On the Transition Dynamics in Endogenous Recombinant Growth Models

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Fabio Privileggi

Abstract This paper is a first attempt at studying the transition dynamics of the Tsur and Zemel (2007) continuous time endogenous growth framework in which knowledge evolves according to the Weitzman (1998) recombinant process. For a specific choice of the probability function characterizing the recombinant process, we find a suitable transformation for the state and control variables diverging to asymptotic constant growth, so that an equivalent 'detrended' system converging to a steady state in the long run can be tackled. Since the dynamical system obtained so far turns out to be analytically intractable, we rely on numerical simulation in order to fully describe the transition dynamics for a set of values of the parameters.

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1.1 Introduction

Tsur and Zemel [8] developed an endogenous growth model in which balanced long-run growth is obtained by assuming that the stock of knowledge evolves according to Weitzman's [9] recombinant expansion process and is used, together with physical capital, as input factor by competitive firms in order to produce a unique physical good. At each instant new knowledge is produced by an independent R&D sector directly controlled by a 'regulator' who aims at maximizing the discounted utility of a representative consumer over an infinite horizon. The optimal resources required for new knowledge production are obtained by the regulator in the form of a tax levied

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on the consumers. The economy, thus, envisages two sectors, a competitive one devoted to the production of the unique physical good, and a regulated R&D sector in which the public good ‘knowledge’ is being directly financed by the regulator and produced according to Weitzman’s production function.

In such framework Tsur and Zemel provide conditions under which the economy performs sustained constant balanced growth in the long run; moreover, when balanced growth occurs, they also characterize the asymptotic optimal tax rate and the common growth rate of all variables. Hence, by endogenizing the optimal choice for investing in knowledge production, their result generalizes Weitzman’s [9] endogenous growth model in which the investment in knowledge production was assumed to be constant and exogenously determined.

In this paper we further extend the Tsur and Zemel results by studying more accurately the transition dynamics along a characteristic turnpike curve in the knowledge-capital state space already discussed in [8]. For a specific parametrization of the model and when the conditions allowing sustained long-run growth are met, we are able to (numerically) compute the optimal policy – in terms of optimal consumption – and thus the optimal time-path trajectories of the stock of knowledge, capital, output and consumption – as well as their transition growth rates – while the economy is being headed along the turnpike curve toward its long-run constant balanced growth behavior.

Our method is based on the standard technique of transforming the state and control variables of the Hamiltonian describing the optimal dynamics of (a slightly generalized version of) the Tsur and Zemel model – all diverging in the long-run – into ‘detrended’ state-like and control-like variables, both converging to a saddle-path stable steady state in the appropriate space as time elapses. To study such detrended system we apply the time-elimination method introduced by Mulligan and Sala-i-Martin [4] (see also [5] and [2], pp. 593-596) so that the optimal detrended consumption policy can be calculated by means of numerical methods for ODEs; then, substituting such policy in the ODE of the state-like variable and solving it – again numerically – with respect to time, the optimal time-path trajectories of both state-like and control-like variables are obtained. Eventually, these trajectories are reconverted into time-path trajectories for the original model, thus allowing for a detailed analysis of the transition dynamics of all relevant variables.

Two technical difficulties had to be overcome: 1) finding a proper probability function for the Weitzman’s recombinant process suitable for the change of variables in the construction of the detrended system of ODEs, and 2) the exploitation of a singular point – other than the saddle-path steady state – which can be used as initial condition for calculating specific solutions for the ODE describing the policy. Due to the high instability of the system of ODEs characterizing the detrended variables, we have been able to fully solve the model only for a set of values of the parameters; more precisely, our approach works satisfactory only on a manifold of dimension one in the parameters’ space (see Remark 2 at the end of Section 1.4).

Section 1.2 discusses the original contribution by Weitzman [9] on the production of new knowledge by combining existing ideas and its generalization to the endogenous recombinant growth framework provided by Tsur and Zemel [8]. The central contribution of this paper is contained in Section 1.3, where, under a suitable choice for the functions of the model – in particular, for the probability of success in matching pairs of ideas – we are able to transform the original diverging dynamics into an equivalent system of two ODEs in two 'detrended' variables converging asymptotically to a steady state in the appropriate space. This allows for numeric computation of the optimal policy of both the detrended system and the original diverging dynamics, which is implemented in Section 1.4 for a specific set of parameters' values. Finally, after using the optimal policy obtained so far to numerically trace out the optimal time-path trajectories, Section 1.5 is dedicated to a qualitative discussion of the transition dynamics thus obtained, while Section 1.6 reports some concluding remarks and topics for future research.

1.2 Endogenous recombinant growth

Weitzman's [9] knowledge production device postulates that originally unprocessed (*seed*) ideas are blended with all other ideas available in order to generate new *hybrid* seed ideas; a costly selection process permits in turn to extract from those a subset of *fertile* seed ideas that are again recombined with all the existent fertile ideas to produce yet new hybrids. This process occurs indefinitely, generating knowledge growth. The hybridization is based on matching m ideas together and then checking whether the resulting new idea is fertile (*i.e.*, successful). If $A(t)$ is the stock of knowledge available at time t (measured as the total number of fertile ideas), let $C_m[A(t)]$ denote the number of different combinations of m elements (hybrids) of $A(t)$; *i.e.*: $C_m[A(t)] = A(t)!/\{m![A(t)-m]!\}$ [*e.g.*, $C_2(A) = A(A-1)/2$]. Therefore, at time t the number of hybrid seed ideas is given by

$$H(t) = C_m[A(t)] - C_m[A(t-1)]. \tag{1.1}$$

If π is the probability of obtaining a successful idea from each matching, the number of new successful ideas at time t is given by (eqn. (2) on p. 337 in [9]):

$$\Delta A(t) = A(t+1) - A(t) = \pi H(t) = \pi\{C_m[A(t)] - C_m[A(t-1)]\}, \tag{1.2}$$

which, in a discrete time framework, defines a *recombinant expansion process* of second order representing the potential knowledge production path. Therefore, the stock of knowledge A has the potential of growing at an increasing rate of growth (Lemma on p. 338 in [9]). However, potentially explosive growth is actually precluded by scarcity of resources employed in the matching process; as a matter of fact, Weitzman [9] reconciles his theory with standard endogenous growth models (see, *e.g.*, [6], [1], or [2]) by showing that knowledge growth – as well as the growth rate of GNP in real economies – is actually bounded. Accordingly, the knowledge generation mechanism envisaged by Weitzman uses two inputs: hybrid seed ideas, H, and physical resources, J. The latter affects the probability π of producing successful ideas by increasing it with larger J for each given H, while J becomes less productive for larger H. To summarize, π results to be increasing in the ratio J/H.

Thus, the *production function for new knowledge* ΔA is:

$$\Delta A = W(J,H) = H\pi(J/H), \tag{1.3}$$

corresponding to (28) on p. 346 in [9]. Note that W in (1.3) is homogeneous of degree 1. In the sequel we shall assume the following.

A. 1 *The function* $\pi : \mathbb{R}_+ \to [0,1]$ *is independent of time and is such that* $\pi' > 0$, $\pi'' < 0$, $\pi(0) = 0$ *and* $\pi(\infty) \leq 1$*; moreover,*[1] $\lim_{x\to 0^+} \pi'(x) < +\infty$.

Provided that J is a constant fraction of the total output y, $J = sy$, Weitzman [9] establishes that in the long run the asymptotic growth rate is a positive constant depending on the exogenously determined saving rate s.

1.2.1 The framework

Tsur and Zemel [8], made an important refinement of Weitzman's analysis by endogenizing the (optimal) resources J employed in the production of new knowledge.[2] Their model features a 'regulator' who has the task of choosing the optimal amount J to be employed in production of new knowledge – which, in turn, is being assigned to all firms producing the amount y of a unique (physical) output – in order to maximize the discounted utility of a representative consumer over an infinite horizon. Output producing firms operate in a competitive environment, while the regulator has the power to levy the exact amount J as a tax on the representative consumer, through which, given all the H hybrid seed ideas freely available, new useful knowledge is being directly generated according to (1.3), and is immediately and freely passed to the output producing firms.

The difficulty in dealing with the second-order dynamic (1.2) is overcome by switching from the Weitzman's discrete time formulation into a continuous time model. This allows the authors to rewrite (1.1) as:

$$H(t) = C_m'[A(t)]\dot{A}(t), \tag{1.4}$$

where $\dot{A}(t)$ is the derivative of the stock of knowledge with respect to time. By replacing $\Delta A(t)$ with $\dot{A}(t)$ in (1.3) we obtain the analogous of (1.3) in continuous time:

$$\dot{A}(t) = H(t)\pi[J(t)/H(t)], \tag{1.5}$$

[1] For simplicity, in the sequel $\lim_{x\to 0^+} \pi'(x)$ will be denoted by $\pi'(0)$.

[2] Our analysis slightly departs from that of Tsur and Zemel by allowing J to be any amount of physical capital available in the economy, while the authors constrain such resources to be only a fraction $0 \leq s \leq 1$ of the total output y. In other words, in our economy the regulator has the power to extract resources also from existing physical capital, in addition to the whole total output y.

where the probability of generating a new fertile idea π still satisfies A.1.

By combining (1.4) and (1.5) the law of motion for the stock of knowledge $A(t)$ is:

$$\dot{A}(t) = J(t) / \varphi [A(t)], \tag{1.6}$$

where

$$\varphi(A) = C'_m(A) \pi^{-1} \left[1/C'_m(A)\right] \tag{1.7}$$

is the *expected unit cost of knowledge production*. Note that $\varphi(\cdot)$ is decreasing and, as knowledge keeps spreading, it converges to

$$\lim_{A\to\infty} \varphi(A) = 1/\pi'(0) > 0, \tag{1.8}$$

where $1/\pi'(0)$ is strictly positive by Assumption A.1.

With no loss of generality, we shall assume that labour is constant and normalized to one:[3] $L \equiv 1$. The output producing firms use a neoclassical production function,

$$y(t) = F[k(t), A(t)], \tag{1.9}$$

depending on aggregate capital and knowledge-augmented labour $A(t)L$, for $L = 1$.

A. 2 $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ *exhibits constant returns to scale and is such that* $F_k > 0$, $F_A > 0$, $F_{kk} < 0$, $F_{AA} < 0$, $F_{kA} > 0$, *and satisfies* $\lim_{k\to 0^+} F(k,A) = +\infty$ *for all* $A > 0$.

Each firm i maximizes instantaneous profit by renting capital k_i and hiring labour $L_i \leq 1$ from the households, taking as given the capital rental rate r, the labour wage w and the stock of knowledge A. Since all firms use the same technology and operate in a competitive market, and all households are the same, the subscript i can be dropped and (1.9) can be rewritten as $y = Af(k/A)$, where

$$f(x) = F(x, 1). \tag{1.10}$$

Since firms act competitively, in equilibrium their profit is zero, that is, households earn $y = Af(k/A) = rk + w$; moreover, the amount of capital demanded, k, satisfies

$$f'(k/A) = r. \tag{1.11}$$

Given that, at each instant t, a fraction $J(t)$ of the whole endowment of the economy, $k(t) + y(t)$, is being employed to finance R&D firms, and a fraction $c(t)$ is being consumed, capital evolves through time according to

$$\dot{k}(t) = y(t) - J(t) - c(t), \tag{1.12}$$

[3] Tsur and Zemel [8] assume that the amount of labour is L, constant through time even if not necessarily equals to one. As stationarity with respect to time of L is the strong assumption here, normalizing labour to $L \equiv 1$ has the advantage of simplifying notation at no cost.

where it is assumed that capital does not depreciate. Since the upper bound[4] for $J(t)$ and $c(t)$ is jointly given by $J(t)+c(t) \leq k(t)+y(t)$, $\dot{k}(t)$ in (1.12) may be negative.

Assuming that all households enjoy an instantaneous utility $u[c(t)]$, with $u : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and strictly concave, the 'regulator' solves

$$\max_{\{c(t),J(t)\}} \int_0^\infty u[c(t)]\, e^{-\rho t} dt \tag{1.13}$$

$$\text{subject to} \begin{cases} \dot{A}(t) = J(t)/\varphi[A(t)] \\ \dot{k}(t) = F[k(t), A(t)] - J(t) - c(t) \\ J(t) + c(t) \leq k(t) + F[k(t), A(t)] \\ k(t) \geq 0,\ J(t) \geq 0,\ c(t) \geq 0 \\ k(0) = k_0 > 0,\ A(0) = A_0 > 0, \end{cases}$$

where $\rho > 0$ is the (constant) discount rate. (1.13) may be interpreted as a maximum welfare problem, where k and A are the state variables and c and J are the controls. Suppressing the time argument, the current-value Hamiltonian associated to (1.13) is

$$H(A, k, J, c, \vartheta_1, \vartheta_2) = u(c) + \vartheta_1 [F(k, A) - J - c] + \vartheta_2 J/\varphi(A), \tag{1.14}$$

where ϑ_1, ϑ_2 are the costates of k and A respectively. Necessary conditions are:

$$u'(c) = \vartheta_1 \tag{1.15}$$

$$J = \begin{cases} 0 & \text{if } \vartheta_2/\varphi(A) < \vartheta_1 \\ \tilde{J} & \text{if } \vartheta_2/\varphi(A) = \vartheta_1 \\ k + F(k, A) - c & \text{if } \vartheta_2/\varphi(A) > \vartheta_1 \end{cases} \tag{1.16}$$

$$\dot{\vartheta}_1 = \rho \vartheta_1 - \vartheta_1 F_k(k, A) \tag{1.17}$$

$$\dot{\vartheta}_2 = \rho \vartheta_2 - \vartheta_1 F_A(k, A) + \vartheta_2 J \varphi'(A) / [\varphi(A)]^2 \tag{1.18}$$

$$\lim_{t \to \infty} H(t)\, e^{-\rho t} = 0, \tag{1.19}$$

where $\tilde{J}$ in (1.16) will be defined in (1.22). The case $J = k + F(k, A) - c$ when $\vartheta_2/\varphi(A) > \vartheta_1$ in (1.16) can be ruled out by the Inada condition of Assumption A.2.

Taking time derivative of $\vartheta_1 = \vartheta_2/\varphi(A)$ in (1.16) and using (1.17) and (1.18) gives

$$F_k(k, A) - F_A(k, A)/\varphi(A) = 0, \tag{1.20}$$

defining the locus on the space (A, k) on which the marginal product of capital equals that of knowledge per unit cost. (1.20) can be rewritten as $z(k/A) = \varphi(A)$, where $z(x) = f(x)/f'(x) - x$, with f defined in (1.10), is increasing in x; thus, (1.20) can be expressed as a function of the only variable A:

$$\tilde{k}(A) = z^{-1}[\varphi(A)]\, A, \tag{1.21}$$

where z^{-1} is the inverse of $z(x)$.

[4] See footnote 2.

Differentiating $\tilde{k}(A)$ with respect to time and using (1.12) and (1.6) yields

$$\tilde{J}(t) = [\tilde{y}(t) - c(t)]\, \varphi[A(t)] \,/\, \left\{\tilde{k}'[A(t)] + \varphi[A(t)]\right\}, \tag{1.22}$$

where $\tilde{y}(t) = F\left\{\tilde{k}[A(t)], A(t)\right\}$, expressing the optimal investment in R&D, $\tilde{J}(t)$, as a function of the optimal consumption $c(t)$, when the economy grows along the curve $\tilde{k}(A)$ defined in (1.21); that is, when $\vartheta_2(t)/\varphi[A(t)] = \vartheta_1(t)$ in (1.16).

We consider the limit of (1.21) for large A, which becomes linear, by defining:

$$\tilde{k}_\infty(A) = \tilde{\eta}A + q, \tag{1.23}$$

where, by (1.8), $\tilde{\eta} = z^{-1}[1/\pi'(0)]$ and q is a non-negative constant. Note that $\tilde{k}(A)$ lies above $\tilde{k}_\infty(A)$ for all $A < \infty$, approaching $\tilde{k}_\infty(A)$ as A increases. The intercept q depends on the number of ideas m being matched at each instant t in (1.4).

Proposition 1. *The intercept q in (1.23) is zero whenever $m > 2$, while $q > 0$ for $m = 2$.*

Proof. Since $\tilde{k}_\infty(A) = \tilde{\eta}A + q$ is the asymptote of $\tilde{k}(A)$,

$$q = \lim_{A\to+\infty}\left[\tilde{k}(A) - \tilde{\eta}A\right] = \lim_{A\to+\infty}\left\{z^{-1}[\varphi(A)] - z^{-1}\left[1/\pi'(0)\right]\right\}A. \tag{1.24}$$

As $\varphi(A)$ is decreasing and, under A.1, bounded away from zero [specifically, $0 < 1/\pi'(0) \le \varphi(A) \le \varphi(A_0)$], by A.2 $z^{-1}[\varphi(A)] - z^{-1}[1/\pi'(0)]$ in (1.24) is $o[\varphi(A)]$. Thus, since, by (1.7), $O[\varphi(A)] = O[C_m'(A)] = O\left(A^{m-1}\right)$ [*i.e.*, $C_m'(A) \sim A^{m-1}$ for large A], if $m > 2$ the limit in (1.24) is zero, while, if $m = 2$, such limit must be nonzero; as $z^{-1}[\varphi(A)] - z^{-1}[1/\pi'(0)] > 0$ for all $A < +\infty$, $q > 0$ holds whenever $m = 2$. □

Another locus will be considered, that on which the marginal product of capital equals the individual discount rate, $f'(k/A) = \rho$, which, by (1.11), implies $r = \rho$. As $f'(\cdot)$ is decreasing, also such curve can be expressed as a function of A:

$$\hat{k}(A) = \hat{\eta}A, \tag{1.25}$$

with $\hat{\eta} = (f')^{-1}(\rho)$; that is, $\hat{k}(A)$ is the linear function with slope $\hat{\eta} > 0$.

The curves $\tilde{k}(A)$, $\tilde{k}_\infty(A)$ and $\hat{k}(A)$ defined in (1.21), (1.23) and (1.25) will be labeled *turnpike*, *asymptotic turnpike* and *stagnation line* respectively. The optimal investment in R&D along the turnpike $\tilde{k}(A)$ defined in (1.22), $\tilde{J}(t)$, will be referred as the *singular policy*. We shall assume the following.

A. 3 *The instantaneous utility is CIES: $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$, with $\sigma \ge 1$.*

Proposition 2 (Tsur and Zemel [8]).

i) *A necessary condition for the economy to sustain long-run growth is*

$$\hat{\eta} > \tilde{\eta}; \tag{1.26}$$

conversely, if $\hat{\eta} \leq \tilde{\eta}$ the economy eventually reaches a steady (stagnation) point on the line $\hat{k}(A)$ corresponding to zero growth.

ii) Under (1.26), for any given initial knowledge stock A_0 there is a corresponding threshold capital stock $k^{sk}(A_0) \geq 0$ such that whenever $k_0 \geq k^{sk}(A_0)$ the economy – possibly after an initial transition outside the turnpike – first reaches the turnpike $\tilde{k}(A)$ in a finite time, and then continues to grow along it as time elapses until the asymptotic turnpike $\tilde{k}_\infty(A)$ is reached in the long-run. Along $\tilde{k}_\infty(A)$ the economy follows a balanced growth path characterized by a common constant growth rate of output, knowledge, capital and consumption given by:

$$\gamma = (r_\infty - \rho)/\sigma > 0, \tag{1.27}$$

where $r_\infty = \lim_{A\to\infty} f'\left[\tilde{k}_\infty(A)/A\right] = f'(\tilde{\eta})$ defines the long-run capital rental rate.[5] *Moreover, $\tilde{J}(t) < \tilde{y}(t)$ for large t, and the income shares devoted to investments in knowledge and capital are constant and given respectively by*

$$s_\infty = \gamma/\left\{r_\infty\left[1+\tilde{\eta}\pi'(0)\right]\right\} \quad \text{and} \quad s_\infty^k = \gamma\tilde{\eta}\pi'(0)/\left\{r_\infty\left[1+\tilde{\eta}\pi'(0)\right]\right\}. \tag{1.28}$$

If $k_0 < k^{sk}(A_0)$ the economy eventually stagnates.

Proposition 2, whose proof can be found in [8], establishes that if (1.26) holds and k_0 is sufficiently high with respect to initial knowledge stock A_0, the economy grows along a turnpike path which, in the long run, converges to a balanced growth path with knowledge and capital growing at the same constant rate and with constant saving rate, thus confirming Weitzman's result in a more general setting.

As the case $\vartheta_2/\varphi(A) > \vartheta_1$ in (1.16) is ruled out, two optimal regimes are possible:

1. zero R&D, corresponding to $J \equiv 0$, which, if maintained forever, eventually leads the economy to some steady state (stagnation point) on the line $\hat{k}(A)$, and
2. a path along the turnpike $\tilde{k}(A)$ – maybe started after a finite period of transition outside the turnpike – corresponding to the singular policy $\tilde{J}$ in (1.22), which envisages growth for all variables as time elapses and, if maintained forever, eventually lead to a balanced growth path along the asymptotic turnpike $\tilde{k}_\infty(A)$.

Under (1.26) and if $k_0 \geq k^{sk}(A_0)$ it can be shown that the turnpike $\tilde{k}(A)$ is 'trapping', *i.e.*, the economy keeps growing along it after it is reached. Hence, there are two types of transitions: one driving the system toward the turnpike starting from outside it, and another characterizing the optimal path along $\tilde{k}(A)$ after it has been entered. We shall focus on the latter; specifically, we shall assume that (1.26) holds, implying that the stagnation line $\hat{k}(A)$ lies strictly above[6] the turnpike $\tilde{k}(A)$ for A sufficiently large, moreover, we shall restrict our attention to the case $k_0 = \tilde{k}(A_0)$. In this scenario $k_0 \geq k^{sk}(A_0)$ is certainly satisfied, as the turnpike $\tilde{k}(A)$ is trapping.

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[5] Note that, under (1.26), $r_\infty = f'(\tilde{\eta}) > f'(\hat{\eta}) = f'\left[(f')^{-1}(\rho)\right] = \rho$.

[6] This holds for all $A > 0$ when $m > 2$, while for A large enough if $m = 2$.

1.2.2 Dynamics along the turnpike

We now adapt the optimal conditions (1.15) - (1.19) to the system's behavior along the turnpike $\tilde{k}(A)$. All variables on the turnpike will be labeled with a '$\sim$' symbol.

Suppressing the time argument and using (1.22), (1.6) becomes

$$\dot{A} = [\tilde{y}(A) - \tilde{c}] / \left[\tilde{k}'(A) + \varphi(A)\right], \tag{1.29}$$

where $\tilde{y}(A) = F\left[\tilde{k}(A), A\right] = Af\left[\tilde{k}(A)/A\right]$ with $f(\cdot)$ defined in (1.10). (1.29) is the unique dynamic constraint as $\dot{\tilde{k}} = \tilde{k}'(A)\dot{A} = \tilde{k}'(A)[\tilde{y}(A) - \tilde{c}] / \left[\tilde{k}'(A) + \varphi(A)\right]$; therefore, now the unique state variable is A, and, by (1.22), the unique control is $\tilde{c}$.

Thus, the 'regulator' solves

$$\max_{\{\tilde{c}(t)\}} \int_0^\infty u[\tilde{c}(t)] e^{-\rho t} dt \tag{1.30}$$

$$\text{subject to } \begin{cases} \dot{A}(t) = \{\tilde{y}[A(t)] - \tilde{c}(t)\} / \left\{\tilde{k}'[A(t)] + \varphi[A(t)]\right\} \\ 0 \leq \tilde{c}(t) \leq \tilde{k}[A(t)] + \tilde{y}[A(t)] \\ A(0) = A_0 > 0. \end{cases}$$

The current-value Hamiltonian for problem (1.30) is

$$\tilde{H}(A, \tilde{c}, \vartheta) = u(\tilde{c}) + \vartheta[\tilde{y}(A) - \tilde{c}] / \left[\tilde{k}'(A) + \varphi(A)\right], \tag{1.31}$$

where ϑ is the costate variable associated with A. Necessary conditions are:

$$\vartheta = u'(\tilde{c})\left[\tilde{k}'(A) + \varphi(A)\right] \tag{1.32}$$

$$\dot{\vartheta} = \left\{\rho - \left[\tilde{y}'(A) - \left(\tilde{k}''(A) + \varphi'(A)\right)\dot{A}\right] / \left[\tilde{k}'(A) + \varphi(A)\right]\right\}\vartheta \tag{1.33}$$

$$\lim_{t\to\infty} \tilde{H}(t) e^{-\rho t} = 0, \tag{1.34}$$

where $\dot{A}$ in (1.33) is given by (1.29).

Since, by (1.20), $F_A\left[\tilde{k}(A), A\right] = F_k\left[\tilde{k}(A), A\right]\varphi(A)$ along the turnpike and, by (1.11), $\tilde{r}(A) = F_k\left[\tilde{k}(A), A\right]$, where $\tilde{r}(A)$ is the capital rental rate on the turnpike, $\tilde{y}'(A) = \tilde{r}(A)\left[\varphi(A) + \tilde{k}'(A)\right]$. Hence, dividing by ϑ, (1.33) can be rewritten as

$$\dot{\vartheta}/\vartheta = \rho - \tilde{r}(A) + \dot{A}\left[\tilde{k}''(A) + \varphi'(A)\right] / \left[\tilde{k}'(A) + \varphi(A)\right]. \tag{1.35}$$

Taking time derivative of (1.32), dividing by ϑ and coupling with (1.35), under Assumption A.3 we get

$$\dot{\tilde{c}}/\tilde{c} = [\tilde{r}(A) - \rho]/\sigma = \left\{f'\left[\tilde{k}(A)/A\right] - \rho\right\}/\sigma, \tag{1.36}$$

where in the second equality (1.11) and (1.10) have been used.

From (1.29) and (1.36) we obtain the following system of ODEs defining the optimal dynamics for $A(t)$ and $\tilde{c}(t)$ along the turnpike under Assumption A.3:

$$\begin{cases} \dot{A} = \left\{ f \left[\tilde{k}(A)/A \right] A - \tilde{c} \right\} / \left[\tilde{k}'(A) + \varphi(A) \right] \\ \dot{\tilde{c}} = \tilde{c} \left\{ f' \left[\tilde{k}(A)/A \right] - \rho \right\} / \sigma, \end{cases} \tag{1.37}$$

Proposition 2 (ii) states that in the long run the ratios $\dot{A}/A$ and $\dot{\tilde{c}}/\tilde{c}$ obtained from (1.37) converge to the balanced growth rate $\gamma = (r_\infty - \rho)/\sigma$.

1.3 Model specif cation and analysis

We now suitably restrict the class of models under investigation.

A. 4 *In addition to Assumption A.3 , the followings hold.*

(i) *Only pairs of ideas will be matched in the recombinant process:* $m = 2$.
(ii) *The probability function* $\pi : \mathbb{R}_+ \to [0,1]$ *of the recombinant process is:*

$$\pi(x) = \beta x / (\beta x + 1), \qquad \beta > 0. \tag{1.38}$$

(iii) *The production function has the Cobb-Douglas form:* $F(k,A) = \theta k^\alpha A^{1-\alpha} = \theta A (k/A)^\alpha$*, with* $\theta > 0$ *and* $0 < \alpha < 1$.

Clearly, $\pi(\cdot)$ in (1.38) satisfies Assumption A.1; parameter β measures the degree of efficiency of the Weitzman matching process, the larger β the higher probability of obtaining a new successful idea out of each (pairwise) matching of seed ideas.

Since, when $m = 2$, $C_2'(A) = (2A-1)/2$, and from (1.38) we get $\pi^{-1}(y) = y/[\beta(1-y)]$, substituting both in (1.7) yields the following explicit form for $\varphi(A)$:

$$\varphi(A) = (2A-1)/[\beta(2A-3)] = (1/\beta)[1 + 2/(2A-3)]. \tag{1.39}$$

As $\pi'(0) = \beta$, Assumption A.4(iii) and (1.39) yields:

$$\tilde{k}(A) = [\alpha/(1-\alpha)]\varphi(A)A = \{\alpha/[\beta(1-\alpha)]\}[1 + 2/(2A-3)]A \tag{1.40}$$

$$\tilde{k}_\infty(A) = \{\alpha/[\beta(1-\alpha)]\}(A+1) \qquad (i.e.,\ \tilde{\eta} = q = \alpha/[\beta(1-\alpha)]) \tag{1.41}$$

$$\hat{k}(A) = (\theta\alpha/\rho)^{1/(1-\alpha)} A \qquad \left(i.e.,\ \hat{\eta} = (\theta\alpha/\rho)^{1/(1-\alpha)}\right), \tag{1.42}$$

and the growth condition (1.26) becomes

$$\rho < \theta\alpha [\beta(1-\alpha)/\alpha]^{1-\alpha}. \tag{1.43}$$

It is seen from (1.40) that the initial condition A_0 must be in the open interval $(3/2, +\infty)$, and the graph of $\tilde{k}(A)$ is a U-shaped curve on it. Since the stock of knowledge A cannot be depleted and the economy is bound to follow the optimal investment in R&D policy $\tilde{J} > 0$ defined in (1.22), along the turnpike A must grow: $\dot{A}(t) > 0$ for all $t \geq 0$. Therefore, a U-shaped $\tilde{k}(A)$ means that capital $\tilde{k}(t)$ decreases [$\dot{\tilde{k}}(t) < 0$] when t is small and increases [$\dot{\tilde{k}}(t) > 0$] for larger t, envisaging that in early times it is optimal to take away some physical capital from the output-producing sector and invest it in R&D, so that the stock of knowledge A can take-off. Moreover, $\dot{A} > 0$ in (1.29) – and thus in (1.37) – has important implications.

Proposition 3. *Under A.4, the* optimal policy *along the turnpike,* $\tilde{c}(A)$*, satisfies*

$$\begin{cases} \tilde{c}(A) > \tilde{y}(A) \ \text{for } 3/2 < A < A^s \\ \tilde{c}(A^s) = \tilde{y}(A^s) \\ \tilde{c}(A) < \tilde{y}(A) \ \text{for } A > A^s, \end{cases} \tag{1.44}$$

where

$$A^s = 1 + (1/2)\left(\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right). \tag{1.45}$$

Moreover, $\tilde{c}'(A) \leq 0$ *in a neighborhood of* A^s*.*

Proof. By differentiating $\tilde{k}(A)$ in (1.40) it is easily seen that the denominator of (1.29), $\tilde{k}'(A) + \varphi(A)$, vanishes on the unique point A^s defined in (1.45), which belongs to the domain $(3/2, +\infty)$ as $A^s > 3/2$ for all $0 < \alpha < 1$; moreover, $\tilde{k}'(A) + \varphi(A) < 0$ for $3/2 < A < A^s$ and $\tilde{k}'(A) + \varphi(A) > 0$ for $A > A^s$. Therefore, $\dot{A}(t) > 0$ for all $t \geq 0$ in (1.29) implies (1.44). Since it can be checked that A^s is also the unique (minimum) stationary point for the optimal output $\tilde{y}(A)$ – *i.e.*, $\tilde{y}'(A^s) = 0$ – and (1.44) states that the graph of the optimal policy $\tilde{c}(A)$ must intersect the graph of the optimal output $\tilde{y}(A)$ from above on $A = A^s$, $\tilde{c}'(A) \leq 0$ must hold in a neighborhood of A^s. □

Proposition 3 will be useful in handling the point corresponding to $(A^s, \tilde{c}(A^s))$ in the 'detrended' system.

1.3.1 State-like and control-like variables

When the economy performs sustained growth in the long run, there are no steady states toward which the system eventually converges. Thus, we transform the state variable A and the control $\tilde{c}$ in a state-like variable, μ, and a control-like variable, χ, respectively, so that $\mu(t)$ and $\chi(t)$ converge to some fixed points μ^* and χ^* in the space (μ, χ) as time elapses. We choose the following transformations:

$$\mu = \tilde{k}(A)/A = [\alpha/(1-\alpha)]\varphi(A) = \{\alpha/[\beta(1-\alpha)]\}[1 + 2/(2A-3)] \tag{1.46}$$

$$\chi = \tilde{c}/A, \tag{1.47}$$

where in (1.46) we used (1.40) and (1.39). Hence, A is related to μ as follows:

$$A = \alpha/[\beta(1-\alpha)\mu - \alpha] + 3/2. \tag{1.48}$$

Given the 'detrended' optimal policy $\chi(\mu)$, the optimal policy of (1.30) is

$$\tilde{c}(A) = \chi[(\alpha/(1-\alpha))\varphi(A)]A. \tag{1.49}$$

Under Assumption A.4(iii), from (1.37) we obtain the following ratios:

$$\dot{A}/A = \left\{\theta \left[\tilde{k}(A)/A\right]^{\alpha} - \tilde{c}/A\right\} / \left[\tilde{k}'(A) + \varphi(A)\right] \tag{1.50}$$

$$\dot{\tilde{c}}/\tilde{c} = \left\{\theta\alpha \left[\tilde{k}(A)/A\right]^{\alpha-1} - \rho\right\}/\sigma. \tag{1.51}$$

The growth rate of μ in (1.46) is $\dot{\mu}/\mu = \tilde{k}'(A)\dot{A}/\tilde{k}(A) - \dot{A}/A$; therefore, $\dot{\mu} = \left[\tilde{k}'(A) - \mu\right]\dot{A}/A$, which, coupled with (1.50) and using (1.47), yields

$$\dot{\mu} = \left[\tilde{k}'(A) - \mu\right](\theta\mu^{\alpha} - \chi) / \left[\tilde{k}'(A) + \varphi(A)\right]. \tag{1.52}$$

As (1.39) equals to $2/(2A-3) = \beta\varphi(A) - 1$ and $\varphi'(A) = -4/\left[\beta(2A-3)^2\right]$, φ' is a function of φ: $\varphi'(A) = -(1/\beta)[2/(2A-3)]^2 = -(1/\beta)[\beta\varphi(A)-1]^2$; moreover, (1.39) may also be rewritten as $A = 1/[\beta\varphi(A)-1] + 3/2$, while (1.46) is equivalent to $\varphi(A) = [(1-\alpha)/\alpha]\mu$. Hence, By differentiating (1.40) and substituting these expressions for $\varphi'(A)$, A and $\varphi(A)$, after a fair amount of algebra $\tilde{k}'(A)$ in (1.52) becomes

$$\tilde{k}'(A) = \{\alpha/[2\beta(1-\alpha)]\}\left\{6\beta[(1-\alpha)/\alpha]\mu - 3\beta^2[(1-\alpha)/\alpha]^2\mu^2 - 1\right\}. \tag{1.53}$$

We can now rewrite (1.52) only in terms of variables μ and χ:

$$\dot{\mu} = \left[1 - \frac{2\beta(1-\alpha)\mu}{2\beta(1-\alpha)(1+2\alpha)\mu - 3\beta^2(1-\alpha)^2\mu^2 - \alpha^2}\right](\theta\mu^{\alpha} - \chi). \tag{1.54}$$

By (1.50), (1.51) and (1.46), the growth rate of χ in (1.47) is $\dot{\chi}/\chi = \left(\theta\alpha\mu^{\alpha-1} - \rho\right)/\sigma - (\theta\mu^{\alpha} - \chi)/\left[\tilde{k}'(A) + \varphi(A)\right]$, which, by replacing $\tilde{k}'(A)$ as in (1.53) and $\varphi(A) = [(1-\alpha)/\alpha]\mu$, yields the following ODE for the control-like variable χ:

$$\dot{\chi} = \left[\frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{2\alpha\beta(1-\alpha)(\theta\mu^{\alpha} - \chi)}{2\beta(1-\alpha)(1+2\alpha)\mu - 3\beta^2(1-\alpha)^2\mu^2 - \alpha^2}\right]\chi. \tag{1.55}$$

Hence, we must study the following system of ODEs:

$$\begin{cases} \dot{\mu} = [1 - 2\beta(1-\alpha)\mu/Q(\mu)](\theta\mu^{\alpha} - \chi) \\ \dot{\chi} = \left[\left(\theta\alpha\mu^{\alpha-1} - \rho\right)/\sigma - 2\alpha\beta(1-\alpha)(\theta\mu^{\alpha} - \chi)/Q(\mu)\right]\chi, \end{cases} \tag{1.56}$$

where

$$Q(\mu) = -3\beta^2(1-\alpha)^2\mu^2 + 2\beta(1-\alpha)(1+2\alpha)\mu - \alpha^2. \tag{1.57}$$

1.3.2 Fixed points and phase diagram

Since $A > 3/2$, from (1.46) one immediately obtains the range $(\mu^*, +\infty)$, with

$$\mu^* = \alpha / [\beta (1-\alpha)], \tag{1.58}$$

for the state-like variable μ, with endpoints corresponding to $A \to +\infty$ and $A \to (3/2)^+$ respectively. In other words, μ^* in (1.58) is the *steady value* for variable μ corresponding to long-run behavior of the economy along the asymptotic turnpike $\tilde{k}_\infty(A)$ [μ^* is the slope of $\tilde{k}_\infty(A)$, as seen in (1.41)]. The feasible set for the detrended variables (μ, χ) therefore is $S = [\mu^*, +\infty) \times \mathbb{R}_{++}$, where we added the boundary μ^* corresponding to the asymptotic dynamics ($A = +\infty$) of the original model.

From the first equation in (1.56), two loci on which $\dot{\mu} = 0$ are found in S: the curve

$$\chi = \theta \mu^\alpha \tag{1.59}$$

and the vertical line $\mu \equiv \mu^*$, with μ^* as in (1.58). Equation (1.59) vanishes the second factor in the RHS of the first equation in (1.56), while μ^* is the largest (and only feasible) solution of $Q(\mu) - 2\beta(1-\alpha)\mu = 0$, with $Q(\mu)$ defined in (1.57), vanishing the first factor in the RHS of the same equation.

From the second equation in (1.56), the unique locus on which $\dot{\chi} = 0$ is given by

$$\chi = \theta \mu^\alpha - Q(\mu) \left(\theta \alpha \mu^{\alpha-1} - \rho\right) / [2\alpha\beta\sigma(1-\alpha)]. \tag{1.60}$$

$Q(\mu)$ turns out to have a unique (admissible) root, call it μ^s, satisfying

$$Q(\mu) = -3\beta^2 (1-\alpha)^2 \mu^2 + 2\beta(1-\alpha)(1+2\alpha)\mu - \alpha^2 = 0, \tag{1.61}$$

with $Q(\mu) > 0$ for $\mu^* \le \mu < \mu^s$ and $Q(\mu) < 0$ for $\mu > \mu^s$. Thus, whether the locus (1.60) lies above or below the locus (1.59) depends on whether $\mu^* \le \mu < \mu^s$ or $\mu > \mu^s$, and on the sign of $\left(\theta \alpha \mu^{\alpha-1} - \rho\right)$. On $\mu = \mu^s$, however, they intersect, and this yields our *first steady state*: (μ^s, χ^s), with $\chi^s = \theta(\mu^s)^\alpha$, which happens to correspond to the point $(A^s, \tilde{c}(A^s))$ discussed in Proposition 3 for the original dynamic (1.37). To see this, recall that, from (1.44), $\tilde{c}(A^s) = \tilde{y}(A^s)$ must hold on the critical point A^s defined in (1.45); by replacing A^s in (1.46) and (1.47), we get,

$$\mu^s = \left(1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right) / [3\beta(1-\alpha)], \qquad \chi^s = \theta(\mu^s)^\alpha, \tag{1.62}$$

where μ^s coincides with the largest (and only admissible) solution of (1.61).

It is immediately seen that $\mu^* < \mu^s$ for all feasible values of parameters α and β, which means that $Q(\mu^*) > 0$ must hold; moreover, using (1.58), the necessary condition for growth (1.43) can be rewritten as $\left[\theta\alpha(\mu^*)^{\alpha-1} - \rho\right] > 0$. Therefore, we conclude that the locus (1.60) intersects the locus $\mu \equiv \mu^*$ strictly below locus (1.59). Since along such vertical line $\dot{\mu} = 0$, we have found the *second steady state* of system (1.56): (μ^*, χ^*), where χ^* is (1.60) evaluated at $\mu = \mu^*$, specifically,

$$\chi^* = \theta \left\{ \alpha / \left[\beta \left(1 - \alpha \right) \right] \right\}^{\alpha} \left(1 - 1/\sigma \right) + \rho / \left[\beta \sigma \left(1 - \alpha \right) \right]. \tag{1.63}$$

Clearly, under Assumption A.4 $\chi^* > 0$. As $\theta \mu^{\alpha}$ in (1.59) is increasing in μ and $\chi^* < \theta \left(\mu^* \right)^{\alpha}$, it follows that $\left(\mu^*, \chi^* \right)$ lies south-west of $\left(\mu^s, \chi^s \right)$. We shall see in short that $\left(\mu^*, \chi^* \right)$ is the saddle-path stable steady state to which system (1.56) converges in the long-run. Hence, χ^* is the asymptotic slope of the optimal consumption $\tilde{c}\left(A \right)$ steadily growing at the constant rate γ in the original model.

As condition (1.43) states that $\rho < \theta \alpha \left(\mu^* \right)^{\alpha - 1}$ must hold and, as $0 < \alpha < 1$, $\theta \alpha \mu^{\alpha - 1}$ is a decreasing function of μ, there must be a unique value $\hat{\mu} > \mu^*$ such that $\left[\theta \alpha \left(\hat{\mu} \right)^{\alpha - 1} - \rho \right] = 0$. It is clear from the last factor in the second term in the RHS of (1.60) that the two loci (1.60) and (1.59) must intersect in $\mu = \hat{\mu}$; hence $\left(\hat{\mu}, \hat{\chi} \right)$, with

$$\hat{\mu} = \left(\theta \alpha / \rho \right)^{\frac{1}{1-\alpha}}, \qquad \hat{\chi} = \theta \left(\theta \alpha / \rho \right)^{\frac{\alpha}{1-\alpha}}, \tag{1.64}$$

is the *third* (and last) *steady state* associated to (1.56). From (1.42), $\hat{\mu}$ in (1.64) corresponds to the value $\hat{A}$ at which $\tilde{k}\left(A \right)$ intersects $\hat{k}\left(A \right)$ in the original model. By equating (1.40) and (1.42) [or by substituting $\hat{\mu}$ as in (1.64) into (1.48)], $\hat{A}$ turns out to be

$$\hat{A} = \alpha / \left[\beta \left(1 - \alpha \right) \left(\theta \alpha / \rho \right)^{\frac{1}{1-\alpha}} - \alpha \right] + 3/2, \tag{1.65}$$

which in turn, if replaced in (1.49) and using $\hat{\chi}$ as in (1.64), yields the value of the optimal policy at the intersection point $\hat{A}$, $\tilde{c}\left(\hat{A} \right) = \hat{\chi} \hat{A}$, of the original model.

The position of the last steady state, $\left(\hat{\mu}, \hat{\chi} \right)$, depends on how large the discount factor ρ is with respect to the parameters α, θ and β. Since $\mu^* < \mu^s$ implies $\theta \alpha \left(\mu^s \right)^{\alpha - 1} < \theta \alpha \left(\mu^* \right)^{\alpha - 1}$, three scenarios may occur, all satisfying condition (1.43).

1. If $\rho < \theta \alpha \left(\mu^s \right)^{\alpha - 1}$, $\mu^s < \hat{\mu}$ and $\left(\hat{\mu}, \hat{\chi} \right)$ lies north-east of $\left(\mu^s, \chi^s \right)$.
2. If $\rho = \theta \alpha \left(\mu^s \right)^{\alpha - 1}$, $\mu^s = \hat{\mu}$ and the two steady states collapse: $\left(\hat{\mu}, \hat{\chi} \right) = \left(\mu^s, \chi^s \right)$.
3. If $\theta \alpha \left(\mu^s \right)^{\alpha - 1} < \rho < \theta \alpha \left(\mu^* \right)^{\alpha - 1}$, $\mu^* < \hat{\mu} < \mu^s$ and $\left(\hat{\mu}, \hat{\chi} \right)$ lies north-east of $\left(\mu^*, \chi^* \right)$ and south-west of $\left(\mu^s, \chi^s \right)$.

We shall focus on the third case, corresponding to a scenario in which A^s lies on the left of $\hat{A}$, on which the turnpike $\tilde{k}\left(A \right)$ intersects the stagnation line $\hat{k}\left(A \right)$.

Proposition 4. *Under A.4 and provided that $\theta \alpha \left(\mu^s \right)^{\alpha - 1} < \rho < \theta \alpha \left(\mu^* \right)^{\alpha - 1}$ holds, the two fixed points $\left(\mu^*, \chi^* \right)$ and $\left(\hat{\mu}, \hat{\chi} \right)$ can be classified as follows.*

1. *$\left(\mu^*, \chi^* \right)$, with coordinates defined in (1.58) and (1.63), is* saddle-path stable, *with the stable arm converging to it from north-east whenever the initial values $\left(\mu\left(t_0 \right), \chi\left(t_0 \right) \right)$ are suitably chosen.*
2. *$\left(\hat{\mu}, \hat{\chi} \right)$, with coordinates defined in (1.64), is an unstable clockwise-rotating spiral.*

Proof. Above the locus (1.59) the term $\left(\theta \mu^{\alpha} - \chi \right)$ in the first equation of (1.56) is negative, while it is positive below. $Q\left(\mu \right)$ in (1.57) is such that $Q\left(\mu \right) > 0$ for $\mu^* < \mu < \mu^s$, while $Q\left(\mu \right) < 0$ for $\mu > \mu^s$; therefore, $\left[1 - 2\beta \left(1 - \alpha \right) \mu / Q\left(\mu \right) \right]$ is negative for $\mu^* < \mu < \mu^s$ and positive for $\mu > \mu^s$. Hence: if $\mu^* < \mu < \mu^s$, $\dot{\mu} > 0$ above locus (1.59) and $\dot{\mu} < 0$ below; while, if $\mu > \mu^s$, $\dot{\mu} < 0$ above locus (1.59) and $\dot{\mu} > 0$ below. Since $\chi > 0$, the sign of $\dot{\chi}$ in the second equation of (1.56) depends on the sign of the term in square brackets in the RHS. From the sign of $Q(\mu)$ we infer that for $\mu^* < \mu < \mu^s$ such term is positive above locus (1.60) and it is negative below, while the converse holds for $\mu > \mu^s$. Thus, when $\mu^* < \mu < \mu^s$, $\dot{\chi} > 0$ above locus (1.60) and $\dot{\chi} < 0$ below; conversely, if $\mu > \mu^s$, $\dot{\chi} < 0$ above locus (1.60) and $\dot{\chi} > 0$ below. The analysis above is sufficient to trace out the phase diagram for the case $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$, *i.e.*, when $\mu^* < \hat{\mu} < \mu^s$, which is reported in Fig. 1.1. Clearly, (μ^*, χ^*) is *saddle-path stable*; it can be guessed that its stable arm is increasing and lying below locus (1.60) on the interval $[\mu^*, \mu^s)$. To check its saddle-path stability, consider the Jacobian of (1.56) evaluated at (μ^*, χ^*):

$$J(\mu^*, \chi^*) = \begin{bmatrix} \frac{\rho-\beta\theta(1-\alpha)(\mu^*)^{\alpha}}{\sigma} & 0 \\ -\frac{(1-\alpha)\left[c_1(\mu^*)^{2\alpha}+c_2(\mu^*)^{\alpha}+\rho^2\right]}{\alpha\sigma^2} & \frac{\rho+\beta\theta(1-\alpha)(\sigma-1)(\mu^*)^{\alpha}}{\sigma} \end{bmatrix}, \qquad (1.66)$$

where $c_1 = (\beta\theta)^2\alpha\sigma(1-\alpha)(\sigma-1)$, $c_2 = \beta\theta\rho(\alpha+\sigma-1)$. By (1.43) the terms on the diagonal have opposite signs; hence, $\det[J(\mu^*, \chi^*)] < 0$ and (μ^*, χ^*) is a saddle.

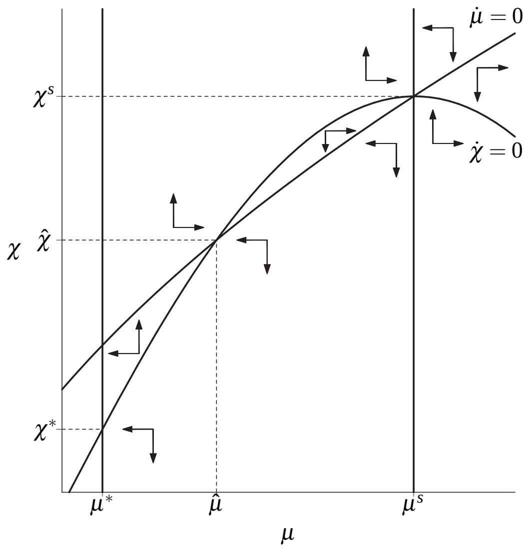

Fig. 1.1 phase diagram of system (1.56) when $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$.

As (μ^*, χ^*) lies strictly below locus (1.59) and the unique intersection point between the loci (1.60) and (1.59) on the interval $[\mu^*, \mu^s)$ is the fixed point $(\hat{\mu}, \hat{\chi})$, it must be the case that (1.60) crosses (1.59) from below on $(\hat{\mu}, \hat{\chi})$. Therefore, $(\hat{\mu}, \hat{\chi})$

is a clockwise rotating *spiral* and the eigenvalues of the Jacobian of (1.56) evaluated at $(\hat{\mu},\hat{\chi})$ are complex. To establish instability we need to show that their real part is positive, or, equivalently, that $\mathrm{tr}\left[J(\hat{\mu},\hat{\chi})\right]>0$. The Jacobian is

$$J(\hat{\mu},\hat{\chi})=\frac{1}{Q(\hat{\mu})}\begin{bmatrix}\left[Q(\hat{\mu})-2\beta(1-\alpha)\hat{\mu}\right]\rho & 2\beta(1-\alpha)\hat{\mu}-Q(\hat{\mu})\\ -\frac{(1-\alpha)\left[\rho Q(\hat{\mu})+2\sigma\alpha^2\beta\theta(\hat{\mu})^\alpha+\rho^2\right]\theta(\hat{\mu})^{\alpha-1}}{\sigma} & 2\alpha\beta(1-\alpha)\theta(\hat{\mu})^\alpha\end{bmatrix},$$

with $Q(\hat{\mu})>0$, as $\mu^*<\hat{\mu}<\mu^s$. Since $\left[\alpha\theta(\hat{\mu})^{\alpha-1}-\rho\right]=0$ on $(\hat{\mu},\hat{\chi})$, it is immediately seen that $\mathrm{tr}\left[J(\hat{\mu},\hat{\chi})\right]=\rho Q(\hat{\mu})>0$, and the proof is complete. □

Remark 1. The critical point (μ^s,χ^s), with coordinates defined in (1.62), cannot be classified analytically, as the Jacobian matrix of (1.56) evaluated at (μ^s,χ^s) has some elements diverging either to $-\infty$ or to $+\infty$ as (μ,χ) approaches (μ^s,χ^s), the sign of infinity depending on the direction along which $(\mu,\chi)\to(\mu^s,\chi^s)$.

We have seen in Section 1.2.1 that $\tilde{k}(A)>\tilde{k}_\infty(A)$ for all A (and thus for all t); this is consistent with $\mu(t)>\mu^*$ for all t. Hence, the stable trajectory must approach (μ^*,χ^*) from the right. We denote by $\chi(\mu)$ such trajectory, which is the *optimal policy expressed in terms of state-like and control-like variables*. Its slope on (μ^*,χ^*) is the slope of the eigenvector associated to the negative eigenvalue of (1.66) (see [2], p. 596), that is,

$$\chi'(\mu^*)=\frac{\beta\theta\alpha\sigma(1-\alpha)(\sigma-1)(\mu^*)^{2\alpha}+\rho(\alpha+\sigma-1)(\mu^*)^\alpha+\left[\rho^2/(\beta\theta)\right]}{\alpha\sigma^2(\mu^*)^\alpha},\tag{1.67}$$

which is clearly positive. Hence, $\chi(\mu)$ approaches (μ^*,χ^*) from north-east in a (right) neighborhood of μ^*; consequently, along the turnpike both ratios $\tilde{k}(A)/A$ and $\tilde{c}/A$ must decline in time when they are approaching the asymptotic turnpike.

Under the assumption that $\theta\alpha(\mu^s)^{\alpha-1}<\rho<\theta\alpha(\mu^*)^{\alpha-1}$, $\mu^*<\hat{\mu}<\mu^s$; by translating $\hat{\mu}$ into $\hat{A}$ through (1.65), it follows that the intersection point between $\tilde{k}(A)$ and $\hat{k}(A)$ lies on the right of the singular point A^s defined in (1.45). Therefore, by condition (1.44) of Proposition 3, $\tilde{c}(\hat{A})<\tilde{y}(\hat{A})$, which is equivalent to $\chi(\hat{\mu})<\theta(\hat{\mu})^\alpha=\hat{\chi}$. Hence, the optimal trajectory $\chi(\mu)$ keeps well below the (unstable) steady state $(\hat{\mu},\hat{\chi})$, which thus happens to be harmless for our analysis, at least for the case[7] $\theta\alpha(\mu^s)^{\alpha-1}<\rho<\theta\alpha(\mu^*)^{\alpha-1}$.

Conversely, the steady state (μ^s,χ^s) is the most problematic as on one hand its stability cannot be checked analytically, while on the other hand the optimal policy $\chi(\mu)$ must actually cross it.[8] However, since in our scenario $\left(A^s,\tilde{k}(A^s)\right)\neq$

[7] A similar situation occurs when $\rho<\theta\alpha(\mu^s)^{\alpha-1}$, in which case $\tilde{c}(\hat{A})>\tilde{y}(\hat{A})$, and thus $\chi(\hat{\mu})>\theta(\hat{\mu})^\alpha=\hat{\chi}$. Only when $\rho=\theta\alpha(\mu^s)^{\alpha-1}$, and the two points $\hat{A}$ and A^s collapse, the optimal trajectory necessarily must cross the (unstable) steady state $(\hat{\mu},\hat{\chi})$; in this case, however, the point $(\hat{\mu},\hat{\chi})=(\mu^s,\chi^s)$ inherits the peculiar singularity properties of (μ^s,χ^s), thus becoming a "supersingular" point to be handled with circumspection.

[8] Condition (1.44) of Proposition 3 states that $\tilde{c}(A^s)=\tilde{y}(A^s)$, which implies $\chi(\mu^s)=\theta(\mu^s)^\alpha=\chi^s$.

$\left(\hat{A}, \tilde{k}\left(\hat{A}\right)\right)$, the system in the original model is not on the stagnation line when it hits $\left(A^s, \tilde{k}\left(A^s\right)\right)$ and thus cannot stop over it; accordingly, the detrended system cannot stop over (μ^s, χ^s). All these 'singularities' attached to (μ^s, χ^s) led us to opt for a qualitative approach based on information gathered on a neighborhood of (μ^s, χ^s). Condition (1.44) of Proposition 3 for $A \neq A^s$ translates into

$$\begin{cases} \chi(\mu) < \theta(\mu)^\alpha \ \text{ for } \mu^* < \mu < \mu^s \\ \chi(\mu) > \theta(\mu)^\alpha \ \text{ for } \mu > \mu^s, \end{cases} \tag{1.68}$$

which, in turn, means that the optimal policy must lie below the locus (1.59) when $\mu^* < \mu < \mu^s$ and above it when $\mu > \mu^s$. A closer inspection of a neighborhood of (μ^s, χ^s) in Fig. 1.1 shows that it is attractive on the area above the locus (1.59) (above $\chi = \theta\mu^\alpha$) and on the right of the vertical line $\mu \equiv \mu^s$, while it is repulsive below $\chi = \theta\mu^\alpha$ and on the left of $\mu \equiv \mu^s$. As $\theta\mu^\alpha$ is increasing in μ, this suggests that the optimal policy $\chi(\mu)$ must be increasing on (μ^s, χ^s) and the optimal trajectory $(\mu(t), \chi(t))$ must cross (μ^s, χ^s) from north-east to south-west as time elapses.

1.3.3 Time elimination, policy function and initial conditions

In order to study the policy function $\chi(\mu)$ – which is the conjugate of $\tilde{c}(A)$ in the original model – we apply the technique developed by Mulligan and Sala-i-Martin [4] and tackle the unique ODE given by the ratio between the equations in (1.56):

$$\chi'(\mu) = \frac{\left[\left(\alpha\theta\mu^{\alpha-1} - \rho\right)/\sigma\right] Q(\mu) - 2\alpha\beta(1-\alpha)\left[\theta\mu^\alpha - \chi(\mu)\right]}{\left[Q(\mu) - 2\beta(1-\alpha)\mu\right]\left[\theta\mu^\alpha - \chi(\mu)\right]}\chi(\mu), \tag{1.69}$$

where $Q(\mu)$ is defined in (1.57).

The natural choice for the initial condition of (1.69) is the saddle-path stable steady state (μ^*, χ^*), while the value of $\chi'(\mu^*)$ in (1.67) will be used to select the stable arm outside (μ^*, χ^*). The previous analysis, however, has endowed us with another reference point, the singular point (μ^s, χ^s), which may be exploited as initial condition as well. Although the Jacobian of (1.56) evaluated on (μ^s, χ^s) is intractable, we are able to compute the slope of the policy at $\mu = \mu^s$ by applying l'Hôpital rule to the RHS of (1.69) evaluated at $\mu = \mu^s$. Since $Q(\mu^s) = 0$ and $\left[\theta(\mu^s)^\alpha - \chi(\mu^s)\right] = 0$, we obtain the following quadratic equation in $\chi'(\mu^s)$:

$$\begin{aligned} &2\beta\sigma(1-\alpha)\mu^s\left[\chi'(\mu^s)\right]^2 - 4\alpha\beta\sigma(1-\alpha)\chi^s\chi'(\mu^s) \\ &-\left\{\left[\alpha\theta(\mu^s)^{\alpha-1} - \rho\right]Q'(\mu^s) - 2\alpha^2\beta\sigma\theta(1-\alpha)(\mu^s)^{\alpha-1}\right\}\chi^s = 0. \end{aligned} \tag{1.70}$$

Substituting μ^s and χ^s as in (1.62) and $Q'(\mu^s) = -2\beta(1-\alpha)\sqrt{1+4\alpha+\alpha^2}$ into (1.70) two positive real solutions appear, the largest being larger than the slope of the locus (1.59) at $\mu = \mu^s$. However, this happens only when $\theta\alpha(\mu^s)^{\alpha-1} < \rho <$

$\theta\alpha(\mu^*)^{\alpha-1}$; this is why we chose to confine our numerical approach to such scenario.

1.4 Numeric simulation of the optimal policy

By applying the *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method (see, *e.g.*, [7]) implemented through Maple 12.02 to ODE (1.69), we were able to find satisfactory result only for single sets of parameters values. We chose values for parameters α, ρ, σ and θ which are often assumed in the macroeconomic literature (see, *e.g.*, [5]): $\alpha = 0.5$, $\rho = 0.04$ and $\theta = \sigma = 1$. Note that $\sigma = 1$ implies logarithmic instantaneous utility. For such parameters' values, β must satisfy the necessary growth condition (1.43), which turns out to be $\beta > 0.0064$.

We plan to exploit the steady state (μ^*, χ^*) and the singular point (μ^s, χ^s) [see (1.58), (1.63) and (1.62)] as initial conditions in order to trace out two different curves as numeric solutions of (1.69) through Maple 12.02. Both curves provide an approximation for the same (unique) trajectory representing the optimal policy[9] $\chi(\mu)$ for $\mu \geq \mu^*$. For the chosen parameters' values, such two curves happen to be sufficiently close to each other for a reasonably large range of μ values only for a unique admissible value of the technological parameter: $\beta = 0.0124$. Since, for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$, each curve provides a reliable approximation of $\chi(\mu)$ around its own initial condition and both match on most of the open interval (μ^*, μ^s), our idea is to approximate the whole $\chi(\mu)$ by using the first curve for μ close to μ^* and the second one for μ close to (and larger than) μ^s, while "joining"them together on some 'intermediate' value on which they almost match.

For our parameters' values, (1.62) yields $\mu^s = 204.4503$, which implies $\rho = 0.04 > 0.035 = \theta\alpha(\mu^s)^{\alpha-1}$, corresponding to the third scenario of Section 1.3.2, in which $A^s < \hat{A}$. Fig. 1.2 portraits the turnpike $\tilde{k}(A)$, the asymptotic turnpike $\tilde{k}_\infty(A)$ and the stagnation line $\hat{k}(A)$ as in (1.40), (1.41) and (1.42); as expected, $A^s = 2.1514 < 2.567 = \hat{A}$.

In view of Proposition 2, the long-run capital rental rate is $r_\infty = f'(\tilde{\eta}) = 0.0557$, the long-run common constant growth rate is $\gamma = 0.0157$, while the long-run income shares invested in knowledge and capital are the same: $s_\infty = s_\infty^k = 0.1408$.

The steady states are $(\mu^*, \chi^*) = (80.6452, 6.4516)$, $(\hat{\mu}, \hat{\chi}) = (156.25, 12.5)$ and $(\mu^s, \chi^s) = (204.4503, 14.2986)$. Fig. 1.3 shows the loci $\dot{\mu} = 0$ and $\dot{\chi} = 0$ in slim black, while the thick curves are the result of the numeric solution of (1.69) representing the policy $\chi(\mu)$: the black one uses (μ^*, χ^*) as initial condition and (1.67),

[9] Such trajectory is the unique true solution of (1.69) corresponding to the stable arm of the saddle point (μ^*, χ^*) and, at the same time, crossing the singular point (μ^s, χ^s). Other solutions of (1.69) may cross at most one of the two points, like, for example, the trajectory corresponding to the unstable arm of (μ^*, χ^*), or other unknown trajectories possibly crossing the singular point (μ^s, χ^s). We owe such clarification to an anonymous referee.

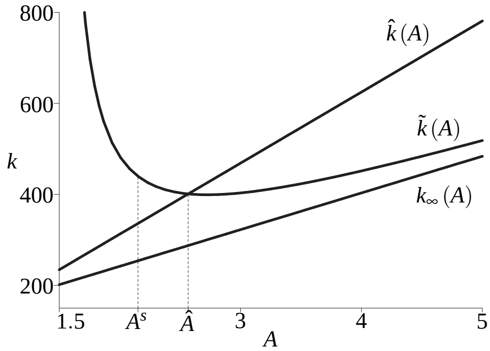

Fig. 1.2 the turnpike $\tilde{k}(A)$, the asymptotic turnpike $k_\infty(A)$ and the stagnation line $\hat{k}(A)$ for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

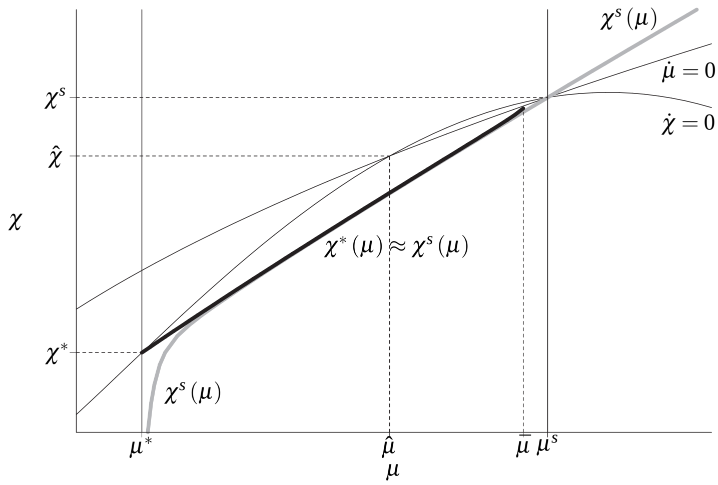

Fig. 1.3 loci $\dot{\mu} = 0$ and $\dot{\chi} = 0$ (slim black curves) and approximate trajectories $\chi^*(\mu)$ and $\chi^s(\mu)$ (black and grey thick curves respectively) for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

$\chi'(\mu^*) = 0.0687$, for the selection of the stable arm; the grey one has (μ^s, χ^s) as initial condition and slope given by the largest solution of (1.70) on $\mu = \mu^s$, $\chi'(\mu^s) = 0.0602$. The two approximate trajectories will be labeled $\chi^*(\mu)$ and $\chi^s(\mu)$ respectively.

Even for our choice of parameters' values the Maple 12.02 algorithm is capable of computing the trajectory $\chi^*(\mu)$ only up to a point: it actually stops at $\bar{\mu} \simeq 197 < 204.4503 = \mu^s$, falling short of the singular point, (μ^s, χ^s). On the other hand, as it is clear from Fig. 1.3, trajectory $\chi^s(\mu)$ heavily underestimates the policy for values of μ approaching μ^* (*i.e.*, far away from μ^s). The two curves, however, seem sufficiently close to each other on most of the interval (μ^*, μ^s), thus suggesting that the numeric approach actually works satisfactorily for these values of parameters.

In order to estimate the whole policy $\chi(\mu)$, for all $\mu \geq \mu^*$, we shall use $\chi^*(\mu)$ for μ values close to μ^*, and $\chi^s(\mu)$ for μ values closer to μ^s. Since from Fig. 1.3 it is clear that $\chi^*(\hat{\mu}) \approx \chi^s(\hat{\mu})$, we shall define the approximated policy as a piecewise function by joining the two trajectories at the point $\hat{\mu} = 156.25 \in (\mu^*, \mu^s)$:

$$\chi(\mu) = \begin{cases} \chi^*(\mu) \text{ for } \mu^* \leq \mu \leq \hat{\mu} \\ \chi^s(\mu) \text{ for } \mu \geq \hat{\mu}. \end{cases} \tag{1.71}$$

Surprisingly, already for $\beta = 0.0123$, or $\beta = 0.0125$, while keeping fixed all other parameters, the curves $\chi^*(\mu)$ and $\chi^s(\mu)$ in Fig. 1.3 split apart, while the range of μ for which the numeric algorithm is able to perform starts to shrink dramatically; this is why we take as reliable only the solution obtained for $\beta = 0.0124$.

Remark 2. We tried different values for the parameters α, ρ, σ and θ; for all feasible set of values for such parameters we found a scenario similar to that described above, at least under condition $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$: only for one specific value of parameter β, related to the choice of α, ρ, σ and θ, the two numerical solutions – $\chi^*(\mu)$ with initial condition (μ^*, χ^*) and $\chi^s(\mu)$ with initial condition (μ^s, χ^s) – turned out to be sufficiently close to each other on a large part of the interval (μ^*, μ^s). We conclude, thus, that the numeric approach works satisfactory only on a manifold of dimension one in the parameters' space.

1.5 Discussion

To get the approximated time-path trajectory of μ we substitute the optimal policy $\chi(\mu)$ as in (1.71) into the first equation of (1.56), yielding the following ODE in t,

$$\dot{\mu}(t) = \{1 - 2\beta(1-\alpha)\mu(t)/Q[\mu(t)]\}\left\{\theta[\mu(t)]^\alpha - \chi[\mu(t)]\right\}, \tag{1.72}$$

with $Q(\cdot)$ defined in (1.57), which can be numerically solved. Since $\chi(\mu)$ in (1.71) is defined piecewise, we need to choose an instant $\hat{t} > 0$ on which the trajectory has the (common) value $\hat{\mu} = 156.25$; then, the initial value $\mu_0 = \mu(0)$ will be given by evaluating in $t = 0$ the solution of (1.72) with $\chi(\cdot) = \chi^s(\cdot)$ and $\mu(\hat{t}) = \hat{\mu}$ as initial condition. For different $\hat{t}$ we can consider any initial value $\mu_0 = \mu(0) > \hat{\mu}$.

In our example we assume $\hat{t} = 36$, corresponding to $\mu_0 = 251.977$ in $t = 0$. According to (1.71), we define $\mu(t)$ as the solution of (1.72) with $\chi(\cdot) = \chi^s(\cdot)$ for $0 \leq t \leq \hat{t}$ [corresponding to $\hat{\mu} \leq \mu(t) \leq \mu_0$], and as the solution of (1.72) with $\chi(\cdot) = \chi^*(\cdot)$ for $t \geq \hat{t}$ [corresponding to $\mu^* \leq \mu(t) \leq \hat{\mu}$]. Fig. 4(a) plots $\mu(t)$ for $0 \leq t \leq 400$ by distinguishing the part (in grey) obtained through $\chi^s(\cdot)$ for $0 \leq t \leq \hat{t} = 36$ from the part eventually converging to μ^* (in black) obtained by means of $\chi^*(\cdot)$ for $t \geq 36$.

The time-path trajectory $\chi(t)$ is then computed by letting $\chi(t) = \chi[\mu(t)]$ in (1.71), with $\mu(t)$ just obtained, for all $0 \leq t \leq 400$. Fig. 4(b) reports the result, again by emphasizing in grey the part for $0 \leq t \leq \hat{t} = 36$. In $t = 0$, $\chi(0) = \chi_0 =$

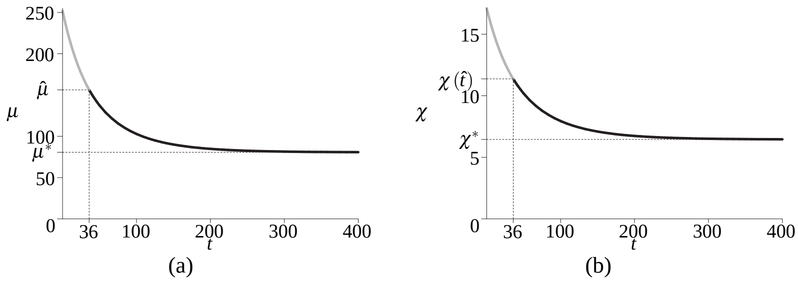

Fig. 1.4 (a) $\mu(t)$ and (b) $\chi(t)$ for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

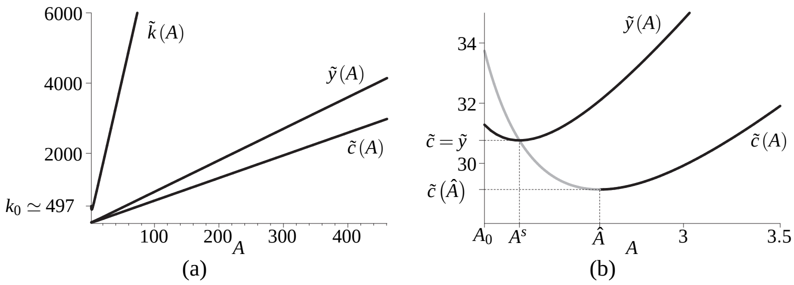

Fig. 1.5 (a) $\tilde{c}$, $\tilde{y}$ and $\tilde{k}$ as functions of A along the turnpike; (b) $\tilde{c}$ and $\tilde{y}$ close to $A_0 = 1.9707$.

17.1194, corresponding to $\mu_0 = 251.977$, while in $t = \hat{t} = 36$, $\chi(36) = 11.3688$; clearly, $\chi(\hat{t}) = 11.3688 < 12.5 = \hat{\chi}$, as expected.

With $\mu(t)$ and $\chi(t)$ at hand, we can compute the optimal consumption $\tilde{c}(A)$ and output $\tilde{y}(A)$ along the turnpike $\tilde{k}(A)$ in the original model as functions of A. By (1.48) we find the initial stock of knowledge $A_0 = 1.9707$ in $t = 0$, corresponding to μ_0. To A_0 corresponds an initial capital $k_0 = \tilde{k}(A_0) = 496.57$ in $t = 0$. $\tilde{c}(A)$ is then obtained through (1.49), with $\chi(\cdot)$ defined in (1.71): $\chi^s(\cdot)$ for $A_0 \leq A \leq \hat{A}$ (corresponding to $\hat{\mu} \leq \mu \leq \mu_0$), and $\chi^*(\cdot)$ for $A \geq \hat{A}$ (corresponding to $\mu^* \leq \mu \leq \hat{\mu}$). Fig. 5(a) reports $\tilde{k}(A)$, $\tilde{y}(A)$ and $\tilde{c}(A)$ just evaluated on a scale larger than in Fig. 1.2. Fig. 5(b) magnifies the intersection point between $\tilde{y}(A)$ and the $\tilde{c}(A)$ occurring on A^s, close to A_0 and to the left of $\hat{A}$. Since on $[A_0, \hat{A}]$ $\tilde{c}(A)$ is being built through $\chi^s(\cdot)$ in (1.72), this portion of its graph is emphasized in grey, as we did in previous figures.

The time-path trajectory of the stock of knowledge $A(t)$ is obtained by evaluating (1.48) at $\mu(t)$ for all t, while time-path trajectories $\tilde{k}(t)$ and $\tilde{y}(t)$ follow by construction. The consumption time-path trajectory $\tilde{c}(t)$ is computed by evaluating (1.49) at $A(t)$ for all t. These trajectories are drawn in Fig. 6(a), while Fig. 6(b) reports the time path-trajectory of the capital rental rate r; once again, their dependence on the $\chi^s(\cdot)$ arm of the policy in (1.71) for $0 \leq t \leq \hat{t} = 36$ is emphasized in grey.

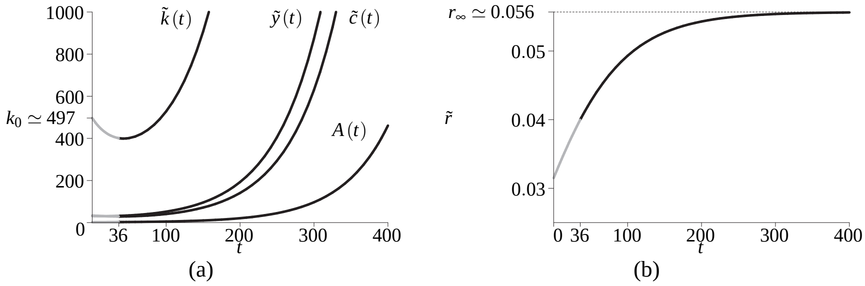

Fig. 1.6 (a) time-path trajectories of A, $\tilde{k}$, $\tilde{y}$ and $\tilde{c}$; (b) time-path trajectory for $\tilde{r}$.

From Figures 1.2, 5(a) and 6(a), emerges that the dynamics along the turnpike are characterized by a much larger amount of physical capital than any other variable. A large initial capital, $k_0 = 496.57$, compared to very few initial ideas, $A_0 = 1.9707$, is required to let the recombinant process to take-off. Such amount, even if for a short time, is partially being 'eaten up' by both consumption [$\tilde{c}(A) > \tilde{y}(A)$ for $A_0 \leq A \leq A^s$] and investment in R&D, thus envisaging an initial period of decline for capital $\tilde{k}$. Fig. 5(b) shows that output and consumption decrease for a short time as well; specifically, output declines until $\tilde{c}(A)$ hits $\tilde{y}(A)$ at $A = A^s$, and consumption decreases until the turnpike crosses the stagnation line on $A = \hat{A}$ (see Fig. 1.2) at $\hat{t} = 36$. For larger t all variables start to increase, with a much higher $\tilde{k}$ with respect to all others, especially to A. For example, when $A \simeq 73$, $\tilde{k} \simeq 6000$ in Fig. 5(a).

In our example, thus, sustained growth requires a large exploitation of physical resources, at least relatively to knowledge, even under a 'balanced' ($\alpha = 0.5$) Cobb-Douglas technology. Such 'asymmetry' is explained by the ratio between the (low) price of capital – numéraire – and the relatively high unit cost of knowledge production: for $\beta = 0.0124$ $\varphi(A)$ turns out to be significantly larger than 1, as $\varphi(A) > \lim_{A\to\infty} \varphi(A) = 1/\pi'(0) = 1/\beta = 80.6452$ [see also next Figures 8(a) and 8(b)].

Fig. 6(a) exhibits a system which actually takes some time to take-off. Provided that our economy starts with very few ideas ($A_0 = 1.9707$) and sufficiently large capital ($k_0 = 496.57$), the initial transient dynamics happen to last quite long; especially $A(t)$ takes no less than 200 periods before becoming significant [note that in the meantime $\tilde{k}(t)$ already started to "blow up"]. For example, it takes around 282 periods to reach the stock $A \simeq 73$, corresponding to $\tilde{k} \simeq 6000$. Similarly, the constant ratio $\tilde{c}(A)/\tilde{y}(A)$ visible in Fig. 5(a) – due to almost linearity of $\tilde{c}(A)$ and $\tilde{y}(A)$ and which can be checked to be close to the asymptotic ratio 0.07184, corresponding to the saving rate $s_\infty + s_\infty^k = 0.2816$ – is actually not reached before at least 300 periods. To conclude, Figures 1.2 and 5(a) should be read carefully when one introduces time: of course the economy grows along the turnpike $\tilde{k}(A)$, but at a very slow pace in early times, while keeps accelerating until it "explodes" along $\tilde{k}_\infty(A)$.

Fig. 6(b) adds more information to the analysis: even if $\tilde{k}$ is always (much) larger than A, its productivity keeps rising in time, as confirmed by its increasing rental rate, $\tilde{r}$, until it reaches its asymptotic value, $r_\infty = 0.0557$.

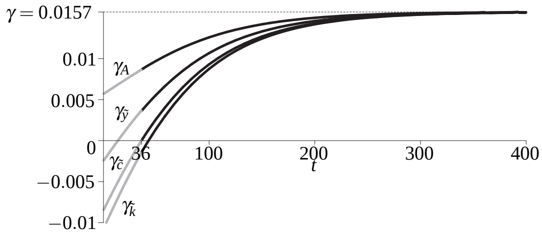

Fig. 1.7 growth rates γ_A, $\gamma_{\tilde{k}}$, $\gamma_{\tilde{y}}$ and $\gamma_{\tilde{c}}$, of A, $\tilde{k}$, $\tilde{y}$ and $\tilde{c}$ as functions of time.

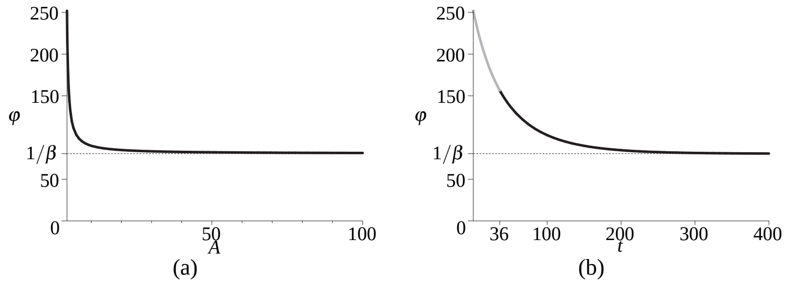

Fig. 1.8 (a) unit cost of knowledge production, φ, as a function of A; (b) its time-path trajectory.

Fig. 1.7 confirms everything in terms of rates of growth. By construction, $A(t)$ is the only variable with rate of growth $\gamma_A = \dot{A}/A$ always positive, while $\tilde{k}(t)$, $\tilde{y}(t)$ and $\tilde{c}(t)$, all experience negative growth at early times, where $\gamma_{\tilde{k}} = \dot{\tilde{k}}/\tilde{k}$, $\gamma_{\tilde{y}} = \dot{\tilde{y}}/\tilde{y}$ and $\gamma_{\tilde{c}} = \dot{\tilde{c}}/\tilde{c}$ are negative. Interestingly, it can be observed that $\tilde{c}(t)$ reaches its absolute minimum in $\hat{t} = 36$ [corresponding to $\tilde{c}(\hat{A})$, as confirmed by Fig. 5(b)].

The striking feature of recombinant growth is evident in Fig. 1.7: all growth rates are increasing in time while approaching their asymptotic common value $\gamma = 0.0157$. This reflects the original Weitzman's [9] hypothesis: in early times ideas are scarce and thus have the potential of growing at increasing rates, in the long-run limited physical resources to be invested in R&D – with respect to the exploding number of ideas – cools down growth to the more realistic case of constant rates.

Fig. 8(a) shows the graph of the unit cost of knowledge production $\varphi(A)$ as in (1.39), which is sharply decreasing in A for A close to A_0. Such jump, however, is to be diluted when time is considered, as shown in Fig. 8(b) where φ is plotted as a function of t, since A starts to grow significantly only after some time [see Fig. 6(a)].

Investment in R&D $\tilde{J}$ and investment in capital $\dot{\tilde{k}}$ as functions of A are plotted in Fig. 1.9; $\tilde{J}$ is computed by using $\tilde{c}(A)$ and $\tilde{y}(A)$, $\varphi(A)$ and $\tilde{k}'(A)$ – obtained by differentiating (1.40) with respect to A – in (1.22). From Figures 9(a) and 9(c), where a large range of A values is considered, we learn that both look linear in A and have the same magnitude, implying that they become the same well before reaching their

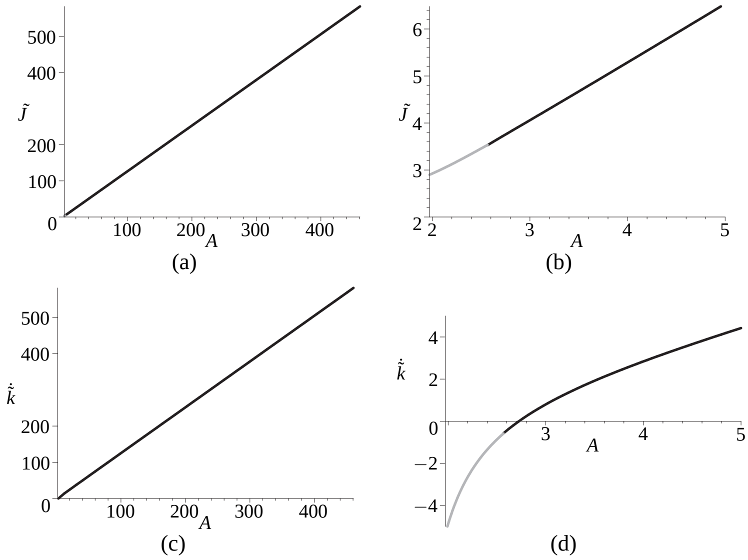

Fig. 1.9 (a) $\tilde{J}(A)$, (b) its detail for A close to A_0; (c) $\dot{\tilde{k}}(A)$, (d) its detail for A close to A_0.

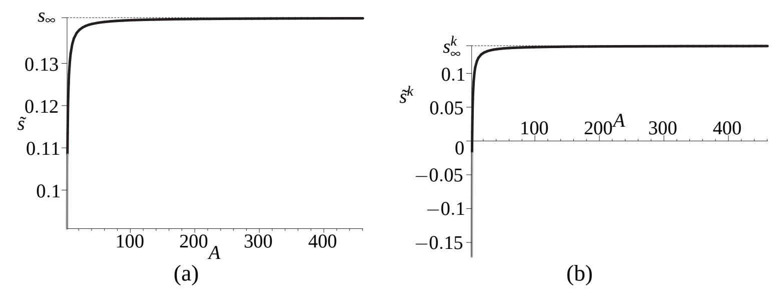

Fig. 1.10 (a) $\tilde{s} = \tilde{J}/\tilde{y}$ as a function of A; (b) $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$ as a function of A.

asymptotic (common) constant share $s_\infty = J_\infty/y_\infty = s^k_\infty = \dot{k}_\infty/y_\infty = 0.1408$. Only for A close to A_0 their behavior differ, as magnified by Figures 9(b) and 9(d).

It is interesting to compare the magnitude of $\tilde{J}(A)$ and $\dot{\tilde{k}}(A)$ in Figures 9(a) and 9(c) with that of $\tilde{c}(A)$ and $\tilde{y}(A)$ in Figures 5(a) and 5(b): for all A – also close to A_0 – the optimal dynamics postulate relatively small investment in both factors with respect to consumption and output. Figures 10(a) and 10(b) confirm this in terms of investment shares, $\tilde{s} = \tilde{J}/\tilde{y}$ and $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$. Both are increasing in A and reach their asymptotic value $s_\infty = s^k_\infty = 0.1408$ quite rapidly, although $\tilde{s}^k < 0$ for small A. Such quick jumps to their asymptotic value is consistent with the linearity exhibited by $\tilde{J}(A)$ and $\dot{\tilde{k}}(A)$ in Figures 9(a) and 9(c).

Also the dynamics of $\tilde{J}$ (or $\tilde{s}$) confirm Weitzman's [9] evolution of knowledge: when A – and thus seed ideas H – is scarce function (1.5) exhibits low productivity; accordingly, only few resources are employed in R&D, while they increase as A – and H – become more abundant. In the long-run are the physical resources that become scarce with respect to knowledge – they grow slower than what (potentially) could do knowledge – and bound the rate of investment $\tilde{s}$ to its asymptotic value s_∞.

The graphs of new (successful) knowledge production, $\dot{A}$, and seed ideas, $\tilde{H}$, as functions of A are reported in Fig. 1.11; the former is given by (1.29), while the latter is computed from (1.4) using $\dot{A}$ and $C_2^{'}(A) = A - 1/2$. Strict convexity of $\tilde{H}$ in Figures 11(c) and 11(d), associated to linearity of $\dot{A}$ (also for A close to A_0) in Figures 11(a) and 11(b), is consistent to formula (1.4), which implies quadratic growth for $\tilde{H}$ when $\dot{A}$ grows linearly. It is worth noting the difference in magnitudes between seed ideas $\tilde{H}$ and the actual successful ideas $\dot{A}$ produced out of $\tilde{H}$: such low returns are justified by the choice of a very small value for the efficiency parameter, $\beta = 0.0124$, in (1.38), requiring abundant seed ideas to guarantee sustained growth of knowledge.

To conclude, Fig. 1.12 shows time-path trajectories of $\tilde{J}$, $\dot{\tilde{k}}$, $\tilde{s}$, $\tilde{s}^k$, $\dot{A}$ and $\tilde{H}$. Due to slow growth of $A(t)$ in early times, linearity of investments $\tilde{J}$ and $\dot{\tilde{k}}$, and of new knowledge $\dot{A}$, evident in Figures 9(a), 9(c) and 11(a), correspond to convex time-path trajectories, as shown in Figures 12(a), 12(b) and 12(e). For the same reason, convexity of $\tilde{H}$ in Fig. 11(c) becomes more accentuated in Fig. 12(f); similarly, the sudden jumps to their asymptotic value of $\tilde{s}$ and $\tilde{s}^k$ in Figures 10(a) and 10(b) is being smoothed in Figures 12(c) and 12(d). Specifically, both need at least 200 periods before approaching their long-run (common) constant value s_∞.

1.6 Conclusions

The exercise performed in this paper is a very preliminary attempt to tackle the transition dynamics in the recombinant growth model introduced by Tsur and Zemel [8]. For CIES instantaneous utility and Cobb-Douglas production in the output sector, we chose a suitable function for the Weitzman's [9] probability of obtaining a successful idea from pairwise matchings of seed ideas, so that the original optimal dynamics along the turnpike, which is diverging in the long-run, can be 'detrended' to an equivalent system converging to a steady state. In the space of the detrended variables we exploit the asymptotic steady state plus a singular point, across which the optimal policy must get through at some early instant, in order to numerically compute two trajectories which, for a specific choice for the parameters' values, happen to be sufficiently close to each other on a large range between such two points. By joining together these trajectories at an intermediate point, we build an approximation of the optimal policy which must be reasonably close to the true policy on all variables' domain. By converting such trajectory into the original state variable (stock of knowledge) and control variable (consumption) trajectories, we obtain a good approximation of the optimal consumption, which in turn, again by

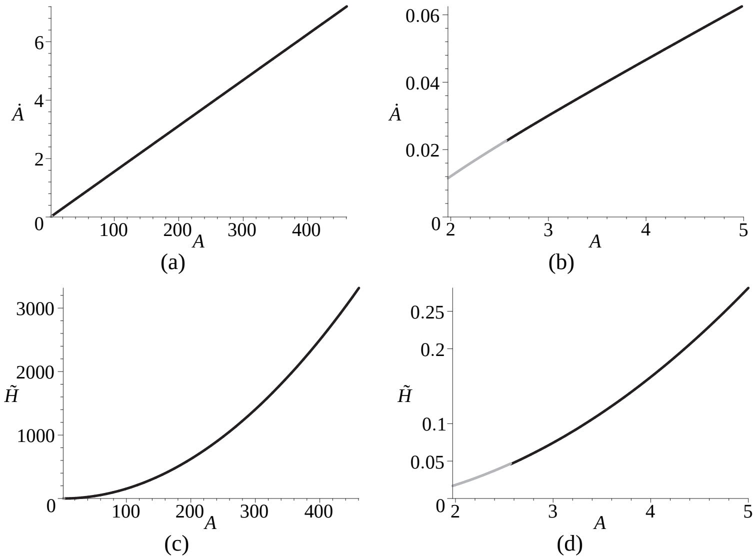

Fig. 1.11 (a) and (b) $\dot{A}$ as a function of A; (c) and (d) $\tilde{H}$ as a function of A.

solving numerically an ODE, yields the transition optimal time-path trajectories of the stock of knowledge, physical capital, output and consumption – as well as their transition growth rates – along the turnpike.

We believe that our main technical contribution is the appropriate form chosen for the Weitzman's probability function defined in Assumption A.4(ii), which allows for 'detrending' the original system (1.37) into the equivalent system (1.56).

If, on one hand the optimal policy obtained in section 1.4, and used to build time-path trajectories in Section 1.5, may clearly be of interest per se, on the other hand it is insufficient for studying how the system's transitional behavior is being affected by changes in the technological parameter β of the probability function π of Assumption A.4(ii), while keeping fixed all other parameters' values. In order to further investigate this topic one needs either to improve the numerical computation of system (1.56) so that the matching of the two aforementioned trajectories in the detrended space is maintained at least on a nontrivial interval of values for parameter β, or trying a completely different approach on either system (1.37) or system (1.56) by means of analytical tools in order to explicitly find the true form of the optimal trajectories. One may tackle the latter by looking for some special function that may prove useful in solving one of (1.37) or (1.56); see, *e.g.*, [3] for a recent application of the Gaussian hypergeometric functions to the Lucas-Uzawa model. Both approaches will be investigated in future research projects.

Acknowledgements We thank Giovanni Ramello for bringing our attention to recombinant growth models, Raouf Boucekkine for precious (and critical) technical suggestions and Mauro

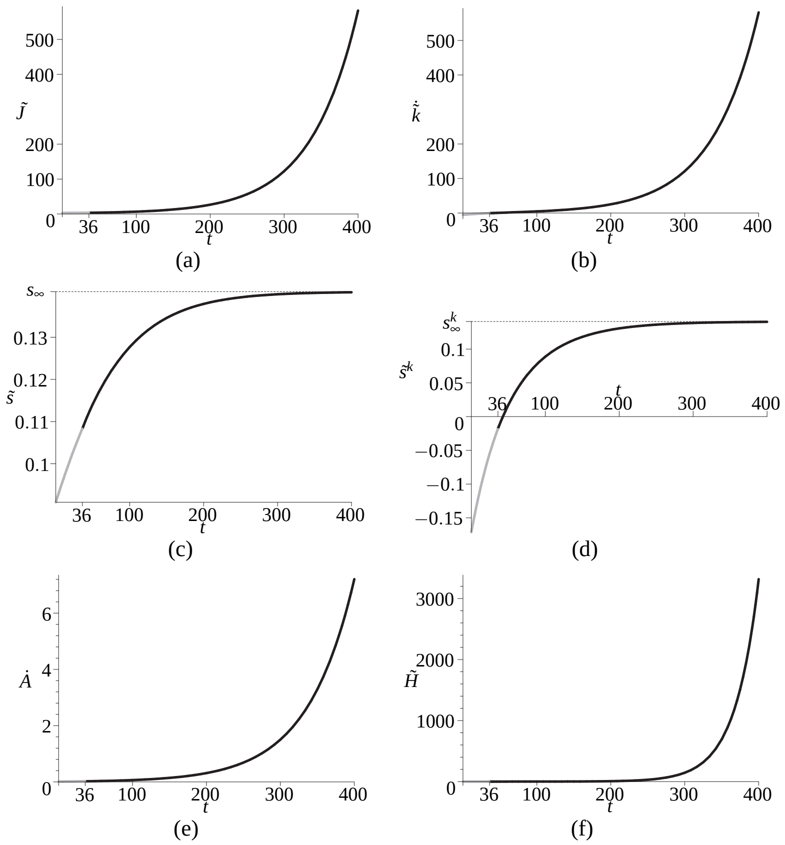

Fig. 1.12 time-path trajectories of (a) $\tilde{J}$, (b) $\dot{\tilde{k}}$, (c) $\tilde{s} = \tilde{J}/\tilde{y}$, (d) $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$, (e) $\dot{A}$, (f) $\tilde{H}$.

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