} \Gamma(v)\, e^{-\int_t^v r(s)\,\mathrm{ds}}\,\mathrm{dv} = \eta(t), \tag{33}$$

where now $\Gamma(t)$ is given by (32). Differentiating both sides with respect to time leads to

$$\dot{V}(t) = r(t)\int_t^{+\infty} \Gamma(v)\, e^{-\int_t^v r(s)\,\mathrm{ds}}\,\mathrm{dv} - \Gamma(t) = \dot{\eta}(t),$$

which, after substituting the integral with (33) and using (32), yields the interest rate

$$r(t) = \frac{\Gamma(t)}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} = \frac{\gamma L(t)}{\eta(t)\,\Omega(t)} + \frac{\dot{\eta}(t)}{\eta(t)}. \tag{34}$$

Using (33), equation (34) can be rewritten in the familiar form of a Hamilton-Jacobi-Bellman equation:

$$r(t)\,V(t) = \frac{\gamma L(t)}{\Omega(t)} + \dot{V}(t). \tag{35}$$

Under the assumption of constant exogenous population growth it is convenient to restate the representative household's problem in per capita terms:

$$\max_{[c(t)]_{t=0}^{\infty}} \int_0^{+\infty} u[c(t)]\, e^{-(\rho-n)t}\,\mathrm{dt} \tag{36}$$
$$\text{subject to } \dot{b}(t) = [r(t) - n]\, b(t) - c(t),$$

where $u(\cdot)$ still satisfies Assumption A.1, $c(t) = C(t)/L(t)$ and $b(t) = B(t)/L(t)$ denote per capita consumption and asset respectively, $r(t)$ is the market rate of returns on assets, and $n = \dot{L}(t)/L(t)$, with the additional constraint $0 \le c(t) \le r(t)\,b(t)$, for a given initial asset level $b(0) = b_0 > 0$. The associated Euler equation is now stated in terms of per capita consumption growth rate:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u[c(t)]}[r(t) - \rho], \tag{37}$$

where again $1/\varepsilon_u(c) = -u'(c)/[u''(c)c]$ is the intertemporal elasticity of substitution. Using (34) it can be rewritten as

$$g_c(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u[c(t)]}\left[\frac{\gamma L(t)}{\eta(t)\Omega(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \rho\right], \tag{38}$$

from which it is apparent that this model exhibits transition dynamics.

From (38) it is clear that in order to achieve an ABGP in the long-run, besides the assumptions of Proposition 3, *i.e.*, $\lim_{t\to+\infty}[1/\varepsilon_u(C)] = 1/\sigma > 0$, and $\lim_{t\to+\infty}\eta(t) = \eta^* > 0$ [which implies that $\dot{\eta}(t)/\eta(t) \to 0$ as $t \to +\infty$], now we need the additional restriction that $\lim_{t\to+\infty}\left[\dot{\Omega}(t)/\Omega(t)\right] = n$; that is, to have an ABGP the term $\Omega(t)$ asymptotically must grow at the same exogenous growth rate of population.

As, under the assumption that $\dot{\Omega}/\Omega = n$, the ratio L/Ω becomes constant in the long-run, it follows that also the ratio $c(t)/a(t)$ must eventually become constant (see the proof of the next Proposition 5 in the Appendix), that is, the asymptotic growth rate of per capita consumption must be equal to that of per capita knowledge: $\lim_{t\to+\infty}[\dot{c}(t)/c(t)] = \lim_{t\to+\infty}[\dot{a}(t)/a(t)] = \lim_{t\to+\infty}\left[\dot{A}(t)/A(t)\right] - n$. Therefore, in order to close the asymptotic equilibrium of our model we need to evaluate the long-run growth rate of knowledge, $\lim_{t\to+\infty}\left[\dot{A}(t)/A(t)\right]$; to this purpose a relationship between the damping term Ω and the stock of knowledge A is required.

A. 5 *The function $\Omega(t)$ depends on the stock of knowledge, i.e., $\Omega(t) = \Omega[A(t)]$, and has elasticity that becomes constant as $A \to +\infty$:*

$$\lim_{A\to+\infty}\frac{\Omega'(A)A}{\Omega(A)} = \varepsilon_\Omega^*, \qquad \textit{with } 0 < \varepsilon_\Omega^* < 1.$$

Proposition 5 *Under the assumptions of Proposition 3 suppose that population grows according to a constant exogenous rate, $\dot{L}/L = n$, and Assumption A.5 holds. Then, if*

$$\frac{1-(1-\varepsilon_\Omega^*)\sigma}{\varepsilon_\Omega^*}n < \rho, \tag{39}$$

the economy admits a unique ABGP along which per capita consumption grows at the constant asymptotic rate

$$g_c^* = \frac{\dot{c}}{c} = \left(\frac{1}{\varepsilon_\Omega^*} - 1\right)n, \tag{40}$$

while (aggregate) knowledge and output grow at the same constant asymptotic growth rate given by

$$g^* = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{n}{\varepsilon_\Omega^*}, \tag{41}$$

and the constant asymptotic interest rate is

$$r^* = \left(\frac{1}{\varepsilon_\Omega^*} - 1\right)\sigma n + \rho. \tag{42}$$

Proof. See the Appendix. ■

According to (40), Assumption A.5 is necessary to have positive per capita consumption growth. In other words, the damping function $\Omega[A(t)]$ asymptotically must grow less than proportionally with respect to the stock of knowledge. This is consistent with the Jones (1995, 1999, 2005) and Kortum (1997) approach as, from (28), one can see that $\Omega(A)$ enters the denominator of a ratio having A in the numerator, thus contributing to slowing down the returns to knowledge for fixed labor L entering the production function of the final sector as time elapses.[15] As a consequence, (40) typically represents the growth rate of a "semi-endogenous" growth model as, if on one hand the economy exhibits sustained growth, on the other hand the per capita growth rate depends only on population growth and the technological parameter ε_Ω^* that determines how effectively knowledge is being employed in the final sector; specifically, the asymptotic growth rate cannot be affected by taxation or other policies. Interestingly, unlike the equilibria obtained in previous sections, here the per capita growth rate g_c^* does not depend either on preferences (parameters σ and ρ), the final good production function (parameter γ), or, most importantly, the new knowledge production function (parameter η^*).

Example 1 *Assume that $\Omega(A) = A^{1-\phi}$, with $0 < \phi < 1$. Then $\varepsilon_\Omega^* \equiv 1-\phi$ for all A (and thus for all $t \geq 0$) and the second argument of the production function in (28) boils down to $A^\phi L$, resembling the familiar form of Jones (1995, 1999 and 2005) new knowledge production function. Indeed, by Proposition 5, the long-run per capita consumption growth rate turns out to be*

$$g_c^* = \frac{\phi n}{1-\phi}, \tag{43}$$

not much different than that found by Jones, which, in its simplified version, amounts to[16] *$g_c^J = n/(1-\phi)$. Note that the appearance of ϕ both in the numerator and in the denominator of (43) lets the consumption growth rate be more sensitive with respect to the technological parameter ϕ than it occurs in Jones's growth rate, as for values of ϕ close to 0 our economy exhibits a per capita consumption growth rate smaller than that in Jones, which vanishes as $\phi \to 0^+$. The intuition might be that while in Jones the technological parameter ϕ affects income growth by slowing only the productivity of researchers, in our case it affects the whole labor. It becomes clear at any rate that the insensitivity to standard policy tools, as subsidies to R&D, arises both here and in semi-endogenous growth models when inefficiencies involve labor.*

7.2 Centralized solution

In this section so far we considered a unit cost of knowledge production, $\eta(t)$, that changes through time along the transitory dynamics. In order to focus on the correction of the market failure generated by the damping function $\Omega(t)$ in (28), we resume Assumption A.3 and assume that the unit cost of knowledge production is constant $\eta(t) \equiv \eta$; therefore, from Proposition 2 we know that no externalities are involved in the knowledge generation process and can examine exclusively how the function $\Omega(t)$ let the decentralized equilibrium characterized above depart from the optimal solution.

Under Assumptions A.1–A.3, suppose that population grows exogenously at a constant rate $n > 0$, *i.e.*, $\dot{L}(t) = nL(t)$, and total output is given by (29), with a damping function

[15] Note, however, that $\varepsilon_\Omega^* < 1$ rules out the case in which the ratio $A(t)/\Omega(t)$ is decreasing in time; this scenario corresponds to $\varepsilon_\Omega^* > 1$, which implies a negative per capita consumption asymptotic growth rate. In other words, to have positive per-capita consumption growth in the long-run a Segerstrom (1998) type of damping function $\Omega(A)$, reducing the productivity of labor as the stock of knowledge increases, cannot be applied in our setting.

[16] This value corresponds to that defined in equation (13.38) on p. 447 in Acemoglu (2009). In the original works of Jones (1995, 1999 and 2005) also a parameter $0 < \lambda \leq 1$ indicating decreasiong returns in researchers' effort appear in the numerator, yielding a per capita consumption growth rate given by $g_c^J = \lambda n/(1-\phi)$.

$\Omega(t) = \Omega[A(t)]$ satisfying assumption A.5. As usual, the social planner first maximizes net output which, using (29) and $X(t) = x(t) L(t)$, amounts to

$$Y(t) - X(t) = \frac{A(t)}{\Omega(t)} L(t) f\left[\frac{\Omega(t) x(t)}{A(t)}\right] - x(t) L(t),$$

with respect to $x(t)$, obtaining the optimal net output $Y^S(t) - X^S(t) = \gamma A(t) L(t) / \Omega(t)$, with γ still defined by (8). Using (6) the dynamic resource constraint can be written as

$$\dot{A}(t) = \frac{1}{\eta}\left[\frac{\gamma L(t) A(t)}{\Omega[A(t)]} - C(t)\right],$$

where all variables, including consumption, are written in aggregate terms. Hence, the social planner problem associated to (36) is:

$$\max_{[c(t)]_{t=0}^{\infty}} \int_0^{+\infty} L(t) u[c(t)] e^{-\rho t} \mathrm{d}t \tag{44}$$

$$\text{subject to } \dot{A}(t) = \frac{L(t)}{\eta}\left[\frac{\gamma A(t)}{\Omega[A(t)]} - c(t)\right],$$

where $c(t) = C(t)/L(t)$ denotes per capita consumption, with the additional constraint $0 \leq c(t) \leq \gamma A(t) / \Omega[A(t)]$, for a given initial knowledge stock $A(0) = A_0 > 0$.

Denoting by $\lambda(t)$ the costate variable associated to the unique dynamic constraint and dropping the time argument for simplicity, the current-value Hamiltonian of the social planner problem is

$$H(A, C, \lambda) = Lu(c) + \frac{\lambda L}{\eta}\left[\frac{\gamma A}{\Omega(A)} - c\right].$$

Necessary conditions are

$$u'(c) = \frac{\lambda}{\eta} \tag{45}$$

$$\dot{\lambda} = \rho\lambda - \frac{\lambda\gamma L[\Omega(A) - A\Omega'(A)]}{\eta[\Omega(A)]^2} \tag{46}$$

$$\lim_{t\to+\infty} \lambda(t) A(t) e^{-\rho t} = 0, \tag{47}$$

where (47) is the transversality condition. Differentiating with respect to time (45) one gets the usual condition $\dot{\lambda}/\lambda = -\varepsilon_u(c)(\dot{c}/c)$,where $\varepsilon_u(c)$ denotes the inverse of the intertemporal elasticity of substitution, which, coupled with (46) and rearranging terms yields the following transitory consumption growth rate:

$$g_c^S = \frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)}\left\{\frac{\gamma L}{\eta\Omega(A)}\left[1 - \frac{\Omega'(A) A}{\Omega(A)}\right] - \rho\right\}. \tag{48}$$

Also this version of the model exhibits transition dynamics; hence, again we look for an ABGP type of equilibrium.

Proposition 6 *Under the assumptions of Proposition 5, if condition (39) holds, the social planner economy admits a unique ABGP along which per capita consumption grows at the asymptotic constant growth rate given by (40), while knowledge and output grow at the asymptotic constant growth rate as in (41) of Proposition 5 for the decentralized economy.*

Furthermore, along the transition dynamics the consumption growth rate in (48) is larger, equal to, or smaller than that in (38), i.e., $g_c^S(t) > g_c(t)$, $g_c^S(t) = g_c(t)$, *or* $g_c^S(t) < g_c(t)$, *provided that* $\varepsilon_\Omega^*(t) = \Omega'[A(t)]A(t)/\Omega[A(t)]$ *satisfies* $\varepsilon_\Omega^*(t) < 0$, $\varepsilon_\Omega^*(t) = 0$, *or* $\varepsilon_\Omega^*(t) > 0$, *respectively.*

Proof. See the Appendix. ■

Proposition 6 states that, like in all previous versions of the model, asymptotically the equilibrium in the decentralized model characterized by Proposition 5 converges to the Pareto optimal solution. However, along the transition dynamics the consumption growth rate under social planner supervision in (48) may be larger or smaller than that in the decentralized market economy described by (38) depending on the transitory value of parameter $\varepsilon_\Omega^*(t)$. This is because a social planner takes into account the negative externalities associated to the function $\Omega[A(t)]$ in the final sector production process. Specifically, depending on whether the damping term $\Omega[A(t)]$ decreases or increases as knowledge A piles up through time – *i.e.*, $\Omega'[A(t)] < 0$ or $\Omega'[A(t)] \geq 0$ – the social planner is able to increase or decrease consumption growth accordingly, pushing it up when knowledge is expected to become more effectively employed by the final sector in the near future, that is, when the term $\Omega[A(t)]$ decreases, while it may be optimal even a negative consumption growth rate if $\varepsilon_\Omega^*(t)$ is sufficiently large, *i.e.*, when $\varepsilon_\Omega^*(t) > 1 - \rho\eta\Omega[A(t)]/[\gamma L(t)]$. Households, on the other hand, can only observe the effect of $\Omega[A(t)]$ through the interest rate, which, according to (34) when η is constant and thus $\dot{\eta}/\eta = 0$, is given by $r(t) = (\gamma/\eta)L(t)/\Omega[A(t)]$, and are not able to discount changes in $\Omega[A(t)]$ that will affect the future effectiveness of the knowledge factor in final production.

Note, however, that Assumption A.5 implies that after some (perhaps large) instant $T > 0$, $\varepsilon_\Omega^*(t)$ must satisfy $\varepsilon_\Omega^*(t) > 0$ for all $t \geq T$. Therefore, after T and before reaching the ABGP – where, according to the first part Proposition 6, the asymptotic growth rate of consumption is the same both in the centralized and decentralized models – the social planner will choose a smaller consumption growth rate than in the decentralized equilibrium because, unlike households, she anticipates the increasing difficulty in knowledge usage by the final sector firms.

8 Conclusions

In this paper we take into account the increasingly immaterial characteristics assumed by technical progress and the implications this has on the ways in which it is transferred to final goods' production. On this basis we assume that both a direct use of ideas in final production and the revelation of firm's willingness to pay for accessing knowledge arise. A further feature that characterizes our model is the reconsideration of the role of human capital. Education has traditionally been identified with a lengthy investment process concentrated in the initial years of life and mainly built through inputs represented by physical and human capital. In the last decades this path has been challenged on the one hand by the emerging of the necessity of permanent and quick updating and adapting of competencies, on the other by the potential disruption that technical progress could bring to the way in which education is delivered. Nowadays non-rival inputs such as educational software, data repositories and standardized routines seem poised to assume an increasing role in enhancing labor performance. Non rival components have also always been paramount in the basic education processes that pass-on social attitudes and the cultural heritage, whose contents become built-in in raw labor supply.

As noted by Romer (1990), knowledge in general can be accessed only through the intermediation of some rival good or factor, but with some simplification we can treat ideas as non-rival goods as long as their circulation means have a tiny cost with respect to that of producing the contents they carry. In this paper we consider a framework in which raw labor too is demanded as a low cost mean for accessing valuable ideas. Thus, both workers and patent holders supply the same type of input – knowledge – while the former provide it jointly with raw labor. As knowledge growth occurs through investment in patented innovations, an important implication of the model is a tendency toward the compression of incomes paid to workers – which can reduce to the sole compensation of the basic competencies – while the remaining income goes to patent holders.

We also showed that when one takes into account the aforementioned features of recent technical progress, competition becomes viable. Hence the economy can reach the first best if knowledge, while being non-rival, is homogeneous and fully excludable when used in final goods production. Even if partial excludability occurs, second best results would be confined to the transitional period, while first best is reached all the same in the long run. Even though non competitive markets in practice are often observed, stressing that competition is logically viable has important policy implications, since it means that, *e.g.*, regulatory and judicial interventions in the field of patents and IPR can foster competition without fearing that a collapse of research activities arises.

While the potential unwanted effects of technical progress have been often identified with the possible growth in unemployment due to the substitution of capital for labor, these dire effects did not materialize in the last decades in advanced countries, where unemployment – bar for the financial crisis years – has not been the more worrying problem. One of the prominent social concerns has actually been instead the shrinking of labor income share. The contribution of this paper is a possible rationale for this stylized fact, which does not fit well into standard growth models where the stability of factors' income shares is a tenet. As long as development disempowers the role of human capital, it raises also potential severe distributive problems. Our simple setting, however, is not suitable for studying them, and thus they are left for future research.

Appendix

Proof of Proposition 1. Clearly, the first condition in (12) implies that, according to (11), $g = \dot{C}/C > 0$. Differentiating with respect to time $\ln(Y)$, with Y as in (3), and recalling that, from (7), $x/A \equiv \delta$ is constant, it is immediately seen that $\dot{Y}/Y = \dot{A}/A$. As the only asset in the economy is expressed in terms of the knowledge stock A owned by households, from (9) $B = VA = \eta A$ must hold; hence, as under Assumption A.3 both η and, by (10), the interest rate, $r = \gamma L/\eta$, are constant the instantaneous budget constraint in (1) can be rewritten as

$$\frac{\dot{A}}{A} = \frac{1}{\eta}\left(\gamma L - \frac{C}{A}\right), \tag{49}$$

which implies that, in order to $\dot{A}/A$ be constant along the BGP, the ratio C/A on the RHS must be constant as well, which is possible if and only if $\dot{A}/A = \dot{C}/C = g$. Next, note that the second condition in (12) implies that $r > g = \dot{A}/A$, so that the transversality condition for problem (1), $\lim_{t\to+\infty} B(t)\, e^{-rt} = \lim_{t\to+\infty} \eta A(t)\, e^{-rt} = 0$, holds. Finally, for each $A(t)$ the amount of the intermediate good is given by (7) as $x(t) = \delta A(t)$; therefore, in $t = 0$, $x(0) = \delta A(0)$ and the economy is immediately put on the BGP. ∎

Proof of Proposition 2. The resource constraint of the economy at instant t is

$$C(t) + J(t) = Y(t) - X(t), \tag{50}$$

where on the RHS we consider total output net of the intermediate goods, $X(t) = x(t)L$. Dropping time dependency for simplicity, in order to obtain a dynamic constraint in the only variables A (state) and C (control) a social planner first considers maximization of the net output $Y - X = [Af(x/A) - x]L$ with respect to x for a given stock A at instant t: the solution is $x^S = (f')^{-1}(1)A = \delta A$, where the superscript "S" denotes the level of per capita intermediate good chosen by the social planner, which happens to be the same as in (7). Hence, net output turns out to be $Y^S(t) - X^S(t) = L[f(\delta) - \delta]A(t) = \gamma LA(t)$, where in the last equality we used (8). Under Assumption A.3 $J = \eta \dot{A}$, and (50) can be rewritten as

$$\dot{A}(t) = \frac{\gamma(t) LA(t) - C(t)}{\eta},$$

which, as $B(t) = \eta A(t)$ with η constant, turns out to be the same as the household's budget constraint in (1). Hence, the social planner problem is the same as (1) and has the same equilibrium of Proposition 1 as solution. ■

Proof of Proposition 3. The arguments are the same as in the previous proof of Proposition 1 and thus we omit them. The only difference is that now from (15) $B(t) = V(t)A(t) = \eta(t)A(t)$ holds, so that $\dot{B}(t) = \dot{\eta}(t)A(t) + \eta(t)\dot{A}(t)$; however, it is immediately seen that the instantaneous budget constraint in (1) at each instant t remains the same as in (49), as, using (16) and rearranging terms,

$$\frac{\dot{A}(t)}{A(t)} = \frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \frac{\dot{\eta}(t)}{\eta(t)} - \frac{C(t)}{\eta(t)A(t)} = \frac{1}{\eta(t)}\left[\gamma L - \frac{C(t)}{A(t)}\right].$$

When $\lim_{t\to+\infty}\eta(t) = \eta^* > 0$, $\dot{\eta}(t) \to 0$ as $t \to +\infty$ and, according to (16), $\lim_{t\to+\infty} r(t) = r^* \equiv \gamma L/\eta^*$; thus, the second condition in (19) implies that $r^* > g^* = \dot{A}/A$, with g^* defined in (20), so that the transversality condition for problem (1), $\lim_{t\to+\infty}\eta(t)A(t)e^{-r(t)t} = \lim_{t\to+\infty}\eta^* A(t)e^{-r^*t} = 0$, holds. ■

Proof of Proposition 4. When $\dot{A}/A > 0$, $\lim_{t\to+\infty}\varphi[A(t)] = \eta^* > 0$ implies that $\lim_{t\to+\infty}\varphi'[A(t)] = 0$, so that it is immediately seen that, as $\lim_{t\to+\infty}[1/\varepsilon_u(C)] = 1/\sigma$, the consumption growth rate in (26) asymptotically converges to that defined in (20). As asymptotically the dynamic constraint (21) becomes equal to (49), the same argument as in the proofs of Proposition 1 applies to establish that $\dot{Y}/Y = \dot{A}/A = \dot{C}/C = g^*$ while the first condition in (19) implies that, according to (20), $g^* > 0$. The second condition in (19) is equivalent to $(1-\sigma)g^* < \rho$, which, according to (25) and, under (13), $\lim_{t\to+\infty}[\dot{\eta}(t)/\eta(t)] = \lim_{t\to+\infty}\left\{\varphi'[A(t)]\dot{A}(t)/\varphi[A(t)]\right\} = 0$, implies

$$\lim_{t\to+\infty}\left(\frac{\dot{\lambda}}{\lambda} + \frac{\dot{A}}{A}\right) = \lim_{t\to+\infty}\left(\frac{\dot{\eta}}{\eta} - \sigma\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\right) = (1-\sigma)g^* < \rho,$$

thus ensuring that the transversality condition (24) holds. ■

Proof of Proposition 5. Under the assumptions $\lim_{t\to+\infty}[1/\varepsilon_u(c)] = 1/\sigma > 0$ and $\lim_{t\to+\infty}\eta(t) = \eta^* > 0$ [implying $\dot{\eta}(t)/\eta(t) \to 0$ as $t \to +\infty$], from (38) it is clear that, in order to achieve an ABGP in the long-run, additionally

$$\lim_{t\to+\infty}\frac{\dot{\Omega}(t)}{\Omega(t)} = \frac{\dot{L}(t)}{L(t)} = n \tag{51}$$

must hold. In view of (29) and recalling that the argument of the function $f(\cdot)$ is constant, $(\Omega x/A) = \delta$, it is immediately seen that, under (51), on the ABGP aggregate output grows at the same growth rate of knowledge:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\partial}{\partial t}\left[\ln A(t) - \ln \Omega(t) + \ln L(t) + \ln f(\delta)\right] = \frac{\dot{A}(t)}{A(t)} - \frac{\dot{\Omega}(t)}{\Omega(t)} + \frac{\dot{L}(t)}{L(t)}$$
$$= \frac{\dot{A}(t)}{A(t)} - n + n = \frac{\dot{A}(t)}{A(t)}. \tag{52}$$

From (33) we obtain the per capita asset as a function of per capita knowledge: $b(t) = B(t)/L(t) = V(t)A(t)/L(t) = \eta(t)a(t)$, where $a(t) = A(t)/L(t)$. Thus, the per capita household's budget constraint in (36) can be rewritten as

$$\frac{\dot{a}(t)}{a(t)} = r(t) - n - \frac{\dot{\eta}(t)}{\eta(t)} - \frac{c(t)}{\eta(t)a(t)} = \frac{\gamma L(t)}{\eta(t)\Omega(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - n - \frac{\dot{\eta}(t)}{\eta(t)} - \frac{c(t)}{\eta(t)a(t)}$$
$$= \frac{\gamma L(t)}{\eta(t)\Omega(t)} - n - \frac{c(t)}{\eta(t)a(t)}, \tag{53}$$

where in the second equality we used (34). Because, by Assumption A.5,

$$\lim_{t\to+\infty}\frac{\dot{\Omega}(t)}{\Omega(t)} = \lim_{t\to+\infty}\frac{\Omega'[A(t)]\dot{A}(t)}{\Omega[A(t)]} = \lim_{t\to+\infty}\left\{\frac{\Omega'[A(t)]A(t)}{\Omega[A(t)]}\cdot\frac{\dot{A}(t)}{A(t)}\right\}$$
$$= \lim_{t\to+\infty}\frac{\Omega'[A(t)]A(t)}{\Omega[A(t)]}\lim_{t\to+\infty}\frac{\dot{A}(t)}{A(t)}$$
$$= \varepsilon_{\Omega}^{*}g^{*},$$

joining it with (51) from (52) one immediately gets (41). It follows from (53) that $g_c^* = \lim_{t\to+\infty}[\dot{c}(t)/c(t)] = g^* - n = (1/\varepsilon_{\Omega}^* - 1)n$, which is (40). The interest rate in (42) is immediately obtained using (40) in the (asymptotic) Euler equation (37). Finally, using (42), condition (39) implies that $r^* > g^* = \dot{A}/A$, so that the transversality condition for problem (36), $\lim_{t\to+\infty}\eta(t)A(t)e^{-r(t)t} = \lim_{t\to+\infty}\eta^* A(t)e^{-r^*t} = 0$, holds. ■

Proof of Proposition 6. Under the same assumptions as in Proposition 5, specifically under Assumption A.5, from (48) it follows that, in order to achieve an ABGP in the long-run, $\Omega(t)$ and $L(t)$ must grow at the same constant rate, n, so that g^* as in (41) is immediately obtained. Rewriting the dynamic resource constrain of (44) as

$$\frac{\dot{A}(t)}{A(t)} = \frac{1}{\eta}\left[\frac{\gamma L(t)}{\Omega[A(t)]} - \frac{L(t)c(t)}{A(t)}\right],$$

it is immediately seen that $\dot{A}/A$ is constant along the ABGP only if the ratio $L(t)c(t)/A(t)$ on the RHS is constant as well, which is possible only if $\dot{A}/A = g^* = \dot{L}/L + \dot{c}/c = n + g_c^*$, which, using (41), yields (40). Using (46) and (41), it holds

$$\lim_{t\to+\infty}\left(\frac{\dot{\lambda}}{\lambda} + \frac{\dot{A}}{A}\right) = \lim_{t\to+\infty}\left\{\rho - \frac{\gamma L}{\eta\Omega(A)}\left[1 - \frac{\Omega'(A)A}{\Omega(A)}\right]\right\} + \lim_{t\to+\infty}\frac{\dot{A}}{A}$$
$$= \lim_{t\to+\infty}\left[\rho - \frac{\gamma L}{\eta\Omega(A)}(1 - \varepsilon_{\Omega}^*)\right] + \frac{n}{\varepsilon_{\Omega}^*}. \tag{54}$$

On the other hand, from (48), asymptotically it holds

$$\lim_{t\to+\infty}\frac{1}{\varepsilon_u(c)}\left\{\frac{\gamma L}{\eta\Omega(A)}\left[1-\frac{\Omega'(A)A}{\Omega(A)}\right]-\rho\right\}=\frac{1}{\sigma}\lim_{t\to+\infty}\left[\frac{\gamma L}{\eta\Omega(A)}\left(1-\varepsilon_\Omega^*\right)-\rho\right]=g_c^*,$$

from which, using (40), one gets

$$\sigma g_c^*=\lim_{t\to+\infty}\left[\frac{\gamma L}{\eta\Omega(A)}\left(1-\varepsilon_\Omega^*\right)-\rho\right]=\sigma\left(\frac{1}{\varepsilon_\Omega^*}-1\right)n$$

which, substituting into (54), yields

$$\lim_{t\to+\infty}\left(\frac{\dot\lambda}{\lambda}+\frac{\dot A}{A}\right)=-\sigma\left(\frac{1}{\varepsilon_\Omega^*}-1\right)n+\frac{n}{\varepsilon_\Omega^*}=\frac{1-\left(1-\varepsilon_\Omega^*\right)\sigma}{\varepsilon_\Omega^*}n,$$

so that the transversality condition (47) holds whenever condition (39) is satisfied.

Finally, recall that under Assumption A.3 $\dot\eta/\eta=0$, so that (38) becomes

$$g_c(t)=\frac{1}{\varepsilon_u[c(t)]}\left\{\left(\frac{\gamma}{\eta}\right)\frac{L(t)}{\Omega[A(t)]}-\rho\right\},$$

which is clearly smaller, equal to, or larger than $g_c^S(t)$ in (48) whenever $\varepsilon_\Omega^*(t)<0$, $\varepsilon_\Omega^*(t)=0$, or $\varepsilon_\Omega^*(t)>0$, respectively. ■

References

[1] Acemoglu, D., *Introduction to Modern Economic Growth*, Princeton, NJ: Princeton University Press, 2009.

[2] Chantrel, É., Grimaud, A. and Tournemaine, F., Pricing Knowledge and Funding Research of New Technology Sectors in a Growth Model, *Journal of Public Economic Theory* **14**: 493–520, 2012.

[3] Corrado, C., Hulten, C. and Sichel, D., Intangible Capital and U.S. Economic Growth, *Review of Income and Wealth* **55**: 661–685, 2009.

[4] Dasgupta, D., Lindahl Pricing, Nonrival Infrastructure, and Endogenous Growth, *Journal of Public Economic Theory* **3**: 413–430, 2001.

[5] Gordon, J. R., The Demise of U.S. Economic Growth: Restatement, Rebuttal and Reflections, *NBER Working Paper No.* **19895**, 2014.

[6] Jones C. I., R&D-Based Models of Economic Growth, *Journal of Political Economy* **103**: 759–784, 1995.

[7] Jones C. I., Growth: With or Without Scale Effects?, *American Economic Review* **89**: 139–144, 1999.

[8] Jones, C. I., Growth and Ideas, in: Aghion, P. and Durlauf, S. (eds.), *Handbook of Economic Growth, Vol. 1B*, Amsterdam: Elsevier, pp. 1063–1111, 2005.

[9] Karabarbounis, L., and Neiman, B., The global decline of the labor share, *Quarterly Journal of Economics* **129**: 61–103, 2014.

[10] Kortum, S. S., Research, patenting, and technological change, *Econometrica* **65**: 1389–1419, 1997.

[11] Kuwahara, S., Dynamical Analysis of the R&D-Based Growth Model with Regime Switch, *Journal of Economics* **108**: 35–57, 2013.

[12] Marchese, C., Marsiglio, S., Privileggi, F., and Ramello G.B., Endogenous Recombinant Growth through Market Production of Knowledge and Intellectual Property Rights, *Dip. "Cognetti de Martiis" Working Paper No.* **13/2014**, 2014 (http://www.unito.it/unitoWAR/ShowBinary/FSRepo/D031/Allegati/WP2014Dip/WP_13_2014.pdf).

[13] Romer, P. M., Endogenous Technological Change, *Journal of Political Economy* **98**: S71–S102, 1990.

[14] Segerstrom, P., Endogenous growth without scale effects, *American Economic Review* **88**: 1290–1310, 1998.

[15] Tsur, Y., and Zemel, A., Towards Endogenous Recombinant Growth, *Journal of Economic Dynamics and Control* **31**: 3459–3477, 2007.

[16] Weitzman, M. L., Recombinant growth, *The Quarterly Journal of Economics* **113**: 331–360, 1998.