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MARTIN'S MAXIMUM AND TOWER FORCING

BY

Sean Cox* and Matteo Viale *,**

Institut für mathematische Logik und Grundlagenforschung
Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
Current address:
Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
1015 Floyd Avenue, Richmond, Virginia 23284, USA
e-mail: scox9@vcu.edu

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ABSTRACT

There are several examples in the literature showing that compactness-like properties of a cardinal κ cause poor behavior of some generic ultrapowers which have critical point κ (Burke [1] when κ is a supercompact cardinal; Foreman–Magidor [6] when $\kappa = \omega_2$ in the presence of strong forcing axioms). We prove more instances of this phenomenon. First, the Reflection Principle (RP) implies that if $\vec{\mathcal{I}}$ is a tower of ideals which concentrates on the class GIC_{ω_1} of ω_1-guessing, internally club sets, then $\vec{\mathcal{I}}$ is not presaturated (a set is ω_1-guessing iff its transitive collapse has the ω_1-approximation property as defined in Hamkins [10]). This theorem, combined with work from [16], shows that if PFA^+ or MM holds and there is an inaccessible cardinal, then there is a tower with critical point ω_2 which is not presaturated; moreover, this tower is significantly different from the non-presaturated tower already known (by Foreman–Magidor [6]) to exist in all models of Martin's Maximum. The conjunction of the Strong Reflection Principle (SRP) and the Tree Property at ω_2 has similar implications for towers of ideals which concentrate on the wider class GIS_{ω_1} of ω_1-guessing, internally stationary sets.

Finally, we show that the word "presaturated" cannot be replaced by "precipitous" in the theorems above: Martin's Maximum (which implies SRP and the Tree Property at ω_2) is consistent with a precipitous tower on GIC_{ω_1}.

1. Introduction

If the universe V of sets satisfies ZFC, there is no elementary embedding $j : V \to N$ where N is wellfounded and the least ordinal moved by j is "small" (like ω_1 or ω_2). Forcing with ideals and towers of ideals are two procedures that can potentially produce such an embedding $j : V \to N$ in some generic extension of V where the least ordinal moved by j is small. A tower of ideals is a sequence of ideals $\vec{I} = \langle I_\lambda \mid \lambda < \delta \rangle$ with a certain coherence property (see Section 2.4). The length of the sequence $\vec{I}$ is called the **height of $\vec{I}$** and, if each ideal in the sequence has the same completeness,[1] this completeness is called the **critical point of $\vec{I}$**. If $\vec{I}$ is a tower then there is a natural poset $\mathbb{P}_{\vec{I}}$ associated with $\vec{I}$, and in the generic extension $V^{\mathbb{P}_{\vec{I}}}$ there is an embedding $j_G : V \to ult(V, G)$ where the least ordinal moved by j equals the critical point of the tower; this embedding is called a **generic ultrapower of V by $\vec{I}$** and $ult(V, G)$ is not necessarily wellfounded.

[1] This will hold for all towers considered in this paper.

Properties of the tower $\vec{I}$ and of its height affect the properties of the generic ultrapower. Woodin proved that if δ is a Woodin cardinal, then many of the natural "stationary towers" of height δ satisfy a very strong property called **presaturation** (see [13] and [18]). Foreman and Magidor [6] proved that if δ is a supercompact cardinal then there are several natural stationary towers of height δ which are precipitous (a property weaker than presaturation). For simplicity let us only consider towers with critical point ω_2. Then one key difference between the Foreman–Magidor stationary towers and the Woodin stationary towers are that with Woodin's examples, there are always some V-regular cardinals which become ω-cofinal in the generic ultrapower, whereas they remain uncountably cofinal in the generic ultrapower in the Foreman–Magidor examples.

On the other hand, compactness-like properties of the critical point of the tower can often prevent nice behavior of the tower. Burke [1] showed that, if κ is supercompact and $\delta > \kappa$ is inaccessible, then there is a tower of height δ with critical point κ which is not precipitous. Strong forcing axioms like the Proper Forcing Axiom and Martin's Maximum are known to make ω_2 behave much like a supercompact cardinal; so in light of Burke's theorem we should expect that strong forcing axioms prevent nice behavior of some towers with critical point ω_2. Foreman and Magidor [6] proved that Martin's Maximum[2] implies that a certain natural tower[3] with critical point ω_2 is not presaturated (see Example 9.16 of [5]).[4] On a related note, they also showed that the Proper Forcing Axiom implies that there is no presaturated ideal on ω_2.

This paper provides more results along the lines of the Burke and Foreman–Magidor theorems, that compactness properties of the critical point of certain towers prevent nice behavior of the tower. We show that under strong forcing axioms, there are certain towers with critical point ω_2 which are not presaturated; these towers are significantly different from the non-presaturated towers produced in Foreman–Magidor [6] in a very strong sense.[5] Specifically:

[2] Really, just the saturation of NS_{ω_1}.

[3] Namely, the stationary tower concentrating on the ω_1-internally approachable structures.

[4] However, their tower can be precipitous, and in fact always is precipitous if its height is a supercompact cardinal.

[5] Namely, while the Foreman–Magidor tower concentrated on internally approachable structures, our ideals concentrate on structures which are definitely not internally approachable.

THEOREM 1: *Assume $RP(\omega_2)$ holds. Whenever $\vec{I}$ is a tower which concentrates on the class GIC_{ω_1} of ω_1-guessing, internally club sets, then $\vec{I}$ is not presaturated.*

In fact we show more: that $RP(\omega_2)$ implies there is no single ideal J such that:

- J concentrates on GIC_{ω_1}; and
- J bounds its completeness.

The latter is a property introduced by the first author in [2] which is closely related to saturation and Chang's Conjecture. See Definition 25 and Theorem 27.

Using Theorem 1 and results from [16], we show:

THEOREM 2: *Assume either PFA^+ or MM, and that δ is inaccessible. Then there is a tower of height δ with critical point ω_2 which is not presaturated (in fact, this tower even fails to have the weak Chang Property; see Definition 22 and Corollary 24).*

The hypotheses of Theorem 1 can be strengthened to obtain a stronger conclusion:

THEOREM 3: *Assume $SRP(\omega_2)$ and the Tree Property at ω_2. Whenever $\vec{I}$ is a tower which concentrates on the class GIS_{ω_1} of ω_1-guessing, internally stationary sets, then $\vec{I}$ is not presaturated.*

Again, similarly to Theorem 1 we actually show more: that $SRP(\omega_2)$ together with the Tree Property at ω_2 implies there is no single ideal which concentrates on GIS_{ω_1} and bounds its completeness.

If we require the tower to be definable, then a theorem of Burke [1] together with the Isomorphism Theorem from [15] yields:

THEOREM 4 (ZFC): *If $2^\omega \le \omega_2$, then there is no precipitous tower of inaccessible height δ which concentrates on GIC_{ω_1} and is definable over $(V_\delta, \in)$.*

Finally, we prove that in Theorems 1 through 3, the conclusion cannot be strengthened to say there is no precipitous tower which concentrates on the relevant class of sets, even if the hypothesis is strengthened to "plus" versions of Martin's Maximum:

Theorem 5: *If κ is supercompact and $\delta > \kappa$ is inaccessible, then if $\mathbb{P}$ is the standard iteration to produce a model of $MM^{+\omega_1}$, there is a precipitous tower in $V^{\mathbb{P}}$ of height δ concentrating on GIC_{ω_1}.*

The paper is organized as follows. Section 2 provides relevant background on guessing models (2.1), forcing axioms and reflection principles (2.2), the key Isomorphism Theorems for ω_1-guessing structures from [15] (2.3), towers of ultrafilters and ideals (2.4), and induced towers (2.5). Those last two subsections (2.4 and 2.5) are used primarily for the consistency proof in Section 6, though Definition 18 and Fact 19 are used throughout the paper. Section 3 provides a brief review of the weak Chang property and relevant theorems from [2] that will be used in the later proofs in Sections 4 and 5 (but not in Section 6). Section 4 proves Theorems 1, 4 and 2. Section 5 proves Theorem 3. The results in Sections 4 and 5 are due to both authors. Section 6 proves Theorem 5, and is due to the first author. Section 7 ends with some open problems.

2. Preliminaries

2.1. The classes GIC_{ω_1}, GIS_{ω_1} and GIU_{ω_1}. Weiss [17] introduced the notion ISP, which is a significant strengthening of the Tree Property. In this paper we use the alternative notion of a δ-**guessing model** from [15]. ZF^- denotes ZF without the Power Set Axiom.

Definition 6: Let H be a transitive ZF^- model and $\delta \in REG^H$. We say that H has the δ-**approximation property** iff (H, V) has the δ-approximation property as in Hamkins [10]. In other words, for every $\eta \in H$: whenever $A \subset \eta$ is such that $z \cap A \in H$ for every $z \in H$ with $|z|^H < \delta$, then $A \in H$.

If M is a set or class which believes "$\aleph_1$ exists" and $\in \restriction (M \times M)$ is extensional (so that M has a transitive collapse)—for example, if $M \prec (H_\theta, \in)$ for some $\theta \geq \omega_2$—we say M is ω_1-**guessing** iff its transitive collapse H_M has the ω_1-approximation property.[6] We let G_{ω_1} denote the class of M such that $|M| = \omega_1 \subset M$ and M is ω_1-guessing; $\sigma_M : H_M \to M$ will always denote the inverse of the Mostowski collapse of M.

[6] This definition is slightly different but equivalent to the definition in [15]. Further, note that since we're assuming $M \prec H_\theta$ and $\omega_1 \subset M$, then $|z|^{H_M} = \omega$ iff $|z| = \omega$ (for any $z \in H_M$).

We use several other common classes of structures (see Foreman and Todorcevic [8]). A set M is ω_1**-internally club** iff $M \cap [M]^\omega$ contains a club in $[M]^\omega$; ω_1**-internally stationary** iff $M \cap [M]^\omega$ is stationary in $[M]^\omega$; and ω_1**-internally unbounded** iff $M \cap [M]^\omega$ is $\subseteq$-cofinal in $[M]^\omega$. Let

$$\Lambda := \{M \mid (M, \in) \text{ satisfies } ZF^-,\ \omega_1 \subseteq M, \text{ and } |M| = \omega_1\}.$$

We let IC_{ω_1}, IS_{ω_1}, and IU_{ω_1} refer respectively to the class of $M \in \Lambda$ which are ω_1-internally club, ω_1-internally stationary, and ω_1-internally unbounded. The classes IC_{ω_1}, IS_{ω_1}, and IU_{ω_1} can be equivalently characterized in ways analogous to internal approachability:

- $M \in IA_{\omega_1}$ iff there is a $\subseteq$-continuous $\in$-chain $\langle N_\xi \mid \xi < \omega_1\rangle$ such that $M = \bigcup_{\xi<\omega_1} N_\xi$ and $\langle N_\xi \mid \xi < \zeta\rangle \in M$ for every $\zeta < \omega_1$;
- $M \in IC_{\omega_1}$ iff there is a $\subseteq$-continuous $\in$-chain $\langle N_\xi \mid \xi < \omega_1\rangle$ such that $M = \bigcup_{\xi<\omega_1} N_\xi$ and $N_\xi \in M$ for every $\xi < \omega_1$;
- $M \in IS_{\omega_1}$ iff there is a $\subseteq$-continuous sequence $\langle N_\xi \mid \xi < \omega_1\rangle$ such that $M = \bigcup_{\xi<\omega_1} N_\xi$ and $N_\xi \in M$ for stationarily many $\xi < \omega_1$. It is straightforward to see that $M \in IS_{\omega_1}$ iff there is some stationary $T_M \subset \omega_1$ such that for every $\subseteq$-continuous sequence $\vec{N}$ with union M, $\{\xi < \omega_1 \mid N_\xi \in M\} =_{NS} T_M$;[7]
- $M \in IU_{\omega_1}$ iff there is a $\subseteq$-continuous sequence $\langle N_\xi \mid \xi < \omega_1\rangle$ such that $M = \bigcup_{\xi<\omega_1} N_\xi$.

Set $GIC_{\omega_1} := G_{\omega_1} \cap IC_{\omega_1}$, $GIS_{\omega_1} := G_{\omega_1} \cap IS_{\omega_1}$, and $GIU_{\omega_1} := G_{\omega_1} \cap IU_{\omega_1}$.

Note that

(1) $\quad G_{\omega_1} \cap \{M \prec (H_\theta, \in) \mid M \cap H_{\omega_2} \in IA_{\omega_1}\}$ is empty for all $\theta \geq \omega_3$.

To see (1): suppose to the contrary that there were some such M, and that $\vec{N} = \langle N_\xi \mid \xi < \omega_1\rangle$ witnessed that $M \cap H_{\omega_2} \in IA_{\omega_1}$. Then for every countable $z \in M$ there is some $\xi < \omega_1$ such that $z \cap \vec{N} = z \cap (\vec{N} \restriction \xi) \in M$. Since this holds for every countable $z \in M$, and since $H_{\omega_2} \in M$ and $M \in G_{\omega_1}$, this would imply that $\vec{N} \in M$ and so $M \cap H_{\omega_2} = \bigcup range(\vec{N}) \in M$. But it is not possible that $M \cap H_{\omega_2} \in M$, because $H_{\omega_2} - M \neq \emptyset$ and $M \prec (H_\theta, \in)$.

Note also that all of the classes mentioned are invariant under isomorphism (i.e., M is in the class iff its transitive collapse H_M is in the class). Viale and Weiss proved:

[7] For $A, B \subset \omega_1$, $A =_{NS} B$ means that $A \Delta B$ is nonstationary.

Theorem 7 (Viale–Weiss [16]): *PFA implies that $GIC_{\omega_1} \cap \wp_{\omega_2}(H_\theta)$ is stationary for all regular $\theta \geq \omega_2$.*

Their proof actually produced models which were not only in GIC_{ω_1}, but persistently so; that is, these models remain in GIC_{ω_1} in any outer model which has the same ω_1. This used the following generalization of a theorem of Baumgartner. For a poset $\mathbb{R}$ and a possibly non-transitive set M, let us say that a filter $g \subset \mathbb{R}$ is $(M, \mathbb{R})$**-generic** iff $g \cap D \cap M \neq \emptyset$ for every $D \in M$ which is dense in $\mathbb{R}$.

Theorem 8 (Viale–Weiss [16]): *For each regular $\delta \geq \omega_2$ there is a proper poset $\mathbb{R}_\delta$ such that:*

(1) $\mathbb{R}_\delta \in H_{\delta^+}$;
(2) $\Vdash_{\mathbb{R}_\delta} \check{H} \in GIC_{\omega_1}$ *where* $H := H_\delta^V$;
(3) *whenever M is a (possibly non-transitive) ZF^- model such that $|M| = \omega_1 \subset M$ and there exists some g which is $(M, \mathbb{R}_\delta)$-generic, then:*
   - $M \cap H_\delta \in GIC_{\omega_1}$;
   - *if W is any transitive ZF model such that $(M, g, \mathbb{R}_\delta) \in W$ and $\omega_1^W = \omega_1^V$, then $W \models$ "$M \in GIC_{\omega_1}$". (Here W could, for example, be any outer model of V which has the same ω_1.)*

Viale proved:

Lemma 9 (Viale): *FA_{ω_1} implies $G_{\omega_1} \subseteq IU_{\omega_1}$; so $G_{\omega_1} = GIU_{\omega_1}$.*

Proof. Viale [15] proved that if M is ω_1-guessing and $|M|$ is strictly less than the so-called pseudo-intersection number, then $M \in IU_{\omega_1}$; FA_{ω_1} implies that the pseudo-intersection number is $\geq \omega_2$. ∎

Finally, we point out the standard fact that all of these classes project:

Lemma 10: *Let Z be any of the classes G_{ω_1}, IA_{ω_1}, IC_{ω_1}, IS_{ω_1}, or IU_{ω_1}. If $M \in Z$ and $\theta \geq \omega_2$ is a regular cardinal then $M \cap H_\theta \in Z$.*

Proof. Here it will be more convenient to work with the following "non-transitivised" characterization of G_{ω_1}: $M \in G_{\omega_1}$ iff for every $\eta \in M$ and every $A \subset \eta \cap M$: if $A \cap z \in M$ for every countable $z \in M$, then there is some $A' \in M$ such that $A' \cap M = A$.

Now suppose $M \in G_{\omega_1}$ and $\theta \geq \omega_2$ is regular; we want to see that $M \cap H_\theta \in G_{\omega_1}$. Let $\eta \in M \cap \theta$ and $A \subset \eta \cap M$, and suppose $z \cap A \in M \cap H_\theta$ for every countable $z \in M \cap H_\theta$. Then clearly $z \cap A \in M$ for every countable $z \in M$, since $A \subseteq \theta$. Since $M \in G_{\omega_1}$, then there is an $A' \in M$ with $A' \cap M = A$. Set $A'' := A' \cap \eta \in M \cap H_\theta$. Then $A'' \cap (M \cap H_\theta) = A$.

For the other classes, we present the argument for IC_{ω_1}; the rest are similar. Suppose $M \in IC_{\omega_1}$ as witnessed by a sequence $\langle N_\xi \mid \xi < \omega_1 \rangle$ where $N_\xi \in M$ for every $\xi < \omega_1$. Let θ be a regular uncountable cardinal. Clearly the sequence $\langle N_\xi \cap H_\theta \mid \xi < \omega_1 \rangle$ is $\subset$-increasing and $\subset$-continuous with union $M \cap H_\theta$; we just need to see that $N_\xi \cap H_\theta \in M \cap H_\theta$ for every $\xi < \omega_1$. If $\theta \in M$ this is trivial. If $\theta \notin M$ then since $M \in IC_{\omega_1}$, $M \cap ORD$ is an ω-closed set of ordinals and so $sup(M \cap \theta)$ has uncountable cofinality. Then for each $\xi < \omega_1$ there is some $\eta_\xi < \theta$, $\eta_\xi \in M$, such that $N_\xi \cap H_\theta = N_\xi \cap H_{\eta_\xi}$; and the latter is in M since both η_ξ and N_ξ are in M. ∎

It is interesting to point out that by the argument of Proposition 2.4 of [6], if Z is any of the classes IA_{ω_1}, IC_{ω_1}, IS_{ω_1}, or IU_{ω_1}, then Z also lifts with respect to the nonstationary ideal; that is, if S is a stationary subset of $Z \cap \wp_{\omega_2}(H_\theta)$ and $\theta' >> \theta$, then $Z \cap \{M \in \wp_{\omega_2}(H_{\theta'}) \mid M \cap H_\theta \in S\}$ is also stationary. This implies that $\langle NS \restriction (Z \cap \wp_{\omega_2}(H_\theta)) \mid \theta \in ORD \rangle$ forms a tower (see Section 3). On the other hand, this can trivially fail for the class $Z = G_{\omega_1}$ because $G_{\omega_1} \cap \wp_{\omega_2}(H_\theta)$ might be nonstationary for large θ. Even if $G_{\omega_1} \cap \wp_{\omega_2}(H_\theta)$ is stationary for every regular $\theta \geq \omega_2$ (as is the case under PFA), it is still not clear—and seems doubtful—that $\langle NS \restriction G_{\omega_1} \cap \wp_{\omega_2}(H_\theta) \mid \theta \in ORD \rangle$ necessarily forms a tower.

2.2. Forcing Axioms, Projective Stationarity, and Reflection Principles. Let Γ be a class of posets and β an ordinal. $FA^{+\beta}(\Gamma)$ means that for every $\mathbb{P} \in \Gamma$, for every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, and for every sequence $\langle \dot{S}_\xi \mid \xi < \beta \rangle$ such that $\Vdash_{\mathbb{P}}$ "$\dot{S}_\xi \subseteq \omega_1$ is stationary" for every $\xi < \beta$, then there is a filter $F \subset \mathbb{P}$ meeting every $D \in \mathcal{D}$ and such that for every $\xi < \beta$: $(\dot{S}_\xi)_F := \{\alpha < \omega_1 \mid (\exists q \in F)(q \Vdash \check{\alpha} \in \dot{S}_\xi)\}$ is stationary. $FA(\Gamma)$ means $FA^0(\Gamma)$ and $FA^+(\Gamma)$ means $FA^{+1}(\Gamma)$. Martin's Axiom is FA(ccc posets), the Proper Forcing Axiom (PFA) is FA(proper posets), and Martin's Maximum (MM) is FA(posets preserving stationary subsets of ω_1). We caution that elsewhere in the literature the notation PFA^{++} and MM^{++} are sometimes used for what we call $PFA^{+\omega_1}$ and $MM^{+\omega_1}$. It is widely known that the standard iteration used to produce a model of MM (resp. PFA) actually produces a model of $MM^{+\omega_1}$ (resp. $PFA^{+\omega_1}$).

For a regular cardinal $\theta \geq \omega_2$, $RP(\theta)$ means that whenever $S \subset [\theta]^\omega$ is stationary, then there is an X such that $\omega_1 \subseteq X$, $|X| = \omega_1$, and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. It is well-known that $FA^+(\sigma\text{-closed})$ implies $RP(\theta)$ for all regular $\theta \geq \omega_2$ (so in particular RP follows from PFA^+); and by [7] this is also implied by MM. A set $P \subset [X]^\omega$ is **projective stationary** iff for every stationary $T \subset \omega_1$, $\{Y \in P \mid Y \cap \omega_1 \in T\}$ is stationary in $[X]^\omega$. For $\theta \geq \omega_2$, the **Strong Reflection Principle at** θ **(**$SRP(\theta)$**)** is the statement: for every projective stationary $P \subset [H_\theta]^\omega$, there is a $\subseteq$-continuous elementary $\in$-chain $\langle N_\xi \mid \xi < \omega_1\rangle$ of countable models such that every N_ξ is an element of P (i.e., there is some ω_1-sized subset X of θ such that $P \cap [X]^\omega$ contains a club in $[X]^\omega$). It was shown in [3] that Martin's Maximum implies $SRP(\theta)$ for all regular $\theta \geq \omega_2$. Extending a result of Gitik [9], Veličković proved the following theorem (see Section 3 of [14]):

Theorem 11: *Whenever $C \subset [\omega_2]^\omega$ is club, $x \in \mathbb{R}$, and $T \subset \omega_1$ is stationary, then there are $a, b, c \in C$ such that $a \cap \omega_1 = b \cap \omega_1 = c \cap \omega_1 \in T$ and $x \in L_{\omega_2}[a, b, c]$.*

Corollary 12: *If W is a transitive ZF^- model with $\omega_2 \subseteq W$ and $\mathbb{R} - W \neq \emptyset$, then $[\omega_2]^\omega - W$ is projective stationary.*

Proof. Let $T \subset \omega_1$ be stationary; we need to show that

$$\{d \in [\omega_2]^\omega \mid d \notin W \text{ and } d \cap \omega_1 \in T\}$$

is stationary in $[\omega_2]^\omega$. Suppose not; then there is a club $C \subset [\omega_2]^\omega$ such that $C \searrow T := \{d \in C \mid d \cap \omega_1 \in T\} \subset W$. Let $x \in \mathbb{R}$ be arbitrary and let $a, b, c \in C$ be as in Theorem 11, so that $x \in L_{\omega_2}[a, b, c]$ and $a \cap \omega_1 = b \cap \omega_1 = c \cap \omega_1 \in T$; so $a, b, c \in C \searrow T \subset W$. Since $\omega_2 \subseteq W \models ZF^-$ and $a, b, c \in W$ then $L_{\omega_2}[a, b, c] \subseteq W$. So $x \in W$; since x was arbitrary we've shown $\mathbb{R} \subset W$, contrary to the assumptions. ∎

2.3. Isomorphism Theorems for GIC_{ω_1} and GIS_{ω_1}. We use the Isomorphism Theorems from Viale [15]. For transitive ZF^- models H and H', we say that H **is a hereditary initial segment of** H' iff $H = H'$ or there is some $\lambda \in Card^{H'}$ such that $H = (H_\lambda)^{H'}$. We provide a slight simplification of both the formulation and the proof of the Isomorphism Theorems from Viale [15]:

THEOREM 13 (Viale [15]): *Assume H and H' are transitive ZF^- models such that:*

(a) $H \cap \mathbb{R} = H' \cap \mathbb{R}$;
(b) *for every transitive $A \in H \cap H'$: the set*

$$H \cap H' \cap \{d \mid (d, \in) \text{ is extensional}\} \tag{2}$$

is $\subseteq$-cofinal in both $([A]^\omega)^H$ and in $([A]^\omega)^{H'}$;[8]

(c) H, H' *are in* G_{ω_1}.

Then one of H, H' is a hereditary initial segment of the other.

Proof. Note that since H and H' are transitive ZF^- models, then item (a) of the hypothesis of the theorem is equivalent to saying that $H \cap H_{\omega_1} = H' \cap H_{\omega_1}$; the latter equality will be more convenient for the current proof.

We first prove the following, by induction on η:

$$V_\eta \cap H = V_\eta \cap H' \text{ for every } \eta \leq \min(ht(H), ht(H')). \tag{3}$$

If η is any limit ordinal and (3) holds at all $\eta' < \eta$, then it clearly holds at η as well. So we only need to show the successor step of the induction: assume (3) holds at some ordinal $\eta < \min(ht(H), ht(H'))$, and prove that (3) holds at $\eta + 1$. Set $A := V_\eta \cap H = V_\eta \cap H'$. Since H is a transitive ZF^- model, its rank function is correct, and so (since we're assuming $\eta < ht(H)$) then $V_\eta \cap H = (V_\eta)^H$; similarly for H'. In particular, $A \in H \cap H'$.

We show $\wp(A) \cap H \subseteq H'$; the proof that $\wp(A) \cap H' \subseteq H$ is similar. Let $Z \in \wp(A) \cap H$. Since $A \in H'$ and H' is ω_1-guessing, then to show that $Z \in H'$ it suffices to show that $Z \cap d \in H'$ for $\subset$-cofinally many d in $([A]^\omega)^{H'}$. By assumption b in the hypothesis of the theorem and the fact that A is transitive, it in turn suffices to show that $Z \cap d \in H'$ for every $d \in H \cap H' \cap ([A]^\omega)^{H'}$ such that $(d, \in)$ is extensional. Pick any such d, let $\bar{d}$ be its transitive collapse, and $\pi_d : (d, \in) \to (\bar{d}, \in)$ be its Mostowski collapsing isomorphism; note that $\pi_d \in H \cap H'$. Then $\bar{Z}_d := \pi_d``(Z \cap d) \in H \cap H_{\omega_1} = H' \cap H_{\omega_1}$. And $Z \cap d = \pi_d^{-1}``\bar{Z}_d$. Since π_d and $\bar{Z}_d$ are both in H', this implies that $Z \cap d \in H'$ and concludes the proof of (3).

[8] Recall that by Mostowski's Collapsing Theorem, every d in the set from (2) has the property that there is a unique transitive $\bar{d}$ and a unique map π_d such that $\pi_d : (d, \in) \to (\bar{d}, \in)$ is an isomorphism.

Suppose now that H and H' have different heights; without loss of generality assume $\beta := ht(H) < ht(H')$. Then β is a cardinal in H'; suppose it were not, and let $\zeta := |\beta|^{H'} < \beta$. Then there is some $b \in H' \cap \wp(\zeta)$ which codes (via Gödel pairing) a bijection between ζ and β. Since β is a limit ordinal then $\zeta + 1 < \beta = ht(H)$; so $V_{\zeta+1} \cap H = V_{\zeta+1} \cap H'$ by (3) and thus $b \in H$. Since H is a ZF^- model it can compute the ordertype coded by b and so $\beta \in H$, a contradiction.

Essentially the same argument of the last paragraph shows that $H = H_\beta^{H'}$: if $a \in H_\beta^{H'}$, then there is some $\zeta < \beta$ and a $b \in H' \cap \wp(\zeta) = H \cap \wp(\zeta)$ which codes a in a way that is absolute across transitive ZF^- models, so, in particular, $a \in H$. ∎

Note that if $2^\omega = \omega_2$, ϕ is some wellorder of $\mathbb{R}$ in order-type ω_2, and M, M' are elementary substructures of $(H_\theta, \in, \phi)$ for some $\theta \geq \omega_2$ such that $M \cap \omega_2 = M' \cap \omega_2 =: \alpha > \omega_1$, then the pair H_M and $H_{M'}$ (the transitive collapses of M and M', respectively) satisfy item (a) from the hypothesis of Theorem 13 (because $H_M \cap \mathbb{R} = \phi``\alpha = H_{M'} \cap \mathbb{R}$).

Corollary 14 (Viale [15]): *Assume H and H' are transitive ZF^- models such that $H \cap \mathbb{R} = H' \cap \mathbb{R}$.*

(1) *Suppose that H and H' are in GIS_{ω_1} where $T_H \subset \omega_1$ and $T_{H'} \subset \omega_1$ witness that H, H' are ω_1-internally stationary (respectively). If $T_H \cap T_{H'}$ is stationary, then one of H, H' is a hereditary initial segment of the other. (See Section 2.1 for the meaning of the notation T_H and $T_{H'}$.)*
(2) *Suppose that H and H' are in GIC_{ω_1}. Then one of H, H' is a hereditary initial segment of the other.*

Proof. We first prove part (1); let H and H' be as in the hypothesis. We will show that H, H' satisfy requirement (b) of the hypothesis of Theorem 13. Fix any $\subset$-increasing and $\subset$-continuous sequence $\langle N_\xi \mid \xi < \omega_1 \rangle$ of countable elementary substructures of $(H, \in)$ such that $H = \bigcup_{\xi<\omega_1} N_\xi$; similarly fix a sequence $\langle N'_\xi \mid \xi < \omega_1 \rangle$ of substructures of $(H', \in)$. Let T_H, $T_{H'}$ witness (respectively) that H, H' are in IS_{ω_1} (i.e., T_H is stationary and $N_\xi \in H$ for every $\xi \in T_H$; similarly for $T_{H'}$). Let A be any transitive set which is an element of $H \cap H'$. Set $B := \{N_\xi \cap A \mid \xi \in T_H \cap T_{H'}\}$. Then $B \subset H \cap H'$ and B is a stationary—in particular $\subset$-cofinal—subset of $[A]^\omega$ (in the sense of V). So then clearly B is also $\subset$-cofinal in each of $([A]^\omega)^H$ and $([A]^\omega)^{H'}$. Note that for each

$\xi < \omega_1$, $(N_\xi, \in)$ is extensional.[9] Since $(N_\xi, \in)$ is extensional and A is transitive, then $(N_\xi \cap A, \in)$ is extensional. So B consists entirely of extensional sets. So we have shown that item b of the hypothesis of Theorem 13 holds for the pair H, H', which completes the proof of part (1).

Now notice that part (2) of the corollary is just a special case of part (1): if H and H' are in IC_{ω_1}, then T_H and $T_{H'}$ each contain a closed unbounded subset of ω_1. So in this case $T_H \cap T_{H'}$ in fact contains a club, so part (1) of the corollary applies. ∎

Finally we state a corollary which will only be used in the proof of Theorem 4:

Corollary 15 (Viale [15]): *Let $B \subset \mathbb{R}$. Set*

$$GIC^B_{\omega_1} := \{H \in GIC_{\omega_1} \mid H \text{ is a transitive } ZF^- \text{ model and } H \cap \mathbb{R} = B\}.$$

Then $GIC^B_{\omega_1}$ is totally ordered by the "hereditary initial segment" relation.

Proof. This follows immediately from Corollary 14. ∎

2.4. Towers of measures and towers of ideals. Suppose Z is a set and $F \subset \wp(Z)$ is a filter. The **support of** F **(**$supp(F)$**)** is the set $\bigcup Z$. For all instances in this paper, the support of a filter will always be a transitive set (typically some H_θ) and Z will always be of the form $\wp_\kappa(H_\theta)$ for some regular $\kappa \le \theta$. **(Ultra) filter** will always mean a normal,[10] countably complete, fine[11] (ultra) filter. If F is a filter then $\breve{F}$ denotes its dual ideal; similarly if I is an ideal then $\breve{I}$ denotes its dual filter. If Γ is a class, we say that a filter F **concentrates on** Γ iff there is an $A \in F$ such that $A \subseteq \Gamma$; if I is an ideal we say that I **concentrates on** Γ iff its dual filter concentrates on Γ. A set $S \subset \bigcup I$ is I**-positive** (written $S \in I^+$) iff $S \notin I$.

Definition 16: If $Z \subseteq Z'$, $I \subset \wp(Z)$ and $I' \subseteq \wp(Z')$ are ideals, we say that I **is the canonical projection of** I' **to** Z iff for every $A \subseteq Z$,

$$A \in I \iff \{M' \in Z' \mid M' \cap \bigcup Z \in A\} \in I'.$$

9 Because $(H, \in)$ is extensional (by transitivity of H) and $(N_\xi, \in) \prec (H, \in)$.

10 F is normal iff for every regressive $g : Z \to V$ there is an $S \in F^+$ such that $g \restriction S$ is constant.

11 Namely, for every $b \in supp(F)$ there is an $A \in F$ such that $b \in M$ for all $M \in A$. Note if F is fine then its support is equal to $\bigcup\bigcup F$.

For filters $F \subset \wp(Z)$ and $F' \subset \wp(Z')$, we say that F is the canonical projection of F' to Z iff $\breve{F}$ is the canonical projection of $\breve{F}'$ to Z. For filters, this is equivalent to saying

$$F = \{\{M' \cap \bigcup Z \mid M' \in A'\} \mid A' \in F'\}.$$

Suppose W is a transitive model of set theory and U is a (possibly external) W-normal,[12] fine ultrafilter, say $U \subset \wp(Z)$ where $Z \in W$ (for example, Z might be $\wp_\kappa^W(H_\lambda^W)$). It is a standard fact that if $j_U : W \to_U ult(W,U) = ({}^Z W \cap W)/U$ is the ultrapower and $\bigcup Z$ is transitive, then:

(4)
- j_U"$(\bigcup Z)$ is an element of $ult(W,U)$ and is represented by $[id_Z]_U$;
- if the wellfounded part of $ult(W,U)$ has been transitivized, then $j \restriction (\bigcup Z)$ is an element of $ult(W,U)$ and is represented by $[f]_U$ where $f(M) :\simeq$ the inverse of the Mostowski collapse map of M.

Of course if $U \in W$ then $ult(W,U)$ is wellfounded, but the comments above show that $\bigcup Z$ is always an element of the (transitivised) wellfounded part of $ult(W,U)$, even when U is external to W. One common example of this "external" case is generic ultrapowers. Suppose $I \subset \wp(Z)$ is an ideal,[13] and let $\mathbb{P}_I := (I^+, \subset)$. If G is $(V, \mathbb{P}_I)$-generic, then G is an ultrafilter on $\wp^V(Z)$ which is normal[14] with respect to sequences from V. In particular, (4) holds and $j_G \restriction (\bigcup Z) \in ult(V,G)$.

Now consider generalizations of these notions to sequences of filters which cohere via the "canonical projection" relation in Definition 16.

Definition 17: Let W be a transitive model of set theory, and δ a regular cardinal in W. Let $\langle Z_\lambda \mid \lambda < \delta \rangle \in W$ and, for simplicity, assume each $\bigcup Z_\lambda \in V_\delta^W$ and each $\bigcup Z_\lambda$ is transitive, and $\bigcup_{\lambda<\delta} \bigcup Z_\lambda = V_\delta^W$. Suppose $\langle U_\lambda \mid \lambda < \delta \rangle$ is a (possibly external to W) sequence of W-normal ultrafilters, where $U_\lambda \subset \wp(Z_\lambda)$ for each $\lambda < \delta$. Also assume there is a fixed $\kappa < \delta$ such that each U_λ has completeness κ. We will call $\vec{U}$ a **tower of W-normal measures** iff for every $\lambda \le \lambda' < \delta$, U_λ is the canonical projection of $U_{\lambda'}$ to Z_λ (as in Definition 16).

12 Namely, normal with respect to sequences from W.

13 Recall we are assuming all ideals are normal, fine, and countably complete.

14 By a density argument and the fact that I was a normal ideal.

If $\vec{U}$ is a tower of W-normal measures, then there is a commutative system of maps obtained by the various ultrapower maps $j_{U_\lambda} : W \to_{U_\lambda} ult(W, U_\lambda)$ and for $\lambda \leq \lambda'$, maps $k_{U_\lambda, U_{\lambda'}} : ult(W, U_\lambda) \to ult(W, U_{\lambda'})$ given by

$$[f]_{U_\lambda} \mapsto [M' \mapsto f(M' \cap \bigcup Z_\lambda)]_{U_{\lambda'}}.$$

The direct limit map of the system is denoted $j_{\vec{U}} : W \to_{\vec{U}} N_{\vec{U}}$. If $\vec{U} \in W$, then this direct limit will always be wellfounded and closed under $< \delta$ sequences from W; so if in addition $j_{\vec{U}}(\kappa) = \delta$, then $j_{\vec{U}}$ can witness the almost-hugeness of $cr(j_{\vec{U}})$ in W.[15]

A (possibly external) direct limit embedding $j_{\vec{U}} : W \to_{\vec{U}} N_{\vec{U}}$ can also be viewed as an ultrapower embedding as follows. Given a (partial) function $f : V_\delta^W \to W$ with $f \in W$, let $\text{supp}(f)$ denote the least cardinal $\lambda \leq \delta$ such that $f(x)$ only depends on $x \cap H_\lambda$. Let

$$B_{<\delta}^W := \{f \in W \mid f : V_\delta^W \to W \text{ and } \text{supp}(f) < \delta\}.$$

Define an equivalence relation $\simeq_{\vec{U}}$ on $B_{<\delta}^W$ by: $f \simeq_{\vec{U}} g$ iff

$$\{M \in Z_\lambda \mid f(M \cap H_\lambda) = g(M \cap H_\lambda)\} \in U_\lambda$$

for all sufficiently large $\lambda < \delta$. Define a relation $\in_{\vec{U}}$ on $B_{<\delta}^W / \simeq_{\vec{U}}$ in the obvious way (this will be well-defined). Then the direct limit $(N_{\vec{U}}, E_{\vec{U}})$ will be isomorphic to $(B_{<\delta}^W / \simeq_{\vec{U}}, \in_{\vec{U}})$; for this reason we will write $ult(W, \vec{U})$ for this direct limit. The Los Theorem will hold in the following form: for each $f_0, \ldots, f_n$ in $B_{<\delta}^W$ and each formula ϕ, $N_{\vec{U}} \models \phi([f_0]_{\vec{U}}, \ldots, [f_n]_{\vec{U}})$ iff

$$\{M \in Z_\lambda \mid W \models \phi(f_0(M), \ldots, f_n(M))\} \in U_\lambda$$

for every sufficiently large $\lambda < \delta$. The following analogues of (4) always hold when taking (possibly external) ultrapowers by a tower of W-normal measures:

For every transitive $X \in V_\delta^W$:

(5)
- $j_{\vec{U}}$ “X is an element of $ult(W, \vec{U})$ and is represented by the function $[M \mapsto M \cap X]_{\vec{U}}$;
- $j_{\vec{U}} \restriction X$ is an element of $ult(W, \vec{U})$ and is represented by $M \mapsto$ the inverse of the Mostowski collapse map of $M \cap X$.

Just as forcing with the positive sets of an ideal gives rise to external ultrapowers of V by a single V-normal measure, forcing with a tower of ideals

[15] See Theorem 24.11 of Kanamori [12] for technical criteria on $\vec{U}$ which will guarantee that $j_{\vec{U}}$ is an almost huge embedding.

(defined below) gives rise to an external ultrapower of V by a tower of V-normal measures.

Definition 18: A sequence $\langle I_\lambda \mid \lambda < \delta\rangle$ is called a **tower of ideals of height** δ iff for every $\lambda \le \lambda' < \delta$, I_λ is the canonical projection (in the sense of Definition 16) of $I_{\lambda'}$ to Z_λ.

We will also require for simplicity that for each λ, if Z_λ is such that $I_\lambda \subset \wp(Z_\lambda)$, then $\bigcup Z_\lambda = H_\lambda$. In this paper Z_λ will always be of the form $\wp_\kappa(H_\lambda)$.

For a class Γ, we say that $\vec{I}$ **concentrates on** Γ iff every ideal in the sequence concentrates on Γ.

If $\vec{I}$ is a tower, there is a natural poset $\mathbb{P}_{\vec{I}}$ associated with $\vec{I}$. Conditions are pairs (λ, S) where $\lambda < \delta$ and $S \in I^+_\lambda$. A condition (λ, S) is strengthened by increasing λ to some λ' and refining the lifting of S to $H_{\lambda'}$. More precisely: $(\lambda', S') \le (\lambda, S)$ iff $\lambda' \ge \lambda$ and

$$S' \subseteq S^{Z_{\lambda'}} := \{M' \in Z_{\lambda'} \mid M' \cap H_\lambda \in S\}.$$

If G is generic for $\mathbb{P}_{\vec{I}}$, let

$$proj(G, \lambda) := \{S \in \wp^V(Z_\lambda) \mid (\lambda, S) \in G\};$$

this is an ultrafilter on $\wp^V(Z_\lambda)$ which is normal with respect to sequences from V (though $proj(G, \lambda)$ need not be $(V, \mathbb{P}_{I_\lambda})$-generic!) and $\langle proj(G, \lambda) \mid \lambda < \delta\rangle$ is a tower of V-normal measures as in Definition 17; in particular, (5) holds and one can prove the following general facts (in the case of towers, we use the notation $j_G : V \to_G ult(V, G)$ to denote the ultrapower embedding $j_{\vec{U}} : V \to_{\vec{U}} ult(V, \vec{U})$ where $\vec{U} = \langle proj(G, \lambda) \mid \lambda < \delta\rangle$):

Fact 19: If $\vec{I}$ is a tower of height δ where δ is inaccessible and G is generic for $\vec{I}$, then:

(1) For every $D \in V_\delta$, $j_G \restriction D \in ult(V, G)$.
(2) For every $\theta < \delta$: $proj(G, \theta) = \{S \in \wp^V(Z_\theta) \mid j_G``H_\theta \in j_G(S)\}$. This fact, combined with item 1 and the assumption that δ is (strongly) inaccessible, implies that $proj(G, \theta) \in ult(V, G)$ for every $\theta < \delta$.
(3) For every $\theta < \delta$ and every $Y \in V_\delta$, the relations $=_{proj(G,\theta)}\restriction ({}^{H_\theta}Y)^V$ and $\in_{proj(G,\theta)}\restriction ({}^{H_\theta}Y)^V$ are elements of $ult(V, G)$ (this follows from the previous bullets: both $({}^{H_\theta}Y)^V$ and $proj(G, \theta)$ are elements of $ult(V, G)$).

(4) If $\vec{I}$ concentrates on $\{M \mid \lambda \subset M \text{ and } M \cap \lambda^+ \in \lambda^+\}$ and $j_{proj(G,\theta)} : V \to ult(V, proj(G,\theta))$ is the ultrapower map by $proj(G,\theta)$, then $k_{proj(G,\theta),proj(G,\theta')} \restriction j_{proj(G,\theta)}(\lambda^+) = \text{id}$.

A tower $\vec{I}$ is called **precipitous** iff $ult(V, G)$ is wellfounded for every generic $G \subset \mathbb{P}_{\vec{I}}$. We refer the reader to Foreman [5] for the general theory of towers, and to Larson [13] and Woodin [18] for the specific cases where all the ideals I_θ in the tower are of the form $NS \restriction Z_\theta$ (towers of this form are called **stationary towers**).

2.5. Induced towers of ideals. We adjust Example 3.30 from [5] to towers: *Definition 20:* Suppose $\mathbb{Q}$ is a poset, δ is inaccessible, and $\langle \dot{U}_\lambda \mid \lambda < \delta \rangle$ is a sequence of $\mathbb{Q}$-names such that $\mathbb{Q} \Vdash$ "$\vec{\dot{U}}$ is a tower of V-normal ultrafilters". For each $\lambda < \delta$, let I_λ be the collection of A such that for every $(V, \mathbb{Q})$-generic object H, $A \notin (\dot{U}_\lambda)_H$. The sequence $\langle I_\lambda \mid \lambda < \delta \rangle$ will be called the **tower of ideals derived from the name $\vec{\dot{U}}$**.

It is straightforward to check that this indeed forms a tower of ideals.

Recall that if $j : V \to N$ is an embedding with critical point κ and $\mathbb{P} \in V$ is a poset such that $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ is a regular embedding,[16] then $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P} * j(\mathbb{P})/j``\dot{G}$ where $\dot{G}$ is the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic. Further, whenever $G * H$ is generic for $\mathbb{P} * j(\mathbb{P})/j``\dot{G}$ then j can be lifted (in $V[G][H]$) to an elementary $j^{G*H} : V[G] \to N[G][H]$; the map j^{G*H} is defined by

$$\tau_G \mapsto j(\tau)_{\hat{G}}, \tag{6}$$

where $\hat{G} \subset j(\mathbb{P})$ is the generic obtained from $G * H$ by the forcing equivalence of $j(\mathbb{P})$ with $\mathbb{P} * j(\mathbb{P})/j``\dot{G}$. Suppose $\delta$ is a $V$-cardinal such that for every $\lambda < \delta$, $j``H_\lambda \in N$. Then for every $\lambda < \delta$,

$$U_\lambda^{G*H} := \{A \mid A \in V[G] \text{ and } j^{G*H}``H_\lambda[G] \in j^{G*H}(A)\} \tag{7}$$

is a $V[G]$-normal ultrafilter. Then from the point of view of $V[G]$, the poset $j(\mathbb{P})/G$ forces that $\langle \dot{U}_\lambda^{G*\dot{H}} \mid \lambda < \delta \rangle$ is a tower of $V[G]$-normal measures (external to $V[G]$, of course). Then in $V[G]$, let $\langle I_\lambda \mid \lambda < \delta \rangle$ be the tower of normal ideals derived from the name $\langle \dot{U}_\lambda^{G*\dot{H}} \mid \lambda < \delta \rangle$ as in Definition 20 (here $V[G]$ is playing the role of V and $j(\mathbb{P})/G$ is playing the role of $\mathbb{Q}$ from Definition 20).

[16] Namely, whenever $A \subset \mathbb{P}$ is a maximal antichain then $j``A$ is a maximal antichain in $j(\mathbb{P})$.

Definition 21: The tower $\vec{I} \in V[G]$ described in the last paragraph will be called the **tower induced by** j.

We caution that if $j_{\vec{U}} : V \to N_{\vec{U}}$ is an embedding by a tower of V-normal measures, $j_{\vec{U}} \restriction \mathbb{P} : \mathbb{P} \to j_{\vec{U}}(\mathbb{P})$ is a regular embedding, G is $(V, \mathbb{P})$-generic, and $\vec{I} \in V[G]$ is the tower induced by $j_{\vec{U}}$ as in Definition 21, then for each $\lambda < \delta$ it will not in general be the case that the dual of I_λ extends U_λ. This is because of the way that the measure U_λ^{G*H} is defined in (7): the measure U_λ^{G*H} concentrates on elementary substructures of $H_\lambda[G]$, not on elementary substructures of H_λ. This is only a minor technical issue, however; generally $N_{\vec{U}} \cap j_{\vec{U}}^{G*H}``H_\lambda[G] = j_{\vec{U}}``H_\lambda$, and it follows that for every $\lambda < \delta$ there are U_λ^{G*H}-many $M \prec H_\lambda[G]$ such that $M \cap V \in V$ (see Corollary 32 for the use of derived towers in this setting).

3. The weak Chang property and ideals which bound their completeness

In this section we discuss presaturation of towers and some concepts introduced by the first author in [2] which will be used in the proofs of Theorems 27 and 29. These concepts are related to Chang's Conjecture, bounding by canonical functions, and saturation. For the reader's convenience all relevant proofs are included here.

A tower of height δ is called **presaturated** iff δ always remains a regular cardinal in generic extensions by the tower. Such a tower is always precipitous and $ult(V, G)$ is closed under $< \delta$ sequences from $V[G]$ (see Proposition 9.2 of [5]). Woodin showed that if δ is a Woodin cardinal, there are several stationary towers of height δ which are presaturated. We use the following weakening of presaturation introduced in [2]:

Definition 22 (Cox [2]): A tower of inaccessible height δ has the **weak Chang property** iff whenever G is generic for the tower, then δ is an element of the wellfounded part of $ult(V, G)$ and is regular in $ult(V, G)$ (though not necessarily in $V[G]$).

LEMMA 23: *Let $\mu = \lambda^+$. If a tower $\vec{I}$ of height δ concentrates on $\Gamma := \{M \mid |M| = \lambda \subset M\}$, then $\vec{I}$ has the weak Chang property iff it forces that $j_G(\mu) = \delta$.*

Proof. The fact that $\vec{I}$ concentrates on Γ implies that μ will be the critical point of j_G and $j_G(\mu) \supseteq \delta$ for any generic G (see [5]). Since $j_G(\mu)$ is the successor of λ in $ult(V, G)$, the equivalence follows easily. ∎

Corollary 24: *Let $\mu = \lambda^+$ and assume $\vec{I}$ is a tower of height δ which concentrates on $\Gamma := \{M \mid |M| = \lambda \subset M\}$. If $\vec{I}$ is presaturated then it satisfies the weak Chang Property.*

For the next lemma we will use the following definition, which is also related to saturation properties of ideals (see [2]):

Definition 25 (Cox [2]): Let J be a normal ideal over $\wp(H)$ where

$$\mu = completeness(J) \subseteq H.$$

We say J **bounds its completeness** iff for every $f : \mu \to \mu$ there are $\breve{J}$-many M such that $otp(M \cap ORD) > f(M \cap \mu)$.

Lemma 26: *Suppose $\vec{I} = \langle I_\theta \mid \theta < \delta \rangle$ is a tower of inaccessible height δ, has completeness $\mu := \lambda^+$, concentrates on $\{M \mid |M| = \lambda \subset M\}$, and has the weak Chang property. Then:*

1. *For every generic G and every $\theta < \delta$, $j_{proj(G,\theta)}(\mu) < \delta$.*
2. *There is a restriction of some ideal in the tower which bounds its completeness.*

Proof. Part 1: Suppose not; let $\mu := \lambda^+$ and let G and θ be such that $j_{proj(G,\theta)}(\mu) \geq \delta$. By assumption, $j_G(\mu) = \delta$; so in fact $\delta = j_{proj(G,\theta)}(\mu) = \{[f]_{proj(G,\theta)} \mid f \in ({}^{Z_\theta}\mu)^V\}$. By Fact 19, $\{[f]_{proj(G,\theta)} \mid f \in ({}^{Z_\theta}\mu)^V\}$ is an element of $ult(V, G)$; moreover

$$|\{[f]_{proj(G,\theta)} \mid f \in ({}^{Z_\theta}\mu)^V\}|^{ult(V,G)} \leq |({}^{Z_\theta}\mu)^V|^{ult(V,G)} \leq |({}^{Z_\theta}\mu)^V|^V < \delta$$

(by inaccessibility of δ in V). This contradicts that δ is regular in $ult(V, G)$.

Part 2: By part 1 with $\theta := \mu$, there is a condition (α, A) in the tower which decides the value of $j_{proj(\dot{G},\mu)}(\mu)$ as some $\eta < \delta$. Without loss of generality, assume

$$\eta < \alpha. \tag{8}$$

We show that $I_\alpha \restriction A$ bounds its completeness; a similar argument shows that $I_\beta \restriction A^{Z_\beta}$ bounds its completeness for every $\beta \in [\alpha, \delta)$.

Let $f : \mu \to \mu$. Suppose for a contradiction that there were some $A' \subseteq A$ such that A' is I_α-positive and for every $M \in A'$, $otp(M \cap ORD) \leq f(M \cap \mu)$ (note also that $M \cap ORD = M \cap \alpha$ for all $M \in A'$). Let G be generic for the tower with $(\alpha, A') \in G$. Then $A' \in proj(G, \alpha)$ and so

$$\eta < \alpha = [M \mapsto otp(M \cap \alpha)]_{proj(G,\alpha)} \leq [M \mapsto f(M \cap \mu)]_{proj(G,\alpha)}.$$

Now f maps into μ, so

$$[f]_{proj(G,\mu)} < j_{proj(G,\mu)}(\mu) = \eta.$$

Hence by part 4 of Fact 19, $[f]_{proj(G,\mu)}$ is not moved by $k_{proj(G,\mu),proj(G,\alpha)}$:

$$[f]_{proj(G,\mu)} = k_{proj(G,\mu),proj(G,\alpha)}([f]_{proj(G,\mu)}) = [M \mapsto f(M \cap \mu)]_{proj(G,\alpha)}.$$

But this implies $\eta < \eta$, a contradiction. ∎

4. RP and towers on GIC_{ω_1}

In this section we prove Theorems 1, 4 and 2.

4.1. Proof of Theorem 1. Theorem 1 follows from Corollary 24, Lemma 26, and the following:

Theorem 27: *Assume $RP(\omega_2)$. Then there is no ideal I which bounds its completeness and concentrates on GIC_{ω_1}.*

Proof. Todorcevic proved that $RP(\omega_2)$ implies $2^\omega \leq \omega_2$ (see Theorem 37.18 of [11]). If CH holds, then for every $\theta \geq \omega_2$ the set of ω_1-guessing submodels of H_θ is nonstationary (see [15]) and the theorem holds trivially.

So suppose from now on that $2^\omega = \omega_2$. Suppose for a contradiction that I is a normal ideal concentrating on some stationary subset S of GIC_{ω_1} (at some H_θ), and that I bounds its completeness (which is ω_2). Without loss of generality we assume that for every $M \in S$, $M \prec (H_\theta, \in, \Delta, \phi)$ where ϕ is some enumeration of the reals of order-type ω_2. For each $\alpha \in proj(S, \omega_2) := \{M \cap \omega_2 \mid M \in S\}$ let $T(\alpha)$ be the collection of all transitive sets of the form H_M, where $M \in S$ and $M \cap \omega_2 = \alpha$.

Let $\bar{I}$ be the projection of I to ω_2.

Claim 27.1: *For $\bar{I}$-measure-one many α, $s_\alpha :=$ $\sup\{ht(H) \mid H \in T(\alpha)\}$ *is at least* ω_2.

Proof of Claim 27.1. Suppose not; so there is some S' which is I-positive and for every $M \in S'$, $s_{M\cap\omega_2} < \omega_2$. Let $f : \omega_2 \to \omega_2$ be defined by sending $\alpha \mapsto s_\alpha$ if $s_\alpha < \omega_2$, and $f(\alpha) = 0$ otherwise. Since I bounds its completeness, there is some $C \in \breve{I}$ such that for every $M \in C$, $otp(M \cap ORD) = ht(H_M) > f(M \cap \omega_2)$. Then for every $M \in C \cap S'$, $f(M \cap \omega_2) = s_{M\cap\omega_2} < ht(H_M)$. Fix any $\hat{M} \in C \cap S'$ and let $\hat{\alpha} := \hat{M} \cap \omega_2$; then

$$s_{\hat{\alpha}} < ht(H_{\hat{M}}) \tag{9}$$

yet $H_{\hat{M}} \in T(\hat{\alpha})$; this is clearly a contradiction to the definition of $s_{\hat{\alpha}}$. ∎

Fix any α such that $s_\alpha = \omega_2$, and let $W := \bigcup T(\alpha)$. Now $S \subseteq GIC_{\omega_1}$, so by Corollary 14, whenever H and H' are elements of $T(\alpha)$ and $ht(H) < ht(H')$, then H is a hereditary initial segment of H'; this implies that W is a transitive ZFC model (of height ω_2). Since $H \in IC_{\omega_1}$ for every $H \in T(\alpha)$, then:

$$\text{For every } \beta < \omega_2,\ W \cap [\beta]^\omega \text{ contains a club.} \tag{10}$$

To see why (10) holds, let $\beta < \omega_2$. Pick an $H \in T(\alpha)$ such that $\beta < H \cap ORD$, and let $\langle N_\xi \mid \xi < \omega_1\rangle$ witness that $H \in IC_{\omega_1}$. Then $\{N_\xi \cap \beta \mid \xi < \omega_1\}$ is a closed unbounded subset of $[\beta]^\omega$, and each $N_\xi \cap \beta$ is an element of $H \subset W$.

Now $\mathbb{R} \cap W = \phi[\alpha]$; in particular $\mathbb{R} - W \neq \emptyset$. By Theorem 11, $R := [\omega_2]^\omega - W$ is stationary (in fact projective stationary). By $RP(\omega_2)$, there is a $\beta < \omega_2$ such that $R \cap [\beta]^\omega$ is stationary.[17] This contradicts (10). ∎

4.2. Proof of Theorem 4. Now we prove Theorem 4; that is, if RP is omitted from the hypothesis of Theorem 1, the Isomorphism Theorem for GIC_{ω_1} prevents precipitous towers on GIC_{ω_1} which are definable.

Proof. If CH holds there are no G_{ω_1} structures so the theorem is trivial. So assume $2^\omega = \omega_2$ and let Δ be a wellorder of $\mathbb{R}$ of order-type ω_2. Suppose $\vec{I} = \langle I_\theta \mid \theta < \delta\rangle$ were such a tower. By Lemma 4.3 of Burke [1], a precipitous tower is not an element of the generic ultrapower.[18] Since we are assuming the tower is definable over V_δ, to obtain a contradiction it suffices to show that V_δ

[17] This uses the fact that $\{\beta \mid \omega_1 \leq \beta < \omega_2\}$ is a club subset of $[\omega_2]^{\omega_1}$ and that $RP(\omega_2)$ implies the following apparently stronger statement (see Theorem 3.1 of Feng–Jech [4]): for every stationary $R \subset [\omega_2]^\omega$, there are stationarily many $Z \subset [\omega_2]^{\omega_1}$ such that $\omega_1 \subset Z$ and $R \cap [Z]^\omega$ is stationary.

[18] If the generic embedding moves δ, which is always the case if the tower concentrates on $\{M \mid |M| = \lambda \subset M\}$.

is an element of some generic ultrapower by the tower. Let G be generic for the tower, let $B := \mathbb{R}^V = j_G(\Delta)``\omega_2^V \in ult(V,G)$, and let $(GIC^B_{\omega_1})^{ult(V,G)}$ be as in Corollary 15, i.e., $(GIC^B_{\omega_1})^{ult(V,G)}$ is the collection of transitive ZF^- models in $ult(V,G)$ which are GIC_{ω_1} (from the point of view of $ult(V,G)$) and whose intersection with $\mathbb{R}^{ult(V,G)}$ equals B.

By Fact 19, $j_G``H^V_\theta \in ult(V,G)$ for each $\theta < \delta$. Since $\vec{I}$ concentrates on $GIC_{\omega_1}$ and $\omega_1 < cr(j_G)$, then by the Los Theorem, $ult(V,G) \models$ "$j_G``H^V_\theta \in GIC_{\omega_1}$" for each $\theta \in REG^V \cap [\omega_2^V, \delta)$. Also note that $\mathbb{R} \cap j_G``H^V_\theta = \mathbb{R} \cap H^V_\theta = B$, so in particular by Corollary 15 we have

$$\forall \theta \in REG^V \cap [\omega_2^V, \delta)\colon\ ult(V,G) \models H^V_\theta \in GIC^B_{\omega_1}. \tag{11}$$

Note that (11) is from the point of view of $V[G]$; we do not know yet that $\langle H^V_\theta \mid \theta \in REG^V \cap \delta\rangle$ is an element of $ult(V,G)$. Let

$$W := \bigcup (GIC^B_{\omega_1})^{ult(V,G)} \in ult(V,G).$$

Fix any V-regular $\theta \in [\omega_2^V, \delta)$; by (11) and Corollary 15, $H^V_\theta = H^W_\theta$. Working inside $V[G]$ (not inside $ult(V,G)$) it follows that $H^W_\delta = \bigcup_{\theta \in REG^V \cap \delta} H^V_\theta = V_\delta$. Since $H^W_\delta \in ult(V,G)$ this implies that $V_\delta \in ult(V,G)$, a contradiction. ∎

4.3. Proof of Theorem 2. Finally we prove Theorem 2. It is well-known that either MM or PFA^+ implies RP; so Theorem 2 will follow from Theorem 1 and the following:

Lemma 28: *Assume PFA and let δ be inaccessible. Then there is a tower of height δ which concentrates on GIC_{ω_1}.*

Proof. In [16] it was shown that PFA implies that $GIC_{\omega_1} \cap \wp_{\omega_2}(H_\theta)$ is stationary for all regular $\theta \geq \omega_2$.

For each $\lambda < \delta$ set $Z_\lambda := \{M \cap H_\lambda \mid M \in GIC_{\omega_1} \cap \wp_{\omega_2}(V_\delta)\}$ and set $I_\lambda :=$ the projection of $NS \restriction GIC_{\omega_1} \cap \wp_{\omega_2}(V_\delta)$ to a normal ideal on Z_λ. It is straightforward to check that a sequence of ideals defined in this way is a tower. By Lemma 10, each $Z_\lambda \subset GIC_{\omega_1} \cap \wp_{\omega_2}(H_\lambda)$.

Alternatively, one can check that the sequence $\langle Z_\lambda \mid \lambda < \delta\rangle$ satisfies Lemma 9.49 of [5], and then use Burke's "stabilization" technique to produce a tower of ideals concentrating on the Z_λs. It is not clear whether this yields the same tower as the previous paragraph. ∎

5. SRP and towers on GIS_{ω_1}

In this section we prove Theorem 3. We actually prove a slightly stronger theorem, namely, the assumption $SRP(\omega_2)$ from Theorem 3 can be weakened to the conjunction of $RP(\omega_2)$ with saturation of NS_{ω_1} (these are both implied by $SRP(\omega_2)$; see Theorems 37.22 and 37.23 of Jech [11]). Recall the Tree Property at κ ($TP(\kappa)$) is the statement that every tree of height κ and width $< \kappa$ has a cofinal branch, and saturation of NS_{ω_1} means that $\wp(\omega_1)/NS_{\omega_1}$ has the ω_2-chain condition.

Theorem 3 follows from Corollary 24, Lemma 26, and the following theorem:

THEOREM 29: *Assume $RP(\omega_2)$, NS_{ω_1} is saturated, and $TP(\omega_2)$. Then there is no ideal concentrating on GIS_{ω_1} which bounds its completeness.*

First we prove:

LEMMA 30: *Assume $RP(\omega_2)$, NS_{ω_1} is saturated, and W is a transitive ZF^- model of height ω_2 such that $W \cap [\beta]^\omega$ is stationary for all[19] $\beta < \omega_2$. Then $\mathbb{R} \subset W$.*

Proof. For each $\beta < \omega_2$, fix any $\subset$-increasing and continuous sequence $\langle a_i^\beta | i < \omega_1 \rangle$ in $[\beta]^\omega$ whose union is $[\beta]^\omega$, and set $T_\beta := \{i < \omega_1 \mid a_i^\beta \in W\}$; note that our assumptions on W imply that each T_β is stationary.[20] Also, if $\beta < \beta'$ then $T_\beta \geq_{NS} T_{\beta'}$. Since $\langle T_\beta \mid \beta < \omega_2 \rangle$ is a $\leq_{NS}$-descending sequence of a stationary subset of ω_1 and NS_{ω_1} is saturated, then the sequence stabilizes, i.e., there is some $\beta < \omega_2$ such that $T := T_\beta =_{NS} T_{\beta'}$ for all $\beta' \in [\beta, \omega_2)$.

Now suppose for a contradiction that $\mathbb{R} - W \neq \emptyset$. Then by Corollary 12, $R := [\omega_2]^\omega - W$ is projective stationary; so in particular, $R \searrow T$ is stationary.[21] By $RP(\omega_2)$, there is some $\beta' \in [\beta, \omega_2)$ such that $(R \searrow T) \cap [\beta']^\omega$ is stationary. But

$$\begin{aligned}(R \searrow T) \cap [\beta']^\omega &=_{NS} \{a_i^{\beta'} \mid a_i^{\beta'} \in R \text{ and } i = a_i^{\beta'} \cap \omega_1 \in T\} \\ &=_{NS} \{a_i^{\beta'} \mid a_i^{\beta'} \notin W \text{ and } i \in T_{\beta'}\}\end{aligned}$$

which, by the definition of $T_{\beta'}$, is nonstationary. Contradiction. ∎

19 Equivalently, cofinally many.

20 Also, modulo NS_{ω_1}, T_β does not depend on the particular sequence $\vec{a}^\beta$. This is not needed in the current proof, however.

21 Recall $R \searrow T$ denotes $\{N \in R \mid N \cap \omega_1 \in T\}$.

Now back to the proof of Theorem 29. Suppose for a contradiction that I concentrates on some stationary $S \subseteq GIS_{\omega_1}$ and bounds its own completeness (which is ω_2). Without loss of generality we can assume that for every $M \in S$, $M \prec (H_\theta, \in, \phi)$ where ϕ is some wellorder of the reals of order-type ω_2 and H_θ is the support of I. For each $M \in S$ let $T_M \subseteq \omega_1$ be the stationary set witnessing that $M \in IS_{\omega_1}$. For each $\alpha \in proj(S, \omega_2)$ define

$$T(\alpha) := \{H_M \mid M \in S \text{ and } \alpha = M \cap \omega_2\};$$

the downward closure of $T(\alpha)$ under the hereditary initial segment relation[22] forms a tree of height $\leq \omega_2$.

Claim 30.1: *For each $\alpha \in proj(S, \omega_2)$, the tree $T(\alpha)$ has width $< \omega_2$.*

Proof. Fix such an α and a level $\eta < \omega_2$ of the tree $T(\alpha)$. Note that if H is at the η-th level, then there is some $M \in S$ such that $H = (H_\lambda)^{H_M}$ where λ is the η-th regular cardinal of H_M (or $\lambda = H_M \cap ORD$). Without loss of generality we assume $\eta \geq 2$; then it is straightforward to show that $\sigma_M[H] = M \cap H_{\sigma_M(\lambda)} \in GIS_{\omega_1}$ and that the set T_M—which witnesses that $M \in IS_{\omega_1}$—also witnesses that $M \cap H_{\sigma_M(\lambda)} \in IS_{\omega_1}$.

Suppose for a contradiction that level η had at least ω_2-many distinct nodes $\langle H_\xi \mid \xi < \omega_2\rangle$, and say $T_\xi \subset \omega_1$ witnesses that $H_\xi \in IS_{\omega_1}$. Note all the H_ξs have the same intersection with the reals (namely $\phi[\alpha]$; so they have the same intersection with H_{ω_1} as well). For any distinct pair ξ and ξ', since $H_\xi \neq H_{\xi'}$ then $T_\xi \cap T_{\xi'}$ is nonstationary by Corollary 14. But then $\{T_\xi \mid \xi < \omega_2\}$ would be an ω_2-sized antichain for NS_{ω_1}, contradicting the fact that NS_{ω_1} is saturated. ∎

Let $\bar{I}$ be the projection of I to ω_2.

Claim 30.2: *For $\bar{I}$-measure-one many $\alpha < \omega_2$, the tree $T(\alpha)$ has height ω_2.*

Proof. The proof of Claim 27.1 can be repeated verbatim. ∎

So by Claims 30.1 and 30.2, for $\bar{I}$-measure-one many $\alpha < \omega_2$, $T(\alpha)$ is a thin tree of height ω_2. Fix such an α. By $TP(\omega_2)$, $T(\alpha)$ has a cofinal branch. The union of this branch is a transitive ZFC model W of height ω_2 such that $\mathbb{R} \cap W = \phi``\alpha$ (so in particular $\mathbb{R} - W \neq \emptyset$) and cofinally many proper initial

[22] Namely, the nodes of $T(\alpha)$ consists of transitive models of the form H_M and models of the form $(H_\lambda)^{H_M}$ where $\lambda \in REG^{H_M}$.

segments of W are in IS_{ω_1}; so in particular $W \cap [\beta]^\omega$ is stationary for all $\beta < \omega_2$. This contradicts Lemma 30.

6. Consistency of MM^+ with a precipitous tower on GIC_{ω_1}

Now we prove Theorem 5. First, we need a "tower" version of Proposition 7.13 from [5].

THEOREM 31 (modification of Proposition 7.13 from [5] for towers): *Suppose $\mathbb{Q} \in V$ is a poset and $1_{\mathbb{Q}} \Vdash$ "δ remains inaccessible, $\langle \dot{U}_\lambda \mid \lambda < \delta \rangle$ is a tower of V-normal measures, and $ult(V, \dot{\vec{U}})$ is wellfounded". Suppose also that in V there are functions Q, h, and for each $q \in \mathbb{Q}$ a function f_q such that:*

- *Q, h, and each f_q each have bounded support in V_δ;*
- *for every $\mathbb{Q}$-generic object H:*
  - *$[Q]_{\dot{\vec{U}}_H} = \mathbb{Q}$;*
  - *$[h]_{\dot{\vec{U}}_H} = H$;*
  - *for every $q \in \mathbb{Q}$, $[f_q]_{\dot{\vec{U}}_H} = q$.*

If $\vec{I} \in V$ is the tower derived from the name $\dot{\vec{U}}$ as in Definition 20, then $\mathbb{P}_{\vec{I}}$ is precipitous, forcing equivalent to $\mathbb{Q}$, and generic ultrapowers by $\vec{I}$ are exactly those maps of the form $j_{\dot{\vec{U}}_H} : V \to ult(V, \dot{\vec{U}}_H)$ where H is $(V, \mathbb{Q})$-generic.

Proof. First, we note that if $\vec{I}$ is a tower where each ideal $I_\lambda \subset \wp(Z_\lambda)$, then the poset $\mathbb{P}_{\vec{I}}$ (as defined in Section 2.4) is forcing equivalent to the poset obtained as follows: Define an equivalence relation on $\mathbb{P}_{\vec{I}} = \{(\lambda, S) \mid \lambda < \delta \text{ and } S \in I_\lambda^+\}$ by

(12) $(\lambda, S) \simeq (\beta, T)$ iff $S^{Z_\eta} \triangle T^{Z_\eta} \in I_\eta$ for some (equivalently: every) $\eta \geq \max(\lambda, \beta)$.

Let $\mathbb{P}'_{\vec{I}} := \mathbb{P}_{\vec{I}} / \simeq$ and partially order $\mathbb{P}'_{\vec{I}}$ in the natural way inherited from the partial ordering of $\mathbb{P}_{\vec{I}}$.[23]

Now let $\vec{I}$ be the tower derived from the name $\dot{\vec{U}}$ as in the statement of the theorem. Similarly to the way Proposition 7.13 from [5] is proved, we define a

[23] Another way to view the poset $\mathbb{P}'_{\vec{I}}$ is to consider the directed system of "canonical liftings" $\iota_{\lambda,\lambda'} : \wp(Z_\lambda)/I_\lambda \to \wp(Z_{\lambda'})/I_{\lambda'}$ (for $\lambda \leq \lambda'$) defined by $[S]_{I_\lambda} \mapsto [S^{Z_{\lambda'}}]_{I_{\lambda'}}$. Then $\mathbb{P}'_{\vec{I}}$ is the direct limit of this system.

map $\phi : \mathbb{P}'_{\vec{I}} \to ro(\mathbb{Q})$ by

$$[(\lambda, S)]_{\simeq} \mapsto \left\| j_{\dot{\vec{U}}_H}\text{“}\left(\bigcup Z_\lambda\right) \in j_{\dot{\vec{U}}_H}(S)\right\|_{ro(\mathbb{Q})}. \tag{13}$$

It is straightforward to check that this map is well-defined and preserves order and incompatibility. Further, identifying $\mathbb{Q}$ with its isomorphic copy in $ro(\mathbb{Q})$, the assumptions of the theorem imply that $\mathbb{Q} \subseteq range(\phi)$: given $q \in \mathbb{Q}$, let f_q be as in the statement of the theorem, and let $\lambda_q < \delta$ be the support of f_q (and without loss of generality assume λ_q is also greater than the support of h and Q). Then ϕ maps the condition $[(\lambda_q, \{M \in Z_{\lambda_q} \mid f_q(M) \in h(M)\})]_{\simeq}$ to q.

Finally, we show why generic ultrapowers by $\vec{I}$ are exactly those embeddings of the form $j_{\dot{\vec{U}}_H}$ for some H which is $(V, \mathbb{Q})$-generic; we thank the referee for pointing out a simpler proof than our original proof. Let G and H be generics for $\mathbb{P}'_I$ and $\mathbb{Q}$, respectively, such that ϕ“$G = H$ (more precisely, such that H is the upward closure of ϕ“G). Then for each $\lambda < \delta$ and each $S \subseteq Z_\lambda$,

$$S \in proj(G, \lambda) \iff [\lambda, S]_{\simeq} \in G \iff \phi([\lambda, S]_{\simeq}) \in H \iff S \in (\dot{U}_\lambda)_H.$$

It follows that $\langle proj(G, \lambda) \mid \lambda < \delta\rangle = \langle (\dot{U}_\lambda)_H \mid \lambda < \delta\rangle$, and so they yield the same ultrapower. ∎

COROLLARY 32: *Suppose $\vec{U} \in V$ is a tower of normal ultrafilters of inaccessible height δ and $j_{\vec{U}} : V \to_{\vec{U}} N_{\vec{U}}$ is the ultrapower. Suppose $\mathbb{P} \in V_\delta$ and that $j_{\vec{U}} \restriction \mathbb{P} : \mathbb{P} \to j_{\vec{U}}(\mathbb{P})$ is a regular embedding. Let G be $(V, \mathbb{P})$-generic, and let $\vec{I} \in V[G]$ be the tower of height δ induced by $j_{\vec{U}}$ as in Definition 21.*

Then in $V[G]$, $\vec{I}$ is precipitous, $\mathbb{P}_{\vec{I}}$ is forcing equivalent to $j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}$“$G$, and generic ultrapowers of $V[G]$ by $\vec{I}$ are exactly those maps of the form $j^{G*H}_{\vec{U}} : V[G] \to N_{\vec{U}}[G][H]$ where H is $j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}$“$G$-generic over $V[G]$.

Proof. Let G be $(V, \mathbb{P})$-generic. We check the conditions of Theorem 31; here $V[G]$ will play the role of the V from Theorem 31 and $j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}$“$G$ will play the role of the $\mathbb{Q}$ from Theorem 31.

Work in $V[G]$. For all H which are $(V[G], j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}$“$G)$-generic, for every $\lambda < \delta$, there are U^{G*H}_λ-many M' such that:

(1) $M' \cap V \in V$;[24] denote this set M.
(2) $V \models$ “$M \cap \mathbb{P}$ is a regular subposet of $\mathbb{P}$”.

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[24] This holds for U^{G*H}_λ-many M' because $N_{\vec{U}} \cap j^{G*H}_{\vec{U}}$“$H_\lambda[G] = j_{\vec{U}}$“$H_\lambda$.

Since we assume $\mathbb{P} \in V_\delta$, there is some $\lambda_{\mathbb{P}} < \delta$ such that $\mathbb{P} \in H_{\lambda_{\mathbb{P}}}$. Now consider the following functions defined in $V[G]$ on

$$A^\delta := \{M' \prec V_\delta[G] \mid M' \cap V \in V \text{ and } M' \cap \mathbb{P} \text{ is a regular subposet of } \mathbb{P}\} :$$

- $Q(M') := \mathbb{P}/(G \cap M')$; note this equals $\mathbb{P}/(G \cap M' \cap H^V_{\lambda_{\mathbb{P}}})$.
- $h(M') :=$ the generic for $\mathbb{P}/(G \cap M')$ obtained from G and the forcing equivalence between $\mathbb{P}$ and $(M' \cap \mathbb{P}) * (\mathbb{P}/(\dot{G} \cap M'))$. Note this only depends on $M' \cap H^V_{\lambda_{\mathbb{P}}}$.
- For any $q \in j_{\vec{U}}(\mathbb{P})/G$, note that $q \in V$ and there is some $f_q : V_\delta \to V$ with support $\lambda_q < \delta$ such that $q = [f_q]_{\vec{U}}$. Then define (in $V[G]$) the function f'_q by $M' \mapsto f_q(M' \cap H^V_{\lambda_q})$.

Note that each of these functions has bounded support in V_δ (Q and h have support $\lambda_{\mathbb{P}}$, and f'_q has support λ_q). Now we check that for every H which is $(V[G], j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}``G)$-generic:

(1) $[Q]_{\vec{U}^{G*H}} = j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}``G$;
(2) $[h]_{\vec{U}^{G*H}} = H$;
(3) for each $q \in j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}``G$, $[f_q]_{\vec{U}^{G*H}} = q$.

We will show (1) to give the reader the idea of how this is done; the others are similar. Let $j := j_{\vec{U}}$ and let $\hat{G} \subset j(\mathbb{P})$ be the $(V, j(\mathbb{P}))$-generic obtained from $G * H$ via the forcing equivalence of $j(\mathbb{P})$ with $\mathbb{P} * j(\mathbb{P})/j``\dot{G}$. By the definition of j^{G*H} (see (6)) and the fact that $\mathbb{P} \subset H_{\lambda_{\mathbb{P}}}$ (so $\hat{G} \subset j(H_{\lambda_{\mathbb{P}}})$),

$$j^{G*H}(G) = \hat{G} \quad \text{and} \quad j^{G*H}``G = j``G = \hat{G} \cap j``H_\lambda.$$

Let $\mathrm{id}_{\lambda_{\mathbb{P}}}$ denote the function with domain A^δ and support $\lambda_{\mathbb{P}}$ defined by $M' \mapsto M' \cap H^V_{\lambda_{\mathbb{P}}}$. Then (in what follows, the equivalence classes are with respect to the equivalence relation $\simeq_{\vec{U}^{G*H}}$ for the tower $\vec{U}^{G*H}$; see Section 2.4):

$$\begin{aligned} j``G =& \hat{G} \cap j``H_\lambda = \hat{G} \cap [\mathrm{id}_{\lambda_{\mathbb{P}}}] = j^{G*H}(G) \cap [\mathrm{id}_{\lambda_{\mathbb{P}}}] \\ =& [M' \mapsto G] \cap [\mathrm{id}_{\lambda_{\mathbb{P}}}] = [M' \mapsto G \cap M' \cap H_{\lambda_{\mathbb{P}}}]. \end{aligned}$$

Note also that $G \subset V$ so $[M' \mapsto G \cap M' \cap H_{\lambda_{\mathbb{P}}}] = [M' \mapsto G \cap M']$. Hence we have shown

$$j``G = [M' \mapsto G \cap M']_{\vec{U}^{G*H}}. \tag{14}$$

Now combining (14) with the fact that $j(\mathbb{P}) = [M' \mapsto \mathbb{P}]$ yields that

$$[Q]_{\vec{U}^{G*H}} = [M' \mapsto \mathbb{P}/(G \cap M')] = [M' \mapsto \mathbb{P}]/[M' \mapsto G \cap M'] = j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}``G$$

which is what we wanted to show. The other equalities are proved in a similar manner. The conclusion then follows by Theorem 31. ∎

It is interesting to note that if $\vec{I} \in V[G]$ is as in Corollary 32 and δ is always moved by generic embeddings of V by $\vec{I}$,[25] then generic ultrapowers of $V[G]$ by $\vec{I}$ do not have $\mathbb{P}_{\vec{I}}$ as an element (by Lemma 4.3 of [1]). However, these generic ultrapowers do have a poset—namely $j_{\vec{U}}(\mathbb{P})/j_{\vec{U}}$"$G$—which, from the point of view of $V[G]$, is forcing equivalent to $\mathbb{P}_{\vec{I}}$ (and all the generic ultrapowers even have a $V[G]$-generic for that poset).

Now back to the proof of Theorem 5. Suppose κ is supercompact and $\delta > \kappa$ is inaccessible. Let $Lav : \kappa \to V_\kappa$ be a Laver function for κ, and $\mathbb{P}$ the standard RCS iteration of length κ which yields a model of Martin's Maximum as in [7]; this actually produces a model of $MM^{+\omega_1}$. In V let U be a normal measure on $\wp_\kappa(H_\eta)$ for some regular $\eta \geq \delta$ such that $j_U(Lav)(\kappa) = \dot{\mathbb{R}}_\delta$, where $\mathbb{R}_\delta$ is the poset from Theorem 8 and $\dot{\mathbb{R}}_\delta$ is the canonical $\mathbb{P}$-name for $(\mathbb{R}_\delta)^{V^{\mathbb{P}}}$. Let $\vec{U} := \langle U_\lambda \mid \lambda < \delta \rangle$ be the tower of normal measures produced from projections of U to $\wp_\kappa(H_\lambda)$ for $\lambda < \delta$. Let $j_{\vec{U}} : V \to N_{\vec{U}}$; recall $N_{\vec{U}}$ is closed under $< \delta$ sequences so in particular $j_{\vec{U}} \restriction H_\lambda \in N_{\vec{U}}$ for every $\lambda < \delta$. Since $\mathbb{P}$ has the κ-cc, then $j_{\vec{U}} \restriction \mathbb{P} = id : \mathbb{P} \to j_{\vec{U}}(\mathbb{P})$ is a regular embedding, so the discussion before Definition 21 applies. Fix some G which is $(V, \mathbb{P})$-generic, and in $V[G]$ let $\vec{I}$ be the tower of ideals induced by $j_{\vec{U}}$ as in Definition 21. By Corollary 32

$$\vec{I} \text{ is precipitous.} \tag{15}$$

So we only have left to show that $\vec{I}$ concentrates on GIC_{ω_1}. First we note:

Claim 33: $j_{\vec{U}}(Lav)(\kappa) = \dot{\mathbb{R}}_\delta$

Proof. By standard arguments there is a $k : N_{\vec{U}} \to ult(V,U)$ such that $k \circ j_{\vec{U}} = j_U$ and $k \restriction \delta = id$. Now $\dot{\mathbb{R}}_\delta = j_U(Lav)(\kappa) = k \circ j_{\vec{U}}(Lav)(k(\kappa)) = k(j_{\vec{U}}(Lav)(\kappa))$; so $\dot{\mathbb{R}}_\delta \in range(k)$. Recall from Theorem 8 that the poset $\mathbb{R}_\delta$ is always an element of H_{δ^+}; so the canonical $\mathbb{P}$-name $\dot{\mathbb{R}}_\delta$ for $\mathbb{R}_\delta^{V^{\mathbb{P}}}$ is an element of $H^V_{\delta^+} = H^{ult(V,U)}_{\delta^+}$. Hence $|\dot{\mathbb{R}}_\delta|^{ult(V,U)} = \delta$. Then since $\dot{\mathbb{R}}_\delta \in range(k)$, we have $\delta = |\dot{\mathbb{R}}_\delta|^{ult(V,U)} \in range(k)$. This implies that $cr(k) > \delta$ (equivalently, that $j_{\vec{U}}(\kappa) > \delta$) and that $k^{-1}(\dot{\mathbb{R}}_\delta) = \dot{\mathbb{R}}_\delta$. ∎

Consider an arbitrary H which is $(V[G], j_{\vec{U}}(\mathbb{P})/G)$-generic. Let H^* denote the κ-th component of H. Now $N_{\vec{U}}[G][H^*] \models V_\delta[G] \in GIC_{\omega_1}$ because H^*

[25] This is always the case if each U_λ in the original tower concentrates on $\wp_\kappa(H_\lambda)$.

is $(N_{\vec{U}}[G], \mathbb{R}_\delta^{N_{\vec{U}}[G]})$-generic (note $V_\delta = V_\delta^{N_{\vec{U}}}$ because $N_{\vec{U}}$ is closed under $< \delta$ sequences from V). Since $N_{\vec{U}}[G][H]$ is an outer model of $N_{\vec{U}}[G][H^*]$ with the same ω_1, then Theorem 8 implies

$$N_{\vec{U}}[G][H] \models V_\delta[G] \in GIC_{\omega_1}. \tag{16}$$

By (16) and (the transitivised variant of) Theorem 10:

(17) For every V-regular $\lambda \in [\kappa, \delta]$: $N_{\vec{U}}[G][H] \models H_\lambda[G] \in GIC_{\omega_1}$.

Since $j_{\vec{U}}^{G*H} \restriction H_\lambda[G]$ is an element of $N_{\vec{U}}[G][H]$ for every $\lambda < \delta$ and the class GIC_{ω_1} is closed under isomorphism, then (17) implies:

(18) For every V-regular $\lambda \in [\kappa, \delta)$, $N_{\vec{U}}[G][H] \models$ "$j_{\vec{U}}^{G*H}$ "$H_\lambda[G]$ is an element of GIC_{ω_1}".

Since (18) holds for arbitrary generic H, then by the definition of each I_λ:

(19) For each $\lambda < \delta$, I_λ concentrates on GIC_{ω_1}.

This concludes the proof of Theorem 5.

7. Questions

We end with some questions.

We proved that under $RP([\omega_2]^\omega)$, there is no presaturated tower which concentrates on GIC_{ω_1}. This suggests a couple of questions:

Question 34: Is it consistent with $RP([\omega_2]^\omega)$ that there is a presaturated tower concentrating on GIS_{ω_1}?

Question 35: Is it consistent with ZFC to have a presaturated tower which concentrates on GIC_{ω_1}?

One way to produce a presaturated tower on GIS_{ω_1} is to perform a "Mitchell collapse" so that an almost-huge cardinal becomes ω_2; however, $RP[\omega_2]^\omega$ fails in this model, so it does not provide an affirmative answer to Question 34.

We also showed that MM implies there is no presaturated tower on GIS_{ω_1}, which suggests:

Question 36: Is it consistent with MM that there is a presaturated tower concentrating on GIU_{ω_1}?

Question 37: If the answer to either of the previous questions is "yes", can this tower be a stationary tower? Or any other kind of "natural" tower?

Finally, in Theorem 4 we showed there is no precipitous tower on GIC_{ω_1} which is definable over V_δ (where δ is the height of the tower).

Question 38: Suppose NS_{ω_1} is saturated. Does this imply that there is no precipitous tower on GIS_{ω_1} which is definable over V_δ (where δ is the height of the tower)?

References

[1] D. R. Burke, *Precipitous towers of normal filters*, Journal of Symbolic Logic **62** (1997), 741–754.

[2] S. Cox and M. Zeman, *Ideal projections as forcing projections*, in preparation.

[3] Qi Feng and T. Jech, *Projective stationary sets and a strong reflection principle*, Journal of the London Mathematical Society **58** (1998), 271–283.

[4] Qi Feng and T. Jech, *Local clubs, reflection, and preserving stationary sets*, Proceedings of the London Mathematical Society **58** (1989), 237–257.

[5] M. Foreman, *Ideals and generic elementary embeddings*, in *Handbook of Set Theory*, Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 885–1147.

[6] M. Foreman and M. Magidor, *Large cardinals and definable counterexamples to the continuum hypothesis*, Annals of Pure and Applied Logic **76** (1995), 47–97.

[7] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and non-regular ultrafilters. I*, Annals of of Mathematics **127** (1988), 1–47.

[8] M. Foreman and S. Todorcevic, *A new Löwenheim–Skolem theorem*, Transaction of the American Mathematical Society **357** (2005), 1693–1715 (electronic).

[9] M. Gitik, *Nonsplitting subset of* $\mathcal{P}_\kappa(\kappa^+)$, The Journal of Symbolic Logic **50** (1985), 881–894.

[10] J. D. Hamkins, *Some second order set theory*, in *Logic and its Applications*, Lecture Notes in Computer Science, Vol. 5378, Springer, Berlin, 2009, pp. 36–50.

[11] T. Jech, *Set Theory*, The third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[12] A. Kanamori, *The Higher Infinite*, Large cardinals in set theory from their beginnings, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[13] P. B. Larson, *The Stationary Tower*, Notes on a course by W. Hugh Woodin, University Lecture Series, Vol. 32, American Mathematical Society, Providence, RI, 2004.

[14] B. Veličković, *Forcing axioms and stationary sets*, Advances in Mathematics **94** (1992), 256–284.

[15] M. Viale, *Guessing models and generalized Laver diamond*, Annals of Pure and Applied Logic **163** (2012), 1660–1678.

[16] M. Viale and C. Weiss, *On the consistency strength of the proper forcing axiom*, Annals of Pure and Applied Logic **228** (2011), 2672–2678.

[17] C. Weiss, *The combinatorial essence of supercompactness*, Annals of Pure and Applied Logic **136** (2012), 1710–1717.

[18] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, de Gruyter Series in Logic and its Applications, Vol. 1, Walter de Gruyter & Co., Berlin, 1999.