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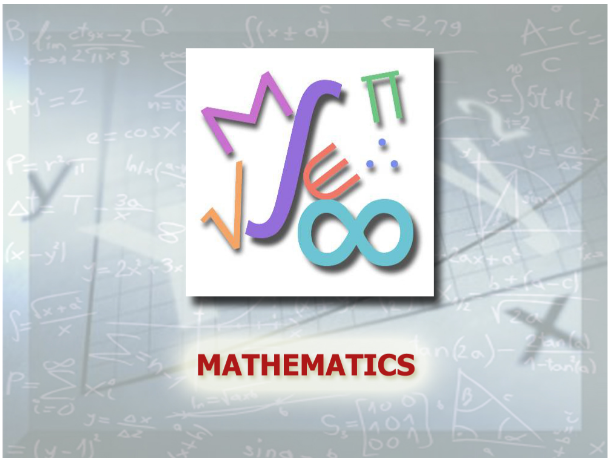

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Abstract

We describe almost abelian Lie algebras and solvmanifolds. In particular we state and use a method to find lattices of almost abelian Lie groups and we find the de Rham cohomology of solvmanifolds arising from Lie groups of this kind. Then we use the description of their minimal models to state properties about formality and symplectic structures.

Regarding Lie algebras, we describe the complex structures in the almost abelian case and the Dolbeault minimal models for general complex structures on nilpotent Lie algebras.

Contents

[1]In Table 1 we impose conditions which become at every step more restrictive. It is therefore implicit that the previous conditions hold only when the more restrictive ones are not satisfied.

Introduction

The object of this thesis is the study of global features and properties of some particular classes of solvable Lie algebras and solvmanifolds.

To describe globally a differential manifold is in general quite difficult. Indeed by definition we can obtain concrete and precise informations only locally, i.e. in the neighborhood of a chosen point. Fortunately there are particular kind of differential manifolds, in our case solvmanifolds and nilmanifolds, for which it is possible to find global properties.

Nilmanifolds are defined as compact homogeneous spaces G/Γ, where G is a connected and simply connected nilpotent Lie group and Γ is a lattice in G. The obvious generalizations of nilmanifolds are solvmanifolds that are obtained taking G solvable.

Nilmanifolds provided in the 1980s the first examples of symplectic manifolds without a Kähler structure. A symplectic structure over a differential manifold M is a closed and not degenerate 2-form, while a Kähler metric is given by a J-hermitian Riemannian metric on a complex manifold (M, J) whose fundamental form is closed [14].

Even if a Kähler structure is richer than a symplectic one, for many years no one was able to find symplectic manifolds with no Kähler structure. The first example of a differential manifold with this feature is due to Thurston [46] and it is a nilmanifold. Indeed in 1988 Benson and Gordon proved that a nilmanifold has a Kähler structure if and only if it is a torus [3].

Many important global properties of nilmanifolds can not be generalized for solvmanifolds, for this reason these manifolds are presently widely studied.

One of the most important features of nilmanifolds G/Γ is that we can always compute their de Rham cohomology in terms of the Lie algebra $\mathfrak{g}$ of G. In 1920 de Rham proved the isomorphism between the de Rham cohomology groups and the singular cohomology ones, then the de Rham cohomology gives us topological and homotopical informations about the manifold, but it is in general difficult to compute because it is a global object.

In 1954 Numizu proved that we can always compute the cohomology of a nilmanifold because it is isomorphic to the cohomology of the associated Lie algebra [35]. Unfortunately this is not true in general for solvmanifolds, but only in particular cases. For example when the Mostow condition holds [34] we are sure that this isomorphism holds. For this reason we consider a technique to compute the de Rham cohomology of solvmanifolds. We will apply it to particular solvmanifolds G/Γ called almost abelian, i.e. $G = \mathbb{R} \ltimes \mathbb{R}^n$ and $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^n$.

In order to study nilmanifolds and solvmanifolds, we need previously to construct them. In particular it is in general not easy to understand when a discrete subgroup of a Lie group is a lattice. Again for nilmanifolds we have a complete theory, indeed Malčev Theorem assures us that we can find a lattice in a nilpotent Lie group if and only if the structure constants of the associated Lie algebra are rational.

As for many other features, also the existence of a lattice is not as much easy to find for a general solvable Lie group. Fortunately if the solvable Lie group is almost abelian we have a method to construct lattices (Proposition 1.3).

Nilmanifolds are important also in relation to minimal models.

Minimal models are objects of rational homotopy theory introduced by Sullivan in the 1960s to describe the rational part of homotopy groups [14], but they provide also informations on the cohomology of differential manifolds.

By definition the minimal model of a nilmanifold can always be computed, but again there is not a generalization of this property for solvmanifolds, so the study of the models of solvmanifolds is quite interesting.

All these helpful properties of nilmanifolds do not hold for the Dolbeault cohomology. For instance, we cannot state general theorems like the one of Nomizu for the Dolbeault cohomology of nilmanifolds, but for nilmanifolds endowed with some classes of complex structures the Dolbeault cohomology can be computed in terms of invariant differential forms [7, 9, 41].

For this reason we are interested in complex structures of solvable Lie algebras and Dolbeault minimal models of nilpotent ones.

In Chapter 1 we give the basic definitions and properties that we will use in the following chapters. In the first section we define solvmanifolds and nilmanifolds and we state properties related to their cohomology, in particular we are interested in understanding when the cohomology of a solvmanifold is isomorphic to the cohomology of the associated Lie algebra, i.e. when the Mostow condition holds (Theorems 1.7, 1.8 and 1.10).

In the second section we describe complex structures of general vector spaces, of differential manifolds and with more details of Lie algebras (Proposition 1.4) stating also a general version of the $\partial\bar{\partial}$-Lemma.

In the third one we define minimal models and we state only the propositions and theorems that we will use in Chapter 6 [14], in particular we are interested in models of fibrations and the concept of formality.
The last section is about symplectic structures and the Hard Lefschetz property.

From Chapter 2 we begin with original material.

In Chapter 2 we describe a symplectic version of the Hodge theory developed by Tseng and Yau [47] related to the Hard Lefschetz property, in particular we are interested in the other cohomology groups that they define. Indeed we prove that if the de Rham cohomology of a solvmanifold is isomorphic to the invariant one, then also these symplectic cohomologies are isomorphic to cohmologies of the Lie algebra (Theorem 2.2) [25].

In Chapter 3 we find Betti numbers and symplectic structures of six dimensional unimodular solvable Lie algebras (Appendices B and C).

Then we compute the dimensions of the invariant cohomologies, finding by Theorem 2.2 symplectic solvmanifolds for which the Hard Lefschetz property holds (Theorem 3.3).

In Chapter 4 we consider solvmanifolds for which the Mostow condition could not hold. In the first section we describe and use a method to compute lattices of

many six dimensional almost abelian solvmanifolds (Theorem 4.1).
Unlike nilpotent Lie groups for which Malčev Theorem (Theorem 1.4) gives a simple criterion for the existence of a lattice, for solvable Lie group it is in general a hard task to find a lattice. In the case of almost abelian Lie groups there is a method (Proposition 1.3) which we apply to determine lattices in six dimensional examples for which the Mostow condition does not hold.

In the second section we describe a technique developed by Kasuya to compute the de Rham cohomology of some solvmanifolds (Proposition 4.1) and in the third one we use it to compute the de Rham cohomology groups of some six dimensional almost abelian solvmanifolds.

In Chapter 5 we describe complex structures of almost abelian Lie algebras.

In the first section we consider the real case $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^n$. First we study when $\mathfrak{g}$ can admit a generic complex structure J and in this case we find a description of J (Theorem 5.3), then we consider two particular cases of complex structures, namely bi-invariant and abelian structures. In particular we prove that almost abelian Lie algebras does not admit bi-invariant complex structures (Theorem 5.4) and that only one kind of almost abelian Lie algebras admits an abelian one (Theorem 5.5). For this last structure we are also able to compute the Dolbeault cohomology.

In the second section we generalize the concept of almost abelian Lie algebra and consider a complex analogue $\mathfrak{g} = \mathbb{C} \ltimes_{\mathrm{ad}} \mathbb{C}^n$ with $\dim_{\mathbb{R}} \operatorname{Im} \mathrm{ad} = 1$. In this case we are again able to study a particular type of complex structure and find similar results to the real case (Theorem 5.7). Moreover we prove that for these complex Lie algebras the $\partial\bar{\partial}$-Lemma does not hold (Theorem 5.8).

In Chapter 6 we study minimal models. In the first section we consider nilmanifolds and the work of Hasegawa [20], in the second one we study minimal models of almost abelian solvmanifolds.
We start from the idea of Oprea and Tralle [36] of using the Mostow fibration and the model of fibrations (Theorem 6.2) to compute the cohomology of almost abelian solvmanifolds. We use the method described by Oprea and Tralle and the cohomology groups found in Chapter 4.3 to compute the minimal models of some six dimensional almost abelian solvmanifolds. Then we use this same method to find properties about

formality (Theorem 6.4) and symplectic structures (Proposition 6.7) [26].

In the third section we define Dolbeault minimal models proving that we have not an existence theorem in this case (Example 6.4). Then we prove that every nilpotent Lie algebra endowed with a complex structure is Dolbeaul minimal (Theorem 6.9), generalizing a result of Cordero, Fernández and Ugarte [10].

Chapter 1

Preliminaries

1.1 Solvmanifolds

We recall some basic definitions of Lie group and Lie algebras, for a complete description of this topic see for example [14, 18, 49].

Definition 1.1. A *Lie group* is a differential manifold G that is endowed with a group structure such that the map

$$\begin{array}{rcl} G\times G & \to & G \\ (a,b) & \mapsto & ab^{-1} \end{array}$$

is $\mathcal{C}^\infty$.

A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear, antisimmetric map called *bracket* $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that $\forall\, X,Y,Z\in\mathfrak{g}$

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0 \quad \textit{Jacobi identity}$$

Example 1.1. Examples of Lie groups are

- $(\mathbb{R}^n,+)$, $(\mathbb{C}^n,+)$.
- $(\mathbb{R}\smallsetminus\{0\},\cdot)$, $(\mathbb{C}\smallsetminus\{0\},\cdot)$, $S^1\subset\mathbb{C}\smallsetminus\{0\}$.
- $(GL_n(\mathbb{R}),\cdot)$.

- The product $G \times H$ of two Lie groups G and H.
- The torus $\mathbb{T}^n$ as product of the unit circle S^1 n times.

Examples of Lie algebras are

- The set of smooth vector fields $\chi(M)$ over a differential manifold M.
- Every vector space V with bracket $[\ ,\ ] \equiv 0$, in particular this Lie algebra is called *abelian.*
- The set of real $n \times n$ matrices $\mathfrak{gl}_n(\mathbb{R})$ with bracket given by

$$[A, B] := AB - BA \qquad \forall\, A, B \in \mathfrak{gl}_n(\mathbb{R}).$$

  In particular this means that for every vector space V of dimension n, $\mathfrak{gl}(V) := End(V)$ is a Lie algebra.

If $\mathfrak{g} = \langle X_1, \cdots, X_n \rangle$, with bracket defined by $[X_i, X_j] = \sum_{k \leq n} c_{i,j}^k X_k \quad \forall\, i, j \leq n$, we call the scalars $c_{i,j}^k$ *structure constants* of $\mathfrak{g}$.

Given a Lie group G and $g \in G$, let L_g and R_g be respectively the left and right translations, then

Definition 1.2. A vector field $X \in \chi(G)$ is *left invariant* if

$$\forall\, a, b \in G \quad (L_a)_* X_b = X_{ab}.$$

Similarly we define right invariant vector fields.
The set of left invariant vector fields is a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

Example 1.2. The Lie algebra of the Lie group $GL_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R})$.

Remark 1.1. The Lie algebra $\mathfrak{g}$ associated to the Lie group G can be identified to tangent space in the identity element e_G by the isomorphism of vector spaces

$$\begin{array}{ccc} \mathfrak{g} & \to & Te_G G \\ X & \mapsto & X_{e_G} \end{array}.$$

Important tools in the study of Lie groups and Lie algebras are the adjoint representations and the exponential map:

Definition 1.3. Given a Lie algebra $\mathfrak{g}$ its *adjoint representation* is

$$\begin{array}{rcl} \mathrm{ad} : \mathfrak{g} & \rightarrow & \mathfrak{gl}(\mathfrak{g}) \\ X & \mapsto & \mathrm{ad}_X \end{array}$$

where $\mathrm{ad}_X(Y) := [Y, X]$.

Given a Lie group G its *adjoint representation* is

$$\begin{array}{rcl} \mathrm{Ad} : G & \rightarrow & \mathfrak{gl}(\mathfrak{g}) \\ g & \mapsto & \mathrm{Ad}_g \end{array}$$

where $\mathrm{Ad}_g(X) := (I_g)_*(X)$ with $I_g := L_g R_{g^{-1}}$.

Definition 1.4. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The *exponential map* is

$$\begin{array}{rcl} \exp : \mathfrak{g} & \rightarrow & G \\ X & \mapsto & \Phi_X(1) \end{array}$$

where Φ_X is the integral curve of the vector field X such that $\Phi_X(0) = e_G$.

Proposition 1.1. *For the exponential map the following properties hold:*

- $\Phi_X(t + s) = \Phi_X(t) \cdot \Phi_X(s) \quad \forall t, s \in \mathbb{R}$,
- $\Phi_X(ts) = \Phi_{tX}(s) \quad \forall t, s \in \mathbb{R}$,
- $Ad(\exp X) = e^{ad_X} \quad \forall X \in \mathfrak{g}$.

In our work Lie groups and Lie algebras will always be sets of matrices, indeed we have the following theorem.

Theorem 1.1. (Ado) *[49] Every finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.*

In particular this theorem implies that every real Lie algebra of finite dimension n is a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. As a consequence for every finite dimensional Lie algebra $\mathfrak{g}$ there is a Lie group $G < GL_n(\mathbb{R})$ such that $\mathfrak{g} = Lie(G)$ [49].

Given a Lie group G we can construct another differential manifold by the quotient map:

Definition 1.5. Let H be a closed subgroup of a Lie group G. The set of left cosets of H, G/H is called a *homogeneous space*.

It inherits the structure of differential manifold by G using the projection map $G \to G/H$.

Solvmanifolds are important types of homogeneous spaces defined as follow.

Definition 1.6. A *solvmanifold* S is a compact homogeneous space $S = G/\Gamma$, where G is a connected and simply connected solvable Lie group and Γ is a lattice in G, i.e. a discrete subgroup with compact quotient space.
If G is nilpotent, the homogeneous space is called *nilmanifold*.

We recall that given a Lie algebra $\mathfrak{g}$ its *derived series* $\mathfrak{d}^*$ and *descending series* $\mathfrak{g}^*$ are defined inductively by

$$\mathfrak{d}^0 = \mathfrak{g}^0 = \mathfrak{g}, \qquad \mathfrak{d}^k = [\mathfrak{d}^{k-1}, \mathfrak{d}^{k-1}], \qquad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$$

A Lie group G and its Lie algebra $\mathfrak{g}$ are called *solvable* or *nilpotent* if there exist $\bar{k}$ such that respectively $\mathfrak{d}^{\bar{k}} = 0$ or $\mathfrak{g}^{\bar{k}} = 0$.
In particular every nilpotent Lie algebra is also solvable.

Example 1.3. Abelian Lie algebras are trivial examples of nilpotent Lie algebras, then the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a nilmanifold.

The set of real strictly upper triangular matrices is a nilpotent Lie algebra, while the set of real upper triangular matrices is a solvable Lie algebra.

The last example is very important indeed it describes all solvable Lie algebras:

Theorem 1.2. (Engel) *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if the endomorphism ad_X is nilpotent for every $X \in \mathfrak{g}$.*

Theorem 1.3. (Lie) *A Lie algebra $\mathfrak{g}$ is solvable if and only if the endomorphism ad_X is solvable for every $X \in \mathfrak{g}$.*

These theorems imply in particular that nilpotent Lie algebras can be represented by strictly upper triangular matrices, while solvable Lie algebras by upper triangular ones. For the proofs see for example [18].

In general it is not very easy to construct a solvmanifold, indeed given a solvable Lie group we do not have a general method to find its lattices.

For nilmanifolds the problem has a straight solution due to Malčev:

Theorem 1.4. (Malčev) *[30] Let G be a nilpotent and simply connected Lie group and let $\mathfrak{g}$ be its Lie algebra. G admits a discrete subgroup Γ such that G/Γ is compact if and only if $\mathfrak{g}$ has rational structure constants.*

For solvmanifolds in general we have just a necessary condition [32]:

Proposition 1.2. *A solvable, connected and simply connected Lie group G can admit a lattice only if its Lie algebra $\mathfrak{g}$ is unimodular, i.e. $\forall\, X \in \mathfrak{g} \quad Tr\, ad_X = 0$.*

For a particular case of solvable Lie groups we have a necessary and sufficient condition. The solvmanifolds associated to these groups are called *almost abelian* and will be studied in details in the following chapters.

Definition 1.7. Given two Lie groups G and H and an action $\varphi : G \times H \to H$ the *semidirect product* $G \ltimes H$ is the Lie group $G \times H$ with the operation of group given by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, h_1 \cdot \varphi(g_1)(h_2)) \quad \forall\, g_1, g_2 \in G, \quad \forall\, h_1, h_2 \in H.$$

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and an action $\psi : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ the *semidirect product* $\mathfrak{g} \ltimes \mathfrak{h}$ is the Lie algebra $\mathfrak{g} \times \mathfrak{h}$ with the bracket given by

$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2], [Y_1, Y_2] + \psi(X_1)(Y_2) - \psi(X_2)(Y_1))$$

$\forall\, X_1, X_2 \in \mathfrak{g} \quad \forall\, Y_1, Y_2 \in \mathfrak{h}$.

In particular the Lie algebra of a semidirect product of Lie groups is the semidirect product of the associated Lie algebras.

Definition 1.8. A solvmanifold $S = G/\Gamma$ is *almost abelian* if the solvable Lie group G and its lattice Γ are semidirect products of the kind $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n, \quad \Gamma = \mathbb{Z} \ltimes_{\varphi|_\mathbb{Z}} \mathbb{Z}^n$, where φ is some action on $\mathbb{R}^n$ depending on the direction $\mathbb{R}$.

In particular if $\mathfrak{g}$ is the Lie algebra of G, then also $\mathfrak{g}$ is called *almost abelian* and $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_{n+1}}} \mathbb{R}^n$, where $\mathbb{R} = \langle X_{n+1} \rangle$ and $\mathbb{R}^n = \langle X_1, \cdots, X_n \rangle$, and $\varphi(t) := e^{t\mathrm{ad}_{X_{n+1}}}$.

A nice feature of almost abelian solvable groups is that there is a criterion on the existence of a lattice [4]:

Proposition 1.3. *Let $G = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^n$ be an almost abelian solvable Lie group. Then G admits a lattice if and only if there exists a $t_0 \neq 0$ for which $\varphi(t_0)$ can be conjugated to an integer matrix.*

In particular the lattice is generated by this value t_0, $\Gamma_{t_0} := \mathbb{Z}_{t_0} \ltimes_{\varphi(t_0)} \mathbb{Z}^n$.

The lattice determine the topology of the solvmanifold because it is its fundamental group. Indeed every solvable connected and simply connected Lie group is diffeomorphic to $\mathbb{R}^n$, then solvmanifolds are Eilenberg-MacLane spaces of type $K(\pi, 1)$, i.e. all their homotopy groups vanish, besides the first. Actually, lattices of solvmanifolds yield their diffeomorphism class:

Theorem 1.5. *[40, Theorem 3.6] Let G_i/Γ_i be solvmanifolds for $i \in \{1, 2\}$ and $\psi : \Gamma_1 \to \Gamma_2$ an isomorphism. Then there exists a diffeomorphism $\Psi : G_1 \to G_2$ such that*

- $\Psi|_{\Gamma_1} = \psi$,
- $\Psi(p\gamma) = \Psi(p)\psi(\gamma)$, *for any $\gamma \in \Gamma_1$ and any $p \in G_1$.*

Much of the rich structure of solvmanifolds is encoded by the Mostow fibration associated to every solvmanifold.

Let $S = G/\Gamma$ be a solvmanifold and let N be the nilradical of G, i.e. the largest nilpotent normal subgroup of G (of course N agrees with G if and only if S is a nilmanifold). Then $\Gamma_N := \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is closed in G and $G/(N\Gamma) =: \mathbb{T}^k$ is a torus. Thus we have the so-called *Mostow fibration* [33]:

$$N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k, \tag{1.1}$$

In general, the Mostow bundle is not principal.

A connected and simply-connected solvable Lie group G with nilradical N is called *almost nilpotent* if its nilradical has codimension one. The group G is then given by the semidirect product $G = \mathbb{R} \ltimes_\varphi N$ of its nilradical with $\mathbb{R}$. From a geometrical point of view, $\varphi(t)$ encodes the monodromy of the Mostow bundle.

Obviously an almost abelian solvable group is an almost nilpotent group whose nilradical is abelian $N = \mathbb{R}^n$. In this case the Mostow fibration (1.1) becomes

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow S \longrightarrow \mathbb{R}/\mathbb{Z}.$$

Homogeneous spaces are very interesting when we want to study the de Rham cohomology of a differential manifold.

We recall that given a differential manifold M with complex of differential forms $(\bigwedge^*(M), d)$ we define for every $k \in \mathbb{N}$ the k-esim group of de Rham cohomology of M as the set of classes of closed forms over the exat ones:

$$H^k(M) := \frac{\{\alpha \in \bigwedge^k(M) \,/\, d\alpha = 0\}}{\{\alpha \in \bigwedge^k(M) \,/\, \exists\, \beta \in \bigwedge^{k-1}(M) \,/\, d\beta = \alpha\}}$$

We can give a definition of cohomology also over a Lie algebra:

Definition 1.9. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual Lie algebra, then we can define a differential $d : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$ over the exterior algebra $\bigwedge^* \mathfrak{g}^*$ by

$$d\omega(X_1, ..., X_{k+1}) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{k+1})$$

$\forall\, \omega \in \bigwedge^k \mathfrak{g}^*,\ \forall\, X_1, \cdots, X_{k+1} \in \mathfrak{g}$.
$(\bigwedge^* \mathfrak{g}^*, d)$ is called *Chevalley-Eilenberg complex*.

Then the *cohomology groups of* $\mathfrak{g}$ are the the cohomology groups associated to this complex:

$$H^k(\mathfrak{g}) := \frac{\{\omega \in \bigwedge^k \mathfrak{g}^* \,/\, d\omega = 0\}}{\{\omega \in \bigwedge^k \mathfrak{g}^* \,/\, \exists\, \eta \in \bigwedge^{k-1} \mathfrak{g}^* \,/\, d\eta = \omega\}}$$

By definition the algebra of differential forms $\bigwedge^*(M)$ of a homogeneous space $M = G/\Gamma$ is the set of differential forms over the Lie group G that are left

Γ-invariant, while if $\mathfrak{g}$ is the Lie algebra associated to G, $\bigwedge^* \mathfrak{g}^*$ is the set of differential forms over the Lie group G that are left G-invariant.

Then we have an inclusion $\bigwedge^* \mathfrak{g}^* \subseteq \bigwedge^*(M)$ that for solvmanifolds is preserved passing to the cohomology [40]:

Theorem 1.6. *[40, Theorem 7.23] For any solvmanifold $S = G/\Gamma$ the inclusion $\bigwedge^* \mathfrak{g}^* \subseteq \bigwedge^*(M)$ induces a natural injection $H^*(\mathfrak{g}) \to H^*(S)$.*

For nilmanifolds and some cases of solvmanifolds this inclusion becomes an isomorphism.

Theorem 1.7. (Nomizu) *[35] Let $N = G/\Gamma$ be a nilmanifold and $\mathfrak{g}$ the Lie algebra associated to G, then $H^*(\mathfrak{g}) \cong H^*(N)$.*

Unfortunately there is not a similar propriety for solvmanifolds in general, but only in particular cases.

Definition 1.10. A solvable Lie group G is *completely solvable* if the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ associated to G has only real eigenvalues.

Theorem 1.8. (Hattori) *[22] Let $S = G/\Gamma$ be a solvmanifold such that the Lie group G is completely solvable, then $H^*(\mathfrak{g}) \cong H^*(S)$.*

Definition 1.11. [8] A subgroup A of $GL_n(\mathbb{R})$ is a *real algebraic group* if it is the set of zeros $\{g = (g_{i,j})\}$ of a family $\{f\}$ of real valued functions on $GL_n(\mathbb{R})$ for which there is a polynomial $p \in \mathbb{R}[X_1, \cdots, X_{n^2+1}]$ such that $f(g) = p(g_{i,j}, \det(g^{-1}))$.

Indeed, $GL(n, R)$ can be viewed as a closed subgroup of $SL_{n+1}(\mathbb{R})$ via the embedding $\rho : GL_n(\mathbb{R}) \to SL_{n+1}(\mathbb{R})$ defined by

$$\rho(A) = \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \frac{1}{\det A} \end{pmatrix}$$

As a subset of $\mathbb{R}^{n^2}$, a real algebraic group has both the Euclidean and Zariski topology.

In general, given a Lie group G, we recall that $\mathrm{Ad}_G(G)$ is the subgroup of $GL(\mathfrak{g})$ generated by e^{ad_X}, for all $X \in \mathfrak{g}$. Since $\mathrm{Ad}\exp X = e^{\mathrm{ad}_X}$, we have that $\mathrm{Ad}_G(G)$ has $\mathrm{ad}(\mathfrak{g})$ as Lie algebra [8].

It turns out that if G is a simply connected solvable Lie group then $\mathrm{Ad}_G(G)$ is a solvable algebraic group, then it is well defined its Zariski closure $\mathcal{A}(\mathrm{Ad}_G(G))$.

If H is a subgroup of a connected Lie group G, we will denote by $\mathcal{A}(\mathrm{Ad}_G(H))$ the (almost) Zariski closure of $\mathrm{Ad}_G(H)$ in the real algebraic group $\mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Theorem 1.9. (Borel Density Theorem) *[40, Theorem 5.5] Let G be a simply connected, solvable Lie group and Γ a lattice of G, then there exists a maximal compact torus $\mathbb{T}_{cpt} \subset \mathcal{A}(Ad_G(G))$ such that*

$$\mathcal{A}(Ad_G(G)) = \mathbb{T}_{cpt}\mathcal{A}(Ad_G(\Gamma)).$$

When this torus $\mathbb{T}_{cpt}$ is trivial Mostow proved that we can compute the cohomology of the solvmanifold $S = G/\Gamma$ by invariant forms:

Definition 1.12. Given a lattice Γ of a simply connected, solvable Lie group G, the *Mostow condition* holds for Γ and G if $\mathcal{A}(\mathrm{Ad}_G(G)) = \mathcal{A}(\mathrm{Ad}_G(\Gamma))$.

Theorem 1.10. (Mostow) *[34] Let G be a simply connected, solvable Lie group, Γ a lattice of G, $S = G/\Gamma$ a solvmanifold and $\mathfrak{g}$ the Lie algebra associated to G. If the Mostow condition holds for Γ and G, then $H^*(\mathfrak{g}) \cong H^*(S)$.*

The Nomizu and Hattori theorems are corollary of the Mostow theorem, indeed if a solvable Lie group is nilpotent or completely solvable, then the Mostow condition holds for each of its lattices.

Even if Theorem 1.10 is very useful, it is difficult to understand if the Mostow condition holds.

1.2 Almost Complex Structures

In this section we give some basic definitions of complex structures on vector spaces and differential manifolds and after we describe with more details complex structures on Lie algebras [14], indeed our study about this topic will be focused on complex structures on almost abelian Lie algebras (Chapter 5).

Definition 1.13. Let V be a real vector space of even dimension, an *almost complex structure* on V is an endomorphism $J : V \to V$ such that $J^2 = -Id$.

An almost complex structure J gives V the structure of complex vector space:

$$i \cdot v := J(v) \quad \forall v \in V.$$

If J is an almost complex structure on V, we can define an almost complex structure on the dual space $V^* = \hom(V, \mathbb{R})$ by

$$\forall f \in V^*, \forall v \in V \quad Jf(v) := f(Jv).$$

Let J be an almost complex structure on the real vector space V, suppose to extend it to the complexification $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$, then by definition J has only eigenvalues $\pm i$ with eigenspaces

$$V^{1,0} := \{z \in V^{\mathbb{C}} / Jz = iz\} = \{v - iJv \ / \ v \in V\}$$

$$V^{0,1} := \{z \in V^{\mathbb{C}} / Jz = -iz\} = \{v + iJv \ / \ v \in V\}$$

then $V^{1,0} \cong V^{0,1}$ and $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.
Vice versa if V is a real vector space, every decomposition of $V^{\mathbb{C}}$ in $V^{\mathbb{C}} = V_1 \oplus V_2$ such that $V_2 \cong \bar{V}_1$ endows V with an almost complex structure with $V^{1,0} \cong V_1$ and $V^{0,1} \cong V_2$.
Moreover a complex basis $(a_1 - ib_1, ..., a_n - ib_n), a_k, b_k \in V$ of V_1 yields a complex basis $(a_1, ..., a_n)$ of V and the almost complex structure J on V is defined by $J(a_k) = b_k$.

Let $\bigwedge V^{\mathbb{C}}$ be the complex exterior algebra of $V^{\mathbb{C}}$, then we define

$$\bigwedge^{p,q} V^{\mathbb{C}} := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$$

and we have

- $\bigwedge V^{\mathbb{C}} = \bigwedge V^{1,0} \otimes \bigwedge V^{0,1} = \sum_{p,q} \bigwedge^{p,q} V^{\mathbb{C}}$,
- $\bigwedge^{p,q} V^{\mathbb{C}} \cong \bigwedge^{q,p} V^{\mathbb{C}}$,
- if $(e_1, ..., e_n)$ is a basis of $V^{1,0}$, then $(\bar{e}_1, ..., \bar{e}_n)$ is a basis of $V^{0,1}$ and $\{e_{j_1} \wedge \cdots \wedge e_{j_p} \wedge \bar{e}_{k_1} \wedge \cdots \wedge \bar{e}_{k_q}\}$ is a basis of $\bigwedge^{p,q} V^{\mathbb{C}}$.

Definition 1.14. Let M be a smooth differential manifold of dimension $2n$, an *almost complex structure* on M is a bundle map $J : TM \to TM$ such that every J_p is an almost complex structure on the real vector space T_pM. The couple (M, J) is called *almost complex manifold.*

Let (M, J) be an almost complex structure of dimension $2n$, then the real tangent space in $p \in M$ $\quad T_p^{\mathbb{R}}M = \mathbb{R}\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle_{j=1,\cdots,n}$ has an almost complex structure J_p. Let $T_p^{\mathbb{C}}M = \mathbb{C}\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle$ be its complexification, called *complex tangent space*, then $T_p^{\mathbb{C}}M$ can be decomposed in the two eigenspaces of J, $T_p^{\mathbb{C}}M = T_p^{1,0} \oplus T_p^{0,1}$ that define the two sub-bundles $T^{1,0}$ and $T^{0,1}$ of $T^{\mathbb{C}}M$.
Considering the dual bundles we obtain the decomposition of the cotangent space $\Omega_p^{\mathbb{C}}M = \Omega_p^{1,0} \oplus \Omega_p^{0,1}$ and of the cotangent bundle $\Omega^{\mathbb{C}}M$.

Then we have the decomposition of the algebra of differential forms with values in $\mathbb{C}$

$$\bigwedge^{\mathbb{C}*}(M) = \bigwedge^*(M) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} \bigwedge^{p,q}(M).$$

If now we consider the differential $d : \bigwedge^r(M) \to \bigwedge^{r+1}(M)$, we have that

$$d\,(\bigwedge^{p,q}(M)) \subset \bigwedge^{p+2,q-1}(M) + \bigwedge^{p+1,q}(M) + \bigwedge^{p,q+1}(M) + \bigwedge^{p-1,q+2}(M).$$

Then we can define two components of d

$$\partial : \bigwedge^{p,q}(M) \to \bigwedge^{p+1,q}(M) \quad \text{and} \quad \bar{\partial} : \bigwedge^{p,q}(M) \to \bigwedge^{p,q+1}(M)$$

but obviously in general we do not have the decomposition $d = \partial + \bar{\partial}$ that occurs when M is a complex manifold [14].

Definition 1.15. Given an almost complex manifold (M, J), J is *integrable* if for every vector fields $X, Y \in \chi(M)$ the Nijenhuis tensor N is zero:

$$N(X, Y) = [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] = 0 \tag{1.2}$$

We have the following theorem [14]:

Theorem 1.11. *An almost complex manifold (M, J) is a complex manifold if and only if J is integrable.*

As a vector space, we can define an almost complex structure over a real Lie algebra $\mathfrak{g}$ and then considering its complexification $\mathfrak{g}^{\mathbb{C}}$ we have

$$\begin{cases} \mathfrak{g}^{\mathbb{C}*} = \bigwedge^{1,0} \mathfrak{g}^{\mathbb{C}*} \oplus \bigwedge^{0,1} \mathfrak{g}^{\mathbb{C}*} = \mathfrak{g}^{1,0*} \oplus \mathfrak{g}^{0,1*} \\ \bigwedge^{k} \mathfrak{g}^{\mathbb{C}*} = \bigoplus_{p+q=k} \bigwedge^{p} \mathfrak{g}^{1,0*} \otimes \bigwedge^{q} \mathfrak{g}^{0,1*} = \bigoplus_{p+q=k} \bigwedge^{p,q} \mathfrak{g}^{\mathbb{C}*} \\ \overline{\bigwedge^{p,q} \mathfrak{g}^{\mathbb{C}*}} = \bigwedge^{q,p} \mathfrak{g}^{\mathbb{C}*} \end{cases}$$

We now consider the Chevalley-Eilenberg complex $(\bigwedge^* \mathfrak{g}^*, d)$.
As for manifolds we say that the almost complex structure J on $\mathfrak{g}$ is *integrable* if equation (1.2) holds, in this case we can refer to J simply as a *complex structure* on $\mathfrak{g}$.

For complex structures on Lie algebras we have the following properties [41]:

Proposition 1.4.

1. *The real Lie algebra $\mathfrak{g}$ has the structure of complex Lie algebra induced by the almost complex structure J if and only if $\forall X, Y \in \mathfrak{g} \quad J[X, Y] = [JX, Y]$ and then J is integrable. These kind of complex structures are called bi-invariant.*

2. *J is integrable if and only if $\mathfrak{g}^{1,0}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with induced bracket.*

3. *J is integrable if and only if $d\mathfrak{g}^{1,0*} \subset \mathfrak{g}^{1,1*} \oplus \mathfrak{g}^{2,0*}$.*

4. *If G is a real Lie group with Lie algebra $\mathfrak{g}$, then giving a left invariant almost complex structure on G is equivalent to assign an almost complex structure J on $\mathfrak{g}$ and J is integrable if and only if it is integrable as almost complex structure on G. In this case it induce a complex structure on G and G becomes a complex Lie group.*

Proof.

1. $\Rightarrow$: as on the vector spaces, the almost complex structure J implies that $\mathfrak{g}$ is a complex vector space by $i \cdot X := J(X)\ \forall X \in \mathfrak{g}$, then we have only to prove the bilinearity over $\mathbb{C}$ of the bracket. But this is obvious using the hypothesis.
   $\Leftarrow$: if $\mathfrak{g}$ is a complex Lie algebra, then the multiplication by i is an almost complex structure on $\mathfrak{g}$ and by bilinearity of the bracket over $\mathbb{C}$ we have the thesis.

2. $\Rightarrow$: we want to prove that given two elements in $\mathfrak{g}^{1,0}$ their bracket is in $\mathfrak{g}^{1,0}$, i.e. if $v = [X - iJX, Y - iJY]$ with $X, Y \in \mathfrak{g}$, then $Jv = iv$.
   By bilinearity of the bracket we have

$$\begin{aligned} Jv &= J([X,Y] - i[JX,Y] - i[X,JY] - [JX,JY]) = \\ &= J[X,Y] - iJ[JX,Y] - iJ[X,JY] - J[JX,JY] \end{aligned}$$

   Equation (1.2) implies $= J[X,Y] + i[X,Y] - i[JX,JY] - J[JX,JY]$ and by $N(X,JY) = 0$ we have

$$\begin{aligned} &= J[X,Y] + i[X,Y] - i[JX,JY] + [X,JY] + [JX,Y] - J[X,Y] = \\ &\qquad = i([X,Y] - [JX,JY] - i[X,JY] - i[JX,Y]) = \quad iv. \end{aligned}$$

   $\Leftarrow$: by hypothesis $\mathfrak{g}^{1,0}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, i.e. if $v = [X - iJX, Y - iJY]$, then $Jv = iv$:

$$\begin{aligned} &J[X,Y] - iJ[X,JY] - iJ[JX,Y] - J[JX,JY] = \\ &\quad = i[X,Y] + [X,JY] + [JX,Y] - i[JX,JY] \end{aligned}$$

   that implies

$$\begin{cases} J[X,Y]-J[JX,JY]=[X,JY]+[JX,Y] \\ -J[X,JY]-J[JX,Y]=[X,Y]-[JX,JY] \end{cases}$$

that is equivalent to equation (1.2).

3. given $\omega \in \mathfrak{g}^{1,0*}$ we want to prove that the component in $\mathfrak{g}^{0,2*}$ of $d\omega$ is zero if and only if $\mathfrak{g}$ is integrable, that by point 2 is equivalent to prove that $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are subalgebras of $\mathfrak{g}^{\mathbb{C}}$.
$\forall\, X, Y \in \mathfrak{g}^{\mathbb{C}}$ we consider their components $X = X^{1,0}+X^{0,1}$ and $Y = Y^{1,0}+Y^{0,1}$, then

$$d\omega(X,Y) = -\omega([X,Y]) = \begin{array}{l} -\overbrace{\omega([X^{1,0},Y^{1,0}])}^{\mathfrak{g}^{2,0*}} - \overbrace{\omega([X^{1,0},Y^{0,1}]+[X^{0,1},Y^{1,0}])}^{\mathfrak{g}^{1,1*}} + \\ -\underbrace{\omega([X^{0,1},Y^{0,1}])}_{\mathfrak{g}^{0,2*}}. \end{array}$$

But $\mathfrak{g}^{0,1}$ is a subalgebra, i.e. $[X^{0,1},Y^{0,1}] \subset \mathfrak{g}^{0,1}$, if and only if $\omega([X^{0,1},Y^{0,1}]) = 0$, because $\omega \in \mathfrak{g}^{1,0*}$ and then $\omega(\mathfrak{g}^{0,1}) = 0$.

4. All the statements are direct consequences of the definition of almost complex structure on a differential manifold and of Remark 1.1.

□

Remark 1.2. We observe that the third property is equivalent to $\bar{\partial}^2 = 0$, then when a real Lie algebra $\mathfrak{g}$ is endowed with a complex structure J, we can define the *Dolbeault complex* $(\bigwedge^{p,q}\mathfrak{g}^{c*}, \bar{\partial})$ associated to $(\mathfrak{g}, J)$ and the *Dolbeault cohomology groups* $H^{p,q}_{\bar{\partial}}(\mathfrak{g})$ associated to this complex.

In general when the differential d can be decomposed in $d = \partial + \bar{\partial}$ we can study if the $\partial\bar{\partial}$-Lemma holds. We enunciate it for a general complex $(\bigwedge^{*,*} V, d = \partial + \bar{\partial})$, if we refer to a complex manifold we have to add the hypothesis of compactness.

Lemma 1.1. ***(**$\partial\bar{\partial}$**-Lemma)*** *Let $v \in \bigwedge^{*,*} V$ such that $\partial v = \bar{\partial} v = 0$, then*

- *if* $v = \bar{\partial} u$, *then there exists* w *such that* $u = \partial w$,
- *if* $v = \partial u$, *then there exists* w *such that* $u = \bar{\partial} w$.

Remark 1.3. A real case of the previous lemma is the dd^*-Lemma, where d^* is defined as $d^* := i(\bar{\partial} - \partial)$. In particular by definition the $\partial\bar{\partial}$-Lemma holds if and only if the dd^*-Lemma does.

For a complete study of this subject see [5] and [12].

If the $\partial\bar{\partial}$-Lemma and the dd^*-Lemma hold we have an important property that we will see in the following section (Theorem 1.13).

1.3 Minimal Models

Minimal models are objects of rational homotopy theory introduced by Quillen and Sullivan in the late 1960s.

We refer to [14, 45] for a deep study of these topics.

Definition 1.16. Let $\mathbb{K}$ ba a field of characteristic 0. A *graded* $\mathbb{K}$*-vector space* is a family of $\mathbb{K}$-vector spaces $\mathcal{A} = \{\mathcal{A}^p\}_{p \geq 0}$. An element of $a \in \mathcal{A}$ has degree p, $|a| = p$, if it belongs to $\mathcal{A}^p$.

Definition 1.17. A *commutative differential graded* $\mathbb{K}$*-algebra, cdga,* $(\mathcal{A}, d)$ is a graded $\mathbb{K}$-vector space $\mathcal{A}$ together with a multiplication $\mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$ that is associative, with unit $1 \in \mathcal{A}^0$ and commutative in the graded sense, i.e.

$$\forall a \in \mathcal{A}^p, b \in \mathcal{A}^q \quad a \cdot b = (-1)^{pq} b \cdot a, \tag{1.3}$$

and with a differential $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$ such that $d^2 = 0$ and

$$\forall\, a \in \mathcal{A}^p, b \in \mathcal{A}^q \quad d(a \cdot b) = da \cdot b + (-1)^p a \cdot db. \tag{1.4}$$

Example 1.4. The complex of differential forms over a differential manifold and the Chevalley-Eilenberg complex over a Lie algebra are cdgas.

Given a $\mathbb{K}$-cdga $(\mathcal{A}, d)$ its cohomology algebra $H^*(\mathcal{A}, \mathbb{K})$ is well defined and it is a $\mathbb{K}$-cdga with $d \equiv 0$.
The *Betti numbers* of $\mathcal{A}$ are the dimensions of the cohomology groups of $\mathcal{A}$,

$$b_i(\mathcal{A}) := \dim H^i(\mathcal{A}, \mathbb{K}).$$

Definition 1.18. A *cdga homomorphism* $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ is a family of homomorphisms $f^p : \mathcal{A}^p \to \mathcal{B}^p$ such that $f^{p+q}(a \cdot b) = f^p(a) \cdot f^q(b)$ and $d_{\mathcal{B}} f^p = f^p d_{\mathcal{A}}$.

Definition 1.19. A cdga $(\mathcal{M}, d)$ is *Sullivan* if it is free commutative, i.e. $\mathcal{M} = \bigwedge V$ with V graded vector space, $V^0 = \mathbb{K}$ and there exist a ordered basis $\{x_\alpha\}$ of V such that $dx_\alpha \in \bigwedge(x_\beta)_{\beta<\alpha}$.
A cdga $(\mathcal{M}, d)$ is *minimal* if it Sullivan and $|x_\beta| \leq |x_\alpha|$ for $\beta < \alpha$ or equivalently $dV \subset \bigwedge^{\geq 2} V$, where with $\bigwedge^{\geq 2} V$ we mean $\bigwedge^i V$ with $i \geq 2$.
A *minimal (Sullivan) model* of the cdga $(\mathcal{A}, d)$ is a minimal (Sullivan) cdga $(\mathcal{M}, d)$ together with a cdga quasi isomorphism $\psi : \mathcal{M} \to \mathcal{A}$, i.e. a morphism that induces an isomorphism on cohomology.

For every topological space T, Sullivan defined a $\mathbb{Q}$-cdga $\mathcal{A}_{PL}(T)$ called the *piecewise linear cdga* associated to T. We refer to [14] for its definition, we only need to know that its cohomology is the cohomology of the space T over the constant sheaf $\mathbb{Q}$ and then we can use all the theory over this cdgas for differential manifolds and their de Rham cohomology only by replacing $\mathbb{Q}$ with $\mathbb{R}$.

In particular from now on, the model of a topological space T is the model of $\mathcal{A}_{PL}(T)$, while the model of a differential manifold M is the model of $\bigwedge^*(M)$.

To understand what minimal models are, we give a fundamental example of their computation:

Example 1.5. Let $\mathcal{A} = \bigwedge^*(S^p)$ be the algebra of differential forms over the sphere of dimension p, then

$$H^k(S^p) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}_\omega & k = p \\ 0 & k \neq 0, p \end{cases}$$

Where ω is the volume form of S^p.

We have to consider two cases:

$p = 2n-1$: Let x be an element of degree $2n-1$ such that $dx = 0$, then the minimal model of $\mathcal{A}$ is $\mathcal{M} = \bigwedge(x)$ the exterior algebra generated by x and

$$\begin{array}{rcl} \rho : \mathcal{M} & \to & \mathcal{A} \\ x & \mapsto & \omega \end{array}$$

Indeed $\mathcal{M}$ is by definition free commutative and the map induced on cohomology is

$$\begin{array}{rcl} H^{2n-1}(\mathcal{M}) & \cong & H^{2n-1}(\mathcal{A}) \\ [x] & \mapsto & [\omega] \end{array}$$

The only generator has degree $2n-1$, then we do not have elements of lower degree and

$$H^k(\mathcal{M}) = 0 = H^k(\mathcal{A}) \quad \forall\, 0 < k < 2n-1.$$

Besides by equation (1.3) in Definition 1.17, $xx = -xx$, that implies $x^2 = 0$, then we do not have either elements of greater degree:

$$H^k(\mathcal{M}) = 0 = H^k(\mathcal{A}) \quad \forall\, k > 2n-1.$$

$p = 2n$: By a similar argument to the previous case, we consider an element x of degree $2n$ such that $dx = 0$, then

$$x \overset{\rho}{\mapsto} \omega \quad \Rightarrow \quad \begin{array}{rcl} H^{2n}(\mathcal{M}) & \cong & H^{2n}(\mathcal{A}) \\ [x] & \mapsto & [\omega] \end{array}$$

But now $x^2 \neq 0$, then to kill the higher cohomology groups we need another generator: let y be such that $|y| = 4n-1$ and $dy = x^2$, then $\mathcal{M} = \bigwedge(x, y)$ and

$$\begin{array}{rcl} \rho : \mathcal{M} & \to & \mathcal{A} \\ x & \mapsto & \omega \\ y & \mapsto & 0 \end{array}$$

Indeed:

- we do not have elements of degree lower of $2n$ then

$$H^k(\mathcal{M}) = 0 = H^k(\mathcal{A}) \quad \forall\, 0 < k < 2n.$$

- All the elements of degree greater then $2n$ that represent cohomology classes, i.e. that are closed, are also exact, then their class is the zero one: $dy \neq 0$, then we have to check only powers of x.
  $\forall\, i > 0 \quad (x^{i-2}y) = x^i$, then $[x^i] = [d(x^{i-2}y)] = [0]$ and

$$H^k(\mathcal{M}) = 0 = H^k(\mathcal{A}) \quad \forall\, k > 2n.$$

We have a theorem of existence and uniqueness of minimal models for *path connected* $\mathbb{K}$-*cdga*, i.e. $\mathcal{A}$ such that $H^0(\mathcal{A}) = \mathbb{K}$:

Theorem 1.12. *[14] A path connected cdga $\mathcal{A}$ admits always a minimal model $\mathcal{M}$ that is unique up to isomorphism.*

Proof of Existence. Let $\mathcal{M}(n) \subset \mathcal{M}$ be the subalgebra generated by elements of degree $\leq n$, then $\mathcal{M}(n) \subset \mathcal{M}(n+1) \subset \cdots \subset \mathcal{M}$. We compute $\mathcal{M}$ by induction on n: Let $\mathcal{M}(0)$ be $\mathbb{K}$ and $\rho_0 : 1 \mapsto 1$.
$\forall\, n$ we compute $\mathcal{M}(n)$ with the map $\rho_n : \mathcal{M}(n) \to \mathcal{A}$ such that

1. $\rho_n^* : H^q(\mathcal{M}(n)) \to H^q(\mathcal{A})$ is an isomorphism $\forall\, q \leq n$ and it is injective for $q = n+1$.

2. $\rho_{n+1|_{\mathcal{M}(n)}} = \rho_n$.

In this way, $\mathcal{M}(n) \subset \mathcal{M}$ $\forall\, n$ implies that $\rho^* : H^q(\mathcal{M}) \cong H^q(\mathcal{A})$ is an isomorphism $\forall\, q$.

Suppose for simplicity that $\mathcal{A}$ is simply connected, i.e. that $H^1(\mathcal{A}) = 0$.

By inductive hypothesis, we suppose to have $\mathcal{M}(n)$ and ρ_n and compute $\mathcal{M}(n+1)$ and ρ_{n+1}: we add to $\mathcal{M}(n)$ new generators $\alpha_1^{n+1}, \cdots, \alpha_k^{n+1}, \beta_1^{n+1}, \cdots, \beta_l^{n+1}$ of degree $n+1$ such that:

- $\{\alpha_i^{n+1}\}_{i=1,\cdots,k}$ is a basis of the cokernel of

$$0 \to H^{n+1}(\mathcal{M}(n)) \overset{\rho_n^*}{\to} H^{n+1}(\mathcal{A}) :$$

  Let $a_i^{n+1} \in \mathcal{A}^{n+1}$ be closed elements that are representative of α_i^{n+1} in cohomology, i.e.

$$\rho_{n+1}(\alpha_i^{n+1}) = a_i^{n+1} \qquad \text{and} \qquad d\alpha_i^{n+1} = 0$$

- Let $\{\eta_1^{n+2}, \cdots, \eta_l^{n+2}\} \in \mathcal{M}^{n+2}(n)$ be a basis of the kernel of

$$H^{n+2}(\mathcal{M}(n)) \overset{\rho_n^*}{\to} H^{n+2}(\mathcal{A}),$$

  and let $b_j^{n+1} \in \mathcal{A}^{n+1}$ be such that $db_j^{n+1} = \rho_n(\eta_j^{n+2})$, then we choose β_j^{n+1} such that

$$\rho_{n+1}(\beta_j^{n+1}) = b_j^{n+1} \qquad \text{and} \qquad d\beta_j^{n+1} = \eta_j^{n+2} \ \in \mathcal{M}^{n+2}(n).$$

If we define $\mathcal{M}(n+1) = \mathcal{M}(n)\{\alpha_i^{n+1}, \beta_j^{n+1}\}$, then we have:

- $\mathcal{M}(n+1)$ is obviously free commutative,
- $d\alpha_i^{n+1} = 0$ and $d\beta_j^{n+1} \in \mathcal{M}^{n+2}(n) \subset [\mathcal{M}(n)]^2 \subset [\mathcal{M}(n+1)]^2$ where we have the first inclusion because every element of degree $n+2$ generated by elements of degree less or equal to n, must be given by products.

Then $\mathcal{M}(n+1)$ is a minimal cdga.
Moreover:

- $\rho_{n+1} : \mathcal{M}(n+1) \to \mathcal{A}$ is a cdga homomorphism and obviously $\rho_{n+1|_{\mathcal{M}(n)}} = \rho_n$,
- $\rho_{n+1}^* : H^q(\mathcal{M}(n+1)) \to H^q(\mathcal{A})$ by inductive hypothesis is injective for $q \leq n+1$ and it is surjective for $q \leq n$.
  We added to $\mathcal{M}(n)$ the cokernel of $H^{n+1}(\mathcal{M}(n)) \overset{\rho_n^*}{\to} H^{n+1}(\mathcal{A})$, then ρ_{n+1}^* is surjective also for $q = n+1$.

  Moreover, what was sent by ρ_n^* in the kernel for $q = n+2$, i.e. η_j^{n+2}, with ρ_{n+1}^* are differentials of β_j^{n+1}, then are 0 in cohomology, and then the kernel is trivial and ρ_{n+1}^* is injective also for $q = n+2$.

We observe that we do not have to check anything else in the kernel of ρ^*_{n+1} because $\mathcal{A}$ is simply connected, then $\mathcal{M}^1(n) = 0$ and so elements of degree $n+2$ can not be generated by those we add of degree $n+1$ multiplied by those of degree 1.

□

A fundamental notion related to minimal models is formality. We have two equivalent definitions of formal cdga [14]:

Definition 1.20. A cdga $(\mathcal{A}, d)$ is *formal* if there exists a cgda homomorphism $\psi : \mathcal{A} \to H^*(\mathcal{A})$ that induces the identity in cohomology.

Definition 1.21. A minimal cdga $(\mathcal{M} = \bigwedge V, d)$ is *formal* if $V = C \oplus N$ such that

- $d(C) = 0$
- d is injective on N
- $\forall n \in I := \bigwedge V \cdot N$ such that $dn = 0$, then n is exact in $\bigwedge V$.

We say that a differential manifold is *formal* if its algebra of differential forms is formal. In particular we have the following fundamental property:

Theorem 1.13. *[5, 12, 14] A compact complex manifold satisfying equivalently the dd^*-Lemma or the $\partial\bar{\partial}$-Lemma is formal.*

Proof. Let $Z^{p,q}_{\partial}(M)$ and $H^{p,q}_{\partial}(M)$ be respectively the spaces of cocycles and the cohomology groups for the differential ∂, we consider the cdga diagram:

$$(H^{p,q}_{\partial}(M), \bar{\partial}) \xleftarrow{\rho} (Z^{p,q}_{\partial}(M), \bar{\partial}) \xrightarrow{j} (\textstyle\bigwedge^{p,q}(M), d)$$

where j is the inclusion and $\rho(\alpha) := [\alpha]$.

1. *j^* is surjective:*
   let $[\alpha] \in H^{p+q}(M)$, then $d\alpha = 0$, $\bar{\partial}(\partial\alpha) = \bar{\partial}(d\alpha - \bar{\partial}\alpha) = -\bar{\partial}^2\alpha = 0$ and $\partial(\partial\alpha) = 0$.

By the lemma there exists β such that $\partial\alpha = \partial\bar{\partial}\beta$, then considering $\gamma := \alpha - d\beta$, we have $\partial\gamma = \partial\alpha - \partial d\beta = \partial\alpha - \partial(\partial + \bar{\partial})\beta = \partial\alpha - \partial\bar{\partial}\beta = 0$, i.e. γ is a ∂-cocycle and $\bar{\partial}\gamma = \bar{\partial}\alpha - \bar{\partial}d\beta = -\partial\alpha - \bar{\partial}\partial\beta = -(\partial\alpha - \partial\bar{\partial}\beta) = 0$, then $j^*[\gamma] = [\gamma]_{H^{p+q}(M)} = [\alpha]_{H^{p+q}(M)}$.

2. *j^* is injective:*
   let $\alpha \in Z_{\partial}^{p,q}(M)$, such that $j^*[\alpha] = [\alpha]_{H^{p+q}(M)} = 0$, i.e. $\alpha = d\beta$. $\partial(\partial\beta) = 0$ and $\bar{\partial}(\partial\beta) = -\partial(\bar{\partial}\beta) = -\partial(d\beta - \partial\beta) = -\partial(d\beta) = -\partial\alpha = 0$, then by the lemma there exists γ such that $\partial\beta = \partial\bar{\partial}\gamma = -\bar{\partial}\partial\gamma$.

   Then $\alpha = d\beta = \partial\beta + \bar{\partial}\beta = -\bar{\partial}\partial\gamma + \bar{\partial}\beta = \bar{\partial}(\beta - \partial\gamma)$ is a $\bar{\partial}$-coboundary and $[\alpha]_{H_{\bar{\partial}}^{p,q}(M)} = 0$.

3. *ρ^* is surjective:*
   let α be such that $\partial\alpha = 0$. $\bar{\partial}(\bar{\partial}\alpha) = 0$ and $\partial(\bar{\partial}\alpha) = -\bar{\partial}(\partial\alpha) = 0$ imply by the lemma that there exists β such that $\bar{\partial}\alpha = \bar{\partial}\partial\beta$. Let consider $\gamma := \alpha - \partial\beta$, then $\partial\gamma = \partial\alpha - \partial^2\beta = 0 - 0 = 0$ and $\bar{\partial}\gamma = \bar{\partial}\alpha - \bar{\partial}\partial\beta = \bar{\partial}\alpha - \bar{\partial}\alpha = 0$ and so $\rho^*[\gamma] = [\gamma]_{H_{\bar{\partial}}^{p,q}(M)} = [\alpha]_{H_{\bar{\partial}}^{p,q}(M)}$.

4. *ρ^* is injective:*
   let α be such that $\rho^*[\alpha] = [\alpha]_{H_{\bar{\partial}}^{p,q}(M)} = 0$ i.e. $\partial\alpha = \bar{\partial}\alpha = 0$ and $\alpha = \partial\beta$. Then by the lemma there exists γ such that $\alpha = \partial\bar{\partial}\gamma = -\bar{\partial}\partial\gamma$, then $[\alpha]_{H_{\bar{\partial}}^{p,q}(M)} = 0$.

5. *The differential induced by $\bar{\partial}$ in $H_{\bar{\partial}}^{p,q}(M)$ is zero:*
   let α be such that $\partial\alpha = 0$, then $\bar{\partial}(\bar{\partial}\alpha) = 0$, $\partial(\bar{\partial}\alpha) = -\bar{\partial}(\partial\alpha) = 0$ and by the lemma there exists γ such that $\bar{\partial}\alpha = \bar{\partial}\partial\gamma = -\partial\bar{\partial}\gamma$. But then $[\bar{\partial}\alpha]_{H_{\bar{\partial}}^{p,q}(M)} = 0$.

Then there exists a homomorphism between $(\bigwedge^{p,q}(M), d)$ and $(H_{\bar{\partial}}^{p,q}(M), 0)$ that induces the identity on cohomology. □

We can generalize the concept of formality. There are two equivalent definitions of s-formality [14–16]:

Definition 1.22. A cdga $(\bigwedge V, d)$ is *s-formal* if there is a cdga homomorphism $\psi : \bigwedge V^{\leq s} \to H^*(\bigwedge V)$, such that the map $\psi^* : H^*(\bigwedge V^{\leq s}) \to H^*(\bigwedge V)$ induced on cohomology is equal to the map $i^* : H^*(\bigwedge V^{\leq s}) \to H^*(\bigwedge V)$ induced by the inclusion $i : \bigwedge V^{\leq s} \to \bigwedge V$.

Definition 1.23. A minimal cdga $(\bigwedge V, d)$ is *s-formal* if for every $i \leq s$
$V^i = C^i \oplus N^i$ such that

- $d(C^i) = 0$
- d is injective on N^i
- $\forall n \in I_s := \bigwedge V^{\leq s} \cdot N^{\leq s}$ such that $dn = 0$, then n is exact in $\bigwedge V$.

In particular a $(\bigwedge V, d)$ is formal if it is s-formal $\forall s \geq 0$.

We can generalize a little the idea of minimal models applying it to homomorphisms and then to fibrations [14].

Definition 1.24. A *relative minimal cdga* is a homomorphism of cdgas of kind

$$i : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{A} \otimes \bigwedge V, d)$$

where

- $i(a) = a \quad \forall\, a \in \mathcal{A}$,
- $d|_{\mathcal{A}} = d_{\mathcal{A}}$,
- $d(V) \subset (\mathcal{A}^+ \otimes \bigwedge V) \oplus \bigwedge^{\geq 2} V$, where with $\mathcal{A}^+$ we mean all the elements in $\mathcal{A}$ with degree greater than 0,
- there exist a ordered basis $\{x_\alpha\}$ of V such that $dx_\alpha \in \mathcal{A} \otimes \bigwedge(x_\beta)_{\beta<\alpha}$.

Remark 1.4. By definition if $\mathcal{A}$ is Sullivan, then also the relative minimal cdga $\mathcal{A} \otimes \bigwedge V$ is Sullivan, but if $\mathcal{A}$ is also minimal $\mathcal{A} \otimes \bigwedge V$ is just Sullivan.

Definition 1.25. (Homotopy lifting property) Given two topological spaces E and B, a *fibration* is a map $p : E \to B$ such that for every topological space X and for every commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "g"] \arrow[d, "(Id,i)"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"] & B \end{tikzcd}$$

there is a continuous map $h : X \times [0,1] \to E$ such that $p \circ h = f$ and $h \circ (Id, i) = g$.

In particular if B is path connected all the fibres $p^{-1}(x)$, with $x \in B$, have the same homotopy type and then we write the fibration $F \to E \xrightarrow{p} B$ with F the fibre.

For a fibration we can define a concept similar to minimal models [14, 36]:

Definition 1.26. Let $F \to E \xrightarrow{p} B$ be a fibration of path connected spaces and let $\mathcal{A}_{PL}(B) \to \mathcal{A}_{PL}(E) \to \mathcal{A}_{PL}(F)$ be the map induced on the piecewise linear cdgas.

The *Sullivan model of the fibration* is the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{PL}(B) \arrow[r] & \mathcal{A}_{PL}(E) \arrow[r] & \mathcal{A}_{PL}(F) \\ (\bigwedge X, d_X) \arrow[u, "\sigma"] \arrow[r, "i"] & (\bigwedge(X \oplus Y), D) \arrow[u, "\tau"] \arrow[r, "q"] & (\bigwedge Y, d_Y) \arrow[u, "\rho"'] \end{tikzcd}$$

where

- $(\bigwedge X, \sigma)$ is the minimal model of B,
- τ is a quasi isomorphism,
- i is a relative minimal cdga,
- $(\bigwedge Y, d_Y)$ is the quotient cdga $(\bigwedge(X \oplus Y), D)/(\bigwedge^+ X \otimes \bigwedge Y)$ and q is the quotient map.

Remark 1.5. We observe that the last point in the definition means that

$$Dy = d_Y y + cx \wedge y', \quad \forall\, y \in Y$$

with $c \in \mathbb{Q}$, $x \in \bigwedge X^+$ and $y' \in \bigwedge Y^{<y}$, where $\bigwedge Y^{<y}$ is the subalgebra of $\bigwedge Y$ generated by all the generators prior to y with respect to an order among the basis of Y.

Definition 1.26 does not describe the map $\rho : (\bigwedge Y, d_Y) \to \mathcal{A}_{PL}(F)$, indeed in general we are not able to feature it. Only in particular cases ρ can be described:

Definition 1.27. A fibration $F \to E \xrightarrow{p} B$ is *quasi nilpotent* if B and F are path connected and the natural action of $\pi_1(B)$ on the homology groups of F is nilpotent.

In particular if B is simply connected every fibration $F \to E \xrightarrow{p} B$ is quasi nilpotent.

We state now a theorem that will be generalized and used in Chapter 6.2.

Theorem 1.14. *[14] Let $F \to E \xrightarrow{p} B$ be a quasi nilpotent fibration, if B and F have finite Betti numbers and the map induced on the first cohmology group $H^1(p)$ is injective, then the map ρ is a quasi isomorphism and the cdga $(\bigwedge Y, d_Y)$ is the minimal model of the fibre F.*

1.4 Symplectic geometry and Hard Lefschetz property

In this section we give some basic definition and properties of symplectic geometry. In particular we are interested in the Hard Lefschetz property and its relation to the symplectic version of the Hodge theory and the dd^*-Lemma [29]. For a complete study of this subject see [14].

Definition 1.28. Let M be a differential manifold of dimension $2n$. A *symplectic structure* on M is a closed 2-form ω in $\bigwedge^*(M)$ such that $\omega^n \neq 0$, i.e. ω is *not degenerate*.

Brylinski developed a symplectic analogue of the Hodge theory for complex manifolds:

Definition 1.29. Let (M, ω) be a symplectic manifold of dimension $2n$, the *symplectic star operator* $*_s : \bigwedge^k(M) \to \bigwedge^{2n-k}(M)$ is defined by the following properties:

- $*_s 1 = \frac{\omega^n}{n!}$.
- $*_s$ is linear.
- $*_s(f\alpha) = f(*_s\alpha)$, for every function f and for every form α.
- $*_s *_s = 1$.
- $\alpha \wedge *_s\alpha = 0$ if and only if $\alpha = 0$.
- $\alpha \wedge *_s\beta = \beta \wedge *_s\alpha$.

Note that the first condition implies that the star operator depends on the symplectic structure of the manifold.

In particular using coordinates $(x_1, .., x_{2n})$ on M it is given by $\forall\, \gamma, \beta \in \bigwedge^k(M)$,

$$\gamma \wedge *_s\beta = (\omega^{-1})^k(\gamma, \beta)\mathrm{dvol} := \frac{1}{k!}(\omega^{-1})^{i_1 j_1}(\omega^{-1})^{i_2 j_2}...(\omega^{-1})^{i_k j_k}\gamma_{i_1 i_2 \dots i_k}\beta_{j_1 j_2 \dots j_k}\frac{\omega^n}{n!}.$$

Definition 1.30. Let (M, ω) be a symplectic manifold of dimension $2n$, the *Lefschetz operator* is

$$L : \bigwedge^k(M) \to \bigwedge^{k+2}(M)$$
$$\eta \mapsto \eta \wedge \omega$$

The *dual Lefschetz operator* $\Lambda : \bigwedge^k(M) \to \bigwedge^{k-2}(M)$ is its dual operator with respect to the scalar product $(\ ,\ )$ defined using the symplectic form ω.

Remark 1.6. (see [47])

1. $\Lambda = *_s L *_s$.
2. Using coordinates $(x_1, .., x_{2n})$ on M the above operators are defined in the following way:
$$\Lambda(\eta) := \frac{1}{2}(\omega^{-1})^{ij} i_{\partial_{x_i}} i_{\partial_{x_j}} \eta$$
where i is the interior product.

Using Λ we can construct another differential $d^\wedge : \bigwedge^k(M) \to \bigwedge^{k-1}(M)$:

$$d^\wedge := (-1)^{k+1} *_s d *_s = d\Lambda - \Lambda d.$$

Remark 1.7. For the differential $d^\wedge$ equation (1.4) does not hold.

We observe that the classical Hodge theory is given in a similar way: if (M, g) is a Riemannian manifold we define the *Hodge operator* $*$ as in Definition 1.29 just by considering the volume form defined by the metric g instead of the volume form $\frac{\omega^n}{n!}$ defined by the symplectic structure. In particular with this notation the operator d^* that we introduced in Section 1.2 can be defined as $d^* := - * d *$ and then it is the analogue of $d^\wedge$ in the symplectic case.

On a symplectic manifold (M, ω) we can always find a compatible almost complex structure J [14, Proposition 4.86], i.e. $\omega(X, JX) > 0$ and $\omega(JX, JY) = \omega(X, Y) \quad \forall\, X, Y \in \chi(M)$. In particular this means that it is well defined the Riemannian metric $g(X, Y) := \omega(X, JY)$. If now we define the Hodge operator $*$ associated to this metric, we have a relation between the two star operators:

$$* = \Im *_s$$

where $\Im := \sum_{p,q} i^{p-q} \prod^{p,q}$.

The differential $d^\wedge$ can be used to state a symplectic analogue of the dd^*-Lemma:

Definition 1.31. A symplectic manifold satisfies the *$dd^\wedge$-Lemma* if

$$\mathrm{Im} d \cap \ker d^\wedge = \mathrm{Im} d^\wedge \cap \ker d = \mathrm{Im} dd^\wedge$$

Definition 1.32. A form $\alpha \in \bigwedge^*(M)$ is *symplectically harmonic* if $d\alpha = d^\wedge \alpha = 0$.

The Lefschetz operator allows us to define the following fundamental property:

Definition 1.33. The *Hard Lefschetz Property* holds if the map induced in cohomology by the Lefschetz operator

$$\begin{array}{rcl} H^k(M) & \to & H^{2n-k}(M) \\ [\alpha] & \mapsto & [\omega^{n-k} \wedge \alpha] \end{array}$$

is an isomorphism $\forall\, k \leq n$.

Theorem 1.15. *[14, 29] Let (M, ω) be a symplectic manifold of dimension $2n$, then the following statements are equivalent:*

1. *Any cohomology class contains at least one symplectically harmonic form.*
2. *(M, ω) satisfies the Hard Lefschetz Property.*

Using Theorem 1.15 Markulov proved the following theorem:

Theorem 1.16. *[14, 31] A compact symplectic manifold satisfies the Hard Lefschetz property if and only if it satisfy the $dd^{\wedge}$-Lemma.*

We observe that Remark 1.7 implies that we do not have an analogue of Theorem 1.13 in the symplectic case, indeed $Z_{d^{\wedge}}$ is not a cdga [5, Remarks pagg. 14 and 83], [14].

Remark 1.8. Complex and symplectic geometries intersect in Kähler manifolds. Indeed Kähler manifolds satisfy both the dd^{*}-Lemma and the $dd^{\wedge}$-Lemma and then are formal and for them the Hard Lefschetz property holds.
Formality of symplectic and Kähler manifolds is deeply studied in [11].

Chapter 2

Tesng-Yau Cohomology

L.S. Tseng and S.T. Yau introduced some classes of finite dimensional cohomologies for symplectic manifolds [47]. These cohomology classes depend on the symplectic form and are in general distinct from the de Rham cohomology, so that they provide new symplectic invariants. As shown in [47] (cf. also Proposition 2.3 below), these new invariants actually agree with the de Rham cohomology if and only if the Hard Lefschetz property holds.

Below we discuss these cohomological invariants, proving that they can be computed using invariant forms, provided this is the case for the Rham cohomology (see Theorem 2.2). This result will allow us to go through the list of symplectic structures on solvable Lie algebras (Appendix C), to see which solvmanifolds, supposing that for them the Mostow condition holds, satisfy the Hard Lefschetz property (Theorem 3.3).

We will give all the definition and properties referring to a differential manifold, but they can similarly be given for a Lie algebra with a symplectic structure.

Let (M, ω) be a symplectic manifold of dimension $2n$, L be the Lefschetz operator, $\mathit{\Lambda}$ be the dual Lefschetz operator and $*_s$ be the symplectic star operators.

We consider another operator $H := \sum_k (n-k) \prod^k$ called *degree count operator*, where $\prod^k : \bigwedge^*(M) \to \bigwedge^k(M)$ projects onto forms of degree k.

L, Λ and H give a representation of the $\mathfrak{sl}_2(\mathbb{R})$ algebra acting on $\bigwedge^*(M)$ by

$$[\Lambda, L] = H, \quad [H, \Lambda] = 2\Lambda, \quad [H, L] = -2L.$$

Using the differentials d and $d^\wedge$ we can obtain another differential

$$dd^\wedge : \bigwedge^k(M) \to \bigwedge^k(M)$$

and for these operator the following lemma holds.

Lemma 2.1. *The differential operators $(d, d^\wedge, dd^\wedge)$ satisfy the following commutation relations with respect to the $\mathfrak{sl}_2(\mathbb{R})$ representation (L, Λ, H):*

$$[d, L] = 0, \quad [d, \Lambda] = d^\wedge, \quad [d, H] = d,$$

$$[d^\wedge, L] = d, \quad [d^\wedge, \Lambda] = 0, \quad [d^\wedge, H] = -d^\wedge,$$

$$[dd^\wedge, L] = 0, \quad [dd^\wedge, \Lambda] = 0, \quad [dd^\wedge, H] = 0.$$

Using these 3 operators we can define, besides the de Rham cohomology ones $H_d^*(M)$, the following cohomology groups

$$H^k_{d^\wedge}(M) := \frac{\ker d^\wedge \cap \bigwedge^k(M)}{\operatorname{im} d^\wedge \cap \bigwedge^k(M)}$$

$$H^k_{d+d^\wedge}(M) := \frac{\ker(d + d^\wedge) \cap \bigwedge^k(M)}{\operatorname{im} dd^\wedge \cap \bigwedge^k(M)}$$

$$H^k_{dd^\wedge}(M) := \frac{\ker dd^\wedge \cap \bigwedge^k(M)}{\operatorname{im} d \cap \bigwedge^k(M) + \operatorname{im} d^\wedge \cap \bigwedge^k(M)}$$

$$H^k_{d\cap d^\wedge}(M) := H^k_d \cap H^k_{d^\wedge} = H^k_{d+d^\wedge} \cap H^k_{dd^\wedge} = \frac{\ker(d + d^\wedge) \cap \bigwedge^k(M)}{\operatorname{im} d \cap \bigwedge^k_0(M) + \operatorname{im} d^\wedge \cap \bigwedge^k_0(M)}$$

where $\bigwedge^k_0(M)$ is $\ker dd^\wedge \cap \bigwedge^k(M)$.

We now analyse these cohmologies separately. Using the Hodge operator $*$ we can define the Hodge adjoint operators $d^{\wedge *} := *d^\wedge *$ and $(dd^\wedge)^* := (-1)^{k+1} * dd^\wedge *$.

Proposition 2.1. (Brylinski) *The operator $*_s$ gives an isomorphism between $H^k_d(M)$ and $H^{2n-k}_{d^\wedge}(M)$.*

This proposition implies in particular that the $H_{d^\wedge}$ cohomology does not lead to new invariants.

We can define the Laplacian associated to $d^\wedge$: $\Delta_{d^\wedge} := d^{\wedge *}d^\wedge + d^\wedge d^{\wedge *}$. A differential form $\alpha \in^* (M)$ is $d^\wedge$-*harmonic* if $\Delta_{d^\wedge}\alpha = 0$ or equivalently $d^\wedge\alpha = d^{\wedge *}\alpha = 0$. We denote the space of $d^\wedge$-harmonic k-forms by $\mathcal{H}^k_{d^\wedge}(M)$.

$\Delta_{d^\wedge}$ is an elliptic differential operator, then we have the Hodge decomposition

$$\bigwedge^k = \mathcal{H}^k_{d^\wedge} \oplus d^\wedge \bigwedge^{k+1} \oplus d^{\wedge *} \bigwedge^{k-1}$$

that implies the isomorphism $\mathcal{H}^k_{d^\wedge}(M) \cong H^k_{d^\wedge}(M)$.

Now we consider the Laplacian operators associate to the other differential that define these particular cohomologies [47]:

- $\Delta_{d+d^\wedge} := dd^\wedge(dd^\wedge)^* + \lambda(d^*d + d^{\wedge *}d^\wedge)$,
- $\Delta_{dd^\wedge} := (dd^\wedge)^*dd^\wedge + \lambda(dd^* + d^\wedge d^{\wedge *})$,
- $\Delta_{d\cap d^\wedge} := dd^* + d^*d + d^\wedge d^{\wedge *} + d^{\wedge *}d^\wedge$.

We define the harmonic spaces $\mathcal{H}^k_{d+d^\wedge}(M)$, $\mathcal{H}^k_{dd^\wedge}(M)$ and $\mathcal{H}^k_{d\cap d^\wedge}(M)$ as the spaces of k-forms on which respectively these Laplacians are zero, then we have the following decompositions

Theorem 2.1. *[47]* **(Tseng-Yau)** *Let M be a compact symplectic manifold. For any compatible triple (ω, J, g) there are the orthogonal decompositions*

- $\bigwedge^k = \mathcal{H}^k_{d+d^\wedge} \oplus dd^\wedge \bigwedge^k \oplus (d^* \bigwedge^{k+1} + d^{\wedge *} \bigwedge^{k-1})$,
- $\bigwedge^k = \mathcal{H}^k_{dd^\wedge} \oplus (d \bigwedge^{k-1} + d^\wedge \bigwedge^{k+1}) \oplus (dd^\wedge)^* \bigwedge^k$,
- $\bigwedge^k = \mathcal{H}^k_{d\cap d^\wedge} \oplus (d \bigwedge_0^{k-1} + d^\wedge \bigwedge_0^{k+1}) \oplus (d^* \bigwedge^{k+1} + d^{\wedge *} \bigwedge^{k-1})$.

These decompositions imply respectively the isomorphisms

$$\mathcal{H}^k_{d+d^\wedge}(M) \cong H^k_{d+d^\wedge}(M),\ \mathcal{H}^k_{dd^\wedge}(M) \cong H^k_{dd^\wedge}(M),\ \mathcal{H}^k_{d\cap d^\wedge}(M) \cong H^k_{d\cap d^\wedge}(M).$$

Since $*\Delta_{d+d^\wedge} = \Delta_{dd^\wedge}*$ there is the following corollary.

Corollary 2.1. *The operator $*_s$ gives an isomorphism between $H^k_{d+d^\wedge}(M)$ and $H^{2n-k}_{dd^\wedge}(M)$.*

We consider the analogue Lefschetz property related to these cohomology groups.

Lemma 2.2. *[47] The Laplacians $\Delta_{d+d^\wedge}$, $\Delta_{dd^\wedge}$ and $\Delta_{d\cap d^\wedge}$ commute with the $\mathfrak{sl}_2(\mathbb{R})$ triple (L, Λ, H).*

Using this lemma we can prove directly the following proposition

Proposition 2.2. *[47]* **(Tseng-Yau)** *On a symplectic manifold of dimension $2n$ and a compatible triple (ω, J, g), the Lefschetz operator defines the isomorphisms*

$$L^{n-k} : H^k_{d+d^\wedge}(M) \cong H^{2n-k}_{d+d^\wedge}(M) \quad \forall\, k \leq n,$$

$$L^{n-k} : H^k_{dd^\wedge}(M) \cong H^{2n-k}_{dd^\wedge}(M) \quad \forall\, k \leq n,$$

$$L^{n-k} : H^k_{d\cap d^\wedge}(M) \cong H^{2n-k}_{d\cap d^\wedge}(M) \quad \forall\, k \leq n.$$

This proposition implies that the Lefschetz operator does not give invariants or other informations if we relate it to these cohomologies. Fortunately we have the following property.

Proposition 2.3. *[47]* **(Tseng-Yau)** *On a compact symplectic manifold (M, ω) the following properties are equivalent:*

- *the Hard Lefschetz property holds.*
- *the canonical homomorphism $H^k_{d+d^\wedge}(M) \to H^k_d(M)$ is an isomorphism for all k.*
- *the canonical homomorphism $H^k_{d\cap d^\wedge}(M) \to H^k_{d+d^\wedge}(M)$ is an isomorphism for all k.*

Remark 2.1. These particular cohomologies are studied in details also in [1]. There are also a complex analogue of these cohomologies, namely the Bott-Chern and the Aeppli cohomolgy. They are very interesting in relation to the $\partial\bar{\partial}$-Lemma because they give a necessary ans sufficient condition to it [2, 12].

We are interested in the Lie groups associated to the six dimensional unimodular solvable non-nilpotent Lie algebras which admit a lattice and for which the de Rham cohomology of the associated solvmanifold can be computed by invariant forms. Indeed the following theorem holds:

Theorem 2.2. *Let G be a Lie group admitting a left invariant symplectic structure and a lattice Γ such that the quotient $Q = G/\Gamma$ is compact. Let $\mathfrak{g}$ be the Lie algebra of G.*

If the inclusion $\bigwedge^* \mathfrak{g}^* \overset{i}{\hookrightarrow} \bigwedge^*(Q)$ is a quasi-isomorphism, i.e. $H^*_d(Q) \cong H^*_d(\mathfrak{g})$, then

$$H^*_{d^\wedge}(Q) \cong H^*_{d^\wedge}(\mathfrak{g}), \quad H^*_{d+d^\wedge}(Q) \cong H^*_{d+d^\wedge}(\mathfrak{g}),$$

$$H^*_{dd^\wedge}(Q) \cong H^*_{dd^\wedge}(\mathfrak{g}), \quad H^*_{d\cap d^\wedge}(Q) \cong H^*_{d\cap d^\wedge}(\mathfrak{g}).$$

Proof. We divide the proof into four steps:

1. *We prove that the invariant cohomologies are well defined, i.e. the algebra of invariant forms $\bigwedge^* \mathfrak{g}^*$ is closed for the operator $d^\wedge$.*

   To this aim it suffices to prove that the operator $*_s$ sends invariant forms to invariant forms. If $L : G \to G$ denotes the left translation, then α and β are invariant if $L^*\alpha = \alpha$ and $L^*\beta = \beta$. Then

$$\begin{aligned} L^*(\alpha \wedge *_s\beta) &= L^*\left(\frac{1}{k!}(\omega^{-1})^{i_1j_1}(\omega^{-1})^{i_2j_2}\cdots(\omega^{-1})^{i_kj_k}\alpha_{i_1i_2\cdots i_k}\beta_{j_1j_2\cdots j_k}\frac{\omega^n}{n!}\right) \\ &= \frac{1}{k!}(L^*(\omega)^{-1})^{i_1j_1}(L^*(\omega)^{-1})^{i_2j_2}\cdots(L^*(\omega)^{-1})^{i_kj_k}L^*(\alpha_{i_1i_2\cdots i_k})L^*(\beta_{j_1j_2\cdots j_k})\frac{L^*(\omega^n)}{n!} \\ &= \frac{1}{k!}(\omega^{-1})^{i_1j_1}(\omega^{-1})^{i_2j_2}\cdots(\omega^{-1})^{i_kj_k}\alpha_{i_1i_2\cdots i_k}\beta_{j_1j_2\cdots j_k}\frac{\omega^n}{n!} = \alpha \wedge *_s\beta\,. \end{aligned}$$

   Therefore, $\alpha \wedge *_s\beta = L^*(\alpha \wedge *_s\beta) = L^*(\alpha) \wedge L^*(*_s\beta) = \alpha \wedge L^*(*_s\beta)$ and so

$$*_s\beta = L^*(*_s\beta)\,.$$

2. *We show that $H^*_{d^\wedge}(Q) \cong H^*_{d^\wedge}(\mathfrak{g})$, $H^*_{d\cap d^\wedge}(Q) \cong H^*_{d\cap d^\wedge}(\mathfrak{g})$ and that $H^*_{d+d^\wedge}(Q) \cong H^*_{d+d^\wedge}(\mathfrak{g})$ if and only if $H^*_{dd^\wedge}(Q) \cong H^*_{dd^\wedge}(\mathfrak{g})$.*

We observe that point 1 and Proposition 2.1 imply the commutativity of the diagram

$$\begin{tikzcd} H^k_d(\mathfrak{g}) \arrow[r, "*_s", "\sim"'] \arrow[d, "\sim"] & H^{2n-k}_{d^\wedge}(\mathfrak{g}) \arrow[d] \\ H^k_d(Q) \arrow[r, "\sim", "*_s"'] & H^{2n-k}_{d^\wedge}(Q) \end{tikzcd}$$

so that, since by assumption the isomorphism holds for H^*_d, it holds for $H^*_{d^\wedge}$, i.e. $H^*_{d^\wedge}(Q) \cong H^*_{d^\wedge}(\mathfrak{g})$. Moreover, since $H^k_{d\cap d^\wedge}(Q) := H^k_d \cap H^k_{d^\wedge}$, the isomorphism holds also for the $d \cap d^\wedge$-cohomology.

Hence Corollary 2.1 implies that if the isomorphism between cohomology and invariant cohomology holds for $H_{d+d^\wedge}$, then it is also true for $H_{dd^\wedge}$ and vice versa.

3. $i^* : H^*_{d+d^\wedge}(\mathfrak{g}^*) \to H^*_{d+d^\wedge}(Q)$ *is injective* (see also [40, page 123]).

   Since Q is compact, there exists an invariant metric $\langle\, , \rangle$ on Q. One can use this metric to define the adjoint operators of d, $d^\wedge$, $d+d^\wedge$ and $dd^\wedge$. Let $\bigwedge^{\perp k}\mathfrak{g}^*$ be the orthogonal complement of $\bigwedge^k \mathfrak{g}^*$ in $\bigwedge^k(Q)$.
   Then $\bigwedge^k(Q) = \bigwedge^k\mathfrak{g}^* \oplus \bigwedge^{\perp k}\mathfrak{g}^*$ and $\bigwedge^k\mathfrak{g}^*$ and $\bigwedge^{\perp k}\mathfrak{g}^*$ are closed under $d+d^\wedge$ and $dd^\wedge$.

   If $i^*[\alpha] := [i(\alpha)] = 0$, then there exists a form $\eta \in \bigwedge(Q)$ such that

$$i(\alpha) = dd^\wedge\eta = dd^\wedge(\tilde\eta + \tilde\eta^\perp) = dd^\wedge\tilde\eta + dd^\wedge\tilde\eta^\perp,$$

   with $\tilde\eta \in \bigwedge^k\mathfrak{g}^*$ and $\tilde\eta^\perp \in \bigwedge^{\perp k}\mathfrak{g}^*$.
   Moreover $dd^\wedge\tilde\eta \in \bigwedge^k\mathfrak{g}^*$, so $i(\alpha - dd^\wedge\tilde\eta) = dd^\wedge\tilde\eta^\perp$ and $[\alpha] = [\alpha - dd^\wedge\tilde\eta]$.

   So we can choose $\tilde\alpha := \alpha - dd^\wedge\tilde\eta$ as a representative of the cohomology class $[\alpha]$ in $H^*_{d+d^\wedge}(\mathfrak{g}^*)$.

   Observe that $\tilde\alpha \in \bigwedge^k\mathfrak{g}^*$ so $(dd^\wedge)^*\tilde\alpha \in \bigwedge^k\mathfrak{g}^*$ and

$$i((dd^\wedge)^*\tilde\alpha) = (dd^\wedge)^* i(\tilde\alpha) = (dd^\wedge)^* dd^\wedge\tilde\eta^\perp \in \bigwedge^k\mathfrak{g}^*,$$

   but then $\tilde\eta^\perp \in \bigwedge^{\perp k}\mathfrak{g}^*$ is orthogonal to $(dd^\wedge)^*dd^\wedge\tilde\eta^\perp \in \bigwedge^k\mathfrak{g}^*$. This implies

$$0 = \langle\tilde\eta^\perp, (dd^\wedge)^*dd^\wedge\tilde\eta^\perp\rangle = \langle dd^\wedge\tilde\eta^\perp, dd^\wedge\tilde\eta^\perp\rangle\,,$$

so $dd^\wedge\tilde{\eta}^\perp = 0$. But then $i(\alpha) = dd^\wedge\tilde{\eta}$, with $\tilde{\eta}$ in $\bigwedge^k \mathfrak{g}^*$, so $\alpha = dd^\wedge\tilde{\eta}$ in $\bigwedge^k \mathfrak{g}^*$, that is $[\alpha] = 0$ belongs to $H^*_{d+d^\wedge}(\mathfrak{g}^*)$.

Remark 2.2. We can similarly prove that also $i^* : H^*_{dd^\wedge}(\mathfrak{g}^*) \to H^*_{dd^\wedge}(Q)$ is injective.

In particular point 3 is always true, independent on the fact that the map i is a quasi-isomorphism.

4. $i^* : H^*_{d+d^\wedge}(\mathfrak{g}^*) \to H^*_{d+d^\wedge}(Q)$ *is surjective.*

   Let $\eta \in \bigwedge^k(Q)$ be such that $d\eta = d^\wedge\eta = 0$. Then the cohomology class $[\eta]^k_{d+d^\wedge}$ is well defined. But also $[\eta]^k_d$ and $[\eta]^k_{d^\wedge}$ exist and by hypothesis they have an invariant representative: $\eta = \tilde{\eta}_1 + d\mu_1$ and $\eta = \tilde{\eta}_2 + d^\wedge\mu_1$ with $\eta_1, \eta_2 \in \bigwedge^* \mathfrak{g}^*$ and $d\tilde{\eta}_1 = d^\wedge\tilde{\eta}_2 = 0$.

   Since $dd^\wedge\eta = 0$, the cohomology class $[\eta]^k_{dd^\wedge}$ exists and

$$\eta = \frac{1}{2}(\tilde{\eta}_1 + \tilde{\eta}_2) + d\frac{\mu_1}{2} + d^\wedge\frac{\mu_2}{2}$$

   then $\frac{1}{2}(\tilde{\eta}_1 + \tilde{\eta}_2)$ is an invariant representative for $[\eta]^k_{dd^\wedge}$.

   Now we apply the isomorphism of Corollary 2.1:

$$[*_s\eta]^{2n-k}_{d+d^\wedge} \cong [\eta]^k_{dd^\wedge} = \left[\frac{\tilde{\eta}_1 + \tilde{\eta}_2}{2}\right]^k_{dd^\wedge} \cong \left[*_s(\frac{\tilde{\eta}_1 + \tilde{\eta}_2}{2})\right]^{2n-k}_{d+d^\wedge}.$$

   Let $*_s\eta = N$, $*_s\tilde{\eta}_1 = N_1$, $*_s\tilde{\eta}_2 = N_2$. Then $\frac{N_1+N_2}{2}$ is an invariant representative in $[N]^{2n-k}_{d+d^\wedge}$.

   To complete the proof we have to show that every $N \in \bigwedge^{2n-k}(Q)$ such that $dN = d^\wedge N = 0$ is of the form $N = *_s\eta$ with $\eta \in \bigwedge^k(Q)$ and $d\eta = d^\wedge\eta = 0$.

   To this aim, it is sufficient to impose $\eta := *_s N$, then $*_s\eta = *_s *_s N = N$. Moreover $d^\wedge := (-1)^{k+1} *_s d *_s$, so $*_s d^\wedge = (-1)^{k+1} d *_s$ and $d^\wedge *_s = (-1)^{k+1} *_s d$. Then for every $\beta \in \bigwedge^k(Q)$ if $d^\wedge\beta = 0$, also $*_s d^\wedge\beta = 0$ and then $d *_s \beta = 0$ and similarly if $d\beta = 0$, then $d^\wedge *_s \beta = 0$.

   Hence $d\eta = d^\wedge\eta = 0$.

□

Remark 2.3. We recall that in particular Theorem 2.2 applies in the following cases:

- If G is nilpotent, using Nomizu theorem [35].
- If G is completely solvable, using Hattori theorem [22].
- If $\mathrm{Ad}_G(G)$ and $\mathrm{Ad}_G(\Gamma)$ have the same algebraic closure, using Mostow theorem [34].

Chapter 3

Low dimensional unimodular solvable Lie algebras

We have seen in Chapter 1.1 that if the Mostow condition holds, we can compute the de Rham cohomology of a solvmanifold using only its associated Lie algebra (Theorem 1.10).

The study of solvable Lie algebras have been developed up to dimension 5, (see for instance [4]), for this reason we want to improve this classification by studying six dimensional solvable Lie algebras. Six dimensional nilpotent Lie algebras were classified in [43] then by Proposition 1.2 we will consider six dimensional unimodular solvable Lie algebras [25].

The complete list of these Lie algebras is given in Appendix A.

3.1 Cohomology of six dimensional unimodular solvable Lie algebras

In this Section we compute the second and third Betti number of six dimensional solvable Lie algebras. Solvable Lie algebras $\mathfrak{g}$ with the property that $b_2(\mathfrak{g}) = b_3(\mathfrak{g})$ are interesting because of a class of manifolds endowed with a closed 3 form, called *Strong geometry*, considered in [28]. Strong geometry is an important example of connection between mathematics and physics, in particular multi-moment maps are used in string theory and one-dimensional quantum mechanics [28].

Let M be a manifold, then (M, γ) is a *Strong geometry* if γ is a closed 3-form on M. Suppose there is a Lie group G that acts on M preserving γ, then we denote by $P_{\mathfrak{g}}$ the kernel of the map $\bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ induced by the Lie bracket of $\mathfrak{g}$.
A *Multi-moment map* is an equivariant map $\nu : M \to P_{\mathfrak{g}}^*$ such that $d\langle \nu, p\rangle = i_p\gamma$, for any $p \in P_{\mathfrak{g}}$, (where i_p denotes the interior product) [27].
We refer to [27] and [28] for details on strong geometry. In particular Madsen and Swann [28] proved the following proposition.

Proposition 3.1. *Let (M, γ) be a Strong geometry and suppose there is a Lie group G that acts on M preserving γ with Lie algebra $\mathfrak{g}$. If $b_2(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$, then there exists a multi-moment map for the action of G on the manifold M.*

Because of this result they listed the Lie algebras with trivial second and third Betti numbers, up to dimension five. We add to their classification the Betti numbers of 6-dimensional solvable, non-nilpotent unimodular Lie algebras.

Remark 3.1. Every Lie algebra $\mathfrak{g}$ whose Lie group is solvable has $b_1(\mathfrak{g}) > 0$ [4].

In Appendix B we list 6-dimensional unimodular, solvable, non-nilpotent Lie algebras $\mathfrak{g}$ together with their first, second and third Betti numbers. The Betti numbers of the 6-dimensional Lie algebras with 5-dimensional nilradical were also computed by M. Freibert and F. F. Schulte-Hengesbach [17].

Comparing the Betti numbers in Appendix B and the structure constants in Appendix A we obtain the following theorem

Theorem 3.1. *Let $\mathfrak{g}$ be a six dimensional unimodular, solvable, non-nilpotent Lie algebra*

- *if $b_1(\mathfrak{g}) = 1$, then its nilradical has codimension 1 and $b_2(\mathfrak{g}) = 0$ if and only if $b_3(\mathfrak{g}) = 0$.*

- *if its nilradical has codimension greater then 1, then $b_1(\mathfrak{g}) \geq 2$ and $b_2(\mathfrak{g}) = 1$ if and only if $b_3(\mathfrak{g}) = 0$.*

Guan studied properties about the steps of nilmanifolds, showing that if a nilmanifold G/Γ admits a symplectic structure then G has to be at most two step as a solvable Lie group [19]. He also conjectured that the Lie group of a solvmanifold admitting a symplectic structure is at most 3-step solvable.

Again looking directly at Appendix A we can prove that this is true for all six dimensional unimodular solvable Lie algebra, regardless of existence or not of a symplectic structure.

Proposition 3.2. *Every six dimensional unimodular, solvable, non-nilpotent Lie algebra $\mathfrak{g}$ is 2 or 3-step solvable, in particular*

- *if its nilradical has codimension 1, it is 3-step solvable unless it is almost abelian, or $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

  $\mathfrak{g}_{6.14}^{a,0}$, $\mathfrak{g}_{6.17}$, $\mathfrak{g}_{6.18}^{0,0}$, $\mathfrak{g}_{6.20}$, $\mathfrak{g}_{6.21}^{0}$, $\mathfrak{g}_{6.23}^{0,0,\varepsilon}$, $\mathfrak{g}_{6.25}^{-1,0}$, $\mathfrak{g}_{6.29}^{0,0,\varepsilon}$, $\mathfrak{g}_{6.36}^{0,0}$, $\mathfrak{g}_{6.54}^{0,-1}$, $\mathfrak{g}_{6.63}$, $\mathfrak{g}_{6.65}^{0,0}$, $\mathfrak{g}_{6.70}^{0,0}$, $\mathfrak{g}_{6.88}^{0,0,0}$.

- *if its nilradical has codimension greater then 1, it is 2-step solvable unless $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

  $\mathfrak{g}_{6.129}$, $\mathfrak{g}_{6.135}$, $\mathfrak{g}_{5.19} \oplus \mathbb{R}$, $\mathfrak{g}_{5.20} \oplus \mathbb{R}$, $\mathfrak{g}_{5.23} \oplus \mathbb{R}$, $\mathfrak{g}_{5.25} \oplus \mathbb{R}$, $\mathfrak{g}_{5.26} \oplus \mathbb{R}$, $\mathfrak{g}_{5.28} \oplus \mathbb{R}$, $\mathfrak{g}_{5.30} \oplus \mathbb{R}$, $\mathfrak{g}_{4.8} \oplus 2\mathbb{R}$, $\mathfrak{g}_{4.9} \oplus 2\mathbb{R}$.

3.2 Symplectic structure and Hard Lefschetz property for six dimensional unimodular solvable Lie algebras

Solvmanifolds up to dimension six admitting an invariant symplectic structure were studied by Bock [4]. In particular, he considered the conditions of being cohomologically symplectic, formality and the Hard Lefschetz property.

Now we consider all the Lie algebras listed in Appendix A and study the existence of a symplectic structure over them.

Similarly to the case of differential manifolds we define a *symplectic structure* on a real Lie algebra of dimension $2n$ as a closed and not degenerate 2-form ω in $\bigwedge^* \mathfrak{g}^*$.

If $\mathfrak{g} = \mathrm{Lie}(G)$ for a Lie group G, then ω is an *left invariant symplectic structure* on G.

Let $\mathfrak{g}$ be a six dimensional real solvable unimodular Lie algebra and let $\{X_1, \cdots, X_6\}$ be an ordered basis of $\mathfrak{g}$, then a 2-form ω is associated in a natural way to a matrix $M = (\omega_{ij}) \in \mathcal{M}_6(\mathbb{R})$, where $\omega_{ij} := \omega(X_i, X_j)$, and $\omega^n \neq 0 \Leftrightarrow \det M \neq 0$.
We use this notation in Appendix C.

Theorem 3.2. *The six dimensional real solvable, non-nilpotent unimodular Lie algebras admitting a symplectic structure are the following:*
$\mathfrak{g}_{6.3}^{0,-1}$, $\mathfrak{g}_{6.10}^{0,0}$, $\mathfrak{g}_{6.13}^{\frac{1}{2},-1,0}$, $\mathfrak{g}_{6.13}^{-1,\frac{1}{2},0}$, $\mathfrak{g}_{6.15}^{-1}$, $\mathfrak{g}_{6.18}^{-1,-1}$, $\mathfrak{g}_{6.21}^{0}$, $\mathfrak{g}_{6.23}^{0,0,\varepsilon}$, *with* $\varepsilon \neq 0$
$\mathfrak{g}_{6.29}^{0,0,\varepsilon}$, $\mathfrak{g}_{6.36}^{0,0}$, $\mathfrak{g}_{6.38}^{0}$, $\mathfrak{g}_{6.54}^{0,-1}$, $\mathfrak{g}_{6.70}^{0,0}$, $\mathfrak{g}_{6.78}$, $\mathfrak{g}_{6.118}^{0,\pm1,-1}$, $\mathfrak{n}_{6.84}^{\pm1}$, $\mathfrak{g}_{5.7}^{p,-p,-1} \oplus \mathbb{R}$,
$\mathfrak{g}_{5.8}^{-1} \oplus \mathbb{R}$, $\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$, $\mathfrak{g}_{5.17}^{0,0,r} \oplus \mathbb{R}$, $\mathfrak{g}_{5.17}^{p,-p,\pm1} \oplus \mathbb{R}$, $\mathfrak{g}_{5.17}^{0,0,\pm1} \oplus \mathbb{R}$, $\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$,
$\mathfrak{g}_{5.19}^{-2,2} \oplus \mathbb{R}$, $\mathfrak{g}_{5.19}^{-\frac{1}{2},-1} \oplus \mathbb{R}$, $\mathfrak{g}_{3.4}^{-1} \oplus 3\mathbb{R}$, $\mathfrak{g}_{3.5}^{0} \oplus 3\mathbb{R}$, $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}^{-1}$, $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.5}^{0}$,
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.4}^{-1}$, $\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.5}^{0}$, $\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_{3.5}^{0}$.

Their symplectic forms are listed in Appendix C.

Proof. To construct the symplectic form we take the generic element $\omega \in \ker d \subset \bigwedge^2 \mathfrak{g}^*$ and we impose it to be not degenerate, that is $\omega^3 \neq 0$.
With this direct computation we can see that the six dimensional solvable unimodular Lie algebras not listed above have always $\omega^3 = 0$ for every $\omega \in \ker d \subset \bigwedge^2 \mathfrak{g}^*$.

We give the computation of the first Lie algebra $\mathfrak{g}_{6.3}^{-\frac{a+1}{3},a}$, for the other cases the idea is similar.

By Appendix A for $(\mathfrak{g}_{6.3}^{-\frac{a+1}{3},a})^*$ we have $d\alpha^1 = \frac{a+1}{3}\alpha^{16} - \alpha^{26}$, $d\alpha^2 = \frac{a+1}{3}\alpha^{26} - \alpha^{36}$, $d\alpha^3 = \frac{a+1}{3}\alpha^{36}$, $d\alpha^4 = -\alpha^{46}$, $d\alpha^5 = -a\alpha^{56}$, $d\alpha^6 = 0$ with $0 < |a| \leq 1$. Then
$d\alpha^{12} = -\frac{2}{3}(a+1)\alpha^{126} + \alpha^{136}$
$d\alpha^{13} = -\frac{2}{3}(a+1)\alpha^{136} + \alpha^{236}$
$d\alpha^{14} = \frac{2-a}{3}\alpha^{146} + \alpha^{246}$
$d\alpha^{15} = \frac{2a-1}{3}\alpha^{156} + \alpha^{256}$
$d\alpha^{16} = 0$
$d\alpha^{23} = -\frac{2}{3}(a+1)\alpha^{236}$
$d\alpha^{24} = \frac{2-a}{3}\alpha^{246} + \alpha^{346}$
$d\alpha^{25} = \frac{2a-1}{3}\alpha^{256} + \alpha^{356}$
$d\alpha^{26} = 0$

$d\alpha^{34} = \frac{2-a}{3}\alpha^{346}$
$d\alpha^{35} = \frac{2a-1}{3}\alpha^{356}$
$d\alpha^{36} = 0$
$d\alpha^{45} = (a+1)\alpha^{456}$
$d\alpha^{46} = 0$
$d\alpha^{56} = 0.$

Let ω be a generic 2-form on $\mathfrak{g}_{6.3}^{-\frac{a+1}{3},a}$. If $a \neq -1, \frac{1}{2}$, then $d\omega = 0$ if and only if

$$\omega = \omega_{1,6}\alpha^{16} + \omega_{2,6}\alpha^{26} + \omega_{3,6}\alpha^{36} + \omega_{4,6}\alpha^{46} + \omega_{5,6}\alpha^{56},$$

but in this case $\det(\omega_{i,j}) = 0$ and ω is degenerate.

If $a = \frac{1}{2}$, then $d\omega = 0$ if and only if

$$\omega = \omega_{1,6}\alpha^{16} + \omega_{2,6}\alpha^{26} + \omega_{3,5}\alpha^{35} + \omega_{3,6}\alpha^{36} + \omega_{4,6}\alpha^{46} + \omega_{5,6}\alpha^{56},$$

but again in this case $\det(\omega_{i,j}) = 0$ and ω is degenerate.

If $a = -1$, then $d\omega = 0$ if and only if

$$\omega = \omega_{1,6}\alpha^{16} + \omega_{2,3}\alpha^{23} + \omega_{2,6}\alpha^{26} + \omega_{3,6}\alpha^{36} + \omega_{4,5}\alpha^{45} + \omega_{4,6}\alpha^{46} + \omega_{5,6}\alpha^{56}$$

and in this case $\det(\omega_{i,j}) \neq 0$ if and only if $\omega_{1,6}\omega_{2,3}\omega_{4,5} \neq 0$. Then for this value of the parameter we have a symplectic form. $\square$

Remark 3.2. Symplectic structures of four dimensional Lie algebras are studied in [39].

Using Theorem 2.2 and Proposition 2.3 we can examine which symplectic solvmanifold G/Γ whose Lie algebra is in Appendix A with G completely solvable, is Hard Lefschetz.

Let $\{\alpha^1, \cdots, \alpha^6\}$ be the dual basis of $\{X_1, ..., X_6\}$. Then a generic element in $\bigwedge^2 \mathfrak{g}^*$ is $\beta = \sum_{i<j} b_{i,j}\alpha^{ij}$, where we use the notation $\alpha^{i_1\cdots i_n} := \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_n}$.

For any such solvmanifold we perform the computation only for a particular choice of the symplectic form. Namely we consider the form composed by the fewest possible generators α^{ij} of $\bigwedge^2 \mathfrak{g}^*$, and we check if the Hard Lefschetz property holds only for this particular choice. This is because computations are very involved for a generic symplectic form.

Proposition 3.3. *[4] The symplectic and completely solvable Lie algebras in Appendix C whose Lie group admits a lattice are the following:*

$\mathfrak{g}_{3.1} \oplus 3\mathbb{R}$, $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}$, $\mathfrak{g}_{3.4} \oplus \mathfrak{g}_{3.4}$, $\mathfrak{g}_{5.7}^{p,-p,-1} \oplus \mathbb{R}$, $\mathfrak{g}_{5.8} \oplus \mathbb{R}$, $\mathfrak{g}_{5.15} \oplus \mathbb{R}$, $\mathfrak{g}_{6.3}$, $\mathfrak{g}_{6.15}$, $\mathfrak{g}_{6.21}^{0}$, $\mathfrak{g}_{6.23}^{0,0,\pm 1}$, $\mathfrak{g}_{6.29}^{0,0,\pm 1}$, $\mathfrak{g}_{6.29}^{0,0,0}$, $\mathfrak{g}_{6.54}^{0,-1}$, $\mathfrak{g}_{6.78}$.

By computing the cohomologies $H^*_{d+d^\Lambda}(\mathfrak{g})$ and $H^*_{d\cap d^\Lambda}(\mathfrak{g})$ we obtain that the Hard Lefschetz property holds only for the solvmanifolds associated to the following symplectic Lie algebras.

- $\mathfrak{g}_{3.4} \oplus 3\mathbb{R}$: $\omega = \omega_{1,2}\alpha^{12} + \omega_{3,6}\alpha^{36} + \omega_{4,5}\alpha^{45}$, $\tilde{\omega} = \omega_{1,2}\alpha^{12} + \omega_{3,4}\alpha^{34} + \omega_{5,6}\alpha^{56}$
  $\hat{\omega} = \omega_{1,2}\alpha^{12} + \omega_{3,5}\alpha^{35} + \omega_{4,4}\alpha^{46}$

$$b^1_d = b^1_{d+d^\Lambda} = b^1_{d\cap d^\Lambda} = 4$$
$$b^2_d = b^2_{d+d^\Lambda} = b^2_{d\cap d^\Lambda} = 7$$
$$b^3_d = b^3_{d+d^\Lambda} = b^3_{d\cap d^\Lambda} = 8$$

- $\mathfrak{g}_{3.4} \oplus \mathfrak{g}_{3.4}$: $\omega = \omega_{1,2}\alpha^{12} + \omega_{3,6}\alpha^{36} + \omega_{4,5}\alpha^{45}$

$$b^1_d = b^1_{d+d^\Lambda} = b^1_{d\cap d^\Lambda} = 2$$
$$b^2_d = b^2_{d+d^\Lambda} = b^2_{d\cap d^\Lambda} = 3$$
$$b^3_d = b^3_{d+d^\Lambda} = b^3_{d\cap d^\Lambda} = 4$$

- $\mathfrak{g}_{5.7}^{p,-p,-1} \oplus \mathbb{R}$:

$p = 1$: $\omega = \omega_{1,4}\alpha^{14} + \omega_{2,3}\alpha^{23} + \omega_{5,6}\alpha^{56}$, $\tilde{\omega} = \omega_{1,3}\alpha^{13} + \omega_{2,4}\alpha^{24} + \omega_{5,6}\alpha^{56}$

$$b^1_d = b^1_{d+d^\Lambda} = b^1_{d\cap d^\Lambda} = 2$$
$$b^2_d = b^2_{d+d^\Lambda} = b^2_{d\cap d^\Lambda} = 5$$
$$b^3_d = b^3_{d+d^\Lambda} = b^3_{d\cap d^\Lambda} = 8$$

$$
\begin{aligned}
p \neq 1:\quad & \omega = \omega_{1,4}\alpha^{14} + \omega_{2,3}\alpha^{23} + \omega_{5,6}\alpha^{56} \\
& b^1_d = b^1_{d+d^\wedge} = b^1_{d\cap d^\wedge} = 2 \\
& b^2_d = b^2_{d+d^\wedge} = b^2_{d\cap d^\wedge} = 3 \\
& b^3_d = b^3_{d+d^\wedge} = b^3_{d\cap d^\wedge} = 4
\end{aligned}
$$

We have then proved:

Theorem 3.3. *There exists a symplectic structure for which the following solvmanifolds are Hard Lefschetz:*

$$(G_{5.7}^{p,-p,-1} \times \mathbb{R})/\Gamma, \quad (G_{3.4} \times 3\mathbb{R})/\Gamma, \quad (G_{3.4} \times G_{3.4})/\Gamma,$$

where Γ are lattices listed in [4].

Remark 3.3. The case of $(G_{5.7}^{p,-p,-1} \times \mathbb{R})/\Gamma$ was already considered in [4].

Chapter 4

Lattices and de Rham cohomology of solvmanifolds

In this chapter we consider solvmanifolds for which we are not sure that the Mostow condition holds.
In this case the invariant cohomology can be strictly included in the cohomology of the solvmanifold (Theorem 1.6), but sometimes even if the Mostow condition does not hold, the de Rham cohomolgy is isomorphic to the invariant one (Proposition 4.1).

We will consider a technique due to Kasuya to understand if this isomorphism holds and in particular we will use it to compute the de Rham cohomology of some almost abelian six dimensional solvmanifolds.

Of course we can apply a method to compute the cohomology only when we have a lattice. For this reason we will first prove, for every case considered, the existence or not of the lattice, also for some value of the parameters for which we can not apply the method, giving examples of many almost abelian solvmanifolds.

4.1 Lattices

In this section we study the existence of lattices for six dimensional, unimodular almost abelian Lie groups which are not completely solvable, since we want to compute the de Rham cohomology of the corresponding solvmanifolds and study the property of formality (see Chapter 6). Our aim is to have situations when the Mostow condition could not hold, in particular we consider cases in which we can apply a proposition that we will state in the following section (Proposition 4.1).

We can show the following

Theorem 4.1. *The simply-connected Lie groups whose Lie algebra is one of the following*

$\mathfrak{g}_{6.8}^{a,b,c,0}$: $[X_1, X_6] = aX_1,\ [X_2, X_6] = bX_2,\ [X_3, X_6] = cX_3,\ [X_4, X_6] = X_5,\ \ [X_5, X_6] = X_4,$
$a + b + c = 0, \quad 0 < |c| \leq |b| \leq |a|.$

$\mathfrak{g}_{6.10}^{0,0}$: $[X_2, X_6] = X_1,\ [X_3, X_6] = X_2,\ [X_4, X_6] = -X_5,\ [X_5, X_6] = X_4.$

$\mathfrak{g}_{6.11}^{a,0,q,s}$: $[X_1, X_6] = aX_1,\ [X_2, X_6] = -x_3,\ [X_3, X_6] = X_2,\ [X_4, X_6] = qX_4 - sX_5,$
$[X_5, X_6] = sX_4 + qX_5, \qquad a + 2q = 0, \quad as \neq 0.$

$\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}$: $[X_1, X_5] = X_1,\ [X_2, X_5] = -X_2,\ [X_3, X_5] = -rX_4,\ [X_4, X_5] = rX_3, \quad r \neq 0.$

$\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$: $[X_2, X_5] = X_1,\ [X_3, X_5] = -X_4,\ [X_4, X_5] = X_3.$

$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$: $[X_1, X_5] = pX_1 - X_2,\ [X_2, X_5] = X_1 + pX_2,\ [X_3, X_5] = -pX_3 - rX_4,$
$[X_4, X_5] = rX_3 - pX_4, \quad r \neq 0.$

$\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$: $[X_1, X_5] = -X_2,\ [X_2, X_5] = X_1,\ [X_3, X_5] = X_1 - X_4,\ [X_4, X_5] = X_2 + X_3.$

$\mathfrak{g}_{3.5}^{0} \oplus \mathbb{R}^3$: $[X_1, X_3] = -X_2,\ [X_2, X_3] = X_1.$

admit a lattice.

Proof. In the indecomposable case the solvable Lie algebras are of the form $\mathbb{R} \ltimes_{\mathrm{ad}_{X_6}} \mathbb{R}^5$, where $\mathbb{R} = \mathrm{span}\langle X_6 \rangle$ and we will give for any Lie algebra the matrix expression of ad_{X_6} with respect to the basis $\{X_1, \ldots, X_5\}$ of $\mathbb{R}^5$. By using Proposition 1.3 if there exists a real number t_0 such that $\exp(t_0 ad_{X_6})$ is conjugate to an integer matrix, then t_0 determines a lattice Γ_{t_0} of the corresponding simply connected almost abelian solvable Lie group.

In particular if the characteristic polynomial and the minimal polynomial of $\exp(t_0 \mathrm{ad}_{X_{n+1}})$ do not have integer coefficients, then Γ_{t_0} is not a lattice. Otherwise a possible choice for the conjugate integer matrix is [4]

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ 0 & 1 & 0 & \dots & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}$$

where a_i are the coefficients of the characteristic polynomial.

We consider all the six dimensional, unimodular almost abelian Lie groups which are not completely solvable. There are eleven such Lie groups that can admit a lattice and their Lie algebras are the following [4] (see Appendix A):
$\mathfrak{g}_{6.8}^{a,b,c,p}$, $\mathfrak{g}_{6.9}^{a,b,p}$, $\mathfrak{g}_{6.10}^{a,-\frac{3}{2}a}$, $\mathfrak{g}_{6.11}^{a,p,q,s}$, $\mathfrak{g}_{6.12}^{-4p,p}$, $\mathfrak{g}_{5.13}^{-1-2q,q,r} \oplus \mathbb{R}$, $\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$, $\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$, $\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$, $\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathbb{R}^2$, $\mathfrak{g}_{3.5}^{0} \oplus \mathbb{R}^3$.

This idea is in general not very simple to use, for this reason we start by considering a value of t_0 such that at least a complex block of the semisimple part of $\exp(t_0 ad_{X_6})$ is of kind $\begin{pmatrix} e^p t_0 \cos(2n\pi) & e^p t_0 \sin(2n\pi) \\ -e^p t_0 \sin(2n\pi) & e^p t_0 \cos(2n\pi) \end{pmatrix}$ with $n \in \mathbb{Z}$, i.e. $\begin{pmatrix} e^p t_0 & 0 \\ 0 & e^p t_0 \end{pmatrix}$. With this choice, the analysis of the characteristic and minimal polynomials becomes operable.

If this is a lattice for some value of the parameters, we continue by studying for the same parameters if also for $t = \frac{t_0}{k}$ with $k \in \mathbb{Z}$ we have a lattice. In this way we can usually use the ideas and construction of the previous case.

We performed some of the computation with the help of the Maple software.

- $\Gamma_{2\pi}$ is a lattice in $G_{6.8}^{a,b,c,p}$ only for $p = 0$:

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -b-c-2p & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & -1 & p \end{pmatrix}$$

So

$$\exp(2\pi \mathrm{ad}_{X_6}) = \begin{pmatrix} e^{2\pi(-b-c-2p)} & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi b} & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi c} & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi p} & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi p} \end{pmatrix}$$

We put $e^{2\pi b} = w,\ e^{2\pi c} = v,\ e^{-2\pi p} = k$, so the matrix becomes

$$\begin{pmatrix} \frac{k^2}{wv} & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{k} \end{pmatrix}$$

Its minimal polynomial is

$$\begin{aligned} m(x) \quad &= \quad k - \frac{k^3wv + wk^2 + k^2v + v^2w^2}{wvk}x + \frac{vk^3 + wk^3 + w^2v^2k + k^2 + wv^2 + w^2v}{wvk}x^2 + \\ &\quad - \frac{k^3 + kwv^2 + kw^2v + w}{wvk}x^3 + x^4 \end{aligned}$$

So the minimal polynomial can have integer coefficients only if $k \in \mathbb{Z}$.

We put $w + v = r,\ wv = s$ and the coefficients become:

$$p_1 = \frac{k^3s + k^2r + s^2}{ks} = k^2 + \frac{k^2r + s^2}{ks}$$

$$p_2 = \frac{k^3r + kr^2 + k^2 + rs}{ks} \qquad p_3 = \frac{k^3 + krs + s}{ks}$$

So $p_1 \in \mathbb{Z}$ if and only if $q_1 = \frac{k^2r+s^2}{ks} \in \mathbb{Z}$ and $p_2 - kq_1 = \frac{k^2+rs}{ks}$.

If $p_1, p_2 \in \mathbb{Z}$ then $h := p_2 - kq_1 \in \mathbb{Z}$. $s = \frac{k^2}{hk-r}$, then $p_3 = \frac{hk^2+1}{k} = hk + \frac{1}{k}$.

So $p_3 \in \mathbb{Z}$ if and only if $\frac{1}{k} \in \mathbb{Z}$, but $k \in \mathbb{Z}$, so $k = 1$ and $p = 0$.

We found out that for $p \neq 0$ $\Gamma_{2\pi}$ is not a lattice.

Now we check for $p = 0$: the characteristic polynomial has coefficients

$$a_0 = -1 \quad a_1 = 2 + \frac{r+s^2}{s} \quad a_2 = -1 - \frac{2s^2+2r+rs+1}{s} = -1 - 2\frac{s^2+r}{s} - \frac{rs+1}{s}$$

$$a_3 = 1 + \frac{2rs+2+s^2+r}{s} = 1 + 2\frac{rs+1}{s} + \frac{s^2+r}{s} \qquad a_4 = -2 - \frac{rs+1}{s}$$

So a_1, a_2, a_3, $a_4 \in \mathbb{Z}$ if and only if $\frac{s^2+r}{s}$, $\frac{rs+1}{s} \in \mathbb{Z}$ and we must check that the solutions are such that w and v are positive:

we solve the system

$$\begin{cases} \frac{s^2+r}{s} = h_1 \\ \frac{rs+1}{s} = h_2 \\ r > 0 \\ 0 < s \leq \frac{r^2}{4} \end{cases}$$

and find that it admits solutions for some values of the integers h_1 and h_2 (for example for $h_1 = 5, h_2 = 6$). In particular we can not accept the solutions $\{s = r-1\}$, because they correspond to $b = 0$ or $c = 0$ and $\{s = 1\}$, because it corresponds to $a = 0$.

Thus, for $p = 0$, we can find values of b and c (and $a = -b - c$) such that the characteristic polynomial of $\exp(2\pi \mathrm{ad}_{X_6})$ has integer coefficients and we can check by direct computation that $\exp(2\pi\, \mathrm{ad}_{X_6})$ is conjugate to $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -h_1 & 0 & 0 \\ 0 & 1 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Therefore, for some choice of the parameters b and c, $\Gamma_{2\pi}$ is a lattice. We denote the group $G_{6.8}^{a,b,c,0}$ for the above choices of the parameters a, b, c by $G_{6.8}^{p=0}$ for short.

•$\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$ is a lattice in $G_{6.8}^{a,b,c,0}$ only if $k = 2, 3, 4, 6$:

Let consider lattices $\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$, then the conditions for the parameters a, b, c that are imposed for $\Gamma_{2\pi}$ to be a lattice must be satisfied also for $\Gamma_{2\pi/k}$.

$$\exp(\frac{2\pi}{k}\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{2\pi(-b-c)/k} & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi b/k} & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi c/k} & 0 & 0 \\ 0 & 0 & 0 & \cos 2\pi/k & \sin 2\pi/k \\ 0 & 0 & 0 & -\sin 2\pi/k & \cos 2\pi/k \end{pmatrix}$$

We put $e^{2\pi b/k} = w,\ e^{2\pi c/k} = v,\ \cos 2\pi/k = u/2$, and $w+v=r,\ wv=s$, then the coefficients of the characteristic polynomial become:

$$a_1 = \frac{us+r+s^2}{s} = u + \frac{r+s^2}{s} \qquad a_2 = -1 - \frac{ur+us^2+1+rs}{s}$$

$$a_3 = 1 + \frac{u+urs+s^2+r}{s} \qquad a_4 = -\frac{1+rs+us}{s} = -\frac{1+rs}{s} - u$$

Then $a_2 = -ua_1+a_4+u+u^2-1$ and $a_3 = -ua_4+a_1-u-u^2+1$, so if $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, then $a_1 + a_4$ and $a_2 + a_3$ are integer and so $u \in \mathbb{Q}$.

We have found out that if $\cos 2\pi/k$ is not rational, then $\Gamma_{2\pi/k}$ is not a lattice.

If $u \in \mathbb{Q}$, then the characteristic polynomial has integer coefficients if and only if $u \in \mathbb{Z}$ and the same system as $t = 2\pi$ admits a solution, with $h_1, h_2 \in \mathbb{Z}$:

$$\begin{cases} \frac{s^2+r}{s} = h_1 \\ \frac{rs+1}{s} = h_2 \\ r > 0 \\ 0 < s \leq \frac{r^2}{4} \end{cases}$$

We know that the solution exists for some conditions on h_1 and h_2 so we can have a lattice for $t = \pi,\ \frac{\pi}{2},\ \frac{\pi}{3}, \frac{2\pi}{3}$.

With direct computation we check that the matrix $\exp(t_0\mathrm{ad}_{X_6})$ is similar to A for $t_0 = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$

and it is similar to $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -h_1 & 0 & 0 \\ 0 & 1 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ for $t_0 = \pi$.

• $\Gamma_{2n\pi}$ is never a lattice in $G_{6.9}^{a,b,p}$ for every $n \in \mathbb{Z}$:

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -2b-2p & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & -1 & p \end{pmatrix}$$

so

$$\exp(2n\pi \mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-4n\pi(b+p)} & 0 & 0 & 0 & 0 \\ 0 & e^{2n\pi b} & 2n\pi e^{2n\pi b} & 0 & 0 \\ 0 & 0 & e^{2n\pi b} & 0 & 0 \\ 0 & 0 & 0 & e^{2n\pi p} & 0 \\ 0 & 0 & 0 & 0 & e^{2n\pi p} \end{pmatrix}$$

We put $e^{2n\pi b} = w,\ e^{-2n\pi p} = k$, so the matrix becomes

$$\begin{pmatrix} \frac{k^2}{w^2} & 0 & 0 & 0 & 0 \\ 0 & w & 2\pi w & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{k} \end{pmatrix}$$

Its minimal polynomial is

$$\begin{aligned} m(x) \quad &= \quad k - \frac{w^3 + 2k^2 + k^3 w}{wk} x + \frac{2w^3 + kw^4 + k^2 + 2k^3 w}{w^2 k} x^2 + \\ & \quad - \frac{k^3 + 2kw^3 + w^2}{w^2 k} x^3 + x^4 \end{aligned}$$

So the $m(x)$ can have integer coefficients only if $k \in \mathbb{Z}$.

The first coefficient is $p_1 = -\dfrac{w^3 + 2k^2}{kw} - k^2$, so it is integer if and only if $h_1 := \dfrac{w^3 + 2k^2}{kw}$ is integer. Then $w^3 = h_1 kw - 2k^2$ and replacing in the other coefficients we have

$$p_2 = \underbrace{\frac{2h_1 kw - 3k^2}{kw^2}}_{X} + h_1 k$$

$$p_3 = \frac{-w^2 - 2h_1k^2w + 3k^3}{kw^2} = -\frac{1}{k} - kX$$

So $p_2 \in \mathbb{Z}$ if and only if $X \in \mathbb{Z}$, then $p_3 \in \mathbb{Z}$ if and only if $k = 1$ and so $p = 0$.

We found out that for $p \neq 0$ $\Gamma_{2\pi}$ is not a lattice.

Now we check for $p = 0$: the characteristic polynomial has coefficients

$$a_0 = -1 \qquad a_1 = 2 + \frac{2 + w^3}{w}$$

$$a_2 = -1 - \frac{2w^3 + 2w^4 + 4w + 1}{w^2} = -1 - \frac{2w^3 + 1}{w^2} - 2\frac{2 + w^3}{w}$$

$$a_3 = 1 + \frac{2w + 2 + 4w^3 + w^4}{w^2} = 1 + 2\frac{2w^3 + 1}{w^2} + \frac{2 + w^3}{w} \qquad a_4 = -2 - \frac{2w^3 + 1}{w^2}$$

So $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ if and only if $\dfrac{2w^3 + 1}{w^2}, \dfrac{2 + w^3}{w} \in \mathbb{Z}$

The solutions of the equation $\dfrac{2 + w^3}{w} = h \in \mathbb{Z}$ are

$$w = \frac{1}{3}\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}} + \frac{h}{\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}}}$$

$$w = -\frac{1}{6}\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}} - \frac{1}{2}\frac{h}{\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}}} +$$

$$\pm\frac{1}{2i}\sqrt{3}\left(\frac{1}{3}\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}} - \frac{h}{\sqrt[3]{-27 + 3\sqrt{-3h^3 + 81}}}\right)$$

If we replace these values in $\dfrac{2w^3 + 1}{w^2} \in \mathbb{Z}$ we obtain that $-3h^3 + 81$ must be a perfect square. Suppose $-3h^3 + 81 = \pm n^2$, in this way we consider all possible real w, then

$$81 = 3h^3 \pm n^2 = \begin{cases} n^2 - (-3h^3) = (n - \sqrt{-3h^3})(n + \sqrt{-3h^3}) \\ 3h^3 - n^2 = (\sqrt{3h^3} - n)(\sqrt{3h^3} + n) \end{cases}$$

If we decompose $81 = \alpha\beta$, with $\alpha \geq \beta$, then $\sqrt{\pm 3h^3} = \frac{\alpha \pm \beta}{2}$.

$81 = 81 \cdot 1,\ 27 \cdot 3,\ 9 \cdot 9$, so $\sqrt{3h^3} = 41,\ 15,\ 9$, then $h^3 = \frac{1681}{3},\ 75,\ 27$, but the only cube is 27, so $h = 3$.

$\sqrt{-3h^3} = 40,\ 12,\ 0$, then $h^3 = -\frac{1600}{3},\ -48,\ 0$, but the only cube is 0, so $h = 0$.

For $h = 3$ we have $w = 1, -2$ and $k = 3, -\frac{15}{4}$, so we can accept only $w = 1$, that is $b = 0$, but we already have $p = 0$, so also $a = 0$ that is not allowed.

For $h = 0$ we have $w = -\sqrt[3]{2}, \frac{1}{2}\sqrt[3]{2} \pm \frac{1}{2}i\sqrt{3}\sqrt[3]{2}$, we are interested only in the real value and for this w, $k = -\frac{3}{2}\sqrt[3]{2}$, not integer, so we have no lattice for $G_{6.9}$ and $t = 2n\pi$.

• $\Gamma_{2\pi}$ is a lattice in $G^a_{6.10}$ if and only if $a = 0$:

$$\mathrm{ad}_{X_6} = \begin{pmatrix} a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2}a & 1 \\ 0 & 0 & 0 & -1 & -\frac{3}{2}a \end{pmatrix}$$

so

$$\exp(2\pi\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{2a\pi} & 2\pi e^{2a\pi} & 2\pi^2 e^{2a\pi} & 0 & 0 \\ 0 & e^{2a\pi} & 2\pi e^{2a\pi} & 0 & 0 \\ 0 & 0 & e^{2\pi b} & 0 & 0 \\ 0 & 0 & 0 & e^{-3a\pi} & 0 \\ 0 & 0 & 0 & 0 & e^{-3a\pi} \end{pmatrix}$$

We put $e^{a\pi} = w$, so the matrix becomes

$$\begin{pmatrix} w^2 & 2\pi w^2 & 2\pi^2 w^2 & 0 & 0 \\ 0 & w^2 & 2\pi w^2 & 0 & 0 \\ 0 & 0 & w^2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{w^3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{w^3} \end{pmatrix}$$

Its minimal polynomial is

$$m(x) = w^3 - (w^6 + 3w)x + 3\frac{1 + w^5}{w}x^2 - \frac{1 + 3w^5}{w^3}x^3 + x^4$$

So $m(x)$ can have integer coefficients only if $w^3 \in \mathbb{Z}$, but then also $w^6 \in \mathbb{Z}$ and so $p_1 \in \mathbb{Z}$ if and only if $3w \in \mathbb{Z}$, then $3w^4 = 3w \cdot w^3 \in \mathbb{Z}$ and so $p_2 = \frac{3}{w} + 3w^4 \in \mathbb{Z}$ if and only if $\frac{3}{w} \in \mathbb{Z}$.

Now put $w = \frac{k}{3}$ with $k \in \mathbb{Z}$, then $\frac{3}{w} = \frac{9}{k} \in \mathbb{Z}$ if and only if $k = \pm 1,\ \pm 3,\ \pm 9$, but w must be positive by definition, so $k = 1, 3, 9$ and $w = \frac{1}{3}, 1, 3$.

We want $w^3 \in \mathbb{Z}$, so $w = 1, 3$ and $p_3 = 4, \frac{730}{27}$, then $w = 1$ and $a = 0$.

We found out that for $a \neq 0$ $\Gamma_{2\pi}$ is not a lattice.

For $a = 0$ the characteristic polynomial is

$$x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

and we can check with direct computation that in this case $\exp(2\pi \mathrm{ad}_{X_6})$ is conjugate to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 1 \end{pmatrix}$$

and then $\Gamma_{2\pi}$ is a lattice.

•$\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$ is a lattice in $G^0_{6.10}$ only if $k = 2, 3, 4, 6$:

Let consider lattices $\Gamma_{2\pi/k}$ with $k \in \mathbb{Z}$:

$$\exp(\frac{2\pi}{k}\mathrm{ad}_{X_6}) = \begin{pmatrix} 1 & \frac{2\pi}{k} & \frac{2\pi^2}{k^2} & 0 & 0 \\ 0 & 1 & \frac{2\pi}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\left(\frac{2\pi}{k}\right) & \sin\left(\frac{2\pi}{k}\right) \\ 0 & 0 & 0 & -\sin\left(\frac{2\pi}{k}\right) & \cos\left(\frac{2\pi}{k}\right) \end{pmatrix}$$

Its characteristic polynomial is

$$x^5 + \left(-3 - 2\cos\left(\frac{2\pi}{k}\right)\right)x^4 + \left(4 + 6\cos\left(\frac{2\pi}{k}\right)\right)x^3 + \left(-4 - 6\cos\left(\frac{2\pi}{k}\right)\right)x^2 + \left(3 + 2\cos\left(\frac{2\pi}{k}\right)\right)x - 1$$

then it has integer coefficients if and only if $2\cos\left(\frac{2\pi}{k}\right) \in \mathbb{Z}$, so we consider $t_0 = \pi,\ \frac{\pi}{2},\ \frac{\pi}{3}$:

for $t_0 = \pi \quad \exp(t\mathrm{ad}_{X_6})$ is conjugate to the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{pmatrix}$$

and then Γ_π is a lattice.

For $t_0 = \frac{\pi}{2} \quad \exp(t\mathrm{ad}_{X_6})$ is conjugate to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{pmatrix}$$

and then $\Gamma_{\frac{\pi}{2}}$ is a lattice.

For $t_0 = \frac{\pi}{3} \quad \exp(t\mathrm{ad}_{X_6})$ is conjugate to the matrix A, then $\Gamma_{\frac{\pi}{3}}$ is a lattice.

Remark 4.1. The lattice Γ_π was found in [4, Proposition 6.18]. In part (ii) it is stated that if there is a lattice in $G^0_{6.10}$ such that the corresponding solvmanifold satisfies $b_1 = 2$ and $b_2 = 3$, then it is symplectic and not formal. Here we show that, for example, Γ_π, is such a lattice. We will deal about symplectic structures and formality later (Chapter 6.2).

• $\Gamma_{2\pi}$ is a lattice in $G_{6.11}^{a,p,q,s}$ only if $p=0$:

$$\mathrm{ad}_{X_6}=\begin{pmatrix} -2(p+q) & 0 & 0 & 0 & 0\\ 0 & p & 1 & 0 & 0\\ 0 & -1 & p & 0 & 0\\ 0 & 0 & 0 & q & s\\ 0 & 0 & 0 & -s & q \end{pmatrix}$$

then

$$\exp(2\pi\mathrm{ad}_{X_6})=\begin{pmatrix} e^{-4(p+q)\pi} & 0 & 0 & 0 & 0\\ 0 & e^{2p\pi} & 0 & 0 & 0\\ 0 & 0 & e^{2p\pi} & 0 & 0\\ 0 & 0 & 0 & e^{2q\pi}\cos 2s\pi & e^{2q\pi}\sin 2s\pi\\ 0 & 0 & 0 & -e^{2q\pi}\sin 2s\pi & e^{2q\pi}\cos 2s\pi \end{pmatrix}$$

We put $e^{2q\pi}=w, \quad e^{-2p\pi}=k, \quad \cos 2s\pi=u$, so the matrix becomes

$$\begin{pmatrix} \frac{k^2}{w^2} & 0 & 0 & 0 & 0\\ 0 & k^{-1} & 0 & 0 & 0\\ 0 & 0 & k^{-1} & 0 & 0\\ 0 & 0 & 0 & wu & w\sqrt{1-u^2}\\ 0 & 0 & 0 & -w\sqrt{1-u^2} & wu \end{pmatrix}$$

Its minimal polynomial is

$$\begin{aligned} m(x) \quad = \quad & k-\frac{\left(2\,k^2u+w^3+k^3w\right)x}{kw}+\frac{\left(2\,k^3wu+2\,w^3u+k^2+kw^4\right)x^2}{kw^2}+\\ & -\frac{\left(2\,w^3uk+k^3+w^2\right)x^3}{kw^2}+x^4 \end{aligned}$$

So $m(x)$ can have integer coefficients only if $k\in\mathbb{Z}$.

$p_1=-\dfrac{2uk^2+w^3}{kw}-k^2$, so it is integer if and only if $h_1:=\dfrac{2uk^2+w^3}{kw}\in\mathbb{Z}$, then $w^3=kh_1w-2uk^2$.

Replacing in the other coefficients we have

$$p_2 = \underbrace{\frac{2h_1kuw - 4k^2u^2 + k^2}{kw^2}}_{X} + h_1k$$

$$p_3 = \frac{1}{k} - kX$$

so $p_2 \in \mathbb{Z}$ if and only if $X \in \mathbb{Z}$ and then $p_3 \in \mathbb{Z}$ if and only if $\frac{1}{k} \in \mathbb{Z}$, that is $k = 1$ and $p = 0$.

We found out that for $p \neq 0$ $\Gamma_{2\pi}$ is not a lattice.

Now we check for $p = 0$: the characteristic polynomial has coefficients

$$a_0 = -1, \quad a_1 = \frac{w^3 + 2u}{w} + 2, \quad a_2 = -2\frac{w^3 + 2u}{w} - \frac{2w^3u + 1}{w^2} - 1$$

$$a_3 = \frac{w^3 + 2u}{w} + 2\frac{2w^3u + 1}{w^2} + 1, \quad a_4 = -\frac{2w^3u + 1}{w^2} - 2$$

then it has integer coefficients if and only if the following system admits solutions

$$\begin{cases} \dfrac{w^3 + 2u}{w} = h_1 \in \mathbb{Z} \\ \dfrac{2w^3u + 1}{w^2} = h_2 \in \mathbb{Z} \\ w > 0 \\ -1 \leq u \leq 1 \end{cases}$$

From the first equation we get $u = \frac{wh - w^3}{2}$, so the system becomes

$$\begin{cases} \dfrac{w^4h - w^6 + 1}{w^2} = h_2 \in \mathbb{Z} \\ w > 0 \\ -1 \leq \frac{wh - w^3}{2} \leq 1 \end{cases}$$

that admits solution for particular values of the integer h_1 and h_2.

So for $p = 0$, we can find values of q and s such that the characteristic polynomial has integer coefficients and we can check with direct computation that in this case

$\exp(2\pi \mathrm{ad}_{X_6})$ is conjugate to the matrix $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -h_1 & 0 & 0 \\ 0 & 1 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

and then $\Gamma_{2\pi}$ is a lattice.

•$\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$ is a lattice in $G_{6.11}^{a,0,q,s}$ only if $k = 2, 3, 4, 6$:
Let consider lattices $\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$:

$$\exp(\frac{2\pi}{k}\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-q\frac{4\pi}{k}} & 0 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{2\pi}{k}\right) & \sin\left(\frac{2\pi}{k}\right) & 0 & 0 \\ 0 & -\sin\left(\frac{2\pi}{k}\right) & \cos\left(\frac{2\pi}{k}\right) & 0 & 0 \\ 0 & 0 & 0 & e^{q\frac{2\pi}{k}}\cos\left(\frac{2\pi}{k}s\right) & e^{q\frac{2\pi}{k}}\sin\left(\frac{2\pi}{k}s\right) \\ 0 & 0 & 0 & -e^{q\frac{2\pi}{k}}\sin\left(\frac{2\pi}{k}s\right) & e^{q\frac{2\pi}{k}}\cos\left(\frac{2\pi}{k}s\right) \end{pmatrix}$$

If we put $e^{\frac{2q\pi}{k}} = w,\ \cos\frac{2\pi}{k} = v,\ \cos\frac{2\pi}{k}s = u$ its characteristic polynomial is

$$-1 + \frac{\left(2\,wv + 2\,u + w^3\right)x}{w} - \frac{\left(4\,uwv + 2\,w^4v + 2\,w^3u + w^2 + 1\right)x^2}{w^2} +$$

$$+\frac{\left(4\,vw^3u + 2\,v + 2\,wu + w^4 + w^2\right)x^3}{w^2} - \frac{\left(2\,vw^2 + 2\,w^3u + 1\right)x^4}{w^2} + x^5$$

Its coefficients can be decomposed in a similar way to the $G_{6.8}$ case, so we obtain that they can be integer only if $v \in \mathbb{Z}$ and under this hypothesis this is equivalent to the following system admitting a solution

$$\begin{cases} \dfrac{1 + 2w^3u}{w^2} = h_1 \in \mathbb{Z} \\ \dfrac{2u + w^3}{w} = h_2 \in \mathbb{Z} \\ w > 0 \\ -1 \leq u \leq 1 \end{cases}$$

Again a solution can be found under particular conditions on the integer h_1 and h_2 and for all the admitted values of t_0 the lattice exists.

• $\Gamma_{2\pi s_2}$ is never a lattice in $G_{6.11}^{a,p,q,\frac{s_1}{s_2}}$ for every value of the parameters:

Let $s = \frac{s_1}{s_2} \in \mathbb{Q}$ and $t = 2\pi s_2$:

$$\exp(2\pi s_2 \mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-4(p+q)\pi s_2} & 0 & 0 & 0 & 0 \\ 0 & e^{2p\pi s_2} & 0 & 0 & 0 \\ 0 & 0 & e^{2p\pi s_2} & 0 & 0 \\ 0 & 0 & 0 & e^{2q\pi s_2} & 0 \\ 0 & 0 & 0 & 0 & e^{2q\pi s_2} \end{pmatrix}$$

We put $e^{-2(p+q)\pi s_2} = \alpha$ and $e^{-2q\pi s_2} + e^{-2p\pi s_2} = \beta$, so its minimal polynomial is

$$m(x) = -\alpha + \frac{\left(\alpha^2\beta + 1\right)x}{\alpha} - \frac{\left(\alpha^3 + \beta\right)x^2}{\alpha} + x^3$$

so it can have integer coefficients only if $\alpha \in \mathbb{Z}$. Then $\frac{\alpha^2\beta+1}{\alpha} = \beta + \frac{1}{\alpha} \in \mathbb{Z}$ implies $\beta \in \mathbb{Q}$.

But then $\frac{\alpha^3+\beta}{\alpha} = \alpha^2 + \frac{\beta}{\alpha} \in \mathbb{Z}$ implies $\frac{\beta}{\alpha} \in \mathbb{Z}$ and so $\beta \in \mathbb{Z}$.

Therefore if α and β are not both integer we have no lattice $\Gamma_{2\pi s_2}$.

Suppose $\alpha, \beta \in \mathbb{Z}$, then $\beta + \frac{1}{\alpha} \in \mathbb{Z}$ only if $\alpha = 1$ that is $a = p + q = 0$, but this value is not acceptable, so $\Gamma_{2\pi s_2}$ is not a lattice.

• $\Gamma_{2m\pi}$ is never a lattice in $G_{6.12}^{-4p,p}$ for every $m \in \mathbb{Z}$:

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -4p & 0 & 0 & 0 & 0 \\ 0 & p & 1 & 1 & 0 \\ 0 & -1 & p & 0 & 1 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & -1 & p \end{pmatrix}$$

so

$$\exp(2m\pi\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-8pm\pi} & 0 & 0 & 0 & 0 \\ 0 & e^{2pm\pi} & 0 & 2m\pi e^{2pm\pi} & 0 \\ 0 & 0 & e^{2pm\pi} & 0 & 2m\pi e^{2pm\pi} \\ 0 & 0 & 0 & e^{2pm\pi} & 0 \\ 0 & 0 & 0 & 0 & e^{2pm\pi} \end{pmatrix}$$

We put $e^{-2pm\pi} = w$, so the matrix becomes

$$\begin{pmatrix} w^4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{w} & 0 & 2m\pi\frac{1}{w} & 0 \\ 0 & 0 & \frac{1}{w} & 0 & 2m\pi\frac{1}{w} \\ 0 & 0 & 0 & \frac{1}{w} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Its minimal polynomial is

$$m(x) = -w^2 + \frac{\left(1 + 2w^5\right)x}{w^2} + \frac{\left(2 + w^5\right)x^2}{w} + x^3$$

So the $m(x)$ can have integer coefficients only if $w^2 \in \mathbb{Z}$, that is $w = \sqrt{n}$ with $n \in \mathbb{N}$.

The coefficients become

$$p_1 = \frac{1}{n} + 2n\sqrt{n}, \qquad p_2 = \frac{2}{\sqrt{n}} + n^2$$

then $p_2 \in \mathbb{Z}$ if and only if $\frac{2}{\sqrt{n}} = k \in \mathbb{Z}$, that is $n = \frac{4}{k^2} \in \mathbb{N}$.
so we have only 2 cases:

$k = \pm 2$ then $n = 1$ that is $w = 1$ and $p = 0$ that is not acceptable.

$k = \pm 1$ then $n = 4$ that is $w = 2$, but then $p_1 = \frac{1}{4} + 8 \cdot 2 \notin \mathbb{Z}$

- $\Gamma_{\frac{2\pi}{r}}$ is a lattice in $G_{5.13}^{-1-2q,q,r} \times \mathbb{R}$ only if $q = 0$:

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1-2q & 0 & 0 \\ 0 & 0 & q & r \\ 0 & 0 & -r & q \end{pmatrix}$$

then

$$\exp(\frac{2\pi}{r}\mathrm{ad}_{X_5}) = \begin{pmatrix} e^{\frac{2\pi}{r}} & 0 & 0 & 0 \\ 0 & e^{\frac{-2(1+2q)\pi}{r}} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi q}{r}} & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi q}{r}} \end{pmatrix}$$

then the minimal polynomial is

$$m(x) = x^3 + (-e^{\frac{2\pi q}{r}} - e^{\frac{-2(1+2q)\pi}{r}} - e^{\frac{2\pi}{r}})x^2 + (e^{\frac{-2\pi(1+q)}{r}} + e^{\frac{2\pi(1+q)}{r}} + e^{\frac{-4\pi q}{r}})x - e^{\frac{2\pi q}{r}}.$$

If we put $e^{\frac{2\pi}{r}} = w \neq 0$ its coefficients become:

$$a_0 = \frac{1}{w^q}, \qquad a_1 = \frac{1}{w^{q+1}} + w^{q+1} + \frac{1}{w^{2q}}, \qquad a_2 = -w^q - \frac{1}{w^{1+2q}} - w$$

$a_0 \in \mathbb{Z}$ implies that also $\frac{1}{w^{2q}} \in \mathbb{Z}$, then if $a_1 \in \mathbb{Z}$ we get $\frac{1}{w^{q+1}} + w^{q+1} \in \mathbb{Z}$, or equivalently that there exists $k \in \mathbb{Z}$ such that $1 + w^{2(q+1)} = kw^{q+1}$.

Then $a_2 = -w^q - \frac{1+w^{2(q+1)}}{w^{1+2q}} = -w^q - \frac{kw^{q+1}}{w^{1+2q}} = -w^q - k\frac{1}{w^q} \in \mathbb{Z}$ implies also $w^q \in \mathbb{Z}$ and so $w^q = 1$ and $q = 0$.

Then for $q \neq 0$ $\quad \Gamma_{\frac{2\pi}{r}}$ is not a lattice.

For $q = 0$

$$\exp(t\mathrm{ad}_{X_5}) = \begin{pmatrix} e^{\frac{2\pi}{r}} & 0 & 0 & 0 \\ 0 & e^{\frac{-2\pi}{r}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and its characteristic and minimal polynomials have integer coefficients if and only if

$$e^{\frac{2\pi}{r}} + e^{\frac{-2\pi}{r}} \in \mathbb{Z}. \tag{4.1}$$

Moreover this matrix is conjugate to the matrix

$$\begin{pmatrix} e^{\frac{2\pi}{r}} + e^{\frac{-2\pi}{r}} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then for values of $r \in \mathbb{R} \setminus \mathbb{Q}$ for which equation (4.1) is satisfied, $\Gamma_{\frac{2\pi}{r}}$ is a lattice.

•$\Gamma_{\frac{2\pi}{k}}$ with $k \in \mathbb{N}$ is a lattice in $G_{5.13}^{-1,0,r} \times \mathbb{R}$ for $k = 2, 4, 6$:
Now we consider $t_0 = \frac{2\pi}{rk}$ with $k \in \mathbb{N}$:

$$\exp(t_0 \mathrm{ad}_{X_5}) = \begin{pmatrix} e^{\frac{2\pi}{rk}} & 0 & 0 & 0 \\ 0 & e^{\frac{-2\pi}{rk}} & 0 & 0 \\ 0 & 0 & \cos(\frac{2\pi}{k}) & \sin(\frac{2\pi}{k}) \\ 0 & 0 & -\sin(\frac{2\pi}{k}) & \cos(\frac{2\pi}{k}) \end{pmatrix}$$

and its characteristic polynomial has coefficients
$a_2 = 2 + 2\cos(\frac{2\pi}{k})e^{\frac{2\pi}{rk}} + 2\cos(\frac{2\pi}{k})e^{-\frac{2\pi}{rk}}$ $\qquad a_1 = a_3 = -2\cos(\frac{2\pi}{k}) - e^{-\frac{2\pi}{rk}} - e^{\frac{2\pi}{rk}}$.
Suppose that equation (4.1) is satisfied, i.e. $e^{\frac{2\pi}{r}} + e^{\frac{-2\pi}{r}} = h \in \mathbb{Z}$, then for $k \in \mathbb{N}$ such that both $2\cos(\frac{2\pi}{k})$ and $e^{-\frac{2\pi}{rk}} + e^{\frac{2\pi}{rk}}$ are integer, we have integer coefficients. This means that we have to consider $k = 2, 4, 6$ and prove that for these values also $e^{-\frac{2\pi}{rk}} + e^{\frac{2\pi}{rk}} \in \mathbb{Z}$.

$k = 2$: $(e^{-\frac{\pi}{r}} + e^{\frac{\pi}{r}})^2 = e^{\frac{2\pi}{r}} + e^{\frac{-2\pi}{r}} + 2$, then we have to consider r such that $h + 2 = n^2$ for some $n \in \mathbb{N}$.

$k = 4$: $(e^{-\frac{\pi}{2r}} + e^{\frac{\pi}{2r}})^4 = e^{\frac{2\pi}{r}} + e^{\frac{-2\pi}{r}} + 6 + 4(e^{-\frac{\pi}{r}} + e^{\frac{\pi}{r}})$, then we want to find an integer n such that $n^4 = h + 6 + 4n^2$, that is possible for r such that $h = n^4 + 4n^2 - 6$ for $n \geq 2$.

$k=6$: if we put $e^{\frac{2\pi}{r}}=w$, equation (4.1) is equivalent to $w=\frac{h\pm\sqrt{h^2-4}}{2}$ and we want to find $n\in\mathbb{Z}$ such that $\sqrt[6]{w}+\frac{1}{\sqrt[6]{w}}=n$ or equivalently $\sqrt[6]{w}=\frac{n\pm\sqrt{n^2-4}}{2}$, then we want $(\frac{n\pm\sqrt{n^2-4}}{2})^6=\frac{h\pm\sqrt{h^2-4}}{2}$ that is possible for r such that $h=n^6-6n^4+9n^2-2$ for $n\geq 2$.

In particular the matrix $\exp(\frac{2\pi}{rk}\mathrm{ad}_{X_5})$ is similar to A for $t_0=\frac{\pi}{2r}$ and $t_0=\frac{\pi}{3r}$ and it is similar to

$$\begin{pmatrix} e^{\frac{\pi}{r}}+e^{\frac{-\pi}{r}} & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

for $t_0=\frac{\pi}{r}$.

• $\Gamma_{2\pi r_2}$ is never a lattice in $G_{5.13}^{-1-2q,q,\frac{r_1}{r_2}}\times\mathbb{R}$ for every value of the parameters: if $r=\frac{r_1}{r_2}$ and $t=2\pi r_2$

$$\exp(t\mathrm{ad}_{X_5})=\begin{pmatrix} e^{2\pi r_2} & 0 & 0 & 0\\ 0 & e^{-2(1+2q)\pi r_2} & 0 & 0\\ 0 & 0 & e^{2\pi q r_2} & 0\\ 0 & 0 & 0 & e^{2\pi q r_2}\end{pmatrix}$$

then the minimal polynomial has coefficients:

$a_2=-e^{2\pi r_2}-e^{-2\pi(1+2q)r_2}-e^{2\pi q r_2}$

$a_1=e^{-4\pi q r_2}+e^{-2\pi(1+q)r_2}+e^{2\pi(1+q)r_2}$

$a_0=-e^{-2\pi q r_2}$

If we impose $a_0=h_0\in\mathbb{Z}$ we get $a_1=h_0^2+h_0^{\frac{1+q}{q}}+h_0^{-\frac{1+q}{q}}$ and $a_2=-\frac{1+h_0^{\frac{1+3q}{q}}+h_0^{\frac{-1+q}{q}}}{h_0}$.

Now from $a_1=h_1\in\mathbb{Z}$ we get $q=\frac{\ln h_0}{-\ln h_0+\ln\left(\frac{h_1\pm\sqrt{h_1^2-4}}{2}\right)}$ and then $a_2=-\frac{1}{h_0}-h_0h_1$, so it can be integer only if $h_0=1$, that is $q=0$, so for $q\neq 0$ $\Gamma_{2\pi r_2}$ can not be a lattice.

If $q=0$ the minimal polynomial has integer coefficients if and only if $e^{2\pi r_2}+e^{-2\pi r_2}=h\in\mathbb{Z}$, but this is not possible for r_2 integer, so $\Gamma_{2\pi r_2}$ can never be a lattice.

• $\Gamma_{2\pi}$ is a lattice in $G^0_{5.14} \times \mathbb{R}$:

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so for $t_0 = 2\pi$ we have

$$\exp(2\pi \mathrm{ad}_{X_5}) = \begin{pmatrix} 1 & 2\pi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the characteristic polynomial $x^4 - 4x^3 + 6x^2 - 4x + 1$ has integer coefficients. In particular $\exp(2\pi \mathrm{ad}_{X_5})$ is conjugate to
$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2\pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and so $\Gamma_{2\pi}$ is a lattice.

•$\Gamma_{\frac{2\pi}{k}}$ with $k \in \mathbb{N}$ is a lattice in $G^0_{5.14} \times \mathbb{R}$ only if $k = 2, 3, 4, 6$:
Let consider $\Gamma_{2\pi/k}$:

$$\exp(\frac{2\pi}{k}\mathrm{ad}_{X_5}) = \begin{pmatrix} 1 & \frac{2\pi}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} \\ 0 & 0 & -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}$$

has characteristic polynomial $x^4 - 2x^3(\cos\frac{2\pi}{k} + 1) + 2x^2(2\cos\frac{2\pi}{k} + 1) - 2x(\cos\frac{2\pi}{k} + 1) + 1$ so we can have lattices for values of $k \in \mathbb{Z}$ such that $\cos\frac{2\pi}{k} = \pm 1, \pm\frac{1}{2}, 0$.
In particular if $\sin\frac{2\pi}{k} \neq 0$ $\exp(2\pi/k\ \mathrm{ad}_{X_5})$ is conjugate to A, otherwise we can use the analogous matrix of the case $k = 1$ to get the same integer matrix.

• $\Gamma_{2\pi}$ is never a lattice in $G_{5.17}^{p,-p,r} \times \mathbb{R}$ with $r \in \mathbb{R} \smallsetminus \mathbb{Q}$:

$$\mathrm{ad}_{X_5} = \begin{pmatrix} p & 1 & 0 & 0 \\ -1 & p & 0 & 0 \\ 0 & 0 & -p & r \\ 0 & 0 & -r & -p \end{pmatrix}$$

If $t = 2\pi$

$$\exp(t\mathrm{ad}_{X_5}) = \begin{pmatrix} e^{2\pi p} & 0 & 0 & 0 \\ 0 & e^{2\pi p} & 0 & 0 \\ 0 & 0 & e^{-2\pi p}\cos(2\pi r) & e^{-2\pi p}\sin(2\pi r) \\ 0 & 0 & -e^{-2\pi p}\sin(2\pi r) & e^{-2\pi p}\cos(2\pi r) \end{pmatrix}$$

then the minimal polynomial has coefficients:
$a_2 = -2e^{-2\pi p}\cos(2\pi r) - e^{2\pi p}, \quad a_1 = e^{-4\pi p} + 2\cos(2\pi r), \quad a_0 = -e^{-2\pi p}$.

$a_0 \in \mathbb{Z}$ if and only if $e^{-2\pi p} \in \mathbb{Z}$, then $a_1 \in \mathbb{Z}$ if and only if $2\cos(2\pi r) \in \mathbb{Z}$ and then $a_2 \in \mathbb{Z}$ if and only if $e^{2\pi p} \in \mathbb{Z}$, but this means $p = 0$, so for $p \neq 0$ we have no lattice in this case.

For $p = 0$ the minimal polynomial is $x^3 + (-2\cos(2\pi r) - 1)x^2 + (2\cos(2\pi r) + 1)x - 1$, so we want $2\cos(2\pi r) \in \mathbb{Z}$, but it is not possible for $r \in \mathbb{R} \smallsetminus \mathbb{Q}$ and $\Gamma_{2\pi}$ is never a lattice.

• $\Gamma_{2\pi r_2}$ is a lattice in $G_{5.17}^{p,-p,\frac{r_1}{r_2}} \times \mathbb{R}$ for some values of the parameters:
if $r = \frac{r_1}{r_2} \in \mathbb{Q}$ and $t_0 = 2\pi r_2$ we have

$$\exp(2\pi r_2 \mathrm{ad}_{X_5}) = \begin{pmatrix} e^{2\pi p r_2} & 0 & 0 & 0 \\ 0 & e^{2\pi p r_2} & 0 & 0 \\ 0 & 0 & e^{-2\pi p r_2} & 0 \\ 0 & 0 & 0 & e^{-2\pi p r_2} \end{pmatrix}$$

has both minimal and characteristic polynomials with integer coefficients if and only if

$$e^{2\pi p r_2} + e^{-2\pi p r_2} = h \in \mathbb{Z}. \tag{4.2}$$

Unfortunately in this case the matrix is not conjugate to the matrix A, so we must find another integer matrix:

from (4.2) we have $pr_2 = \frac{1}{2\pi}\ln\left(\frac{h\pm\sqrt{h^2-4}}{2}\right)$ for every $2 \leq h \in \mathbb{Z}$, then $\exp(2\pi r_2 \mathrm{ad}_{X_5})$ becomes

$$\begin{pmatrix} \frac{h+\sqrt{h^2-4}}{2} & 0 & 0 & 0 \\ 0 & \frac{h+\sqrt{h^2-4}}{2} & 0 & 0 \\ 0 & 0 & \frac{h-\sqrt{h^2-4}}{2} & 0 \\ 0 & 0 & 0 & \frac{h-\sqrt{h^2-4}}{2} \end{pmatrix}$$

Using the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{h^2-4}} & 0 & -\frac{1}{\sqrt{h^2-4}} & 0 \\ \frac{\sqrt{h^2-4}-h}{2\sqrt{h^2-4}} & 0 & \frac{\sqrt{h^2-4}+h}{2\sqrt{h^2-4}} & 0 \\ 0 & \frac{1}{\sqrt{h^2-4}} & 0 & -\frac{1}{\sqrt{h^2-4}} \\ 0 & \frac{\sqrt{h^2-4}-h}{2\sqrt{h^2-4}} & 0 & \frac{\sqrt{h^2-4}+h}{2\sqrt{h^2-4}} \end{pmatrix}$$

we obtain the conjugate integer matrix

$$\begin{pmatrix} h & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & h & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so for these values of p and r_2 $\Gamma_{2\pi r_2}$ is a lattice.

• $\Gamma_{\frac{2\pi}{k}}$ is a lattice in $G_{5.17}^{p,-p,r} \times \mathbb{R}$ for some values of $k, r \in \mathbb{Z}$:

to study other lattices we consider for the sake of simplicity the case $r \in \mathbb{Z}$ and $t_0 = \frac{2\pi}{k}$ with $k \in \mathbb{N}$.

The characteristic polynomial of $\exp(t_0 \mathrm{ad}_{X_5})$ is

$$x^4 + \Big(-2e^{\frac{-2\pi p}{k}}\cos\frac{2\pi r}{k} - 2e^{\frac{2\pi p}{k}}\cos\frac{2\pi}{k}\Big)x^3 + \Big(e^{\frac{-4\pi p}{k}} + 4\cos\frac{2\pi r}{k}\cos\frac{2\pi}{k} + e^{\frac{4\pi p}{k}}\Big)x^2 +$$
$$+\Big(-2e^{\frac{2\pi p}{k}}\cos\frac{2\pi r}{k} - 2e^{\frac{-2\pi p}{k}}\cos\frac{2\pi}{k}\Big)x + 1$$

The coefficients of this polynomial depend strongly on the relation between k and r, so it is difficult to determine in general for which values of k they are integer.

For this reason we consider only particular cases:

$\circ$ $k = 2$:

If r is even the coefficients become:

$a_1 = -a_3 = 2\sqrt{\frac{h+\sqrt{h^2-4}}{2}} - 2\sqrt{\frac{h-\sqrt{h^2-4}}{2}} = 2\sqrt{h-2}$

$a_2 = \sqrt{\frac{h+\sqrt{h^2-4}}{2}} - 4 + \sqrt{\frac{h-\sqrt{h^2-4}}{2}} = h - 4 \in \mathbb{Z}$.

So we consider only $h \in \mathbb{Z}$ such that $h - 2 = n^2$ for some $n \in \mathbb{Z}$ and the matrix becomes

$$\begin{pmatrix} \frac{n+\sqrt{n^2-4}}{2} & 0 & 0 & 0 \\ 0 & \frac{n+\sqrt{n^2-4}}{2} & 0 & 0 \\ 0 & 0 & \frac{n-\sqrt{n^2-4}}{2} & 0 \\ 0 & 0 & 0 & \frac{n-\sqrt{n^2-4}}{2} \end{pmatrix}$$

that is conjugate to an integer matrix (see $k = 1$), then in these cases we have a lattice.

If r is odd the coefficients become $a_1 = a_3 = 2\sqrt{h+2}$ $\quad a_2 = h + 4 \in \mathbb{Z}$ and similarly we have a lattice if there exists an integer n such that $h + 2 = n^2$.

$\circ$ $k = 4$:

If $r \equiv 0 \mod 4$ then $a_1 = -2e^{p\frac{\pi}{2}}$, $\quad a_2 = e^{p\pi} + e^{-p\pi}$, $\quad a_3 = -2e^{-p\frac{\pi}{2}}$ that are integer if and only if $p = 0$ and for this value our matrix is integer, so there is the lattice.

If $r \equiv 1 \mod 4$ then $a_1 = a_3 = 0$ $\quad a_2 = e^{-p\pi} + e^{p\pi} = \sqrt{h+2}$ so again we have a lattice only if $h + 2 = n^2$ for some $n \in \mathbb{Z}$:

$$\exp(t_0 \mathrm{ad}_{X_5}) = \begin{pmatrix} 0 & \frac{\sqrt{n+2}+\sqrt{n-2}}{2} & 0 & 0 \\ -\frac{\sqrt{n+2}+\sqrt{n-2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{n+2}-\sqrt{n-2}}{2} \\ 0 & 0 & -\frac{\sqrt{n+2}-\sqrt{n-2}}{2} & 0 \end{pmatrix}$$

is conjugate to

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -n-2 & 0 & 0 \end{pmatrix}$$

so we have a lattice.

If $r \equiv 2 \mod 4$ then $a_1 = 2e^{p\frac{\pi}{2}}, \quad a_2 = e^{p\pi} + e^{-p\pi}, \quad a_3 = 2e^{-p\frac{\pi}{2}}$ so again there is a lattice only if $p = 0$.

If $r \equiv 3 \mod 4$ we get same coefficients as if $r \equiv 1 \mod 4$ and then we have a lattice only if $h + 2 = n^2$ for some $n \in \mathbb{Z}$.

∘ $k = 6$:

If $r \equiv 0 \mod 6$ then $\quad a_1 = -e^{-p\frac{\pi}{3}} - 2e^{p\frac{\pi}{3}}, \quad a_2 = 2 + e^{-p\frac{2\pi}{3}} + e^{p\frac{2\pi}{3}}$, $a_3 = -e^{p\frac{\pi}{3}} - 2e^{-p\frac{\pi}{3}}$.

$a_1 = n \in \mathbb{Z}$ if and only if $e^{p\frac{\pi}{3}} = \frac{-n\pm\sqrt{n^2-8}}{4}$, then $a_3 = \frac{5n\pm3\sqrt{n^2-8}}{4}$ can be integer only if $n^2 - 8 = x^2$ for some integer x, that is $\{x = \pm1, n = \pm3\}$. Then $e^{p\frac{\pi}{3}} = 1, \frac{1}{2}$, but only for $p = 0 \quad a_3 \in \mathbb{Z}$.
In this case $\exp(t\mathrm{ad}_{X_5})$ is conjugate to the matrix A, then there is a lattice.

If $r \equiv 1 \mod 6$ then $\quad a_1 = a_3 = -e^{-p\frac{\pi}{3}} - e^{p\frac{\pi}{3}}, \quad a_2 = 1 + e^{-p\frac{2\pi}{3}} + e^{p\frac{2\pi}{3}}$.

$a_1 = a_3 = n \in \mathbb{Z}$ if and only if $e^{p\frac{\pi}{3}} = \frac{n\pm\sqrt{n^2-4}}{2}$, but from (4.2) we know that $e^{2\pi p} = \frac{h\pm\sqrt{h^2-4}}{2}$, this means that $\left(\frac{n\pm\sqrt{n^2-4}}{2}\right)^6 = \frac{h\pm\sqrt{h^2-4}}{2}$, that is possible only for integer h of kind $h = n^6 - 6n^4 + 9n^2 - 2$ for every $2 \leq n \in \mathbb{Z}$.

Also in this case $\exp(t\mathrm{ad}_{X_5})$ is conjugate to the matrix A, then there is a lattice.

If $r \equiv 2 \mod 6$ then $\quad a_1 = -a_3 = -e^{-p\frac{\pi}{3}} + e^{p\frac{\pi}{3}}, \quad a_2 = -1 + e^{-p\frac{2\pi}{3}} + e^{p\frac{2\pi}{3}}$, then with the same computation of the last case we get a lattice if h is of kind $h = n^6 + 6n^4 + 9n^2 + 2$ for every $n \in \mathbb{Z}$.

If $r \equiv 3 \mod 6$ then $\quad a_1 = -e^{-p\frac{\pi}{3}} + 2e^{p\frac{\pi}{3}}, \quad a_2 = -2 + e^{-p\frac{2\pi}{3}} + e^{p\frac{2\pi}{3}}$, $a_3 = -e^{p\frac{\pi}{3}} + 2e^{-p\frac{\pi}{3}}$, then in a similar manner to the first case we have a lattice if and only if $p = 0$.

If $r \equiv 4 \mod 6$ the coefficients of the characteristic polynomial are just opposite to those in the case $r \equiv 2 \mod 6$, while if $r \equiv 5 \mod 6$ the coefficients are equal to those in the case $r \equiv 1 \mod 6$, then there is a lattice under the same conditions.

∘ Now we consider separately the particular case $p = 0$ and we try to find in general for which values of $k \in \mathbb{N}$ $\Gamma_{2\pi/k}$ is a lattice:

$$\exp(t_0 \mathrm{ad}_{X_5}) = \begin{pmatrix} \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} & 0 & 0 \\ -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi r}{k} & \sin\frac{2\pi r}{k} \\ 0 & 0 & -\sin\frac{2\pi r}{k} & \cos\frac{2\pi r}{k} \end{pmatrix}$$

has characteristic polynomial

$$x^4 + \Big(-2\cos\frac{2\pi r}{k} - 2\cos\frac{2\pi}{k}\Big)x^3 + \Big(2 + 4\cos\frac{2\pi r}{k}\cos\frac{2\pi}{k}\Big)x^2 + \Big(-2\cos\frac{2\pi r}{k} - 2\cos\frac{2\pi}{k}\Big)x + 1$$

If we now impose

$$\begin{cases} a_1 = a_3 = h_1 \in \mathbb{Z} \\ a_2 - 2 = h_2 \in \mathbb{Z} \end{cases} \tag{4.3}$$

we get

$$\cos\frac{2\pi}{k} = \frac{-h_1 \pm \sqrt{h_1^2 - 4h_2}}{4} \qquad \cos\frac{2\pi r}{k} = \frac{-h_1 \mp \sqrt{h_1^2 - 4h_2}}{4}.$$

The limitations $-4 \le a_1,\ a_2 - 2 \le 4$ imply $h_1, h_2 \in [-4,4] \cap \mathbb{Z}$, but these integers must also satisfy $-1 \le \frac{-h_1 - \sqrt{h_1^2-4h_2}}{4} \le \frac{-h_1 + \sqrt{h_1^2-4h_2}}{4} \le 1$.

So we get the system $\begin{cases} h_1, h_2 \in [-4,4] \cap \mathbb{Z} \\ \frac{-h_1 - \sqrt{h_1^2-4h_2}}{4} \ge -1 \\ \frac{-h_1 + \sqrt{h_1^2-4h_2}}{4} \le 1 \end{cases}$ that admits solutions

$$\{h_1 = 0,\ h_2 = -4,-3,-2,-1,0\}, \quad \{h_1 = \pm 1,\ h_2 = -2,-1,0\},$$

$$\{h_1 = \pm 2,\ h_2 = 0,1\}, \quad \{h_1 = \pm 3,\ h_2 = 2\}, \quad \{h_1 = \pm 4,\ h_2 = 4\}$$

We now use these values in (4.3) to get $r \in \mathbb{Z},\ k \in \mathbb{Z} \setminus \{2,4,6\}$ and find $\{r = 0,\ k = 3\}$ (not acceptable), $\{r = 1,\ k = 3\}, \{r = 3,\ k = 8\}, \{r = 5,\ k = 12\}$.

For all these values the matrix $\exp(t_0 \mathrm{ad}_{X_5})$ is conjugate to the matrix A, so we have lattices.

• $\Gamma_{2\pi}$ is a lattice in $G^0_{5.18} \times \mathbb{R}$:

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so for $t_0 = 2\pi$ we have:

$$\exp(2\pi \mathrm{ad}_{X_5}) = \begin{pmatrix} 1 & 0 & 2\pi & 0 \\ 0 & 1 & 0 & 2\pi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the characteristic polynomial $x^4 - 4x^3 + 6x^2 - 4x + 1$ has integer coefficients.
In particular $\exp(2\pi \mathrm{ad}_{X_5})$ is conjugate to $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} \frac{1}{\pi} & 0 & 0 & 0 \\ \frac{1}{\pi} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \pi \end{pmatrix}$
so we have a lattice $\Gamma_{2\pi}$.

•$\Gamma_{\frac{2\pi}{k}}$ with $k \in \mathbb{N}$ is a lattice in $G^0_{5.18} \times \mathbb{R}$ only if $k = 2, 3, 4, 6$:
let consider the other lattices $\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$,

$$\exp(\frac{2\pi}{k} \mathrm{ad}_{X_5}) = \begin{pmatrix} \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} & \frac{2\pi}{k}\cos\frac{2\pi}{k} & \frac{2\pi}{k}\sin\frac{2\pi}{k} \\ -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} & -\frac{2\pi}{k}\sin\frac{2\pi}{k} & \frac{2\pi}{k}\cos\frac{2\pi}{k} \\ 0 & 0 & \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} \\ 0 & 0 & -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}$$

has characteristic polynomial $x^4 - 4\cos\frac{2\pi}{k}x^3 + (4\cos\frac{2\pi}{k} + 2)x^2 - 4\cos\frac{2\pi}{k}x + 1$
so we can have lattices for $k = 2, 3, 4, 6$.

For $k = 2$ we get the conjugate matrix $\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ by the same matrix as $k = 1$, while for the other values of k our matrix is conjugate to A.

• $\Gamma_{2\pi}$ is never a lattice in $G_{4.6}^{-2p,p} \times \mathbb{R}^2$:

$$\mathrm{ad}_{X_4} = \begin{pmatrix} -2p & 0 & 0 \\ 0 & p & 1 \\ 0 & -1 & p \end{pmatrix}.$$

$$\exp(2\pi \mathrm{ad}_{X_4}) = \begin{pmatrix} e^{-4\pi p} & 0 & 0 \\ 0 & e^{2\pi p} & 0 \\ 0 & 0 & e^{2\pi p} \end{pmatrix}$$

has minimal polynomial $x^2 - x(e^{2\pi p} + e^{-4\pi p}) + e^{-2\pi p}$, so $a_0 \in \mathbb{Z}$ if and only if $e^{-2\pi p} \in \mathbb{Z}$, then $e^{-4\pi p} \in \mathbb{Z}$ and $a_1 \in \mathbb{Z}$ if and only if $e^{2\pi p} \in \mathbb{Z}$. But this means $p = 0$ that is not admitted, so we have no lattice in this case.

• $\Gamma_{2\pi}$ is a lattice in $G_{3.5}^{0} \times \mathbb{R}^3$:

$$\mathrm{ad}_{X_3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so for $t_0 = 2\pi$ we have

$$\exp(2\pi \mathrm{ad}_{X_3}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has integer coefficients, so we have a lattice $\Gamma_{2\pi}$.

•$\Gamma_{\frac{2\pi}{k}}$ with $k \in \mathbb{N}$ is a lattice in $G_{3.5}^{0} \times \mathbb{R}^3$ only if $k = 2, 3, 4, 6$:
Let consider now $t = \frac{2\pi}{k}$ with $k \in \mathbb{N}$:

$$\exp(t_0 \mathrm{ad}_{X_3}) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

has characteristic polynomial $x^2 - 2x\cos t + 1$ so we must consider $k = 6, 4, 3, 2$:

for $k = 2, 4$ $\exp(\frac{2\pi}{k}\mathrm{ad}_{X_3})$ has integer coefficients, while for $k = 6, 3$ it is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 2\cos\frac{2\pi}{k} \end{pmatrix}$ so we have a lattice for all these values of k.

□

4.2 Kasuya's techniques

We will state a proposition that allows us to understand when the invariant cohomology is isomorphic to the non-invariant one, also when the Mostow condition does not hold.

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{n}$ its nilradical, then there exists a vector space $V \cong R^k$ such that $\mathfrak{g} = V \oplus \mathfrak{n}$ as vector spaces and $\mathrm{ad}_s(A)(B) = 0 \quad \forall A, B \in V$ where $\mathrm{ad}_s(A)$ is the semisimple part of $\mathrm{ad}(A)$ [13].

We can define the map $\mathrm{ad}_s : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ by $\mathrm{ad}_s(A + X)(Y) = (\mathrm{ad}_A)_s(Y)$, for $A \in V$, $X \in \mathfrak{n}$ and $Y \in \mathfrak{g}$ [23].

Therefore ad_s is linear and $[\mathrm{ad}_s, \mathrm{ad}_s] = 0$. Since $\mathfrak{g}^1 \subset \mathfrak{n}$, ad_s is a representation of $\mathfrak{g}$ and its image $\mathrm{ad}_s(\mathfrak{g})$ is abelian and consists of semisimple elements.

We will denote by $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ the extension of ad_s to G, then $\mathrm{Ad}_s(G)$ is diagonalizable.

Let $\mathbb{T} = \mathcal{A}(\mathrm{Ad}_s(G))$ be the Zariski closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$, then $\mathbb{T}$ is diagonalizable and it is a torus in $\mathcal{A}(\mathrm{Ad}_G(G))$.

Lemma 4.1. *The Zariski closure $\mathbb{T} = \mathcal{A}(Ad_s(G))$ of $Ad_s(G)$ is a maximal torus of the Zariski closure $\mathcal{A}(Ad_G(G))$ of $Ad_G(G)$.*

Proposition 4.1. *[23, Corrolary 10.1] Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. Suppose that the semisimple part of $Ad_G(G)$ is represented by diagonal matrices as $(Ad_g)_s = diag(\alpha_1(g), \cdots, \alpha_n(g))$ and that the following condition is satisfied:*

- *for any* $\{i_1, \cdots, i_p\} \subset \{1, \cdots, n\}$ *if the character* $\alpha_{i_1 \cdots i_p}$ *is non-trivial then its restriction on* Γ $\alpha_{i_1 \cdots i_p}|_\Gamma$ *is also non-trivial, with* $\alpha_{i_1 \cdots i_p}$ *the product of characters* $\alpha_{i_1}, \cdots, \alpha_{i_p}$.

Then an isomorphism $H^*(G/\Gamma, \mathbb{C}) \cong H^*(\mathfrak{g}^{\mathbb{C}}, \mathbb{C})$ *holds.*
In particular this implies also $H^*(G/\Gamma) \cong H^*(\mathfrak{g})$.

We observe that in this proposition the Mostow condition does not appear, then we can consider it also when we do not have this information, that usually is quite difficult to obtain.

4.3 Six dimensional almost abelian solvmanifolds

We want to apply this method to almost abelian solvmanifolds $\mathbb{R} \ltimes_\varphi \mathbb{R}^n / \mathbb{Z} \ltimes_{\varphi|_\mathbb{Z}} \mathbb{Z}^n$.

Remark 4.2. We observe that in the almost abelian case $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_{n+1}}} \mathbb{R}^n$, the vector space V such that $\mathfrak{g} = V \oplus \mathfrak{n}$ is isomorphic to $\mathbb{R}$, then in this case ad_s is the semisimple part of $\mathrm{ad}_{X_{n+1}}$.

If for $t = t_0$ we have a lattice, we usually have by similar arguments a lattice also for $t = \frac{t_0}{k}$ with $k \in \mathbb{N}$, as we have seen in the previous section. Moreover to use Proposition 4.1 it seems a good choice t_0 such that the complex eigenvalues of $\varphi|_\mathbb{Z}$ are of kind $\rho(\cos(2h\pi) + i\sin(2h\pi))$ with $h \in \mathbb{N}$. In this way also the other lattices with $t = \frac{t_0}{k}$ can be easily studied and we can give a good description of these kind of solvmanifolds.

We consider the almost abelian Lie groups of Theorem 4.1.

$G_{6.8}^{a,b,c,p}$

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -b-c-2p & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & -1 & p \end{pmatrix}$$

If we consider $t_0 = 2\pi$, by Theorem 1.3, for $p \neq 0$ we do not have a lattice, then we consider $p = 0$.

$$\exp(t\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{t(-b-c)} & 0 & 0 & 0 & 0 \\ 0 & e^{tb} & 0 & 0 & 0 \\ 0 & 0 & e^{tc} & 0 & 0 \\ 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & -\sin t & \cos t \end{pmatrix}.$$

and

$$\exp(2\pi\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{2\pi(-b-c)} & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi b} & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi c} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously the character $\cos t + i \sin t \neq 1$ on the group, but it is the identity on the lattice $\Gamma_{2\pi}$, then in this case we can not apply the method.

Let consider lattices $\Gamma_{2\pi/k}$ with $k \in \mathbb{N}$: by Theorem 4.1 we can have a lattice for $t = \pi,\ \frac{\pi}{2},\ \frac{\pi}{3}, \frac{2\pi}{3}$.

We can easily verify that for all these values of t the condition of Proposition 4.1 is satisfied, then

$H^1(G_{6.8}^{p=0}/\Gamma_t) = H^1(\mathfrak{g}_{6.8}^{p=0}) = \langle \alpha^6 \rangle$

$H^2(G_{6.8}^{p=0}/\Gamma_t) = H^2(\mathfrak{g}_{6.8}^{p=0}) = \langle \alpha^{45} \rangle$

$H^3(G_{6.8}^{p=0}/\Gamma_t) = H^3(\mathfrak{g}_{6.8}^{p=0}) = \langle \alpha_{123},\ \alpha_{456} \rangle$.

$G^a_{6.10}$

$$\mathrm{ad}_{X_6}=\begin{pmatrix} a & 1 & 0 & 0 & 0\\ 0 & a & 1 & 0 & 0\\ 0 & 0 & a & 0 & 0\\ 0 & 0 & 0 & -\frac{3}{2}a & 1\\ 0 & 0 & 0 & -1 & -\frac{3}{2}a \end{pmatrix},$$

by Theorem 4.1 for $t_0 = 2\pi$ we have to consider $a = 0$.

$$\exp(t\mathrm{ad}_{X_6})=\begin{pmatrix} 1 & -t & \frac{1}{2}t^2 & 0 & 0\\ 0 & 1 & -t & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \cos t & -\sin t\\ 0 & 0 & 0 & \sin t & \cos t \end{pmatrix}$$

and

$$\exp(2\pi\mathrm{ad}_{X_6})=\begin{pmatrix} 1 & 2\pi & 2\pi^2 & 0 & 0\\ 0 & 1 & 2\pi & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again in this case the complex block does not allow us to apply the method.

Let consider lattices Γ_t with $t = \pi,\ \frac{\pi}{2},\ \frac{\pi}{3}$: using Proposition 4.1 we obtain for all these t

$$\begin{aligned} H^1(G^0_{6.10}/\Gamma_t) &= H^1(\mathfrak{g}^0_{6.10}) = \langle \alpha^3,\ \alpha^6\rangle\\ H^2(G^0_{6.10}/\Gamma_t) &= H^2(\mathfrak{g}^0_{6.10}) = \langle \alpha^{16},\ \alpha^{23},\ \alpha^{45}\rangle\\ H^3(G^0_{6.10}/\Gamma_t) &= H^3(\mathfrak{g}^0_{6.10}) = \langle \alpha^{123},\ \alpha^{126},\ \alpha^{345},\ \alpha^{456}\rangle. \end{aligned}$$

$G_{6.11}^{a,p,q,s}$

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -2(p+q) & 0 & 0 & 0 & 0 \\ 0 & p & 1 & 0 & 0 \\ 0 & -1 & p & 0 & 0 \\ 0 & 0 & 0 & q & s \\ 0 & 0 & 0 & -s & q \end{pmatrix}$$

If $t_0 = 2\pi$ by Theorem 4.1 we take again $p = 0$ and

$$\exp(t\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-2\,qt} & 0 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 & 0 \\ 0 & -\sin t & \cos t & 0 & 0 \\ 0 & 0 & 0 & e^{qt}\cos(ts) & e^{qt}\sin(ts) \\ 0 & 0 & 0 & -e^{qt}\sin(ts) & e^{qt}\cos(ts) \end{pmatrix}$$

and

$$\exp(2\pi\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-4q\pi} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{2q\pi}\cos(2\pi s) & e^{2q\pi}\sin(2\pi s) \\ 0 & 0 & 0 & -e^{2q\pi}\sin(2\pi s) & e^{2q\pi}\cos(2\pi s) \end{pmatrix}$$

then also in this case we can not apply the method.

If we consider the other lattices $\Gamma_{2\pi/k}$ with $k = 2, 4, 6$, thanks to Proposition 4.1 we have

$$\begin{aligned} H^1(G_{6.11}^{p=0}/\Gamma_{\bar t}) &= \langle \alpha^6 \rangle \\ H^2(G_{6.11}^{p=0}/\Gamma_{\bar t}) &= \langle \alpha^{23} \rangle \\ H^3(G_{6.11}^{p=0}/\Gamma_{\bar t}) &= \langle \alpha^{145},\, \alpha^{236} \rangle \,. \end{aligned}$$

If $s = \frac{s_1}{s_2} \in \mathbb{Q}$ another good choice is $t_0 = 2\pi s_2$, but by Theorem 4.1 $\Gamma_{2\pi s_2}$ is never a lattice.

$G_{5.13}^{-1-2q,q,r} \times \mathbb{R}$

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1-2q & 0 & 0 \\ 0 & 0 & q & r \\ 0 & 0 & -r & q \end{pmatrix}$$

As in the previous case we would consider two different cases: again by Theorem 4.1 we have $q = 0$, otherwise $\Gamma_{\frac{2\pi}{r}}$ is not a lattice and if $r = \frac{r_1}{r_2} \in \mathbb{Q}$, $t_0 = 2\pi r_2$ is never a lattice.

If $t_0 = \frac{2\pi}{r}$ we can not apply the method.

Now we consider $t = \frac{2\pi}{rk}$ with $k = 2, 4, 6$: for all these values we can apply Proposition 4.1 and the cohomology groups are:

$$\begin{aligned} H^1(G_{5.13}^{-1,0,r} \times \mathbb{R}/\Gamma_{\frac{2\pi}{rk}}) &= H^1(\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}) = \langle \alpha^5, \, \alpha^6 \rangle \,, \\ H^2(G_{5.13}^{-1,0,r} \times \mathbb{R}/\Gamma_{\frac{2\pi}{rk}}) &= H^2(\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}) = \langle \alpha^{12}, \, \alpha^{34}, \, \alpha^{56} \rangle \,, \\ H^3(G_{5.13}^{-1,0,r} \times \mathbb{R}/\Gamma_{\frac{2\pi}{rk}}) &= H^3(\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}) = \langle \alpha^{125}, \, \alpha^{126}, \, \alpha^{345}, \, \alpha^{346} \rangle \,. \end{aligned}$$

$G_{5.14}^{0} \times \mathbb{R}$

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so for $t = 2\pi$ again we can not apply the method.

Let consider the cases $\Gamma_{2\pi/k}$ with $k = 2, 4, 6$: we can use Proposition 4.1 and the cohomology groups are:

$$
\begin{aligned}
H^1(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= H^1(\mathfrak{g}^0_{5.14}\oplus\mathbb{R}) = \langle\alpha^2,\,\alpha^5,\,\alpha^6\rangle\,,\\
H^2(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= H^2(\mathfrak{g}^0_{5.14}\oplus\mathbb{R}) = \langle\alpha^{12},\,\alpha^{15},\,\alpha^{26},\,\alpha^{34},\,\alpha^{56}\rangle\,,\\
H^3(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= H^3(\mathfrak{g}^0_{5.14}\oplus\mathbb{R}) = \langle\alpha^{125},\,\alpha^{126},\,\alpha^{156},\,\alpha^{234},\,\alpha^{345},\,\alpha^{346}\rangle\,.
\end{aligned}
$$

$G^{p,-p,r}_{5.17}\times\mathbb{R}$

$$
\mathrm{ad}_{X_5} = \begin{pmatrix} p & 1 & 0 & 0\\ -1 & p & 0 & 0\\ 0 & 0 & -p & r\\ 0 & 0 & -r & -p \end{pmatrix}
$$

If $r=\frac{r_1}{r_2}\in\mathbb{Q}$ and $t_0=2\pi r_2$ we know by the prove of Theorem 4.1 that we have a lattice for $e^{2\pi p r_2}+e^{-2\pi p r_2}\in\mathbb{Z}$.

$$
\exp(t\mathrm{ad}_{X_5}) = \begin{pmatrix} e^{tp}\cos t & e^{tp}\sin t & 0 & 0\\ -e^{tp}\sin t & e^{tp}\cos t & 0 & 0\\ 0 & 0 & e^{-tp}\cos(tr) & e^{-tp}\sin(tr)\\ 0 & 0 & -e^{-tp}\sin(tr) & e^{-tp}\cos(tr) \end{pmatrix}
$$

and

$$
\exp(2\pi r_2\mathrm{ad}_{X_5}) = \begin{pmatrix} e^{2\pi p r_2} & 0 & 0 & 0\\ 0 & e^{2\pi p r_2} & 0 & 0\\ 0 & 0 & e^{-2\pi p r_2} & 0\\ 0 & 0 & 0 & e^{-2\pi p r_2} \end{pmatrix}
$$

For $p=0$, again $\cos t+i\sin t\neq 1$, but on the lattice $\Gamma_{2\pi r_2}$ it is trivial.

For $p\neq 0$ the character $e^{tp}(\cos t+i\sin t)\cdot e^{-tp}(\cos(rt)+i\sin(rt))$ can not be trivial for every t, but for $t_0=2\pi r_2$ it becomes $e^{2\pi p r_2}\cdot e^{-2\pi p r_2}=1$.

Then Proposition 4.1 does not allow us to compute the cohomology of the solvmanifold.

Now we consider all the other lattices considered in Theorem 4.1.

$p = 0$: the only characters $\alpha_{i_1\cdots i_p}$ that are trivial for every t are $(\cos t + i\sin t)\cdot(\cos t - i\sin t)$ and $(\cos rt + i\sin rt)\cdot(\cos rt - i\sin rt)$.

But for $r = \frac{2\pi m}{t}$ for some $m \in \mathbb{Z}$ we have that $(\cos rt + i\sin rt)$ and $(\cos rt - i\sin rt)$ are trivial, for $r = \frac{2\pi m}{t} - 1$ for some $m \in \mathbb{Z}$ we have that $(\cos t + i\sin t)\cdot(\cos rt + i\sin rt)$ and $(\cos t - i\sin t)\cdot(\cos rt - i\sin rt)$ are trivial and for $r = \frac{2\pi m}{t} + 1$ for some $m \in \mathbb{Z}$ we have that $(\cos t + i\sin t)\cdot(\cos rt - i\sin rt)$ and $(\cos t - i\sin t)\cdot(\cos rt + i\sin rt)$ are trivial.

Since in our computation $t = \frac{2\pi}{k}$ we have that

- if $r \equiv 0 \mod k$ or $r \equiv 1 \mod k$ or $r \equiv -1 \mod k$, then we can not apply Proposition 4.1,
- if $r \equiv j \mod k$ with $j \neq 0, \pm 1$, then by Proposition 4.1 we have
$H^1(G^{0,0,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^1(\mathfrak{g}^{0,0,r}_{5.17} \oplus \mathbb{R}) = \langle \alpha^5, \alpha^6 \rangle$,
$H^2(G^{0,0,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^2(\mathfrak{g}^{0,0,r}_{5.17} \oplus \mathbb{R}) = \langle \alpha^{13} + \alpha^{24},\ \alpha^{14} - \alpha^{23},\ \alpha^{56} \rangle$,
$H^3(G^{0,0,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^3(\mathfrak{g}^{0,0,r}_{5.17} \oplus \mathbb{R}) = \langle \alpha^{135} + \alpha^{245},\ \alpha^{145} - \alpha^{235},\ \alpha^{146} - \alpha^{236},\ \alpha^{136} + \alpha^{246} \rangle$.

$p \neq 0$: any character $\alpha_{i_1\cdots i_p}$ is not trivial.

But for $r = \frac{2\pi m}{t} - 1$ for some $m \in \mathbb{Z}$ we have that $(\cos t + i\sin t)\cdot(\cos rt + i\sin rt)$ and $(\cos t - i\sin t)\cdot(\cos rt - i\sin rt)$ are trivial and for $r = \frac{2\pi m}{t} + 1$ for some $m \in \mathbb{Z}$ we have that $(\cos t + i\sin t)\cdot(\cos rt - i\sin rt)$ and $(\cos t - i\sin t)\cdot(\cos rt + i\sin rt)$ are trivial.

Then we have that

- if $r \equiv 1 \mod k$ or $r \equiv -1 \mod k$, then we can not apply Proposition 4.1,
- if $r \equiv j \mod k$ with $j \neq \pm 1$, then by Proposition 4.1 we have
$H^1(G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^1(\mathfrak{g}^{p,-p,r}_{5.17} \oplus \mathbb{R}) = \langle \alpha^5, \alpha^6 \rangle$,
$H^2(G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^2(\mathfrak{g}^{p,-p,r}_{5.17} \oplus \mathbb{R}) = \langle \alpha^{56} \rangle$,
$H^3(G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^3(\mathfrak{g}^{p,-p,r}_{5.17} \oplus \mathbb{R}) = \{0\}$.

$G^0_{5.18} \times \mathbb{R}$

$$\mathrm{ad}_{X_5} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so for $t = 2\pi$ as usual we can not apply the method.

Let consider the other lattices $\Gamma_{2\pi/k}$ with $k = 2, 4, 6$: we can apply Proposition 4.1 only for $k \neq 2$.

Indeed we have characters $\cos t \pm i \sin t, \cos t \pm i \sin t$, then the product $(\cos t \pm i \sin t) \cdot (\cos t \pm i \sin t)$, that for t generic is not trivial, for $t = m\pi$ for some $m \in \mathbb{Z}$ is trivial, then we can not compute the cohomology for Γ_π.

For $k = 3, 4, 6$, by Proposition 4.1 we have
$H^1(G^0_{5.18} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^1(\mathfrak{g}^0_{5.18} \oplus \mathbb{R}) = \langle \alpha^5,\, \alpha^6 \rangle$
$H^2(G^0_{5.18} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^2(\mathfrak{g}^0_{5.18} \oplus \mathbb{R}) = \langle \alpha^{13} + \alpha^{24},\, \alpha^{34},\, \alpha^{56} \rangle$
$H^3(G^0_{5.18} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = H^3(\mathfrak{g}^0_{5.18} \oplus \mathbb{R}) = \langle \alpha^{125},\, \alpha^{135} + \alpha^{245},\, \alpha^{136} + \alpha^{246},\, \alpha^{346} \rangle$

$G^0_{3.5} \times \mathbb{R}^3$

$$\mathrm{ad}_{X_3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so for $t = 2\pi$ we can not apply the method.

For $t = \frac{2\pi}{k}$ with $k = 2, 4, 6$ by Proposition 4.1 we have that the cohomology groups are

$$H^1(G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = H^1(\mathfrak{g}^0_{3.5} \oplus \mathbb{R}) = \langle \alpha^3, \alpha^4, \alpha^5, \alpha^6 \rangle,$$
$$H^2(G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = H^2(\mathfrak{g}^0_{3.5} \oplus \mathbb{R}) = \langle \alpha^{12}, \alpha^{34}, \alpha^{35}, \alpha^{36}, \alpha^{45}, \alpha^{46}, \alpha^{56} \rangle,$$
$$H^3(G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = H^3(\mathfrak{g}^0_{3.5} \oplus \mathbb{R}) = \langle \alpha^{123}, \alpha^{124}, \alpha^{125}, \alpha^{126}, \alpha^{345}, \alpha^{346}, \alpha^{356}, \alpha^{456} \rangle.$$

Chapter 5

Complex structures on almost abelian Lie algebras

We have seen in Chapter 1.2 that the existence of a complex structure on a differential manifold or a Lie algebra allows to define the Dolbeault cohomology groups (Theorem 1.11 and Remark 1.2).

A complex structure on an homogeneous space $M = G/\Gamma$ is *invariant* if it comes from a complex structure on the associated Lie algebra $\mathfrak{g}$ or equivalently from a left invariant complex structure on G (Proposition 1.4).

Like for the de Rham cohomology, the inclusion $\bigwedge^{*,*}\mathfrak{g}^* \subseteq \bigwedge^{*,*}(S)$ of the exterior algebras induces for solvmanifolds S only an inclusion in the Dolbeault cohmology up to isomorphism $H^{*,*}_{\bar{\partial}}(\mathfrak{g}) \subseteq H^{*,*}_{\bar{\partial}}(S)$ [7].

Unfortunately all the theorems that (in the real case) assure us that this inclusion is an isomorphism (Theorems 1.7, 1.8 and 1.10) do not hold in general for the Dolbeault cohmology.

We have an analogue of the Nomizu theorem in a large group of nilmanifolds (see Theorem 6.7) and it is an open problem to prove that it is always true [41]. We will discuss this subject in Chapter 6.3.1.

There is not a similar result for solvmanifolds, but for dimension 4 we have a result due to Hasegawa:

Theorem 5.1. *[21] Every complex structure on a four dimensional solvmanifold is invariant.*

In view of these results and with the hope that more can be developed, we are interested in describing complex structures on almost abelian solvable Lie algebras. In particular we find a description for a general complex structure on $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^n$ and for the complex analogue $\mathfrak{g} = \mathbb{C} \ltimes_{\text{ad}} \mathbb{C}^n$ with $\dim_{\mathbb{R}} \operatorname{Im} \text{ad} = 1$. In this case we also find a property related to the $\partial\bar{\partial}$-Lemma.

5.1 $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^{2n-1}$

5.1.1 General case

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^{2n-1}$ be an almost abelian Lie algebra, we want to study the complex structures J on $\mathfrak{g}$.

The main result of this section is the following theorem:

Theorem 5.2. *An almost abelian Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes_{ad_{X_{2n}}} \mathbb{R}^{2n-1}$ can admit a complex structure J, only if $ad_{X_{2n}}$ has at least a real eigenvalue and its complex part is $\mathbb{C}$-diagonalizable.*

Proof. This theorem is direct consequence of the following lemmas and propositions and then a corollary of Theorem 5.3. □

Suppose that $\text{ad}_{X_{2n}}$ can be written in Jordan form, i.e. there exists a basis of $\mathfrak{g}$ such that

$$\mathfrak{g} = \langle X_1^1, \cdots, X_{m_1}^1, \cdots, X_1^p, \cdots, X_{m_p}^p, Y_1^1, \cdots, Y_{2n_1}^1, \cdots, Y_1^q, \cdots, Y_{2n_q}^q, X_{2n} \rangle$$

with

$$\begin{array}{ll}
\left[X_1^t, X_{2n}\right] = a_t X_1^t & \forall\, t = 1, \cdots, p \\
\left[X_i^t, X_{2n}\right] = X_{i-1}^t + a_t X_i^t & \forall\, i = 2, \cdots, m_t \\
\left[Y_1^s, X_{2n}\right] = b_s Y_1^s - c_s Y_2^s & \forall\, s = 1, \cdots, q \\
\left[Y_2^s, X_{2n}\right] = c_s Y_1^s + b_s Y_2^s & \\
\left[Y_{2j-1}^s, X_{2n}\right] = Y_{2j-3} + b_s Y_{2j-1}^s - c_s Y_{2j}^s & \forall\, j = 2, \cdots, n_s \\
\left[Y_{2j}^s, X_{2n}\right] = Y_{2j-2} + c_s Y_{2j-1}^s + b_s Y_{2j}^s. &
\end{array}$$

Consider J on $\mathbb{R}^{2n-1}$ given by a general almost complex structure

$$JX_i^t = \sum_{\substack{u=1,\cdots,p \\ v=1,\cdots,m_u}} \varphi_{v,i}^{u,t} X_v^u + \sum_{\substack{u=1,\cdots,q \\ v=1,\cdots,n_u}} \psi_{v,i}^{u,t} Y_v^u + \eta_i^t X_{2n}$$

$$JY_j^s = \sum_{\substack{u=1,\cdots,p \\ v=1,\cdots,m_u}} \alpha_{v,j}^{u,s} X_v^u + \sum_{\substack{u=1,\cdots,q \\ v=1,\cdots,n_u}} \beta_{v,j}^{u,s} Y_v^u + \rho_j^s X_{2n}$$

such that $J^2 = -\mathrm{Id}$.

Lemma 5.1. *If $\eta_i^t \neq 0$ then, by the integrability of J, $i = m_t$ and $a_t = a_1$.*

Proof. If we consider the coefficient of X_{2n} in $N(X_i^t, X_j^s)\quad \forall\, s,t$ we obtain

$$\eta_1^t \eta_1^s (a_t - a_s) = 0 \qquad i = j = 1 \tag{5.1}$$

$$\eta_1^t \eta_i^s (a_t - a_s) - \eta_1^t \eta_{i-1}^s = 0 \quad i \neq 1, j = 1 \tag{5.2}$$

$$\eta_j^t \eta_i^s (a_t - a_s) + \eta_i^s \eta_{j-1}^t - \eta_j^t \eta_{i-1}^s = 0 \quad i \neq 1, j \neq 1 \tag{5.3}$$

$\forall\, t$ such that $m_t > 1$ we have that for $s = t$ and $i = 2$, equation (5.2) becomes $\eta_1^t = 0$, then in (5.3) we have for $j = 2, i = 3\quad \eta_2^t = 0$, then by induction for $i = j+1\quad \eta_j^t = 0\quad \forall\, j < m_t$ and equations (5.1) and (5.3) are satisfied.

Equation (5.3) becomes $\eta_{m_t}^t \eta_{m_s}^s (a_t - a_s) = 0$, then if $a_s \neq a_t$ it implies $\eta_{m_t}^t = 0$ or $\eta_{m_s}^s = 0$. So we can suppose to fix $s = 1$ and then for $a_t \neq a_1$ we have also $\eta_{m_t}^t = 0$. $\square$

Lemma 5.2. *For every t, i such that $\eta^t_i = 0$ we have*

$$JX^t_i = \sum_{\substack{a_u = a_t \\ v \leq i \\ m_u - v \geq m_t - i}} \varphi^{u,t}_{1,i-v+1} X^u_v. \tag{5.4}$$

Proof. Let $s \leq p$ be such that $\eta^s_{m_s} \neq 0$, then $N(X^s_{m_s}, X^t_i) = 0$ is equivalent $\forall\, u$ to

$$\varphi^{u,t}_{v,i}(a_t - a_u) - \varphi^{u,t}_{v+1,i} + \underline{\varphi^{u,t}_{v,i-1}} = 0 \qquad \forall\, v < m_u \tag{5.5}$$

$$\varphi^{u,t}_{m_u,i}(a_t - a_u) + \underline{\varphi^{u,t}_{m_u,i-1}} = 0 \tag{5.6}$$

$$\psi^{u,t}_{2v-1,i}(a_t - b_u) - \psi^{u,t}_{2v,i} c_u - \psi^{u,t}_{2v+1,i} + \underline{\psi^{u,t}_{2v-1,i-1}} = 0 \qquad \forall\, v < n_u \tag{5.7}$$

$$\psi^{u,t}_{2v,i}(a_t - b_u) + \psi^{u,t}_{2v-1,i} c_u - \psi^{u,t}_{2v+2,i} + \underline{\psi^{u,t}_{2v,i-1}} = 0 \qquad \forall\, v < n_u \tag{5.8}$$

$$\psi^{u,t}_{2n_u-1,i}(a_t - b_u) - \psi^{u,t}_{2n_u,i} c_u + \underline{\psi^{u,t}_{2n_u-1,i-1}} = 0 \tag{5.9}$$

$$\psi^{u,t}_{2n_u,i}(a_t - b_u) + \psi^{u,t}_{2n_u-1,i} c_u + \underline{\psi^{u,t}_{2n_u,i-1}} = 0 \tag{5.10}$$

where the underlined parts are those we do not have when $i = 1$.

Starting by equations (5.9) and (5.10) and applying then induction on v in equations (5.7) and (5.8) we obtain $\psi^{u,t}_{2v-1,1} = \psi^{u,t}_{2v,1} = 0 \quad \forall\, v$ and by induction on i in all these 4 equations we have $\psi^{u,t}_{2v-1,i} = \psi^{u,t}_{2v,i} = 0 \quad \forall\, v, i$.

By a similar argument on equations (5.5) and (5.6) we have $\varphi^{u,t}_{v,i} = 0 \quad \forall u, v$ such that $a_u \neq a_t$.

If $a_u = a_t$, then equations (5.5) and (5.6) become

$$\varphi^{u,t}_{v+1,1} = 0 \qquad \forall\, v < m_u \tag{5.11}$$

$$\varphi^{u,t}_{v+1,i} = \varphi^{u,t}_{v,i-1} \qquad \forall\, v < m_u,\ \forall\, i > 1 \tag{5.12}$$

$$\varphi^{u,t}_{m_u,i-1} = 0 \qquad \forall\, i > 1 \tag{5.13}$$

Using equations (5.11) and (5.12) we obtain that $\varphi^{u,t}_{v,i} = 0 \quad \forall\, v > i$, while using (5.13) and (5.12) we have $\varphi^{u,t}_{v,i} = 0 \quad \forall\, v$ such that $m_t - i > m_u - v$.

In particular (5.12) implies that $\varphi^{u,t}_{v,i} = \varphi^{u,t}_{1,i-v+1}$. $\square$

In a similar way to the previous proof we obtain the following lemma:

Lemma 5.3. *$\forall\, t, s$ such that $\eta^t_{m_t} \neq 0$, $\eta^s_{m_s} \neq 0$ we have*

$$\varphi^{u,s}_{v,m_s}\eta^t_{m_t} - \varphi^{u,t}_{v,m_t}\eta^s_{m_s} = 0 \qquad \forall\, a_u \neq a_1,\ \forall\, v \tag{5.14}$$
$$\varphi^{u,s}_{v,m_s}\eta^t_{m_t} - \varphi^{u,t}_{v,m_t}\eta^s_{m_s} = 0 \qquad \forall\, a_u = a_1,\ \forall\, v > 1 \tag{5.15}$$
$$\psi^{u,s}_{v,m_s}\eta^t_{m_t} - \psi^{u,t}_{v,m_t}\eta^s_{m_s} = 0 \qquad \forall\, u, v. \tag{5.16}$$

Proposition 5.1. *Let $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^{2n-1}$ be an almost abelian Lie algebra,*

1. *if $ad_{X_{2n}}$ has 2 real eigenvalues a_{t_1} and a_{t_2} of multiplicity greater then 1 such that for every real eigenvalue a_u of multiplicity greater then 1, $a_{t_1} \neq a_u$ and $a_{t_2} \neq a_u$, then $\mathfrak{g}$ does not admit a complex structure,*

2. *if $ad_{X_{2n}}$ has two different real eigenvalues a_{t_1} and a_{t_2} such that for every real eigenvalue a_u with $a_u = a_{t_i}$ we have $m_{t_i} > m_u$ for $i = 1, 2$, then $\mathfrak{g}$ does not admit a complex structure,*

3. *if $ad_{X_{2n}}$ has 2 different real eigenvalues with odd algebraic multiplicity, then $\mathfrak{g}$ does not admit a complex structure.*

Proof.

1. If $a_{t_1} \neq a_u$ and $a_{t_2} \neq a_u$, then at least one of a_{t_1} and a_{t_2} is different from a_1. Suppose that $a_{t_1} \neq a_1$, then by equation (5.4) the coefficient of $X^{t_1}_i$ in $J^2(X^{t_1}_i)$ is $(\varphi^{t_1,t_1}_{1,1})^2$ that can not be -1.

2. At least one of a_{t_1} and a_{t_2} is different from a_1. Suppose that $a_{t_1} \neq a_1$. For every real eigenvalue a_u such that $a_u = a_{t_1}$ consider i such that $m_u = i - 1$. Then equation (5.13) becomes $\varphi^{u,t_1}_{i-1,i-1} = 0$ and then (5.12) implies $\varphi^{u,t_1}_{i,i} = 0\ \forall\, i < m_{t_1}$. In particular we have $\varphi^{u,t_1}_{1,1} = 0$, then $JX^{t_1}_1 = \varphi^{t_1,t_1}_{1,1}X^{t_1}_1$ and then $J^2 = -Id$ is impossible.

3. At least one of the two eigenvalues is different from a_1, then equation (5.4) implies that its algebraic multiplicity must be even, because it is equivalent to $J : V \to V$ almost complex structure with $V = \langle X_1^u \rangle$.

$\square$

Lemma 5.4. $\rho_j^s = 0$ *for any* $s = 1, \cdots, q$, *for any* $j = 1, \cdots, 2n_s$.

Proof. We use induction on j:

If we consider the coefficient of X_{2n} in $N(Y_1^s, Y_2^s) \quad \forall\, s$ we obtain

$$-c_s[(\rho_1^s)^2 + (\rho_2^s)^2]$$

that can be 0 only if $\rho_1^s = \rho_2^s = 0$.

If we now generalize to $N(Y_{2j-1}^s, Y_{2j}^s) \quad \forall\, j = 1, \cdots, n_s$ we have

$$\rho_{2j-1}^s \rho_{2j-2}^s - \rho_{2j}^s \rho_{2j-3}^s - c_s[(\rho_{2j-1}^s)^2 + (\rho_{2j}^s)^2]$$

but by induction hypothesis it becomes $-c_s[(\rho_{2j-1}^s)^2 + (\rho_{2j}^s)^2]$ that is zero only if $\rho_{2j-1}^s = \rho_{2j}^s = 0$. $\square$

Proposition 5.2. *Let* $\mathfrak{g}$ *be an almost abelian Lie algebra. If* $\mathrm{ad}_{X_{2n}}$ *has only complex eigenvalues, then* $\mathfrak{g}$ *does not admit a complex structure.*

Proof. By Lemma 5.4 we have that if $\mathrm{ad}_{X_{2n}}$ has not real eigenvalues, then $J^2 X_{2n} = -X_{2n}$ implies that the square of the coefficient of X_{2n} in JX_{2n} is equal to -1 that is impossible in $\mathbb{R}$. $\square$

Now we consider the complex blocks:

Lemma 5.5. $\forall\, s = 1, \cdots, q \quad \forall\, j = 1, \cdots, 2n_s$

$$JY_j^s = \sum_{\substack{b_u = b_s \\ c_u = c_s}} (\beta_{1,j}^{u,s} Y_1^u + \beta_{2,j}^{u,s} Y_2^u).$$

Proof. Let $X^t_{m_t}$ such that $\eta^t_{m_t} \neq 0$, then $N(X^t_{m_t}, Y^s_{2j-1}) = 0$ imply $\forall\, u, v$

$$(a_u - b_s)\alpha^{u,s}_{v,2j-1} + c_s\alpha^{u,s}_{v,2j} - \underline{\alpha^{u,s}_{v,2j-3}} + \alpha^{u,s}_{v+1,2j-1} = 0 \quad (5.17)$$

$$(a_u - b_s)\alpha^{u,s}_{m_u,2j-1} + c_s\alpha^{u,s}_{m_u,2j} - \underline{\alpha^{u,s}_{m_u,2j-3}} = 0 \quad (5.18)$$

$$(b_u - b_s)\beta^{u,s}_{2v-1,2j-1} + c_u\beta^{u,s}_{2v,2j-1} + c_s\beta^{u,s}_{2v-1,2j} - \underline{\beta^{u,s}_{2v-1,2j-3}} + \beta^{u,s}_{2v+1,2j-1} = 0 \quad (5.19)$$

$$(b_u - b_s)\beta^{u,s}_{2v,2j-1} - c_u\beta^{u,s}_{2v-1,2j-1} + c_s\beta^{u,s}_{2v,2j} - \underline{\beta^{u,s}_{2v,2j-3}} + \beta^{u,s}_{2v+2,2j-1} = 0 \quad (5.20)$$

$$(b_u - b_s)\beta^{u,s}_{2n_u-1,2j-1} + c_u\beta^{u,s}_{2n_u,2j-1} + c_s\beta^{u,s}_{2n_u-1,2j} - \underline{\beta^{u,s}_{2n_u-1,2j-3}} = 0 \quad (5.21)$$

$$(b_u - b_s)\beta^{u,s}_{2n_u,2j-1} - c_u\beta^{u,s}_{2n_u-1,2j-1} + c_s\beta^{u,s}_{2n_u,2j} - \underline{\beta^{u,s}_{2n_u,2j-3}} = 0 \quad (5.22)$$

and $N(X^t_{m_t}, Y^s_{2j}) = 0$ imply $\forall\, u, v$

$$(a_u - b_s)\alpha^{u,s}_{v,2j} - c_s\alpha^{u,s}_{v,2j-1} - \underline{\alpha^{u,s}_{v,2j-2}} + \alpha^{u,s}_{v+1,2j} = 0 \quad (5.23)$$

$$(a_u - b_s)\alpha^{u,s}_{m_u,2j} - c_s\alpha^{u,s}_{m_u,2j-1} - \underline{\alpha^{u,s}_{m_u,2j-3}} = 0 \quad (5.24)$$

$$(b_u - b_s)\beta^{u,s}_{2v-1,2j} + c_u\beta^{u,s}_{2v,2j} - c_s\beta^{u,s}_{2v-1,2j-1} - \underline{\beta^{u,s}_{2v-1,2j-2}} + \beta^{u,s}_{2v+1,2j} = 0 \quad (5.25)$$

$$(b_u - b_s)\beta^{u,s}_{2v,2j} - c_u\beta^{u,s}_{2v-1,2j} - c_s\beta^{u,s}_{2v,2j-1} - \underline{\beta^{u,s}_{2v,2j-2}} + \beta^{u,s}_{2v+2,2j} = 0 \quad (5.26)$$

$$(b_u - b_s)\beta^{u,s}_{2n_u-1,2j} + c_u\beta^{u,s}_{2n_u,2j} - c_s\beta^{u,s}_{2n_u-1,2j-1} - \underline{\beta^{u,s}_{2n_u-1,2j-2}} = 0 \quad (5.27)$$

$$(b_u - b_s)\beta^{u,s}_{2n_u,2j} - c_u\beta^{u,s}_{2n_u-1,2j} - c_s\beta^{u,s}_{2n_u,2j-1} - \underline{\beta^{u,s}_{2n_u,2j-2}} = 0 \quad (5.28)$$

where the underlined parts are those we do not have when $j = 1$.

Using induction both on j and v we obtain by equations (5.17), (5.18), (5.23) and (5.24) $\alpha^{u,s}_{v,j} = 0 \quad \forall\, u, v$, while by the other 8 equations that $\beta^{u,s}_{v,j} \neq 0$ only for $v = 1, 2$ and $b_u + ic_u = b_s + ic_s$. $\square$

This Lemma implies the following proposition

Proposition 5.3. *Let $\mathfrak{g}$ be an almost abelian Lie algebra, if $ad_{X_{2n}}$ has complex Jordan blocks of dimension greater then 1, then $\mathfrak{g}$ does not admit a complex structure.*

Proof. For $j > 2$ we have that Y_j^s does not appear in $J^2(Y_j^s)$, then it can not be equal to $-Y_j^s$. $\square$

We now consider also JX_{2n}:

Proposition 5.4. *Let $\mathfrak{g} = \mathbb{R} \ltimes_{ad_{X_{2n}}} \mathbb{R}^{2n-1}$ be an almost abelian Lie algebra endowed with an almost complex structure $J : \mathfrak{g} \to \mathfrak{g}$.*
If $N|_{\mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1}} = 0$, then J is integrable.

Proof. We consider $\mathfrak{g} = \mathfrak{h}^1 \oplus \mathfrak{h}^2 \oplus \langle X_{2n} \rangle$, where $\mathfrak{h}^2$ is the bigger J-invariant subspace of $\mathfrak{g}$, i.e. $J(\mathfrak{h}^2) \subset \mathfrak{h}^2$. In particular by the previous notation in $\mathfrak{h}^1$ we have only all the $X_{m_t}^t$ such that $\eta_{m_t}^t \neq 0$.
By hypothesis $J^2 = -Id$ and $\forall X, Y \in \mathfrak{h}^1 \oplus \mathfrak{h}^2 \quad N(X, Y) = 0$. In particular if $X \in \mathfrak{h}^2$ and $Y \in \mathfrak{h}^1$ we have $JX \in \mathfrak{h}^2$ and $JY = Y^1 + Y^2 + yX_{2n}$ where $Y^i \in \mathfrak{h}^i$, $i = 1, 2$, then $N(X, Y) = -y[JX, X_{2n}] + yJ[X, X_{2n}]$ that is zero only if

$$[JX, X_{2n}] - J[X, X_{2n}] = 0. \tag{5.29}$$

Using $J^2 = -Id$ this is equivalent to

$$J[JX, X_{2n}] + [X, X_{2n}] = 0. \tag{5.30}$$

Suppose that $JX_{2n} = X^1 + X^2 + xX_{2n}$ where $X^i \in \mathfrak{h}^i$, $i = 1, 2$.

We consider 2 cases:

$Z \in \mathfrak{h}^2$:

$$\begin{aligned} N(Z, X_{2n}) &= [Z, X_{2n}] - x[JZ, X_{2n}] + J[JZ, X_{2n}] + xJ[Z, X_{2n}] = \\ &= [Z, X_{2n}] + J[JZ, X_{2n}] + x(J[Z, X_{2n}] - [JZ, X_{2n}]) \end{aligned}$$

that by equations (5.29) and (5.30) is zero.

$Z \in \mathfrak{h}^1$: let $JZ = Z^1 + Z^2 + zX_{2n}$ where $Z^i \in \mathfrak{h}^i$, $i = 1, 2$, then
$JZ^1 = Y^1 + Y^2 + yX_{2n}$ where $Y^i \in \mathfrak{h}^i$, $i = 1, 2$ and $JZ^2 = \tilde{Z}^2 \in \mathfrak{h}^2$.

We now consider the 2 hypothesis: $J^2 Z = -Z$ is equivalent to

$$\begin{cases} Y^1 + zX^1 = -Z \\ Y^2 + \tilde{Z}^2 + zX^2 = 0 \\ y + zx = 0 \end{cases} \tag{5.31}$$

and $N(Z, Z^1) = 0$ is equivalent to

$$-y[JZ, X_{2n}] + z[JZ^1, X_{2n}] - zJ[Z^1, X_{2n}] + yJ[Z, X_{2n}] = 0.$$

By (5.31) $y = -xz$, then this equation becomes

$$x[JZ, X_{2n}] + [JZ^1, X_{2n}] - J[Z^1, X_{2n}] - xJ[Z, X_{2n}] = 0. \tag{5.32}$$

$$\begin{aligned} N(Z, X_{2n}) = {} & [Z, X_{2n}] - x[JZ, X_{2n}] + z[X^1, X_{2n}] + z[X^2, X_{2n}] + \\ & +J[Z^1, X_{2n}] + J[Z^2, X_{2n}] + xJ[Z, X_{2n}] \end{aligned}$$

but by equation (5.29) $J[Z^2, X_{2n}] = [JZ^2, X_{2n}] = [\tilde{Z}^2, X_{2n}]$, then

$$= [Z, X_{2n}] - x[JZ, X_{2n}] + z[X^1, X_{2n}] + J[Z^1, X_{2n}] + xJ[Z, X_{2n}] + [zX^2 + \tilde{Z}^2, X_{2n}].$$

By (5.31) $[zX^2 + \tilde{Z}^2, X_{2n}] = [-Y^2, X_{2n}] = [-JZ^1 + Y^1 + yX_{2n}, X_{2n}] =$
$= -[JZ^1, X_{2n}] + [Y^1, X_{2n}] = -[JZ^1, X_{2n}] - [Z, X_{2n}] - z[X^1, X_{2n}]$, then

$$= -x[JZ, X_{2n}] + J[Z^1, X_{2n}] + xJ[Z, X_{2n}] - [JZ^1, X_{2n}]$$

that by (5.32) is zero.

$\square$

We summarize all the results in the following theorem.

Theorem 5.3. *Let $\mathfrak{g} = \mathbb{R} \ltimes_{ad_{X_{2n}}} \mathbb{R}^{2n-1}$ be an almost abelian Lie algebra endowed with an almost complex structure J.*
J is integrable if and only if the hypothesis of Propositions 5.1, 5.2 and 5.3 do not hold and $\mathfrak{g}$ has a basis
$\langle X_1^1, \cdots, X_{m_1}^1, \cdots, X_1^p, \cdots, X_{m_p}^p, Y_1, \cdots, Y_{2q}, X_{2n} \rangle$

$$\begin{array}{ll} [X_1^t, X_{2n}] = a_t X_1^t & \forall\, t \leq p \quad \text{with not necessarily } a_t \neq a_s \text{ for } t \neq s \leq p \\ [X_i^t, X_{2n}] = X_{i-1}^t + a_t X_i^t & \forall\, t \text{ such that } m_t > 1,\ \forall\, 1 < i \leq m_t \\ [Y_{2j-1}, X_{2n}] = b_j Y_{2j-1} - c_j Y_{2j} & \forall\, j \leq q \\ [Y_{2j}, X_{2n}] = c_j Y_{2j-1} + b_j Y_{2j}. & \end{array}$$

In particular we have

$$If\ a_t = a_1 \quad JX_{m_t}^t = \sum_{\substack{u \leq p \\ v \leq m_u}} \psi_{v,t}^u X_v^u + \sum_{k \leq 2q} \rho_{k,t} Y_k + \eta_t X_{2n} \quad \forall\, i \leq r,$$

$$If\ a_t \neq a_1 \quad JX_{m_t}^t = \sum_{\substack{a_u = a_t \\ v \leq m_t}} \varphi_{1,m_t-v+1}^{u,t} X_v^u \quad \forall\, t \leq p$$

$$\forall i < m_t \quad JX_i^t = \sum_{\substack{a_u = a_t \\ v \leq i \\ m_u - v \geq m_t - i}} \varphi_{1,i-v+1}^{u,t} X_v^u \quad \forall\, t \leq p$$

$$JY_{2j-1} = \sum_{\substack{b_k = b_j \\ c_k = c_j}} (\beta_{k,j} Y_{2k-1} + \gamma_{k,j} Y_{2k}) \quad \forall\, j \leq q$$

$$JY_{2j} = \sum_{\substack{b_k = b_j \\ c_k = c_j}} (-\gamma_{k,j} Y_{2k-1} + \beta_{k,j} Y_{2k}) \quad \forall\, j \leq q$$

with $J^2 = -Id$ and the equations of Lemma 5.3 satisfied.

Remark 5.1. We observe that in [38, Prop. 3] this result was found for dimension 4.

5.1.2 Bi-invariant and abelian cases

Next we consider two particular cases of complex structures.

Let $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_{2n}}} \mathbb{R}^{2n-1}$ be an almost abelian Lie algebra, with $\mathfrak{g} = \langle X_1, \cdots, X_{2n} \rangle$ and $[X_i, X_{2n}] = \sum_{k<2n} c_{ik} X_k$.

Suppose that $\mathfrak{g}$ admits a *bi-invariant* complex structure J (see Proposition 1.4).

This is a much stronger condition than integrability requested by equation (1.2), indeed we have the following theorem:

Theorem 5.4. *An almost abelian Lie algebra can not be endowed with a bi-invariant complex structure.*

Proof. $\forall\, i, j < 2n$ we have $J[X_i, X_j] = J(0) = 0$.

Suppose $\forall i \leq 2n$ $JX_i = \sum_{k \leq 2n} \varphi_{k,i} X_k$, then

$$0 = [JX_i, X_j] = \sum_{k \leq 2n} \varphi_{k,i} [X_k, X_j] = -\varphi_{2n,i} [X_j, X_{2n}] = -\sum_{k<2n} \varphi_{2n,i} c_{jk} X_k$$

then $\forall\, k < 2n$ $\varphi_{2n,i} c_{jk} = 0$, this means that $\forall\, i, j, k < 2n$ (also $i = j = k$ or $i = j$ or $i = k$ or $j = k$) we have $\varphi_{2n,i} c_{jk} = 0$, but if $\mathfrak{g}$ is not abelian, then there exist $j, k < 2n$ such that $c_{jk} \neq 0$.
If we fix these j and k than $\varphi_{2n,i} c_{jk} = 0$ implies $\varphi_{2n,i} = 0$ $\forall i < 2n$, then $\forall i < 2n$ $JX_i = \sum_{k<2n} \varphi_{k,i} X_k$ that, as in Proposition 5.2, implies $J^2 X_{2n} \neq -X_{2n}$. □

Now suppose that $\mathfrak{g}$ is endowed with an *abelian* complex structure J, i.e. J is an almost complex structure on $\mathfrak{g}$ and $[JX, JY] = [X, Y] \quad \forall X, Y \in \mathfrak{g}$ or equivalently

$$[JX, Y] = -[X, JY]. \tag{5.33}$$

Again this is a stronger condition than integrability, then if we want to study when $\mathfrak{g}$ admits an abelian complex structure, we can suppose that we have the same setting as in the statements of Proposition 5.4 and Theorem 5.3.

Theorem 5.5. *An almost abelian Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes_{ad_{X_{2n}}} \mathbb{R}^{2n-1}$ can be endowed with an abelian complex structure J only if $ad_{X_{2n}}$ has a unique non zero eigenvalue*

a, that is real and has geometric muliplicity equal to 1.
In particular if $[X_1, X_{2n}] = aX_1$ is the only not zero bracket, we have for $i = 1, 2n$ $JX_i = \sum_{k\leq 2n} \varphi_{k,i} X_k$ with $\varphi_{2n,1} \neq 0$ and for $1 < i < 2n$ $JX_i = \sum_{1<k<2n} \psi_{k,i} X_k$ such that $J^2 = -Id$.

Proof. We use the notation of Proposition 5.4: if both X and Y are in $\mathfrak{h}^2$, then also $JX, JY \in \mathfrak{h}^2$ and $[JX, Y] = [X, JY] = 0$.

Suppose now that $X \in \mathfrak{h}^1$ and $Y \in \mathfrak{h}^2$, then $[X, JY] = 0$ and if $JX = X^1 + X^2 + xX_{2n}$ we have $[JX, Y] = -x[Y, X_{2n}]$ and then $[Y, X_{2n}] = 0$, i.e. $[\mathfrak{h}^2, X_{2n}] = 0$. In particular because of Lemmas 5.1, 5.2 and 5.4 this implies that we do not have real eigenvalues a_i with $m_i > 1$ and we do not have complex eigenvalues at all.
In $\mathfrak{h}^1$ we have $X^t_{m_t}$ such that $a_t = a_1$ and $\eta_t \neq 0$, then this implies that $\forall\, a_i \neq a_1 \quad a_i = 0$ and $\forall\, a_i = a_1 \quad m_i = 1$.

Now consider $Z_i, Z_j \in \mathfrak{h}^1$: $[JZ_i, Z_j] = -\eta_i a_1 Z_j$ and $[Z_i, JZ_j] = \eta_j a_1 Z_i$, then equation (5.33) becomes $\eta^i Z_j - \eta^j Z_i = 0$ that implies $\eta_i = \eta_j = 0$, but then $J^2 X_{2n} \neq -X_{2n}$, so we have just one generator in $\mathfrak{h}^1$, i.e. $\mathrm{ad}_{X_{2n}}$ has a unique eigenvalue a not zero, that is real and has geometric multiplicity equal to 1.

Suppose then $\mathfrak{h}^1 = \langle X \rangle$ with $[X, X_{2n}] = aX$, $JX = X^1 + X^2 + xX_{2n}$ and $JX_{2n} = Z^1 + Z^2 + zX_{2n}$ with $X^i, Z^i \in \mathfrak{h}^i$, in particular $X^1 = uX$ and $Z^1 = vX$ with $u, v \in \mathbb{R}$.
$[JX, X_{2n}] = uaX \quad [X, JX_{2n}] = zaX$, then equation (5.33) is satisfied for $u = -z$. $\square$

In the general case we were not able to describe the Dolbeault complex $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \bar{\partial})$ associated to $\mathfrak{g}$ without knowing specifically J, but when J is an abelian complex structure we can.

Suppose that $\mathfrak{g}^{1,0}$ is generated by $\{Z_1 := X_1 - iJX_1, \cdots, Z_{2n} := X_{2n} - iJX_{2n}\}$ and $\mathfrak{g}^{0,1}$ by their complex conjugate elements.

The complex structure J is abelian, then we have $\forall\, 1 \leq r, s \leq 2n$

$$[Z_r, Z_s] = [X_r, X_s] - [JX_r, JX_s] + i([X_r, JX_s] + [JX_r, X_s]) = 0.$$

Similarly we get $[\bar{Z}_r, \bar{Z}_s] = 0$ and $[Z_r, \bar{Z}_s] = 2[X_r, X_s] - 2i[X_r, JX_s]$, then
$\forall\, 1 < r, s < 2n \qquad [Z_r, \bar{Z}_s] = 0,$

$\forall\, 1 < s < 2n \qquad [Z_1, \bar{Z}_s] = [Z_s, \bar{Z}_1] = [Z_{2n}, \bar{Z}_s] = [Z_s, \bar{Z}_{2n}] = 0,$
$[Z_1, \bar{Z}_{2n}] = \lambda(1 + i\varphi_{1,1})(Z_1 + \bar{Z}_1),$
$[Z_{2n}, \bar{Z}_1] = \lambda(-1 + i\varphi_{1,1})(Z_1 + \bar{Z}_1),$
$[Z_1, \bar{Z}_1] = -i\lambda\varphi_{2n,1}(Z_1 + \bar{Z}_1),$
$[Z_{2n}, \bar{Z}_{2n}] = i\lambda\varphi_{1,2n}(Z_1 + \bar{Z}_1).$

To can define the differential on $\mathfrak{g}^{*\mathbb{C}}$ we need to extract a basis $\mathcal{B}$ from the set of generators $\{Z_1, \cdots, Z_{2n}\}$ of $\mathfrak{g}^{1,0}$: we impose $Z_1 \in \mathcal{B}$, if for every $A \in \mathbb{C} \quad Z_{2n} \neq A \cdot Z_1$, we can impose also $Z_{2n} \in \mathcal{B}$, otherwise $Z_{2n} \notin \mathcal{B}$.

Let $0 \neq A = A_1 + iA_2 \in \mathbb{C}$ such that $Z_{2n} \neq A \cdot Z_1$. Then

$$X_{2n} - iJX_{2n} = A(X_1 - iJX_1) \;\Leftrightarrow\; X_{2n} - i\sum_{k \leq 2n} \varphi_{k,2n} X_k = A(X_1 - i\sum_{k\leq 2n}\varphi_{k,1}X_k)$$

$$\Leftrightarrow\; -i\varphi_{1,2n}X_1 - \sum_{1<k<2n} i\varphi_{k,2n}X_k + (1 - i\varphi_{2n,2n})X_{2n} =$$

$$= A((1 - i\varphi_{1,1})X_1 - \sum_{1<k<2n} i\varphi_{k,1}X_k - i\varphi_{2n,1}X_{2n})$$

$$\Leftrightarrow \begin{cases} -i\varphi_{1,2n} = A(1 - i\varphi_{1,1}) \\ \forall\, 1 < k < 2n \quad -i\varphi_{k,2n} = -iA\varphi_{k,1} \\ 1 + i\varphi_{1,1} = -iA\varphi_{2n,1} \end{cases} \tag{5.34}$$

By the second equation we have that if there exists $1 < k < 2n$ such that $\varphi_{k,2n} \neq 0$, then also $\varphi_{k,1} \neq 0$ and then $A \in \mathbb{R}$ and the first equation implies $A = 0$, then $\forall\, 1 < k < 2n \quad \varphi_{k,2n} = \varphi_{k,1} = 0$ and the sistem (5.34) becomes

$$\begin{cases} -i\varphi_{1,2n} = (A_1 + iA_2)(1 - i\varphi_{1,1}) \\ 1 + i\varphi_{1,1} = -i\varphi_{2n,1}(A_1 + iA_2) \end{cases} \quad \Rightarrow \quad \begin{cases} A_1 = -\frac{\varphi_{1,1}}{\varphi_{2n,1}} \\ A_2 = \frac{1}{\varphi_{2n,1}} \end{cases}$$

Remark 5.2. $\varphi_{k,2n} = 0 \quad \forall\, 1 < k < 2n \quad \Leftrightarrow \quad \varphi_{k,1} = 0 \quad \forall\, 1 < k < 2n$

Indeed if $\varphi_{k,2n} = 0 \quad \forall\, 1 < k < 2n, \quad J^2X_{2n} = -X_{2n}$ implies $\varphi_{1,2n}\varphi_{p,1} = 0$ $\forall\, 1 < p < 2n$ and then $\varphi_{p,1} = 0 \quad \forall\, 1 < p < 2n$.
Similarly if $\varphi_{k,1} = 0 \quad \forall\, 1 < k < 2n, \quad J^2X_1 = -X_1$ implies $\varphi_{p,2n} = 0 \quad \forall\, 1 < p < 2n$.

We have then two cases:

1. if $\forall\ 1 < k < 2n\ \ \varphi_{k,2n} = 0$, then $Z_{2n} = \left(\frac{-\varphi_{1,1}+i}{\varphi_{2n,1}}\right) Z_1, Z_{2n} \notin \mathcal{B}$ and we can choose $\mathcal{B} = \{Z_1, Z_{k_2}, \cdots, Z_{k_n}\}$ with $k_2, \cdots, k_n \in \{2, \cdots, 2n-1\}$.
   If $\{\omega^1, \cdots, \omega^n\}$ is its dual basis in $\mathfrak{g}^{1,0*}$ then

$$\begin{cases} d\omega^1 = i\lambda\varphi_{2n,1}\omega^1 \wedge \bar{\omega}^1 \\ d\omega^k = 0 \quad \forall k > 1 \end{cases}$$

   Now we compute the Dolbeault cohomology groups $H^{*,*}(\mathfrak{g})$:

   $\mathfrak{g}^{\mathbb{C}*} = \langle \omega^1, \cdots, \omega^n, \bar{\omega}^1, \cdots, \bar{\omega}^n \rangle$ with

$$\begin{cases} \bar{\partial}\omega^1 = c\omega^1 \wedge \bar{\omega}^1, \quad \text{with } c = i\lambda\varphi_{2n,1} \\ \bar{\partial}\omega^k = 0 \quad \forall k > 1 \\ \\ \bar{\partial}\bar{\omega}^k = 0 \quad \forall k \geq 1 \end{cases}$$

   Let α be a generator of $\bigwedge^{p,q} \mathfrak{g}^{\mathbb{C}*}$, then $\alpha = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}$.

$$\text{If } \forall\, 1 \leq r \leq p \quad i_r \neq 1, \quad \bar{\partial}\alpha = 0 \tag{5.35}$$

   If $\exists\, 1 \leq r \leq p$ such that $i_r = 1$, let say $i_1 = 1$, then
   $\bar{\partial}(\omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}) = 0$ and then

$$\bar{\partial}\alpha = c\omega^1 \wedge \bar{\omega}^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}$$

$$\text{If } \exists\, 1 \leq s \leq q \text{ such that } j_s = 1, \quad \bar{\partial}\alpha = 0 \tag{5.36}$$

$$\text{If } \forall\, 1 \leq s \leq q \ \ j_s \neq 1, \quad \bar{\partial}\alpha = \pm c\alpha \wedge \bar{\omega}^1 \neq 0 \tag{5.37}$$

   Then the space of closed (p,q)-forms is generated by elements of kind (5.35) and (5.36).

Now let β be a generator of $\bigwedge^{p,q-1} \mathfrak{g}^{\mathbb{C}*}$, then

$$\beta = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_{q-1}}.$$

$\bar{\partial}\beta = \alpha$ implies $\bar{\partial}\beta \neq 0$, then β is of kind (5.37) and then $\alpha = \bar{\partial}\beta = c\omega^1 \wedge \bar{\omega}^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_{q-1}}$.

In particular α is of kind (5.36) and then the space of exact (p,q)-forms is generated by elements of kind (5.36).

This means that $H^{p,q}(\mathfrak{g}) = \langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} \rangle$ with $i_r \neq 1 \quad \forall\, 1 \leq r \leq p$.

2. if $\exists\ 1 < k < 2n$ such that $\varphi_{k,2n} \neq 0$, then $Z_{2n} \neq AZ_1 \quad \forall A \in \mathbb{C}$ and we can choose $\mathcal{B} = \{Z_1, Z_{k_3}, \cdots, Z_{k_n}, Z_{2n}\}$ with $k_3, \cdots, k_n \in \{2, \cdots, 2n-1\}$.
If $\{\omega^1, \cdots, \omega^n\}$ is its dual basis in $\mathfrak{g}^{1,0*}$ then

$$\begin{cases} d\omega^1 = & -\lambda(1 + i\varphi_{1,1})\omega^1 \wedge \bar{\omega}^n + i\lambda\varphi_{2n,1}\omega^1 \wedge \bar{\omega}^1 + \\ & +\lambda(1 - i\varphi_{1,1})\omega^n \wedge \bar{\omega}^1 - i\lambda\varphi_{1,2n}\omega^n \wedge \bar{\omega}^n \\ d\omega^k = & 0 \quad \forall k > 1 \end{cases}$$

Again we compute the Dolbeault cohomology:

$\mathfrak{g}^{\mathbb{C}*} = \langle \omega^1, \cdots, \omega^n, \bar{\omega}^1, \cdots, \bar{\omega}^n \rangle$ with

$$\begin{cases} \bar{\partial}\omega^1 = iA\omega^1 \wedge \bar{\omega}^1 + iB\omega^n \wedge \bar{\omega}^n + C\omega^1 \wedge \bar{\omega}^n - \bar{C}\omega^n \wedge \bar{\omega}^1 \\ \bar{\partial}\omega^k = 0 \quad \forall k > 1 \\ \\ \bar{\partial}\bar{\omega}^k = 0 \quad \forall k \geq 1 \end{cases}$$

where $A = \lambda\varphi_{2n,1}, \quad B = -\lambda\varphi_{1,2n}, \quad C = -\lambda(1 + i\varphi_{1,1})$.

A generator of $\bigwedge^{p,q} \mathfrak{g}^{\mathbb{C}*}$ is of kind $\alpha = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}$, then we have to consider 9 different cases:

- if $\forall\, 1 \leq r \leq p \quad i_r \neq 1$, then $\bar{\partial}\alpha_1 = 0$.

- if $i_1 = 1$, but $\forall\ 2 \le r \le p,\ 1 \le a, b \le q \quad i_r \ne n,\ j_a \ne 1,\ j_b \ne n$, then $\alpha_2 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\begin{aligned}\bar\partial\alpha_2 = \quad & iA\omega^1 \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & +iB\omega^n \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & +C\omega^1 \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & -\bar C\omega^n \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} \ne 0\end{aligned}$$

- if $i_1 = 1,\ i_p = n$, but $\forall\ 1 \le a, b \le q \quad j_a \ne 1,\ j_b \ne n$, then $\alpha_3 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\begin{aligned}\bar\partial\alpha_3 = \quad & iA\omega^1 \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & +C\omega^1 \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{j_q} \ne 0\end{aligned}$$

- if $i_1 = 1,\ j_1 = 1$, but $\forall\ 2 \le r \le p \quad i_r \ne n$ and $\forall\ 2 \le a \le q$ $j_a \ne n$, then $\alpha_4 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\begin{aligned}\bar\partial\alpha_4 = \quad & iB\omega^n \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & +C\omega^1 \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q} \ne 0\end{aligned}$$

- if $i_1 = 1,\ j_1 = n$, but $\forall\ 2 \le r \le p \quad i_r \ne n$ and $\forall\ 2 \le a \le q$ $j_a \ne 1$, then $\alpha_5 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^n \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\begin{aligned}\bar\partial\alpha_5 = \quad & iA\omega^1 \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^n \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q} + \\ & -\bar C\omega^n \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^n \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q} \ne 0\end{aligned}$$

- if $i_1 = 1,\ i_p = n,\ j_1 = 1$, but $\forall\ 2 \le a \le q\ j_a \ne n$, then $\alpha_6 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\bar\partial\alpha_6 = C\omega^1 \wedge \bar\omega^n \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$$

- if $i_1 = 1,\ i_p = n,\ j_1 = n$, but $\forall\ 2 \le a \le q\ j_a \ne 1$, then $\alpha_7 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^n \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$ and

$$\bar\partial\alpha_7 = iA\omega^1 \wedge \bar\omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar\omega^n \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_q}$$

- if $i_1 = 1,\ j_1 = 1,\ j_q = n$, but $\forall\ 2 \le r \le p\ i_r \ne n$, then $\alpha_8 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^1 \wedge \bar\omega^{j_2} \wedge \cdots \wedge \bar\omega^{j_{q-1}} \wedge \bar\omega^n$ and $\bar\partial\alpha_8 = 0$

- if $i_1 = 1$, $i_p = n$, $j_1 = 1$, $j_q = n$, then
  $\alpha_9 = \omega^1 \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{p-1}} \wedge \omega^n \wedge \bar{\omega}^1 \wedge \bar{\omega}^{j_2} \wedge \cdots \wedge \bar{\omega}^{j_{q-1}} \wedge \bar{\omega}^n$ and $\bar{\partial}\alpha_9 = 0$

Then the space of closed (p, q)-forms is generated by elements of kind $\alpha_1,\ iA\alpha_4 + C\alpha_5,\ iA\alpha_6 + C\alpha_7,\ \alpha_8,\ \alpha_9$. But

- $iA\alpha_4 + C\alpha_5 = \bar{\partial}\alpha_2 - iB\alpha_1 + \bar{C}\alpha_1'$,
- $iA\alpha_6 + C\alpha_7 = \bar{\partial}\alpha_3$,
- $\alpha_8 = \frac{1}{C}(\bar{\partial}\alpha_4 - iB\alpha_1)$,
- $\alpha_9 = \frac{1}{C}\bar{\partial}\alpha_6$,

then again the cohomology is given only by elements of kind 1, i.e.
$H^{p,q}(\mathfrak{g}) = \langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} \rangle$ with $i_r \neq 1 \quad \forall\, 1 \leq r \leq p$.

5.2 $\mathfrak{g} = \mathbb{R}^2 \ltimes_{\text{ad}} \mathbb{R}^{2n}$

In order to study complex structures, it seems more natural consider Lie algebras of kind $\mathfrak{g} = \mathbb{R}^2 \ltimes_{\text{ad}} \mathbb{R}^{2n}$.

Because of the isomorphism $\mathbb{C} \cong \mathbb{R}^2$ we will call these Lie algebras *complex almost abelian Lie algebras*, i.e. $\mathfrak{g} = \mathbb{C} \ltimes_{\text{ad}} \mathbb{C}^n$.
We will restrict to complex almost abelian Lie algebra $\mathfrak{g}$ such that $\dim_{\mathbb{R}} \operatorname{Im} \operatorname{ad} = 1$. In particular this means that for every real basis $\{Y_1, Y_2\}$ of $\mathbb{R}^2$ there exists $k \in \mathbb{R}$ such that $\operatorname{ad}_{Y_1} = k \operatorname{ad}_{Y_2}$. By a simple basis change in $\mathbb{R}^2$ we can suppose that there exists a real basis $\{X_1, \cdots, X_{2n}\}$ of $\mathbb{R}^{2n}$ such that

$$\forall\, i = 1, \cdots, 2n \quad [X_i, Y_1] = [X_i, Y_2].$$

We want to study a particular type of complex structures J on $\mathfrak{g}$: $J = J_1 \oplus J_2$ with J_1 and J_2 complex structures defined in terms of the basis $\langle Y_1, Y_2, X_1, \cdots, X_{2n} \rangle$

by

$$\begin{array}{llll} J_1: & \mathbb{C} & \to & \mathbb{C} \\ & Y_1 & \mapsto & \chi_1 Y_1 + \chi_2 Y_2 \\ & Y_2 & \mapsto & -\frac{1+\chi_1^2}{\chi_2} Y_1 - \chi_1 Y_2 \end{array} \qquad \begin{array}{llll} J_2: & \mathbb{C}^n & \to & \mathbb{C}^n \\ & X_i & \mapsto & \sum_{k\leq 2n} \varphi_{k,i} X_k \end{array} \tag{5.38}$$

The main result is the following theorem. As in the real case it is direct consequence of the properties and lemmas below.

Theorem 5.6. *Let $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ be a complex almost abelian Lie algebra such that $\dim_{\mathbb{R}} Im\, ad = 1$. If $\mathfrak{g}$ is endowed with a complex structure J of type (5.38) then in Jordan form with respect to the basis $\langle Y_1, Y_2, X_1, \cdots, X_{2n}\rangle$ we have*

$$ad_{Y_1} = ad_{Y_2} = \begin{pmatrix} a_1 & 1 & & & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & & & \\ & & \ddots & 1 & & & & & & & & & \\ & & & a_1 & & & & & & & & & \\ & & & & \ddots & & & & & & & & \\ & & & & & a_t & 1 & & & & & & \\ & & & & & & \ddots & \ddots & & & & & \\ & & & & & & & \ddots & 1 & & & & \\ & & & & & & & & a_t & & & & \\ & & & & & & & & & b_1 & c_1 & & \\ & & & & & & & & & -c_1 & b_1 & & \\ & & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & & b_q & c_q \\ & & & & & & & & & & & & -c_q & b_q \end{pmatrix}$$

with not necessarily $a_j \neq a_k$ or $b_j + ic_j \neq b_k + ic_k$ for $j \neq k$.

Proposition 5.5. *Let $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ be a complex almost abelian Lie algebra such that $\dim_{\mathbb{R}} Im\, ad = 1$. An almost complex structure J on $\mathfrak{g}$ of type defined by (5.38) is integrable if and only if*

$$J[X_i, Y_1] - [JX_i, Y_1] = 0 \quad \forall\, i \leq 2n. \tag{5.39}$$

Proof. By definition of $\mathfrak{g}$, for every almost complex structure of type (5.38) we have $N(Y_1, Y_2) = N(X_i, X_j) = 0 \quad \forall\, i, j$, then we have to prove the proposition only considering $N(X_i, Y_j) \quad \forall\, i, j$.

$\Rightarrow$: J is integrable, then by equation (1.2) and $[X_i, Y_1] = [X_i, Y_2]$, $\forall\, i \leq 2n$

$$\begin{cases} N(X_i, Y_1) = & [X_i, Y_1] + J[JX_i, Y_1] + \\ & +(\chi_1 + \chi_2)(J[X_i, Y_1] - [JX_i, Y_1]) = 0 \\ & \\ N(X_i, Y_2) = & [X_i, Y_1] + J[JX_i, Y_1] + \\ & +(\frac{1+\chi_1^2+\chi_1\chi_2}{\chi_2})(J[X_i, Y_1] - [JX_i, Y_1]) = 0 \end{cases}$$

Then

$$N(X_i, Y_1) - N(X_i, Y_2) = \left(\frac{1 + (\chi_1 + \chi_2)^2}{\chi_2}\right)(J[X_i, Y_1] - [JX_i, Y_1]) = 0$$

that implies $J[X_i, Y_1] - [JX_i, Y_1] = 0$.

$\Leftarrow$: Suppose that equation (5.39) holds. Since $J^2 = -Id$, it is equivalent to $[X_i, Y_1] + J[JX_i, Y_1] = 0 \quad \forall\, i \leq 2n$, then

$$\begin{cases} N(X_i, Y_1) = & [X_i, Y_1] + J[JX_i, Y_1] + \\ & +(\chi_1 + \chi_2)(J[X_i, Y_1] - [JX_i, Y_1]) = \\ & = 0 + (\chi_1 + \chi_2)(0) = 0 \\ & \\ N(X_i, Y_2) = & [X_i, Y_1] + J[JX_i, Y_1] + \\ & +(\frac{1+\chi_1^2+\chi_1\chi_2}{\chi_2})(J[X_i, Y_1] - [JX_i, Y_1]) = \\ & 0 + (\frac{1+\chi_1^2+\chi_1\chi_2}{\chi_2})(0) = 0 \end{cases}$$

that is that J is integrable.

$\square$

Suppose that ad is given in Jordan form, i.e.

$$\mathfrak{g} = \langle X_1^1, \cdots, X_{m_1}^1, \cdots, X_1^p, \cdots, X_{m_p}^p, Z_1^1, \cdots, Z_{2n_1}^1, \cdots, Z_1^q, \cdots, Z_{2n_q}^q, Y_1, Y_2 \rangle$$

with

$$
\begin{array}{ll}
\left[X_1^t, Y_1\right] = \left[X_1^t, Y_2\right] = a_t X_1^t & \forall\, t = 1, \cdots, p \\
\left[X_i^t, Y_1\right] = \left[X_i^t, Y_2\right] = X_{i-1}^t + a_t X_i^t & \forall\, i = 2, \cdots, m_t \\
\left[Z_1^s, Y_1\right] = \left[Z_1^s, Y_2\right] = b_s Z_1^s - c_s Z_2^s & \forall\, s = 1, \cdots, q \\
\left[Z_2^s, Y_1\right] = \left[Z_2^s, Y_2\right] = c_s Z_1^s + b_s Z_2^s & \\
\left[Z_{2j-1}^s, Y_1\right] = \left[Z_{2j-1}^s, Y_2\right] = Z_{2j-3} + b_s Z_{2j-1}^s - c_s Z_{2j}^s & \forall\, j = 2, \cdots, n_s \\
\left[Z_{2j}^s, Y_1\right] = \left[Z_{2j}^s, Y_2\right] = Z_{2j-2} + c_s Z_{2j-1}^s + b_s Z_{2j}^s &
\end{array}
$$

Consider J_2 give by

$$
JX_i^t = \sum_{\substack{u=1,\cdots,p \\ v=1,\cdots,m_u}} \varphi_{v,i}^{u,t} X_v^u + \sum_{\substack{u=1,\cdots,q \\ v=1,\cdots,n_u}} \psi_{v,i}^{u,t} Z_v^u
$$

$$
JZ_j^s = \sum_{\substack{u=1,\cdots,p \\ v=1,\cdots,m_u}} \alpha_{v,j}^{u,s} X_v^u + \sum_{\substack{u=1,\cdots,q \\ v=1,\cdots,n_u}} \beta_{v,j}^{u,s} Z_v^u
$$

We do not give the proofs of the following lemmas and propositions because they are similar to those we proved in the real case.

Lemma 5.6. *$\forall\, t, i$ we have*

$$
JX_i^t = \sum_{\substack{a_u = a_t \\ v \leq i \\ m_u - v \geq m_t - i}} \varphi_{1,i-v+1}^{u,t} X_v^u. \tag{5.40}
$$

Proposition 5.6. *Let $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ be a complex almost abelian Lie algebra such that $\dim_{\mathbb{R}} Im\, ad = 1$,*

1. *if ad has a real eigenvalues a_t such that $a_t \neq a_u$ for every real eigenvalue a_u, then $\mathfrak{g}$ does not admit a complex structure of type (5.38).*

2. *if ad has a real eigenvalues a_t of multiplicity $m_t > m_u$ for every real eigenvalue a_u such that $a_u = a_t$, then $\mathfrak{g}$ does not admit a complex structure of type (5.38).*

3. *if ad has a real eigenvalue with odd algebraic multiplicity, then $\mathfrak{g}$ does not admit a complex structure of type (5.38).*

Lemma 5.7. $\forall\, s = 1, \cdots, q \quad \forall\, j = 1, \cdots, 2n_s$

$$JZ_j^s = \sum_{\substack{b_u = b_s \\ c_u = c_s}} (\beta_{1,j}^{u,s} Z_1^u + \beta_{2,j}^{u,s} Z_2^u).$$

Proposition 5.7. *Let* $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ *be a complex almost abelian Lie algebra such that* $\dim_{\mathbb{R}} Im\, ad = 1$*, if ad has complex Jordan blocks of dimension greater then 1, then* $\mathfrak{g}$ *does not admit a complex structure of type (5.38).*

We summarize all the results in the following theorem.

Theorem 5.7. *Let* $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ *be a complex almost abelian Lie algebra such that* $\dim_{\mathbb{R}} Im\, ad = 1$*. Suppose that* $\mathfrak{g}$ *is endowed with an almost complex structure* J *of type (5.38).*
J is integrable if and only if the hypothesis of Propositions 5.6 and 5.7 does not hold and if given $\mathfrak{g} = \langle X_1^1, \cdots, X_{m_1}^1, \cdots, X_1^p, \cdots, X_{m_p}^p, Z_1, \cdots, Z_{2q}, Y_1, Y_2 \rangle$ *with*

$$\begin{array}{ll}
[X_1^t, Y_1] = [X_1^t, Y_2] = a_t X_1^t & \forall\, t = 1, \cdots, p \\
[X_i^t, Y_1] = [X_i^t, Y_2] = X_{i-1}^t + a_t X_i^t & \forall\, i = 2, \cdots, m_t \\
[Z_{2j-1}, Y_1] = [Z_{2j-1}, Y_2] = b_j Z_{2j-1} - c_j Z_{2j} & \forall\, j = 1, \cdots, q \\
[Z_{2j}, Y_1] = [Z_{2j}, Y_2] = c_j Z_{2j-1} + b_j Z_{2j} &
\end{array}$$

we have

$$JX_i^t = \sum_{\substack{a_u = a_t \\ v \leq i \\ m_u - v \geq m_t - i}} \varphi_{1,i-v+1}^{u,t} X_v^u \quad \forall\, t \leq p,\ \forall\, i \leq m_t$$

$$JZ_{2j-1} = \sum_{\substack{b_k = b_j \\ c_k = c_j}} (\beta_{k,j} Z_{2k-1} + \gamma_{k,j} Z_{2k}) \quad \forall\, j \leq q$$

$$JZ_{2j} = \sum_{\substack{b_k = b_j \\ c_k = c_j}} (-\gamma_{k,j} Z_{2k-1} + \beta_{k,j} Z_{2k}) \quad \forall\, j \leq q.$$

Now we want to describe the complex of Dolbeault forms on the dual of the complexification of $\mathfrak{g}$. Suppose that $\mathfrak{g}^{1,0}$ is generated by

$$\{Z_1 := X_1 - iJX_1, \cdots, Z_{2n} := X_{2n} - iJX_{2n}, A_1 := Y_1 - iJY_1, A_2 := Y_2 - iJY_2\}$$

and $\mathfrak{g}^{0,1}$ by their complex conjugate elements.

Proposition 5.5 implies that structure constants of $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ are basically the same:

Lemma 5.8. *If* $[X_i, Y_1] = [X_i, Y_2] = \sum_k c_{i,k} X_k$, *then*

$$[Z_i, A_1] = \sum_k \varepsilon c_{i,k} Z_k, \qquad [Z_i, A_2] = \sum_k \sigma c_{i,k} Z_k,$$

$$[Z_i, \bar{A}_1] = \sum_k \bar{\varepsilon} c_{i,k} Z_k, \qquad [Z_i, \bar{A}_2] = \sum_k \bar{\sigma} c_{i,k} Z_k$$

$$[\bar{Z}_i, A_1] = \sum_k \varepsilon c_{i,k} \bar{Z}_k, \qquad [\bar{Z}_i, A_2] = \sum_k \sigma c_{i,k} \bar{Z}_k,$$

$$[\bar{Z}_i, \bar{A}_1] = \sum_k \bar{\varepsilon} c_{i,k} \bar{Z}_k, \qquad [\bar{Z}_i, \bar{A}_2] = \sum_k \bar{\sigma} c_{i,k} \bar{Z}_k$$

where $\varepsilon := 1 - i(\chi_1 + \chi_2)$ *and* $\sigma := 1 + i\left(\frac{1+\chi_1^2}{\chi_2} + \chi_1\right)$

Proof. By Proposition 5.5 we have that $[Z_i, Y_1] = \sum_k c_{i,k} Z_k$ and $[\bar{Z}_i, Y_1] = \sum_k c_{i,k} \bar{Z}_k$ and because of $[X_i, Y_1] = [X_i, Y_2]$ we have the thesis. $\square$

This Lemma implies that the study of the $\partial\bar{\partial}$-lemma for these algebras is very simple. We observe that we can not describe in general a basis extracted among the generators $\{Z_i, \bar{Z}_i,\}$ of $\mathfrak{g}^{\mathbb{C}}$ without knowing a particular description of the complex structure J, but we are able to study this lemma all the same.

Suppose that $\mathcal{B}$ is a basis of $\mathfrak{g}^{1,0} \smallsetminus \langle A_1, A_2 \rangle$, then $\mathcal{B} \oplus \langle A_1 \rangle$ is a basis of $\mathfrak{g}^{1,0}$, because by definition of J, $\langle A_1, A_2 \rangle$ is generated by A_1.

A basis of $\mathfrak{g}^{\mathbb{C}*}$ is

$$\langle \omega^i, \bar{\omega}^i, \eta, \bar{\eta} \rangle_{i=1,\cdots,n}$$

where $\omega^i = Z_i^*$, $\eta = A_1^*$ and $Z_i \in \mathcal{B}$.

By Lemma 5.8 we have the following description of the complexes $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \partial)$ and $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \bar{\partial})$:

$$\begin{array}{r}\partial\omega^i = \sum_k C_{i,k}\varepsilon\omega^k \wedge \eta \\ \partial\bar{\omega}^i = \sum_k \bar{C}_{i,k}\varepsilon\bar{\omega}^k \wedge \eta \\ \partial\eta = \partial\bar{\eta} = 0 \\ \\ \bar{\partial}\omega^i = \sum_k C_{i,k}\bar{\varepsilon}\omega^k \wedge \bar{\eta} \\ \bar{\partial}\bar{\omega}^i = \sum_k \bar{C}_{i,k}\bar{\varepsilon}\bar{\omega}^k \wedge \bar{\eta} \\ \bar{\partial}\eta = \bar{\partial}\bar{\eta} = 0.\end{array} \tag{5.41}$$

where constants $C_{i,k}$ depends on $c_{i,k}$, J and the choice of the basis $\mathcal{B}$.

Theorem 5.8. *Let $\mathfrak{g} = \mathbb{C} \ltimes_{ad} \mathbb{C}^n$ be a complex almost abelian Lie algebra such that* $\dim_{\mathbb{R}}$ *Im ad* $= 1$*. Suppose that $\mathfrak{g}$ is endowed with a complex structure J of type (5.38), then for $\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}$ the $\partial\bar{\partial}$-Lemma does not hold.*

Proof. By equations (5.41) we have that $\forall\, \alpha \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}$

$$\partial\alpha = \varepsilon\tilde{\alpha} \wedge \eta \qquad \bar{\partial}\alpha = \bar{\varepsilon}\tilde{\alpha} \wedge \bar{\eta}$$

for some $\tilde{\alpha} \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}$, eventually $\tilde{\alpha} = 0$.

Let $\alpha = \varepsilon\alpha_1 \wedge \eta \wedge \bar{\eta}$ with $\alpha_1 \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*} \setminus \langle \eta, \bar{\eta} \rangle$ such that there exists $\beta \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*} \setminus \langle \eta, \bar{\eta} \rangle$ with $\partial\beta = \varepsilon\alpha_1 \wedge \eta$ (if $\mathfrak{g}$ is not abelian there exists always such β).

Then $\partial(\beta \wedge \bar{\eta}) = \partial\beta \wedge \bar{\eta} = \varepsilon\alpha_1 \wedge \eta \wedge \bar{\eta} = \alpha$ and $\partial\alpha = \bar{\partial}\alpha = 0$ and we have the hypothesis of the lemma satisfied.

If $\beta \wedge \bar{\eta} \neq \bar{\partial}\gamma \quad \forall\, \gamma \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}$, then the lemma does not hold.

If there exists $\gamma \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}$ such that $\beta \wedge \bar{\eta} = \bar{\partial}\gamma$, then let consider $\forall\, \beta' \in \bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*} \setminus \langle \eta, \bar{\eta} \rangle \quad \tilde{\beta} := \beta \wedge \bar{\eta} + \beta' \wedge \eta$, then $\partial\tilde{\beta} = \partial(\beta \wedge \bar{\eta}) + 0 = \alpha$, but $\forall\, \gamma \quad \bar{\partial}\gamma = \bar{\varepsilon}\tilde{\gamma} \wedge \bar{\eta}$, then $\bar{\partial}\gamma \neq \tilde{\beta}$ and the lemma does not hold. $\square$

Chapter 6

Minimal models and formality

In this chapter we want to study minimal models and formality of nilmanifolds and solvmanifolds.

De Rham models of nilmanifolds were completely described by Hasegawa in [20] and we will only state its result. About solvmanifolds we will use a result of Oprea and Tralle [36] to compute the minimal models of the almost abelian solvmanifolds studied in Chapter 4 and to study formality and symplectic structures in the almost abelian case [26]. In particular we will find a necessary condition to formality and a method to define symplectic forms.

In the last part we will define Dolbeault minimal models and we will generalize results of Cordero, Fernández and Ugarte about the Dolbeault model of a nilpotent Lie algebra [10]. In particular we will prove that the Dolbeault complex of a nilpotent Lie algebra endowed with a complex structure is always minimal.

6.1 de Rham models of nilmanifolds

Let $N = G/\Gamma$ be a nilmanifold. Nomizu theorem implies that the de Rham minimal model of N is the model of the nilpotent Lie algebra $\mathfrak{g}$ associated to G.

In $\bigwedge^0 \mathfrak{g}$ we have only constant functions, then $H^1(\mathfrak{g})$ is generated by 1-forms α such that $d(\alpha) = 0$, that is $d(\alpha)(X, Y) := -\alpha([X, Y]) = 0\ \forall X, Y \in \mathfrak{g}$, then by

1-forms that are null on $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$.

In particular these 1-forms are well defined on the quotient $\mathfrak{g}/\mathfrak{g}^1$ and we can consider

$$k = b_1(\mathfrak{g}) = \dim H^1(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}^1).$$

$\mathfrak{g} \cong \mathfrak{g}^1 \oplus \mathfrak{g}/\mathfrak{g}^1$, then we can choose a basis $\{X_1, \cdots, X_k, \cdots, X_m\}$ of $\mathfrak{g}$, $m \geq k \geq 2$ such that

$$\mathfrak{g}^1 = \langle X_{k+1}, \ldots, X_m \rangle.$$

Lemma 6.1. $[X_i, X_j] = \sum_{i<j<p} a_{ij}^p X_p, \qquad a_{ij}^p \in \mathbb{Q}.$

Proof. We consider the descending series

$$\{0\} \subset \cdots \subset \mathfrak{g}^k \subset \mathfrak{g}^{k-1} \subset \mathfrak{g}^1 \subset \mathfrak{g} :$$

$\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] \subset \mathfrak{g}^1$, then we can choose a basis of $\mathfrak{g}^1$ such that

$$\underbrace{X_{k+1}, ..., X_h}_{\in \mathfrak{g}^1 \backslash \mathfrak{g}^2}, \underbrace{X_{h+1}, ..., X_m}_{\in \mathfrak{g}^2}.$$

If $k+1 \leq i, j \leq h$, $X_i, X_j \in \mathfrak{g}^1 \setminus \mathfrak{g}^2$, then $[X_i, X_j] \in \mathfrak{g}^2$ and it is linear combination of $X_{h+1}, ..., X_m$, then $i < j < p$.

Otherwise if one of the index, name j, is between $k+1$ and h, i.e. $X_j \in \mathfrak{g}^2$, then $[X_i, X_j] \in \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2]$, and we can choose a basis of $\mathfrak{g}^2$ such that

$$\underbrace{X_{h+1}, ..., X_t}_{\in \mathfrak{g}^2 \backslash \mathfrak{g}^3}, \underbrace{X_{t+1}, ..., X_m}_{\in \mathfrak{g}^3}.$$

We repeat this computation for every terms if the series and we always have $i < j < p$. □

If $\{\omega^1, \cdots, \omega^m\}$ is the dual basis of $\{X_1, \cdots, X_m\}$ we have

$$d\omega_p = \sum_{i<j<p} a_{ij}^p \omega_i \wedge \omega_j \tag{6.1}$$

where $d\omega_p = 0$ for $p \leq k$ and $d\omega_p \neq 0$ for $p > k$.

Equation 6.1 and Theorem 1.7 imply directly the following property.

Proposition 6.1. *If $N = G/\Gamma$ is nilmanifold with Lie algebra $\mathfrak{g}$, then the minimal model $(\mathcal{M}, d)$ of N is isomorphic to $(\bigwedge^* \mathfrak{g}^*, d)$.*

Also about formality in the case of nilmanifold we have a complete theory [20].

Lemma 6.2. $d\mathcal{M}^{m-1} = 0$

Proof. $\{x_1 \cdot x_2 \cdots \hat{x}_q \cdots x_m\} \quad \forall\, q = 1, ..., m$ is a basis of $\mathcal{M}^{m-1}$, then we have to prove that $d(x_1 x_2 \cdots \hat{x}_q \cdots x_m) = 0$. But this is consequence of equation (6.1). □

Remark 6.1. This Lemma implies that $\mathcal{M}^n = \langle x_1 \cdots x_m \rangle$ and then $b_m(\mathcal{M}) = 1$.

Theorem 6.1. (Hasegawa) *[20] Let N be a nilmanifold with minimal model $\mathcal{M}$, then $\mathcal{M}$ is formal if and only if N is a torus.*

Proof. N is a torus if and only if its Lie algebra $\mathfrak{g}$ is abelian, i.e. $\mathfrak{g}^1 = 0$.
Then $\dim(\mathfrak{g}/\mathfrak{g}^1) = \dim(\mathfrak{g})$ and $k = m$ with $k = \dim H^1(\mathfrak{g}) = \dim H^1(\mathcal{M})$. This implies that we have to prove that $\mathcal{M}$ is formal if and only if $m = k$.

Let suppose by contradiction that $\mathcal{M}$ is formal for $k < m$, then there exists $\psi : \mathcal{M} \to H^*(\mathcal{M})$ that induces the identity in cohomology. We consider the restriction ψ^1 of ψ to $\mathcal{M}^1$. ψ sends every closed form of $\mathcal{M}$ in its cohomology class, then $\psi^1(x_q) = [x_q] \in H^1(\mathcal{M}) \quad \forall q = 1, .., k$.
$\{[x_q]\}_{q=1,\cdots,k}$ is a basis of $H^1(\mathcal{M})$ then, to have the identity in cohomology, $\mathcal{M}^1$ must be generated by $x_1, \cdots, x_k$ together with a basis $\{y_{k+1}, ..., y_m\}$ of $\ker \psi^1$.
But then in dimension m, where there is only one generator, we have $x_1 x_2 \cdots x_m = a x_1 \cdots x_k y_{k+1} \cdots y_m$ with $a \neq 0$.

$y_i \in \ker \psi^1$ then $\psi(x_1 x_2 \cdots x_m) = a\psi(x_1 \cdots x_k y_{k+1} \cdots y_m) = 0$, but $x_1 x_2 \cdots x_m$ is a closed element, then $\psi(x_1 x_2 \cdots x_m) = [x_1 x_2 \cdots x_m]$ and then $[x_1 x_2 \cdots x_m] = 0$ that by Remark 6.1 is not possible. □

6.2 de Rham models of almost abelian solvmanifolds

Proposition 6.1 states that the Chevalley-Eilenberg complex associated to a nilpotent Lie algebra is a minimal cdga. This property does not generalize to solvmanifolds, then can be interesting to compute their minimal model.

In particular we will use a method developed by Oprea and Tralle, that consists in applying a generalization of Theorem 1.14 due to Felix and Thomas to the Mostow fibration [36].

Definition 6.1. A cdga $\mathcal{A}$ is of *finite type* if it is a finite dimensional vector space.

Theorem 6.2. *[36], [37, Theorem 4.6] Let $F \to E \to B$ be a fibration and let U be the largest $\pi_1(B)$-submodule of $H^*(F, \mathbb{Q})$ on which $\pi_1(B)$ acts nilpotently. Suppose that $H^*(F, \mathbb{Q})$ is a vector space of finite type and that B is a nilpotent space, then in the Sullivan model of the fibration*

$$\begin{tikzcd}
\mathcal{A}_{PL}(B) \arrow[r] & \mathcal{A}_{PL}(E) \arrow[r] & \mathcal{A}_{PL}(F) \\
(\bigwedge X, d_X) \arrow[u, "\sigma"] \arrow[r, "i"] & (\bigwedge(X \oplus Y), D) \arrow[u, "\tau"] \arrow[r, "q"] & (\bigwedge Y, d_Y) \arrow[u, "\rho"]
\end{tikzcd}$$

the cdga homomorphism $\rho : (\bigwedge Y, d_Y) \to \mathcal{A}_{PL}(F)$ induces an isomorphism

$$\rho^* : H^*(\bigwedge Y, d_Y) \to U.$$

We recall that by definition of Sullivan model of a fibration (Definition 1.26), we have that in the commutative diagram of Theorem 6.2

- $(\bigwedge X, d_X)$ and $(\bigwedge Y, d_Y)$ are minimal cdga,
- σ and τ are quasi isomorphisms,
- i is the inclusion and q is the projection,
- $\forall x \in X \ Dx = d_X x$ and $\forall y \in Y \ Dy = d_Y y + cx \wedge y'$ with $c \in \mathbb{Q}$, $x \in \bigwedge X^+$ and $y' \in \bigwedge Y^{<y}$, where with $\bigwedge X^+$ we mean all the elements in $\bigwedge X$ with degree greater than 0 and with $\bigwedge Y^{<y}$ the subalgebra of $\bigwedge Y$ generated by all the generators prior to y with respect to an order among the basis of Y.

If we apply this theorem to the Mostow fibration (1.1), we can construct the minimal model $(\bigwedge(X \oplus Y), D)$ of the solvmanifold using the models of the base $\mathbb{T}^k$ and of the fibre N/Γ_N (actually of its submodule U).

In general, finding U is very difficult, because the action named in Theorem 6.2 is not easily described. But when the solvmanifold $S = \mathbb{R} \ltimes_\varphi \mathbb{R}^n/\mathbb{Z} \ltimes_{\varphi|_\mathbb{Z}} \mathbb{Z}^n$ is almost abelian Oprea and Tralle found an easy computation of this action (see [36]). In this case the Sullivan model of the Mostow fibration is

$$\begin{tikzcd} \bigwedge^*(\mathbb{R}/\mathbb{Z}) \arrow[r] & \bigwedge^*(S) \arrow[r] & \bigwedge^*(\mathbb{R}^n/\mathbb{Z}^n) \\ (\bigwedge(A), 0) \arrow[u, "\sigma"] \arrow[r, "i"] & (\bigwedge(X \oplus Y), D) \arrow[u, "\tau"] \arrow[r, "q"] & (\bigwedge Y, d_Y) \arrow[u, "\rho"] \end{tikzcd}$$

with $|A| = 1$. Moreover for degree reason also $(\bigwedge(X \oplus Y), D)$ is minimal and the following proposition holds.

Proposition 6.2. *[37, Theorem 3.8] For an almost abelian solvmanifold $S = \mathbb{R} \ltimes_\varphi \mathbb{R}^n/\mathbb{Z} \ltimes \mathbb{Z}^n$ the action of $\pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$ on $H^*(\mathbb{R}^n/\mathbb{Z}^n)$ is given by:*

- *restricting $\varphi : \mathbb{R} \to Aut(\mathbb{R}^n)$ to $\varphi : \mathbb{Z} \to Aut(\mathbb{R}^n)$,*
- *taking the dual automorphism $\varphi^t : \mathbb{Z} \to Aut(\mathbb{R}^n)$,*
- *extending to the exterior algebra $\bigwedge \varphi^t : \mathbb{Z} \to Aut(\bigwedge \mathbb{R}^n)$ as cdga map,*
- *taking the induced automorphism on cohomology $(\bigwedge \varphi^t)^* : \mathbb{Z} \to Aut(H^*(\bigwedge \mathbb{R}^n))$.*

To simplify the notation we denote the action $(\bigwedge \varphi^t)^*$ with φ.
By definition of nilpotent action we have that a form α is in U if and only if there exists a constant $k \in \mathbb{N}^+$ such that $(\varphi - \mathrm{Id})^k(\alpha) = 0$, where Id is the identity map. Even with Proposition 6.2 to compute U could be quite tough, fortunately the following properties simplify the computation.

Proposition 6.3. *$\alpha \in U$ if and only if $\varphi_s(\alpha) = \alpha$, where φ_s is the semisimple part of φ.*

Proof. We give the proof in 4 steps:

1. *we can prove the proposition on the complexification:*
   let V a generic real vector space generated by $\{v_1, \cdots, v_n\}$, then its complexification $V^{\mathbb{C}}$ is generated by elements $w_{jk} := v_j + iv_k$. Given an endomorphism φ of V, we can extend it to the complexification, $\varphi^{\mathbb{C}}$, and we can define the unipotent spaces:

$$U := \{v \in V / \exists p, (\varphi - \mathrm{Id})^p(v) = 0\}$$

$$U^{\mathbb{C}} := \{w \in V^{\mathbb{C}} / \exists p, (\varphi^{\mathbb{C}} - \mathrm{Id})^p(w) = 0\}$$

$$w_{jk} \in U^{\mathbb{C}} \Leftrightarrow (\varphi^{\mathbb{C}} - \mathrm{Id})^p(w_{jk}) = 0 \Leftrightarrow (\varphi - \mathrm{Id})^p(v_j) + i(\varphi - \mathrm{Id})^p(v_k) = 0$$

$$\Leftrightarrow \begin{cases} (\varphi - \mathrm{Id})^p(v_j) = 0 \\ (\varphi - \mathrm{Id})^p(v_k) = 0. \end{cases} \Leftrightarrow \begin{cases} v_j \in U \\ v_k \in U \end{cases}$$

$$\varphi_s^{\mathbb{C}}(w_{jk}) = w_{jk} \Leftrightarrow \varphi_s(v_j) + i\varphi_s(v_k) = v_j + iv_k \Leftrightarrow \begin{cases} \varphi_s(v_j) = v_j \\ \varphi_s(v_k) = v_k \end{cases}$$

   Then $w \in U^{\mathbb{C}} \Leftrightarrow \varphi_s^{\mathbb{C}}(w) = w$ implies $v \in U \Leftrightarrow \varphi_s(v) = v$.

2. $\varphi^{\mathbb{C}}$ *has a canonic form:*
   let $\mathrm{ad}_{X_{n+1}}$ be in Jordan form. Then we can consider $\varphi^{\mathbb{C}}$ on $\bigwedge^k \mathbb{C}^n$ for every k to be associated to a matrix made of blocks

$$\begin{pmatrix} e^{\lambda t} & & * \\ & \ddots & \\ 0 & & e^{\lambda t} \end{pmatrix}$$

   Let α be a generator of $\bigwedge^k \mathbb{C}^n$ such that the coefficients of $\varphi^{\mathbb{C}}(\alpha)$ belong to this block, then $\varphi^{\mathbb{C}}(\alpha) = e^{\lambda t}\alpha + \beta$, where β is combination of elements belonging to this same block, (the * part).

   Now we decompose $\varphi^{\mathbb{C}}$ in the unipotent and semisimple part:

   $\varphi^{\mathbb{C}} = \varphi_u^{\mathbb{C}} \cdot \varphi_s^{\mathbb{C}}$ where $\varphi_u^{\mathbb{C}}$ is made of blocks $\begin{pmatrix} 1 & & \star \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ and the semisimple part $\varphi_s^{\mathbb{C}}$ consists of diagonal blocks of the form $e^{\lambda t}\mathrm{Id}$.

This means that $\varphi_u^{\mathbb{C}}(\alpha) = \alpha + \beta'$, where β' is combination of elements belonging to this same block, (the $\star$ part), $\varphi_s^{\mathbb{C}}(\alpha) = e^{\lambda t}\alpha$ and $\beta = e^{\lambda t}\beta'$.
Then $\varphi^{\mathbb{C}}(\alpha) = e^{\lambda t}\varphi_u^{\mathbb{C}}(\alpha)$ and in general $\varphi^{\mathbb{C}} = e^{\lambda t}\varphi_u^{\mathbb{C}}$ for some λ.

3. $\forall p \quad (\varphi^{\mathbb{C}})^p(\alpha) = e^{p\lambda t}(\varphi_u^{\mathbb{C}})^p(\alpha)$:

   we use induction: for $p = 2$ we have

$$(\varphi^{\mathbb{C}})^2(\alpha) = \varphi^{\mathbb{C}}(e^{\lambda t}\varphi_u^{\mathbb{C}}(\alpha)) = e^{\lambda t}\varphi^{\mathbb{C}}(\varphi_u^{\mathbb{C}}(\alpha)),$$

   but β' is combination of elements belonging to the same block, then

$$= e^{\lambda t}(e^{\lambda t}\varphi_u^{\mathbb{C}}(\varphi_u^{\mathbb{C}}(\alpha))) = e^{2\lambda t}(\varphi_u^{\mathbb{C}})^2(\alpha).$$

   If now suppose that the property holds for $p-1$ we can prove it for p in a similar way.

4. $(\varphi^{\mathbb{C}} - \mathrm{Id})^k(\alpha) = 0 \;\Leftrightarrow\; \varphi_s^{\mathbb{C}}(\alpha) = \alpha$:

   let j be the dimension of the block to which α belong, then
   $(\varphi_u^{\mathbb{C}} - \mathrm{Id})^j(\alpha) = 0$.

   "$\Rightarrow$": Let $h \geq \max\{j, k\}$, then

$$0 = (\varphi^{\mathbb{C}} - \mathrm{Id})^h(\alpha) = (e^{\lambda t}\varphi_u^{\mathbb{C}} - \mathrm{Id})^h(\alpha) =$$

$$= [e^{\lambda t}(\varphi_u^{\mathbb{C}} - \mathrm{Id}) + (e^{\lambda t} - \mathrm{Id})]^h(\alpha) =$$

$$= \sum_{p=0}^{h} \binom{h}{p} (e^{\lambda t} - \mathrm{Id})^{h-p}(\alpha) \cdot e^{p\lambda t}(\varphi_u^{\mathbb{C}} - \mathrm{Id})^p(\alpha) =$$

$$= \sum_{p=0}^{h-1} \binom{h}{p} (e^{\lambda t} - \mathrm{Id})^{h-p}(\alpha) \cdot e^{p\lambda t}(\varphi_u^{\mathbb{C}} - \mathrm{Id})^p(\alpha) + e^{h\lambda t}(\varphi_u^{\mathbb{C}} - \mathrm{Id})^h(\alpha)$$

   but $h \geq j$, then the last summand is 0 and

$$= (e^{\lambda t} - \mathrm{Id})(\alpha) \left(\sum_{p=0}^{h-1} \binom{h}{p} (e^{\lambda t} - \mathrm{Id})^{h-p-1}(\alpha) \cdot e^{p\lambda t}(\varphi_u^{\mathbb{C}} - \mathrm{Id})^p(\alpha) \right)$$

   then $(e^{\lambda t} - \mathrm{Id})(\alpha) = 0$, i.e. $\varphi_s^{\mathbb{C}}(\alpha) = e^{\lambda t}\alpha = \alpha$.

"$\Leftarrow$": $\varphi_s^{\mathbb{C}}(\alpha) = \alpha \Leftrightarrow e^{\lambda t} = 1 \Leftrightarrow \varphi^{\mathbb{C}}(\alpha) = \varphi_u^{\mathbb{C}}(\alpha)$, then

$$(\varphi^{\mathbb{C}} - \mathrm{Id})^j(\alpha) = (\varphi_u^{\mathbb{C}} - \mathrm{Id})^j(\alpha) = 0.$$

$\square$

This proposition gives also a geometrical meaning to the complexification of U, $U^{\mathbb{C}}$: let V_λ be the subspace of $\mathbb{C}^n$ generated by the generators α of $\mathbb{C}^n$ such that the coefficients of $\varphi^{\mathbb{C}}(\alpha)$ belong a block of eigenvalue λ,
i.e. $\varphi^{\mathbb{C}}(\alpha) = e^{\lambda t}\varphi_u^{\mathbb{C}}(\alpha)$, then

$$U^{\mathbb{C}} = \bigoplus_{\{i_1,\cdots,i_k\}\subseteq\{1,\cdots,n\},\, \sum_p \lambda_{i_p} t = 0} V_{\lambda_{i_1}} \bigwedge \cdots \bigwedge V_{\lambda_{i_k}}$$

Now we prove a property of U that we will use after to study formality of S.

Proposition 6.4. *Let $\alpha, \beta \in H^*(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the n-dimensional abelian Lie algebra, and suppose that $\alpha \in U$, then $\beta \in U$ if and only if $\alpha \wedge \beta \in U$.*

Proof. Due to Proposition 6.3 this proof is very simple.

$\Rightarrow$: α and $\beta \in U$ is equivalent to $\varphi_s(\alpha) = \alpha$ and $\varphi_s(\beta) = \beta$, then

$$\varphi_s(\alpha \wedge \beta) = \varphi_s(\alpha) \wedge \varphi_s(\beta) = \alpha \wedge \beta.$$

$\Leftarrow$: α and $\alpha \wedge \beta \in U$ is equivalent to $\varphi_s(\alpha) = \alpha$ and $\varphi_s(\alpha \wedge \beta) = \alpha \wedge \beta$, then

$$\alpha \wedge \beta = \varphi_s(\alpha \wedge \beta) = \varphi_s(\alpha) \wedge \varphi_s(\beta) = \alpha \wedge \varphi_s(\beta),$$

then $\varphi_s(\beta) = \beta + \gamma$ for some $\gamma \in H^*(\mathbb{R}^n)$ such that $\alpha \wedge \gamma = 0$, but this is true for every α, then $\gamma \equiv 0$ and $\varphi_s(\beta) = \beta$ that is equivalent to $\beta \in U$.

$\square$

Remark 6.2. U is a submodule of $H^*(\mathbb{R}^n)$, then also in U the zero class is represented only by the zero form in $\bigwedge^*(\mathbb{R}^n)$.

We now compute the minimal model of the solvmanifolds we found in Chapter 4.3 using this method and then we study its formality. Unfortunately, with this method, in some of our examples we cannot find the model uniquely, because we can have different choices for the construction of $(\bigwedge(X \oplus Y), D)$. However, we can identify the right one, knowing the cohomology groups from the previous computations.

In all the following computation we denote the degree of an element by its subscript and by $(\mathcal{M}_U, d)$ the minimal cdga $(\bigwedge Y, d_Y)$ and by $(\mathcal{M}, D)$ the minimal model $(\bigwedge(X \oplus Y), D)$ of S.

$G^{p=0}_{6.8}/\Gamma_{2\pi}$:

$$U = \begin{cases} \langle \alpha^4, \alpha^5 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{45} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1234}, \alpha^{1235} \rangle \subset H^4(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases},$$

and a minimal model for U is $\mathcal{M}_U = (\bigwedge(x_1, y_1, z_3), 0)$.

The minimal model of the base $\mathbb{R}/\mathbb{Z}$ is $(\bigwedge(A), 0)$. So the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(A, x_1, y_1, z_3), D)$ with 2 possible choices for the differential: $D \equiv 0$ or $DA = Dx = Dz = 0,\ Dy = Ax$.

Since we do not know the cohomology groups of this solvmanifold, we are not able in this case to identify the right model.

$G^{p=0}_{6.8}/\Gamma_{\pi, \frac{\pi}{2}, \frac{\pi}{3}}$:

$$U = \begin{cases} \langle \alpha^{45} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases}$$

So $\mathcal{M}_U = (\bigwedge(x_2, \beta_3, y_3), d)$, $dx = dy = 0$, $d\beta = x^2$ and the minimal model of the solvmanifold is

$$\mathcal{M} = (\bigwedge(A, x_2, \beta_3, y_3), D), \qquad DA = Dx = Dy = 0,\ D\beta = x^2 .$$

To study formality we consider the cdga map

$$\begin{array}{rcl} \psi : \mathcal{M} & \to & H^*(\mathcal{M}) \\ A & \mapsto & [A] \\ x & \mapsto & [x] \\ y & \mapsto & [y] \\ \beta & \mapsto & 0 \end{array}$$

that gives the identity in cohomology, then $\mathcal{M}$ is formal.

$G_{6.10}^{a=0}/\Gamma_{2\pi}$:

$$U = H^*(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_1, y_1, z_1, p_1, q_1), 0)$$

The minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(A, x_1, y_1, z_1, p_1, q_1), D)$, but we have 7 different choices for D:

1. $D \equiv 0$
2. $DA = Dx = Dy = Dz = Dp = 0,\ Dq = Ax$
3. $DA = Dx = Dy = Dz = 0\ Dp = Ax,\ Dq = Ay$
4. $DA = Dx = Dy = Dz = 0\ Dp = Ax,\ Dq = Ap$
5. $DA = Dx = Dy = 0\ Dz = Ax\ Dp = Ay,\ Dq = Az$
6. $DA = Dx = Dy = 0\ Dz = Ax\ Dp = Az,\ Dq = Ap$
7. $DA = Dx = 0\ Dy = Ax\ Dz = Ay\ Dp = Az,\ Dq = Ap$

Again we do not know the cohomology groups of this solvmanifold and then we are not able to identify the right model.

$G_{6.10}^{a=0}/\Gamma_{\pi, \frac{\pi}{2}, \frac{\pi}{3}}$:

$$U = \begin{cases} \langle \alpha^1, \alpha^2, \alpha^3 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{12}, \alpha^{13}, \alpha^{23}, \alpha^{45} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123}, \alpha^{145}, \alpha^{245}, \alpha^{345} \rangle \subset H^3(\mathfrak{n}) & \Rightarrow \qquad \mathcal{M}_U = (\bigwedge(x_1, y_1, z_1, t_2, \beta_3), d), \\ \langle \alpha^{1245}, \alpha^{1345}, \alpha^{2345} \rangle \subset H^4(\mathfrak{n}) & \qquad\quad dx = dy = dz = dt = 0,\ d\beta = t^2 \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases}$$

The minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(A, x_1, y_1, z_1, t_2, \beta_3), D)$, but we have 13 different choices for D. Fortunately, only the following are not isomorphic with each other:

1. $DA = Dx = Dy = Dz = Dt = 0,\ D\beta = t^2$
2. $DA = Dx = Dy = 0,\ Dz = Ay\ Dt = 0,\ D\beta = t^2$
3. $DA = Dx = 0,\ Dy = Ax,\ Dz = Ay\ Dt = 0,\ D\beta = t^2$

Computing the cohomology groups of these c.d.g.a. and comparing with those of $G_{6.10}^{a=0}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$, we find that (3) is the right one.

$\mathcal{M}$ is not formal, indeed if

$$\begin{array}{rcl} \psi : \mathcal{M} & \rightarrow & H^*(\mathcal{M}) \\ A & \mapsto & [A] \\ x & \mapsto & [x] \\ y & \mapsto & 0 \\ z & \mapsto & 0 \\ t & \mapsto & [t] \\ \beta & \mapsto & 0 \end{array}$$

we have that $[Az] \neq 0$, but $\psi^*([Az]) = 0$.

$G_{6.11}^{p=0}/\Gamma_{2\pi}$:

$$U = \begin{cases} \langle \alpha^2, \alpha^3 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{23} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{145} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1245}, \alpha^{1345} \rangle \subset H^4(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases} \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_1, y_1, z_3), 0),$$

The minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(A, x_1, y_1, z_3), D)$, but we have 2 different choices for D:

1. $D \equiv 0$

2. $DA = Dx = 0,\ Dy = Ax,\ Dz = 0$

Again we do not know the cohomology groups of this solvmanifold, so we can not choose the right model.

$G^{p=0}_{6.11}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$:

$$U = \begin{cases} \langle \alpha^{23} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{145} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases} \Rightarrow \quad \begin{array}{l} \mathcal{M}_U = (\bigwedge(x_2, \beta_3, y_3), d), \\ \quad dx = dy = 0,\ d\beta = x^2 \end{array}$$

The minimal model of the solvmanifold is

$\mathcal{M} = (\bigwedge(A, x_2, \beta_3, y_3), D), \qquad DA = Dx = Dy = 0,\ D\beta = x^2$

and it is formal as in the case of $G^{p=0}_{6.8}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$.

$G^{-1,0,r}_{5.13} \times \mathbb{R}/\Gamma_{\frac{2\pi}{r}}$:

$$U = \begin{cases} \langle \alpha^3, \alpha^4 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{12}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123}, \alpha^{124} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

then $\mathcal{M}_U = (\bigwedge(x_1, y_1, z_2, \beta_3), d)$ with $dx = dy = dz = 0,\ d\beta = z^2$ and the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_1, y_1, z_2, \beta_3), D)$ with 4 different choices:

1. $Du = DA = Dx = Dy = Dz = 0$, $D\beta = z^2$

2. $Du = DA = Dx = Dy = 0$, $Dz = Axy$, $D\beta = z^2$

3. $Du = DA = Dx = 0$, $Dy = Ax$, $Dz = 0$, $D\beta = z^2$

4. $Du = DA = Dx = 0$, $Dy = Ax$, $Dz = Axy$, $D\beta = z^2$

We are not able to know the right model.

$G^{-1,0,r}_{5.13} \times \mathbb{R}/\Gamma_{\frac{2\pi}{rk}}$:

$$U = \begin{cases} \langle \alpha^{12}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

then $\mathcal{M}_U = (\bigwedge(x_2, y_2, \beta_3, \gamma_3), d)$ with $dx = dy = 0,\ d\beta = x^2,\ d\gamma = y^2$ and the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_2, y_2, \beta_3, \gamma_3), D)$ with $Du = DA = Dx = Dy = 0,\ D\beta = x^2,\ D\gamma = y^2$.

This model is formal.

$G^0_{5.14} \times \mathbb{R}/\Gamma_{2\pi}$:

$U = H^*(\mathfrak{n})$ then $\mathcal{M}_U = (\bigwedge(x_1, y_1, z_1, t_1), 0)$ and the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_1, y_1, z_1, t_1), D)$ with 4 different choices:

1. $Du = DA = Dx = Dy = Dz = Dt = 0$
2. $Du = DA = Dx = Dy = Dz = 0,\ Dt = Ax$
3. $Du = DA = Dx = Dy = 0,\ Dz = Ax, Dt = Ay$
4. $Du = DA = Dx = 0,\ Dy = Ax, Dz = Ay, Dt = Az$

Again we can not make a choice.

$G^0_{5.14} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}$:

$$U = \begin{cases} \langle \alpha^{1}, \alpha^{2} \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{12}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{134}, \alpha^{234} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

then $\mathcal{M}_U = (\bigwedge(x_1, y_1, z_2, \beta_3), d),\ dx = dy = dz = 0,\ d\beta = z^2$ and the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_1, y_1, z_2, \beta_3), D)$ with 4 different choices:

1. $Du = DA = Dx = Dy = Dz = 0,\ D\beta = z^2$
2. $Du = DA = Dx = Dy = 0,\ Dz = Axy, D\beta = z^2$

3. $Du = DA = Dx = Dz = 0,\ Dy = Ax, D\beta = z^2$
4. $Du = DA = Dx = 0,\ Dy = Ax, Dz = Axy, D\beta = z^2$

Using the cohomology we know that the third is the right one, then because of $[Ay]$, $\mathcal{M}$ is not formal.

$G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{2k_p\pi r_2} \quad (r = \frac{r_1}{r_2} \in \mathbb{Q})$:

$$\text{if}\, p = 0 \quad U = H^*(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_1, y_1, z_1, t_1), 0)$$

The minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_1, y_1, z_1, t_1), D)$, with different choices for D, but again we are not able to choose the right one.

$$\text{if}\, p \neq 0 \quad U = \begin{cases} \langle \alpha^{13}, \alpha^{14}, \alpha^{23}, \alpha^{24} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

To list all the generators in $\mathcal{M}_U$ is almost impossible in this case: to every degree we need to add several generators to get the isomorphism in cohomology, but in this way we improve the number of generators needed.

Let us denote with $\mathcal{M}^n$ the subalgebra of $\mathcal{M}$ generated by all generators of $\mathcal{M}$ of degree n. Then $\mathcal{M}_U^1 = \{0\}$, $\mathcal{M}_U^2 = (\bigwedge(x_2, y_2, z_2, t_2), 0)$ and for any $n > 2 \quad \mathcal{M}_U^n$ can be computed by induction (Theorem 1.12).
Then the minimal model of the solvmanifold is

$$\mathcal{M} = \quad (\bigwedge(u_1, A, \mathcal{M}_U), D),$$

but we can not describe D in this case.

Now we consider all the solvmanifolds of kind $G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}$ with $r \in \mathbb{Z}$ for which we were able to compute the cohomology groups using Proposition 4.1.

$G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\pi}$ with r even and $p \neq 0$:

$$U = \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_4, \beta_7), d),\ dx = 0, d\beta = x^2$$

then the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_4, \beta_7), D)$,
$Du = DA = Dx = 0,\ D\beta = x^2$.

It is formal as in the case of $G_{6.8}^{p=0}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$.

$G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}$ with $r \equiv 2 \mod 4$:

$p = 0$ and $U = \begin{cases} \langle \alpha^{12}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases}$ then

$\mathcal{M}_U = (\bigwedge(x_2, y_2, \beta_3, \gamma_3), d),\ dx = dy = 0, d\beta = x^2, d\gamma = y^2$ and using the cohomology we find that the model of the solvmanifold is

$\mathcal{M} = (\bigwedge(u_1, A, x_2, y_2\beta_3, \gamma_3), D),\ Du = DA = Dx = Dy = 0, D\beta = x^2,\ D\gamma = y^2$.

Again we have formality as in the case of $G_{6.8}^{p=0}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$.

$G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{3}}$:

• $r \equiv 2 \mod 6$:

$$\text{if } p \neq 0 \quad U = \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_4, \beta_7), d),\ dx = 0, d\beta = x^2$$

then the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(u_1, A, x_4, \beta_7), D)$, $Du = DA = Dx = 0,\ D\beta = x^2$.

It is formal as in the case of $G_{6.8}^{p=0}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$.

if $p = 0 \quad U = \begin{cases} \langle \alpha^{12}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases}$ then

$\mathcal{M}_U = (\bigwedge(x_2, y_2, \beta_3, \gamma_3), d),\ dx = dy = 0, d\beta = x^2, d\gamma = y^2$ and using the cohomology we find that the model of the solvmanifold is

$\mathcal{M} = (\bigwedge(u_1, A, x_2, y_2\beta_3, \gamma_3), D),\ Du = DA = Dx = Dy = 0, D\beta = x^2,\ D\gamma = y^2$.

Again we have formality as in the case of $G_{6.8}^{p=0}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$.

• $r \equiv 3 \mod 6$: $p = 0$ and we have the same computation of the case $t = \frac{\pi}{2}$ with $r \equiv 2 \mod 4$.

• $r \equiv 4 \mod 6$: we have the same computation of the case $r \equiv 2 \mod 6$.

$G_{5.17}^{0,0,r} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}$ with $r = 1, k = 3$ or $r = 3, k = 8$ or $r = 5, k = 12$:

$$\text{if } p \neq 0 \quad U = \begin{cases} \langle \alpha^{13} + \alpha^{24}, \alpha^{14} - \alpha^{23} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases}$$

$$\text{if } p = 0 \quad U = \begin{cases} \langle \alpha^{12}, \alpha^{13} + \alpha^{24}, \alpha^{14} - \alpha^{23}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

The computation of the minimal model $\mathcal{M}_U$ is complicated, in particular we have no generators in degree 1 and

$\mathcal{M}_U^2 = (\bigwedge(x, y), 0)$ for $p \neq 0$,

$\mathcal{M}_U^2 = (\bigwedge(x, y, z, t), 0)$ for $p = 0$,

and for any $n > 2$ $\quad \mathcal{M}_U^n$ can be computed by induction (Theorem 1.12).

In both cases we have $\mathcal{M} = (\bigwedge(u_1, A, \mathcal{M}_U), D),\ Du = DA = 0,\ D|_{\mathcal{M}_U} \equiv d$.

To study formality of this solvmanifold we use the following theorem.

Theorem 6.3. *[15] Let M be a connected and orientable compact differentiable manifold of dimension $2n$, or $2n-1$. Then M is formal if and only if is $(n-1)$-formal.*

We can apply this theorem to the c.d.g.a. $\mathcal{M}_U$ because the manifold M in the hypothesis can be replaced by a real c.d.g.a. $\mathcal{A}$ with the following properties:

- $H^0(\mathcal{A}) = \mathbb{R}$;
- for any $i > \dim(\mathcal{A}) \quad H^i(\mathcal{A}) = 0$;
- $H^{\dim(M)-i}(\mathcal{A}) \cong H^i(\mathcal{A})$ (Poincaré duality).

$\mathcal{M}_U$ has dimension 4 and it has these three characteristics, so to prove that it is formal we must only prove that it is 1-formal. In particular in this case $\mathcal{M}_U$ is simply connected because $U^1 = \{0\}$, so it is 1-formal and then the theorem states that it is always formal.

Now we use formality of $(\mathcal{M}_U, d)$ to study formality of the model of the solvmanifold $(\mathcal{M}, D)$: since $\mathcal{M}$ has differential D such that $D|_{\mathcal{M}_U} \equiv d$, then it is obviously formal.

$G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$:

$U = H^*(\mathfrak{n})$ then we have the same computation of $G^0_{5.14}$ with $t = 2\pi$, and again we can not make a choice.

$G^0_{5.18} \times \mathbb{R}/\Gamma_{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}}$:

$$U = \begin{cases} \langle \alpha^{12}, \alpha^{13} + \alpha^{24}, \alpha^{14} - \alpha^{23}, \alpha^{34} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases},$$

then U has the same model of the case last case of $G_{5.17}$.

Again we have different choices for the model of the solvmanifold, but the right one is

$$\mathcal{M} = \quad (\bigwedge(u_1, A, \mathcal{M}_U), D), \quad Du = DA = Dx = Dy = 0,\ Dz = Ax,\ Dt = Ay,$$
$$D|_{\mathcal{M}^n_U} \equiv d\ \forall n > 2\,.$$

Again it is not formal.

$G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$:
$U = H^*(\mathfrak{n})$, then $\mathcal{M}_U = (\bigwedge(x_1, y_1), 0)$.

The minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(w_1, v_1, u_1, A, x_1, y_1), D)$, but we can not describe D.

$G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}$:

$$U = \langle \alpha^{12} \rangle = H^2(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\bigwedge(x_2, \beta_3), d),\ dx = 0, d\beta = x^2$$

then the minimal model of the solvmanifold is $\mathcal{M} = (\bigwedge(w_1, v_1, u_1, A, x_2, \beta_3), D)$, $Dw = Dv = Du = DA = Dx = 0$, $D\beta = x^2$ and it is formal.

6.2.1 Formality and symplectic structures on almost abelian solvmanifolds

The previous computations shows that this method can be used to find the minimal model of an almost abelian solvmanifold if we know its cohomology groups.

In [36] the aim of this method was to find the cohmology groups of an almost abelian solvmanifold by computing its minimal model and then its cohomology groups that by Definition 1.19 are isomorphic to those of the solvmanifold.

Because of the different possible choices that we usually have to compute the model of the solvmanifold, we want to use this idea to find properties of the solvmanifold related to formality and symplectic structures.

Proposition 6.5. *$(\mathcal{M}_U, d)$ is always formal.*

Proof. We use the definition of formality given in Definition 1.21: $\mathcal{M}_U = \bigwedge Y$ with $Y = C \oplus N$, $d(C) = 0$ and d is injective on N.

We observe that Proposition 6.4 implies that if $m \in \mathcal{M}_U$ such that $dm \neq 0$, then $\rho(m) = 0$ with ρ the cdga homomorphism that defines the minimal model. Indeed we never need to kill a class that is sent by ρ^* in a product in $H^*(\mathbb{R}^n)$ that is not in U, we only have to consider products that are zero in $H^*(\mathbb{R}^n)$ and by Remark 6.2 that are zero in $\bigwedge^*(\mathbb{R}^n)$.

In particular this means that for every generator $y \in N$ we have $\rho(y) = 0$. Suppose that there exists a closed element n in $\mathcal{M}_U$ that lies in $I = \bigwedge V \cdot N$, then $dn = 0$ and $n = \sum_i n_1^i \cdot n_2^i$ with at least one of the two factors in N. If for example $n_2^i \in N$, then $\rho(n_2^i) = 0$ and so $\rho(n) = \sum_i \rho(n_1^i) \cdot \rho(n_2^i) = 0$. This implies that $\rho^*([n]) = 0$, then to keep the isomorphism in cohomology also $[n] = 0$, i.e. n is exact.

Then by Definition 1.21 $(\mathcal{M}_U, d)$ is formal. □

Remark 6.3. We observe that as in [44] Sullivan used that the product of harmonic forms is harmonic to prove the formality of a Riemannian manifold [14, 24], we used only Proposition 6.4 to prove that $\mathcal{M}_U$ is formal.

Now consider the minimal model $(\mathcal{M}, D)$ of the solvmanifold S.
By definition $DA = 0$ and

$$\forall x \in Y \ Dx = \langle \begin{array}{ll} dx & \text{or} \\ dx + yA & \text{with } y \in \Lambda Y^{<x} \end{array} . \tag{6.2}$$

A generic element in $(\mathcal{M}, D)$ has form $s = x + yA$ with $x, y \in \mathcal{M}_U$, then s is closed if and only if $Dx + Dy \cdot A = 0$.

Suppose $Dx = dx + x'A$ and $Dy = dy + y'A$ (x' and y' can be also zero and we will use this notation from now on), then

$$Ds = dx + (x' + dy)A = 0 \text{ if and only if } \begin{cases} dx = 0 \\ x' + dy = 0. \end{cases} \tag{6.3}$$

If s is also exact, i.e. there exists $r = p + qA$ with p and $q \in \mathcal{M}_U$ such that $Dr = s$, then $\begin{cases} x = dp \\ y = p' + dq \end{cases}$

Definition 6.2. A cdga $\mathcal{A}$ is of *k-finite type* if $\forall i \leq k$ $\mathcal{A}^i$ is a finite dimensional vector space.

Remark 6.4. Obviously $\mathcal{M}$ is of k-finite type if and only if $\mathcal{M}_U$ is of k-finite type.

We can now prove results about the formality:

Theorem 6.4. *If $\mathcal{M}$ is of k-finite type and S is k-formal then*

$$\ker D_i|_{\mathcal{M}_U} = \ker d_i \quad \forall i \leq k,$$

where with d_i we mean $d|_{\mathcal{M}_U^i}$.

Proof. Suppose that for some $i \leq k$ $\ker D_i|_{\mathcal{M}_U} \subsetneq \ker d_i$, then there exists $x \in \mathcal{M}_U^i$ such that $dx = 0$, but $Dx \neq 0$. This means for (6.2) that $Dx = yA$ with $0 \neq y \in \mathcal{M}_U^{<x}$, then $D(Ax) = 0$ and $x \in N^i$, so $Ax \in I_k$ is closed.

If it is not exact, then $\mathcal{M}$ is not k-formal, otherwise there exists an element of degree i $x^1 \in \mathcal{M}^{>x}$ such that $Dx^1 = Ax$, then $x^1 \in N^i$ and again $Ax^1 \in I_k$ is closed. If it is not exact $\mathcal{M}$ is not k-formal, otherwise there exists another element of degree i $x^2 \in \mathcal{M}^{>x^1>x}$ such that $Dx^2 = Ax^1$ and so on, but $\mathcal{M}$ is of k-finite type, then exists $p \in \mathbb{N}$ such that $D(Ax^p) = 0$ not exact and so $\mathcal{M}$ is not k-formal.

□

We also have a sufficient condition to formality:

Proposition 6.6. *If $D_i|_{\mathcal{M}_U} = d_i \quad \forall i \leq k$, then S is k-formal.*

Proof. If $D_i|_{\mathcal{M}_U} = d_i \ \forall i \leq k$, then $\mathcal{M}^i = \mathcal{M}_U^i \oplus \mathcal{M}_U^{i-1} \wedge \langle A \rangle \ \forall i \leq k$, then by Proposition 6.5 $\mathcal{M}$ is k-formal. □

Remark 6.5. We observe that in all the examples above if $D_i|_{\mathcal{M}_U} \neq d_i$ for some i, then in particular $\ker D_i|_{\mathcal{M}_U} \subsetneq \ker d_i$, then with these two results we have a good description of formality of almost abelian solvmanifolds.

Example 6.1. Let consider the almost abelian Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_8}} \mathbb{R}^7$ of dimension 8 defined by

$$\mathrm{ad}_{X_8} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -p \end{pmatrix} \qquad p \neq 0.$$

The map on the Lie group G is

$$\exp(t\,\mathrm{ad}_{X_8}) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{pt} & te^{pt} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{pt} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-pt} & te^{-pt} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-pt} \end{pmatrix},$$

then choosing $p \in \mathbb{R}$ and $\bar{t} \in \mathbb{R}$ such that $e^{p\bar{t}} + e^{-p\bar{t}} \in \mathbb{Z}$ we can prove that its characteristic polynomial has integer coefficients and that this matrix is conjugate to the integer matrix A (see Chapter 4). Then by Proposition 1.3 $\Gamma_{\bar{t}}$ is a lattice and $S = G/\Gamma_{\bar{t}}$ is an almost abelian solvmanifold.

G is completely solvable, then by Hattori theorem $H^*(S) \cong H^*(\mathfrak{g})$. In particular we have $H^1(S) = \langle \alpha^3, \alpha^8 \rangle$.

To study the formality of S we do not need to compute all the module U or the minimal model $\mathcal{M}_U$:

we just compute U^1 using Proposition 6.3:

$$\begin{aligned}
\varphi_s(\alpha^1) &= \alpha^1, \\
\varphi_s(\alpha^2) &= \alpha^2, \\
\varphi_s(\alpha^3) &= \alpha^3, \\
\varphi_s(\alpha^4) &= e^{pt}\alpha^4, \\
\varphi_s(\alpha^5) &= e^{pt}\alpha^5, \\
\varphi_s(\alpha^6) &= e^{-pt}\alpha^6, \\
\varphi_s(\alpha^7) &= e^{-pt}\alpha^7.
\end{aligned}$$

Then $U^1 = \langle \alpha^1, \alpha^3, \alpha^3 \rangle$ and in particular $\mathcal{M}_U^1 = (\bigwedge(x,y,z), 0)$, but $H^1(S) = \langle \alpha^3, \alpha^8 \rangle$, then

$$\mathcal{M}^1 = (\bigwedge(A,x,y,z), D) \text{ with } DA = Dx = 0, Dy = xA,\ Dz = yA$$

and so for Theorem 6.4 S is not 1-formal.

Now we analyse how this method allows us to find symplectic structures on almost abelian solvmanifolds.

Suppose that $S = \mathbb{R} \ltimes \mathbb{R}^{2n-1}/\mathbb{Z} \ltimes \mathbb{Z}^{2n-1}$ has dimension $2n$ and is endowed with a symplectic structure ω. We denote with $\{\alpha^1, \cdots, \alpha^{2n-1}\}$ the basis of $\bigwedge^1(\mathbb{R}^{2n-1})$ and with $\{\alpha^{2n}\}$ the basis of $\bigwedge^1(\mathbb{R})$.

The concept of symplectic structure can be transferred in odd dimension.

Definition 6.3. If M is a $(2n-1)$-dimensional manifold a *co-symplectic structure* on M is a couple (F, η) where F is a 2-form, η is a 1-form on M, both are closed and $F^{n-1} \wedge \eta \neq 0$.

For a complete study of co-symplectic structures see [6].

We call a *co-symplectic structure on* U a co-symplectic structure (F, η) on $\mathbb{R}^{2n-1}$ such that $[F], [\eta] \in U$. Observe that every form on $\mathbb{R}^{2n-1}$ is closed, so the only necessary condition to get this structure is the non-degeneracy.

Let (F, η) be a co-symplectic structure on U. This means that

$$F := \sum_{1\leq i<j\leq 2n-1} a_{ij}\alpha^{ij}, \quad \eta := \sum_{1\leq k\leq 2n-1} b_k\alpha^k$$

$[F], [\eta] \in U$ and $F^{n-1} \wedge \eta \neq 0$.

Now consider the minimal model $\mathcal{M}$ of S. If A is the generator we add to U from $\bigwedge^*(\mathbb{R})$, then with the notation of Theorem 6.2 we have $\sigma(A) = \alpha^{2n}$ and then also

$$\begin{array}{rcl} \tau : \mathcal{M} & \to & \bigwedge^*(S) \\ A & \mapsto & \alpha^{2n} \\ \mathcal{M}_U & \mapsto & \rho(\mathcal{M}_U) \subset \bigwedge^*(\mathbb{R}^{2n-1}) \end{array}$$

$[F], [\eta] \in U$ then there exist $x \in \mathcal{M}_U^2$ and $y \in \mathcal{M}_U^1$ such that $\rho^*([x]) = [F]$ and $\rho^*([y]) = [\eta]$.
But in $U \subset H^*(\mathbb{R}^{2n-1})$ we do not have exact forms. So $\rho(x) = F$ and $\rho(y) = \eta$.

Therefore $dx = dy = 0$ and if $s := x + yA \in \mathcal{M}^2$, $Ds = Dx = x'A$.

$$s^n = (x + yA)^n = \sum_{p=0}^{n} \binom{n}{p} x^{n-p}y^pA^p = x^n + nx^{n-1}yA$$

because both y and A have odd degree and then their powers are 0. But

$$\rho(x^{n-1}y) = (\rho(x))^{n-1}\rho(y) = F^{n-1} \wedge \eta \neq 0,$$

then $x^{n-1}y \neq 0$ in $\mathcal{M}_U$ and so $x^{n-1}yA \neq 0$ in $\mathcal{M}$.

$x^n \in \mathcal{M}_U$, then $x^n \neq -nx^{n-1}yA \in \mathcal{M}$, then $s^n \neq 0$ in $\mathcal{M}$.

In particular $\omega := \tau(s) = \tau(x) + \tau(y)\tau(A) = F + \eta \wedge \alpha^{2n}$ is a 2-form on S and

$$\omega^n = \tau(s^n) = (\tau(x))^n + n(\tau(x))^{n-1}\tau(y)\tau(A) = F^n + nF^{n-1} \wedge \eta \wedge \alpha^{2n}.$$

$F^n = 0$ because it is in $\bigwedge(\alpha^1, \cdots, \alpha^{2n-1})$ and $F^{n-1} \wedge \eta \neq 0$ by hypothesis, then also $\omega^n = nF^{n-1} \wedge \eta \wedge \alpha^{2n} \neq 0$.
Since $d\omega = \tau(Ds)$ by Definition 1.18, if $x' = 0$, ω is closed and we have a symplectic structure on S.

We have then proved the following proposition:

Proposition 6.7. *If* $\ker D_2|_{\mathcal{M}_U} = \ker d_2$ *and there exists a co-symplectic structure* (F, η) *on* U*, then there exists a symplectic structure* $\omega := F + \eta \wedge \alpha^{2n}$ *on* S*.*

Example 6.2. Let consider $S = G_{6.10}^{a=0}/\Gamma_{\frac{2\pi}{k}}$ studied in Chapter 4.
In this case the generic co-symplectic structure on U is given by

$$F = a_{12}\alpha^{12} + a_{13}\alpha^{13} + a_{23}\alpha^{23} + a_{45}\alpha^{45} \quad \text{and} \quad \eta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$$

$$\text{with } F^2 \wedge \eta \neq 0 \quad \Leftrightarrow \quad 2a_{45}(a_{12}b_3 - a_{13}b_2 + a_{13}b_1) \neq 0.$$

Let $x \in \mathcal{M}_U^2$ and $y \in \mathcal{M}_U^1$ such that $\tau(x) = F$ and $\tau(y) = \eta$, then

$$x = a_{12}zy + a_{13}zx + a_{23}yx + a_{45}t$$

and $y = b_1z + b_2y + b_3x$.

The element $s := x + yA \in \mathcal{M}^2$ is closed if and only if

$$x' = a_{12}zx + a_{13}yx = 0$$

that is if and only if $a_{12} = a_{13} = 0$.

Then if we consider $F = a_{23}\alpha^{23} + a_{45}\alpha^{45}$ and $\eta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$ with $b_1 \neq 0$, $a_{23} \neq 0$, $a_{45} \neq 0$, we have a symplectic structure on S given by $\omega := F + \eta \wedge \alpha^6$.

Remark 6.6. We observe that the symplectic form found in this example is invariant and then listed in Appendix C, but in general this method allows us to find also non-invariant symplectic structures.

6.3 Dolbeault models of Lie algebras

We want to modify the concept of cdga and its minimal models to associate them to Dolbeault cohomology. In this way we can define minimality and formality also in the Dolbeault cohomology case.

Definition 6.4. Let $\mathbb{K}$ ba a field of characteristic 0. A *bigraded* $\mathbb{K}$*-vector space* is a family of $\mathbb{K}$-vector spaces $\mathcal{A} = \{\mathcal{A}^{p,q}\}_{p,q\geq 0}$. An element of $\mathcal{A}$ has degree (p,q) if it belongs to $\mathcal{A}^{p,q}$.

Definition 6.5. A *commutative differential bigraded* $\mathbb{K}$*-algebra, cdba,* $(\mathcal{A}, d)$ is a bigraded $\mathbb{K}$-vector space $\mathcal{A}$ together with a multiplication

$$\mathcal{A}^{p,q} \otimes \mathcal{A}^{r,s} \to \mathcal{A}^{p+r,q+s}$$

that is associative, with unit $1 \in \mathcal{A}^{0,0}$ and commutative in the graded sense, i.e. $\forall a \in \mathcal{A}^{p,q}, b \in \mathcal{A}^{r,s} \quad a \cdot b = (-1)^{(p+q)\cdot(r+s)} b \cdot a$, and with a differential of *bidegree* $(0,1)$ $d : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ such that $d^2 = 0$ and

$$\forall a \in \mathcal{A}^{p,q}, b \in \mathcal{A}^{r,s} \quad d(a \cdot b) = da \cdot b + (-1)^{p+q} a \cdot db. \tag{6.4}$$

We observe that given a cdba $(\mathcal{A}, d)$ also its cohomology algebra is a cdba $(H^{*,*}(\mathcal{A}), 0)$.

Example 6.3. The Dolbeault complex of complex manifolds and Lie algebras endowed with a complex structure are $\mathbb{C}$-cdba's.

Definition 6.6. A *cdba morphism* $f : (\mathcal{A}, d) \to (\mathcal{B}, d)$ is a family of homomorphisms $f : \mathcal{A}^{p,q} \to \mathcal{B}^{p,q}$ such that $df = fd$ and $f(a \cdot b) = f(a) \cdot f(b)$.

Suppose that a cdba $(\mathcal{A}^{*,*}, d_1)$ is endowed also with another differential of bidegree $(1,0)$

$$d_2 : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q},$$

then $(\mathcal{A}, d = d_1 + d_2)$ is a cdga and for these kind of cdba's $(\mathcal{A}^{*,*}, d_1, d_2)$ we can state the $\partial\bar{\partial}$-Lemma (Lemma 1.1).

We can now state a rational homotopy version of Theorem 1.13:

Theorem 6.5. *If for the cdba* $(\mathcal{A}, d_1, d_2)$ *the* $\partial\bar{\partial}$*-Lemma holds, then* $(\mathcal{A}, d_1 + d_2)$ *is a formal cdga.*

We consider now minimality of cdba's:

Definition 6.7. [10] A cdba $(\mathcal{M}^{*,*}, d)$ is *minimal* if it is free commutative, i.e. $\mathcal{M}^{*,*} = \bigwedge^{*,*} V$ with V bigraded vector space, and there exist a ordered basis $\{x_\alpha\}$ of V such that $dx_\alpha \in \bigwedge^{*,*}(x_\beta)_{\beta<\alpha}$ and the total degree is respected, $|x_\beta| \leq |x_\alpha|$.
A *minimal model* of the cdba $(\mathcal{A}^{*,*}, d)$ is a minimal cdba $(\mathcal{M}^{*,*}, d)$ together with a cdba quasi-isomorphism $\varphi : (\mathcal{M}, d) \to (\mathcal{A}, d)$.

Observe that obviously if the cdba is minimal, it is itself its minimal model with a quasi-isomorphism the identity map.

It is well known that for every path connected cdga there exists a minimal model (Theorem 1.12), but it is not true in general for cdba, i.e. in the bigraded case.
The problem comes from the fact that d has bidegree $(0, 1)$, indeed if we compute a cdba minimal model following the usual construction of models for cdga in some cases we cannot proceed.
We recall how it goes for cdga's: to compute a cdga minimal model $\mathcal{M}_\mathcal{A}$ we start considering the first cohomology group of the cdga $\mathcal{A}$ that is not trivial $H^i = \langle [a_1], \cdots, [a_k] \rangle$ and taking a number of generators of that degree equal to the dimension of the group, $\mathcal{M}^i_\mathcal{A} = \bigwedge(x_1, \cdots, x_k)$ with $dx_j = 0 \; \forall j = 1 \cdots k$ and

$$\begin{aligned} \psi : \mathcal{M}_\mathcal{A} &\to \mathcal{A} \\ x_j &\mapsto a_j \end{aligned}$$

Then we consider all the products and powers of these generators and we check the cohomology classes that they generate. If these classes are sent by ψ^* in classes not zero in $H^*(\mathcal{A})$, then we have a quasi isomorphism, otherwise we have to "kill" these classes to maintain the cohomology isomorphism and then we add new generators to make these elements exact.
We continue considering the following not zero cohomology group and adding the number of generators in $\mathcal{M}_\mathcal{A}$ needed to have the cohomology isomorphism also in this dimension and then again we check powers and products and so on for every cohomology group.

If now we apply the same idea to compute the minimal model $\mathcal{M}$ of a cdba $\mathcal{A}$ the only obstruction appear when we have to "kill" a cohomology class in $H^{p,0}(\mathcal{M})$.

Indeed in this case we have a closed element $x_{p,0} \in \mathcal{M}$ and we want to make it exact, but we cannot add $y \in \mathcal{M}$ such that $dy = x$ because of the degree of x, since d has bidegree $(0,1)$, so we cannot have the cohomology isomorphism and then we cannot have a minimal model.

In particular this can happen in the computation of the Dolbeault minimal model of a complex manifold M or a Lie algebra $\mathfrak{g}$ endowed with a complex structure J. Suppose to have a closed generator $x_{p,0} \in \mathcal{M}$ with p even, then for every power r we have total degree $|x_{p,0}^r| = rp$ even and then $x_{p,0}^r \neq 0$. But if $\psi(x) = \alpha \in \bigwedge^{*,*}(\mathfrak{g}^{\mathbb{C}*})$, then must exist a power r such that $\alpha^r = 0$ and then we have to "kill" $[x_{p,0}^r]$ but it is not possible.

Example 6.4. Let $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_{2n}}} \mathbb{R}^{2n-1}$ with

$$\mathrm{ad}_{X_{2n}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & 0 & 0 \\ 0 & -c_1 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{2n-2} & c_{2n-2} \\ 0 & 0 & 0 & 0 & -c_{2n-2} & b_{2n-2} \end{pmatrix}$$

and $b_j + ic_j \neq b_k + ic_k$ for $j \neq k$.

Then by Theorem 5.3 a complex structure J is given by $\forall\, j = 1, \cdots, 2n-2$

$$JX_1 = \varphi_1 X_1 + \varphi_n X_{2n},\; JY_{2j-1} = Y_{2j},\; JY_{2j} = -Y_{2j-1},\; JX_{2n} = \frac{-1-\varphi_1^2}{\varphi_n} X_1 - \varphi_1 X_{2n}$$

A basis for $\mathfrak{g}^{1,0}$ is $\{X_1 - i(\varphi_1 X_1 + \varphi_n X_{2n}), Y_{2j-1} - iY_{2j}\}_{j=1,\cdots,2n-2}$, then if $\{\eta, \omega^j\}_{j=1,\cdots,2n-2}$ is the dual basis of $\mathfrak{g}^{1,0*}$ we have in particular that

$$\bar{\partial}\omega^j = \frac{\varphi_1 b_j - c_j + i(\varphi_1 c_j + b_j)}{2\varphi_n} \omega^j \wedge \bar{\eta}.$$

Then

$$\bar{\partial}(\omega^j \wedge \omega^k) = \frac{\varphi_1(b_j + b_k) - (c_j + c_k) + i(\varphi_1(c_j + c_k) + (b_j + b_k))}{2\varphi_n} \omega^j \wedge \omega^k \wedge \bar{\eta}$$

that for $b_k = -b_j$ and $c_k = -c_j$ is zero.

In this case to compute the Dolbeault minimal model we would need to add a generator $x_{2,0}$ with $\psi(x) = \omega^j \wedge \omega^k$ which is impossible. Thus the model does not exists.

6.3.1 Minimality and Formality of nilpotent Lie algebras

Next we consider nilpotent Lie algebras. We already observed in Chapter 5 that the properties about the isomorphism between the de Rham cohomology of a solvmanifold and its invariant one do not hold in general for the Dolbeault cohomology.

The only theorem that has a complex version is the Nomizu theorem [7, 9, 41]:

Definition 6.8. [41] Let $\mathfrak{g}$ be a nilpotent Lie algebra.

- A *rational structure* for $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_{\mathbb{Q}}$ defined over the rational such that $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{g}$.
- A complex structure J on $\mathfrak{g}$ is a *rational complex structure* if it maps $\mathfrak{g}_{\mathbb{Q}}$ into itself.

Theorem 6.6. *[7, Theorem 2] Let $N = G/\Gamma$ be a nilmanifold endowed with an invariant complex structure J. If J is rational then $H^{*,*}_{\bar{\partial}}(\mathfrak{g}) \cong H^{*,*}_{\bar{\partial}}(N)$.*

There are few examples of nilpotent Lie algebras that are not endowed with a rational complex structure, moreover we have a more general result.

Theorem 6.7. *[41, Theorem 1.10] Let $N = G/\Gamma$ be a nilmanifold endowed with an invariant complex structure J. The inclusion $i : H^{*,*}_{\bar{\partial}}(\mathfrak{g}) \to H^{*,*}_{\bar{\partial}}(N)$ is an isomorphism if*

- *the complex structure J is bi-invariant, G is a complex Lie group and N is complex parallelizable [42];*
- *the complex manifold N is an iterated principal holomorphic torus bundle [9];*
- *the complex structure J is rational [7, Theorem 2];*
- *the complex structure J is abelian [7];*

- $\mathfrak{g}$ *admits a torus bundle series for* J *compatible with the rational structure induced by* Γ.

Moreover, there is a dense open subset U *of the space of all invariant complex structures on* N *such that* i *is an isomorphism for all* $J \in U$ *[7, Theorem 1].*

Because of these theorems the Dolbeult minimal model of a nilpotent Lie algebra is often also the Dolbeault model of a nilmanifold and then the study of these models is quite interesting.

Suppose that $\mathfrak{g}$ is a nilpotent Lie algebra, then the following theorem holds [43]:

Theorem 6.8. *A nilpotent Lie algebra* $\mathfrak{g}$ *of dimension* n *admits a complex structure if and only if* $\mathfrak{g}^{\mathbb{C}*}$ *has a basis* $(\omega^1, \cdots, \omega^n, \bar{\omega}^1, \cdots, \bar{\omega}^n)$ *such that* $d\omega^i \in I(\omega^1, \cdots, \omega^{i-1})$.

In particular this means that there exist a basis $(\omega^1, \cdots, \omega^n)$ of $\mathfrak{g}^{1,0*}$ and constants $a^i_{j,k}, b^i_{j,k} \in \mathbb{C}$ such that

$$d(\omega^i) = \sum_{1\le j<i,\ 1\le k\le n} [\omega^j \wedge (a^i_{jk}\omega^k + b^i_{jk}\bar{\omega}^k)]$$

or equivalently

$$\begin{cases} \bar{\partial}(\omega^i) = \sum_{1\le j<i,\ 1\le k\le n} b^i_{jk}\omega^j \wedge \bar{\omega}^k \\ \\ \bar{\partial}(\bar{\omega}^i) = \overline{\partial\omega^i} = \sum_{1\le j<i,\ 1\le k\le n} \bar{a}^i_{jk}\bar{\omega}^j \wedge \bar{\omega}^k \end{cases}$$

Definition 6.9. [10] A complex structure on a nilpotent Lie algebra is *nilpotent* if there exist a basis $(\omega^1, \cdots, \omega^n)$ of $\mathfrak{g}^{1,0*}$ such that

$$d\omega^i \in \bigwedge(\omega^1, \cdots, \omega^{i-1}, \bar{\omega}^1, \cdots, \bar{\omega}^{i-1}).$$

In particular this means that $\forall k \ge i \quad a^i_{jk} = b^i_{jk} = 0$ and then $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}, \bar{\partial})$ is a minimal cdba with both the ordered basis $(\omega^1, \bar{\omega}^1, \cdots, \omega^n, \bar{\omega}^n)$ and $(\bar{\omega}^1, \cdots, \bar{\omega}^n, \omega^1, \cdots, \omega^n)$.

Particular cases of nilpotent complex structures are the bi-invariant complex structures, i.e. $b^i_{jk} = 0$ also for $k < i$ and the abelian complex structures, i.e. $a^i_{jk} = 0$ also for $k < i$.

In [10] the authors proved that if a nilpotent Lie algebra $\mathfrak{g}$ is endowed with a nilpotent complex structure, then the cdba $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}, \bar\partial)$ is minimal, similarly to the de Rham case.

We want to improve this result: if we ask that only $a^i_{jk} = 0 \quad \forall k \geq i$, then again we get minimality with the order $(\bar\omega^1, \cdots, \bar\omega^n, \omega^1, \cdots, \omega^n)$ and more in general if there exist an order $i_1 < \cdots < i_n$ for which the cdga $(\bigwedge^* \mathfrak{g}^{0,1*}, \bar\partial)$ is minimal with the ordered basis $(\bar\omega^{i_k})_{k=1,\cdots,n}$, then considering the basis $(\bar\omega^{i_1}, \cdots, \bar\omega^{i_n}, \omega^1, \cdots, \omega^n)$ we get minimality for the cdba $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}, \bar\partial)$. This idea can be further generalized.

Theorem 6.9. *Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra with complex structure J, then there exists an ordered basis for which the cdba $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}, \bar\partial)$ is minimal.*

Proof. Because of Theorem 6.8 we know that if there exists a basis $(\bar\eta^1, \cdots, \bar\eta^n)$ of $\mathfrak{g}^{0,1*}$ for which $(\bigwedge^* \mathfrak{g}^{0,1*}, \bar\partial)$ is a minimal cdga, then $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}, \bar\partial)$ is a minimal cdba with respect to the basis $(\bar\eta^1, \cdots, \bar\eta^n, \omega^1, \cdots, \omega^n)$ where $(\omega^1, \cdots, \omega^n)$ is the basis of $\mathfrak{g}^{1,0*}$ used in Theorem 6.8.

We recall that on a nilpotent Lie algebra $\mathfrak{g}$ we can always find a basis $(X_1, \cdots, X_n)$ such that the structure constants are $[X_i, X_j] = \sum_{i<j<p} c^p_{ij} X_p$, or equivalently a basis $(\alpha^1, \cdots, \alpha^n)$ of $\mathfrak{g}^*$ such that $d\alpha^p = -\sum_{i<j<p} c^p_{ij}\alpha^{ij}$. In particular this implies that if $\mathfrak{g}$ is a nilpotent Lie algebra, then $(\bigwedge^* \mathfrak{g}^*, d)$ is a minimal cdga [20] (Section 6.1). Then we need to prove that $\mathfrak{g}^{0,1*}$ is nilpotent.

From Theorem 6.8 we know that $\bar\partial(\bar\omega^i) = \sum_{1\leq j<i,\ 1\leq k\leq n} \bar a^i_{jk}\bar\omega^j \wedge \bar\omega^k$.
Let consider the Lie algebra $\mathfrak{h}$ such that in the cdga $(\bigwedge^* \mathfrak{h}^*, d)$ we have

$$d\alpha^i = \sum_{1\leq j<i,\ 1\leq k\leq n} \bar a^i_{jk}\alpha^j \wedge \alpha^k$$

for a given basis $(\alpha^1, \cdots, \alpha^n)$ of $\mathfrak{h}^*$, then the cdga's $(\bigwedge^* \mathfrak{g}^{0,1*}, \bar\partial)$ and $(\bigwedge^* \mathfrak{h}^*, d)$ are isomorphic.

$d\alpha^i = \sum_{1\leq j<i,\ 1\leq k\leq n} \bar a^i_{jk}\alpha^j \wedge \alpha^k$ is equivalent to $[X_k, X_j] = -[X_j, X_k] = \sum_{i>j} \bar a^i_{jk} X_i$ for the dual basis $(X_1, \cdots, X_n)$ of $\mathfrak{h}$. Then the endomorphism ad_{X_k} of $\mathfrak{h}$ is associated to the strictly triangular matrix

$$\begin{pmatrix} 0 & & & & \\ a^2_{1k} & \ddots & & & \\ \vdots & a^3_{2k} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ a^n_{1k} & a^n_{2k} & \cdots & a^n_{n-1k} & 0 \end{pmatrix}$$

that is that ad_{X_k} is a nilpotent endomorphism for every k.

Theorem 1.2 implies then that $\mathfrak{h}$ is a nilpotent Lie algebra and so $(\bigwedge^* \mathfrak{h}^*, d)$ and $(\bigwedge^* \mathfrak{g}^{0,1*}, \bar{\partial})$ are minimal cdga's. □

Remark 6.7. Because of Theorems 6.6 and 6.7 we have that if N is a complex nilmanifold, there exist J invariant complex structure over N such that $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \bar{\partial})$ is the Dolbeault minimal model of (N, J).

Example 6.5.

dim 6: The only not nilpotent complex structure on a Nilpotent Lie algebra of dimension 6 is given by the following basis of $\mathfrak{g}^{1,0*}$ [48]

$$\begin{cases} d\omega_1 = 0 \\ d\omega_2 = E\omega_1 \wedge \omega_3 + \omega_1 \wedge \bar{\omega}_3 \\ d\omega_3 = A\omega_1 \wedge \bar{\omega}_1 + ib\omega_1 \wedge \bar{\omega}_2 - ib\bar{E}\omega_2 \wedge \bar{\omega}_1 \end{cases}$$

where $A, E \in \mathbb{C}$ with $|E| = 1$ and $b \in \mathbb{R} \setminus \{0\}$.

In particular we have

$$\begin{cases} \bar{\partial}\omega_1 = 0 \\ \bar{\partial}\omega_2 = \omega_1 \wedge \bar{\omega}_3 \\ \bar{\partial}\omega_3 = A\omega_1 \wedge \bar{\omega}_1 + ib\omega_1 \wedge \bar{\omega}_2 - ib\bar{E}\omega_2 \wedge \bar{\omega}_1 \\ \bar{\partial}\bar{\omega}_1 = 0 \\ \bar{\partial}\bar{\omega}_2 = \bar{E}\bar{\omega}_1 \wedge \bar{\omega}_3 \\ \bar{\partial}\bar{\omega}_3 = 0 \end{cases}$$

Then taking the order $(\bar{\omega}^1, \bar{\omega}^3, \bar{\omega}^2, \omega^1, \omega^2, \omega^3)$ the cdba $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \bar{\partial})$ is minimal.

dim 8: [41] Let $(\bigwedge^* \mathfrak{g}^*, d)$ be the 8-dimensional cdga with
$d\alpha^1 = 0,\ d\alpha^2 = 0,\ d\alpha^3 = 0,\ d\alpha^4 = 0,\ d\alpha^5 = 0,\ d\alpha^6 = \alpha^{12},\ d\alpha^7 = \alpha^{16} + \alpha^{23}$,
$d\alpha^8 = \alpha^{26} - \alpha^{13}$ and consider the complex structure J given by
$J\alpha^1 = \alpha^2,\ J\alpha^4 = \alpha^6,\ J\alpha^5 = \alpha^3,\ J\alpha^7 = \alpha^8.$
Then J is not nilpotent and the basis of $\mathfrak{g}^{\mathbb{C}*}$ is given by

$$\begin{cases} \bar{\partial}\omega_1 = 0 \\ \bar{\partial}\omega_2 = \omega_1 \wedge \bar{\omega}_2 \\ \bar{\partial}\omega_3 = \frac{1}{2}\omega_1 \wedge \bar{\omega}_1 \\ \bar{\partial}\omega_4 = 0 \\ \bar{\partial}\bar{\omega}_1 = 0 \\ \bar{\partial}\bar{\omega}_2 = 0 \\ \bar{\partial}\bar{\omega}_3 = 0 \\ \bar{\partial}\bar{\omega}_4 = 0 \end{cases}$$

then taking the order $(\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4, \omega^1, \omega^2, \omega^3, \omega^4)$ we get minimality.

dim 10: [41] Let $(\bigwedge^* \mathfrak{g}^*, d)$ be the 10-dimensional cdga with
$d\alpha^1 = 0,\ d\alpha^2 = 0,\ d\alpha^3 = 0,\ d\alpha^4 = 0,\ d\alpha^5 = 0,\ d\alpha^6 = 0,\ d\alpha^7 = 0$,
$d\alpha^8 = \alpha^{15} + \alpha^{16} + \alpha^{35} + \alpha^{36},\ d\alpha^9 = \alpha^{25} + \alpha^{26} + \alpha^{45} + \alpha^{46}$,
$d\alpha^{10} = \alpha^{18} + \alpha^{38} + \alpha^{29} + \alpha^{49}$ and consider the complex structure J
given by $J\alpha^1 = \alpha^2,\ J\alpha^3 = \alpha^4,\ J\alpha^5 = \alpha^7,\ J\alpha^6 = \alpha^{10},\ J\alpha^8 = \alpha^9.$

Then J is not nilpotent and the basis of $\mathfrak{g}^{\mathbb{C}*}$ is given by

$$\begin{cases} \bar\partial\omega_1 = 0 \\ \bar\partial\omega_2 = 0 \\ \bar\partial\omega_3 = 0 \\ \bar\partial\omega_4 = \frac{1}{2}\omega_1 \wedge \bar\omega_3 + \frac{1}{2}\omega_1 \wedge \bar\omega_5 + \frac{1}{2}\omega_2 \wedge \bar\omega_5 \\ \bar\partial\omega_5 = -\frac{i}{2}\omega_1 \wedge \bar\omega_4 + \frac{i}{2}\omega_4 \wedge \bar\omega_1 - \frac{i}{2}\omega_2 \wedge \bar\omega_4 + \frac{i}{2}\omega_4 \wedge \bar\omega_2 \\ \bar\partial\bar\omega_1 = 0 \\ \bar\partial\bar\omega_2 = 0 \\ \bar\partial\bar\omega_3 = 0 \\ \bar\partial\bar\omega_4 = \frac{1}{2}\bar\omega_1 \wedge \bar\omega_3 + \frac{1}{2}\bar\omega_1 \wedge \bar\omega_5 + \frac{1}{2}\bar\omega_2 \wedge \bar\omega_5 \\ \bar\partial\bar\omega_5 = 0 \end{cases}$$

then taking the order $(\bar\omega^1, \bar\omega^2, \bar\omega^3, \bar\omega^5, \bar\omega^4, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ we get minimality.

The definition of formality on cdba is equal to the definition given on cdga:

Definition 6.10. A cdba $(\mathcal{A}^{*,*}, d)$ is *formal* if there exist a cdba morphism

$$\psi : \mathcal{A}^{*,*} \to H^{*,*}(\mathcal{A})$$

that induces the identity on cohomology.

For the real case Hasegawa proved that a nilpotent Lie algebra $\mathfrak{g}$ is formal if and only if it is abelian (Theorem 6.1).

The proof is based on the minimality of $(\bigwedge^* \mathfrak{g}^*, d)$, then we can follow the same idea also for $(\bigwedge^{*,*} \mathfrak{g}^{\mathbb{C}*}, \bar\partial)$.

In particular if J is nilpotent a proof can be found in [10].

Lemma 6.3. $\bar\partial(\bigwedge^{n,n-1} \mathfrak{g}^{\mathbb{C}*}) \equiv 0$.

Proof. Minimality implies that the $\bar\partial$ of a generator is always combination of wedge of two prior generators, but in every element of $\bigwedge^{n,n-1} \mathfrak{g}^{\mathbb{C}*}$ all but one generator appear, i.e. it is generated by elements of kind $\omega^1 \wedge \cdots \wedge \omega^n \wedge \bar\omega^1 \wedge \cdots \wedge \hat{\bar\omega}^i \wedge \cdots \wedge \bar\omega^n$, then we always have wedge of a generator with itself in the $\bar\partial$ and then it must be zero. $\square$

Remark 6.8. Lemma 6.3 implies in particular that $b^{n,n}=1$.

Theorem 6.10. *$(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*},\bar\partial)$ is formal if and only if $\mathfrak{g}$ is abelian.*

Proof. Suppose by contradiction that $\mathfrak{g}$ is not abelian and $(\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*},\bar\partial)$ is formal, then $H^{1,0}(\mathfrak{g}^{\mathbb{C}*})\oplus H^{0,1}(\mathfrak{g}^{\mathbb{C}*})\subsetneq \mathfrak{g}^{\mathbb{C}*}$, but there exists $\psi:\bigwedge^{*,*}\mathfrak{g}^{\mathbb{C}*}\to H^{*,*}(\mathfrak{g}^{\mathbb{C}*})$ such that $\psi^*\equiv Id$.

Let $\{\alpha^1,\cdots,\alpha^k\}$ and $\{\beta^1,\cdots,\beta^h\}$ be respectively a basis of $H^{1,0}(\mathfrak{g}^{\mathbb{C}*})$ and $H^{0,1}(\mathfrak{g}^{\mathbb{C}*})$ and let ψ^1 be $\psi|_{\mathfrak{g}^{\mathbb{C}*}}$, then if $\{\gamma^1,\cdots,\gamma^{2n-h-k}\}$ is a basis of $\ker\psi^1$ we have

$$
\begin{array}{rccl}
\psi^1: & \mathfrak{g}^{\mathbb{C}*} & \to & H^{1,0}(\mathfrak{g}^{\mathbb{C}*})\oplus H^{0,1}(\mathfrak{g}^{\mathbb{C}*})\\
& \alpha^i & \mapsto & [\alpha^i]\\
& \beta^j & \mapsto & [\beta^j]\\
& \gamma^s & \mapsto & 0
\end{array}
$$

Now consider $\bigwedge^{n,n}\mathfrak{g}^{\mathbb{C}*}$: by definition of cdba morphism we have $\psi^1(\alpha^1)\cdots\psi^1(\alpha^h)\cdot\psi^1(\beta^1)\cdots\psi^1(\beta^k)\cdot\psi^1(\gamma^1)\cdots\psi^1(\gamma^{2n-h-k}) =: \psi(\alpha^1\wedge\cdots\wedge\alpha^h\wedge\beta^1\wedge\cdots\wedge\beta^k\wedge\gamma^1\wedge\cdots\wedge\gamma^{2n-h-k}) = [\alpha^1\wedge\cdots\wedge\alpha^h\wedge\beta^1\wedge\cdots\wedge\beta^k\wedge\gamma^1\wedge\cdots\wedge\gamma^{2n-h-k}]$, but $\psi^1(\gamma^i)=0$, then $[\alpha^1\wedge\cdots\wedge\alpha^h\wedge\beta^1\wedge\cdots\wedge\beta^k\wedge\gamma^1\wedge\cdots\wedge\gamma^{2n-h-k}]=0$ that is a contradiction of Lemma 6.3. $\square$

Corollary 6.1. *Let $N=G/\Gamma$ be a nilmanifold endowed with an invariant complex structure J such that $H^{*,*}_{\bar\partial}(\mathfrak{g})\cong H^{*,*}_{\bar\partial}(N)$, then (N,J) is formal if and only if it is a complex torus.*

Appendix A

Six dimensional solvable (non nilpotent) unimodular Lie algebras

Algebra	Structure equations
$\mathfrak{g}_{6.1}^{a,b,c,e}$ $0 < \|e\| \leq \|c\| \leq \|b\| \leq \|a\| \leq 1,$ $a+b+c+e=-1$	$d\alpha^1 = -\alpha^{16},\ d\alpha^2 = -a\alpha^{26},\ d\alpha^3 = -b\alpha^{36}$ $d\alpha^4 = -c\alpha^{46},\ d\alpha^5 = -e\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.2}^{a,c,e}$ $0 < \|e\| \leq \|c\| \leq 1,$ $2a+c+e=-1$	$d\alpha^1 = -a\alpha^{16} - \alpha^{26},\ d\alpha^2 = -a\alpha^{26},\ d\alpha^3 = -\alpha^{36},$ $d\alpha^4 = -c\alpha^{46},\ d\alpha^5 = -e\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.3}^{-\frac{a+1}{3},a}$ $0 < \|a\| \leq 1$	$d\alpha^1 = \frac{a+1}{3}\alpha^{16} - \alpha^{26},\ d\alpha^2 = \frac{a+1}{3}\alpha^{26} - \alpha^{36},$ $d\alpha^3 = \frac{a+1}{3}\alpha^{36},\ d\alpha^4 = -\alpha^{46},$ $d\alpha^5 = -a\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.4}^{-\frac{1}{4}}$	$d\alpha^1 = \frac{1}{4}\alpha^{16} - \alpha^{26},\ d\alpha^2 = \frac{1}{4}\alpha^{26} - \alpha^{36},$ $d\alpha^3 = \frac{1}{4}\alpha^{36} - \alpha^{46},\ d\alpha^4 = \frac{1}{4}\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.6}^{a,b}$ $a \leq b, \quad a+b = -\frac{1}{2}$	$d\alpha^1 = -\alpha^{16},\ d\alpha^2 = -a\alpha^{26} - \alpha^{36},\ d\alpha^3 = -a\alpha^{36},$ $d\alpha^4 = -b\alpha^{46} - \alpha^{56},\ d\alpha^5 = -b\alpha^{56}, d\alpha^6 = 0$
$\mathfrak{g}_{6.7}^{a,-\frac{2}{3}a}$ $a \neq 0$	$d\alpha^1 = -a\alpha^{16} - \alpha^{26},\ d\alpha^2 = -a\alpha^{26} - \alpha^{36},\ d\alpha^3 = -a\alpha^{36},$ $d\alpha^4 = \frac{2}{3}a\alpha^{46} - \alpha^{56},\ d\alpha^5 = \frac{2}{3}a\alpha^{56}, d\alpha^6 = 0$
$\mathfrak{g}_{6.8}^{a,b,c,p}$ $0 < \|c\| \leq \|b\| \leq \|a\|,$	$d\alpha^1 = -a\alpha^{16},\ d\alpha^2 = -b\alpha^{26},\ d\alpha^3 = -c\alpha^{36},$ $d\alpha^4 = -p\alpha^{46} - \alpha^{56},\ d\alpha^5 = \alpha^{46} - p\alpha^{56},\ d\alpha^6 = 0$

$a+b+c+2p=0$	
$\mathfrak{g}_{6.9}^{a,b,p}$ $a\neq 0,\ a+2b+2p=0$	$d\alpha^1=-a\alpha^{16},\ d\alpha^2=-b\alpha^{26}-\alpha^{36},\ d\alpha^3=-b\alpha^{36},$ $d\alpha^4=-p\alpha^{46}-\alpha^{56}, d\alpha^5=\alpha^{46}-p\alpha^{56}, d\alpha^6=0$
$\mathfrak{g}_{6.10}^{a,-\frac{3}{2}a}$	$d\alpha^1=-a\alpha^{16}-\alpha^{26},\ d\alpha^2=-a\alpha^{26}-\alpha^{36},\ d\alpha^3=-a\alpha^{36},$ $d\alpha^4=\frac{3}{2}a\alpha^{46}-\alpha^{56},\ d\alpha^5=\alpha^{46}+\frac{3}{2}a\alpha^{56}, d\alpha^6=0$
$\mathfrak{g}_{6.11}^{a,p,q,s}$ $as\neq 0,\ a+2p+2q=0$	$d\alpha^1=-a\alpha^{16},\ d\alpha^2=-p\alpha^{26}-\alpha^{36},\ d\alpha^3=\alpha^{26}-p\alpha^{36},$ $d\alpha^4=-q\alpha^{46}-s\alpha^{56},\ d\alpha^5=s\alpha^{46}-q\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.12}^{-4p,p}$ $p\neq 0$	$d\alpha^1=4p\alpha^{16},\ d\alpha^2=-p\alpha^{26}-\alpha^{36}-\alpha^{46},$ $d\alpha^3=\alpha^{26}-p\alpha^{36}-\alpha^{56},\ d\alpha^4=-p\alpha^{46}-\alpha^{56},$ $d\alpha^5=\alpha^{46}-p\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.13}^{a,b,h}$ $a\neq 0,\ 2a+2b+h=-1$	$d\alpha^1=-\alpha^{23}-(a+b)\alpha^{16},\ d\alpha^2=-a\alpha^{26},\ d\alpha^3=-b\alpha^{36},$ $d\alpha^4=-\alpha^{46},\ d\alpha^5=-h\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.14}^{a,b}$ $a\neq 0,\quad a+b=-\frac{1}{3}$	$d\alpha^1=-\alpha^{23}-(a+b)\alpha^{16}-\alpha^{56},\ d\alpha^2=-a\alpha^{26},$ $d\alpha^3=-b\alpha^{36},\ d\alpha^4=-\alpha^{46},\ d\alpha^5=-(a+b)\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.15}^{-1}$	$d\alpha^1=-\alpha^{23},\ d\alpha^2=-\alpha^{26},\ d\alpha^3=\alpha^{36},$ $d\alpha^4=-\alpha^{26}-\alpha^{46},\ d\alpha^5=-\alpha^{36}+\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.17}^{-\frac{1}{2},0}$	$d\alpha^1=-\alpha^{23}+\frac{1}{2}\alpha^{16},\ d\alpha^2=\frac{1}{2}\alpha^{26},\ d\alpha^3=0,$ $d\alpha^4=-\alpha^{36},\ d\alpha^5=-\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.18}^{a,-2a-3}$ $a\neq -\frac{3}{2}$	$d\alpha^1=-\alpha^{23}-(1+a)\alpha^{16},\ d\alpha^2=-a\alpha^{26},\ d\alpha^3=-\alpha^{36},$ $d\alpha^4=-\alpha^{36}-\alpha^{46},\ d\alpha^5=(2a+3)\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.19}^{-\frac{4}{3}}$	$d\alpha^1=-\alpha^{23}+\frac{1}{3}\alpha^{16}-\alpha^{56},\ d\alpha^2=\frac{4}{3}\alpha^{26},\ d\alpha^3=-\alpha^{36},$ $d\alpha^4=-\alpha^{36}-\alpha^{46},\ d\alpha^5=\frac{1}{3}\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.20}^{-3}$	$d\alpha^1=-\alpha^{23}-\alpha^{16}-\alpha^{46},\ d\alpha^2=0,\ d\alpha^3=-\alpha^{36},$ $d\alpha^4=-\alpha^{36}-\alpha^{46},\ d\alpha^5=3\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.21}^{a}$ $a\neq -\frac{1}{4}$	$d\alpha^1=-\alpha^{23}-2a\alpha^{16},\ d\alpha^2=-a\alpha^{26},\ d\alpha^3=-\alpha^{26}-a\alpha^{36},$ $d\alpha^4=-\alpha^{46},\ d\alpha^5=(4a+1)\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.22}^{-\frac{1}{6}}$	$d\alpha^1=-\alpha^{23}+\frac{1}{3}\alpha^{16}-\alpha^{56},\ d\alpha^2=\frac{1}{6}\alpha^{26},$ $d\alpha^3=-\alpha^{26}+\frac{1}{6}\alpha^{36},\ d\alpha^4=-\alpha^{46},$ $d\alpha^5=\frac{1}{3}\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.23}^{a,-7a,\varepsilon}$ $a\varepsilon=0$	$d\alpha^1=-\alpha^{23}-2a\alpha^{16}-\varepsilon\alpha^{56},\ d\alpha^2=-a\alpha^{26},$ $d\alpha^3=-\alpha^{26}-a\alpha^{36},\ d\alpha^4=-\alpha^{36}-a\alpha^{46},$ $d\alpha^5=5a\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.25}^{b,-1-b}$	$d\alpha^1=-\alpha^{23}+b\alpha^{16},\ d\alpha^2=-\alpha^{26},\ d\alpha^3=(1+b)\alpha^{36},$ $d\alpha^4=-b\alpha^{46},\ d\alpha^5=-\alpha^{46}-b\alpha^{56},\ d\alpha^6=0$
$\mathfrak{g}_{6.26}^{-1}$	$d\alpha^1=-\alpha^{23}-\alpha^{56},\ d\alpha^2=-\alpha^{26},\ d\alpha^3=\alpha^{36},$ $d\alpha^4=0,\ d\alpha^5=-\alpha^{46},\ d\alpha^6=0$
$\mathfrak{g}_{6.27}^{-2b,b}$ $b\neq 0$	$d\alpha^1=-\alpha^{23}+b\alpha^{16},\ d\alpha^2=2b\alpha^{26},\ d\alpha^3=-b\alpha^{36},$ $d\alpha^4=-\alpha^{36}-b\alpha^{46},\ d\alpha^5=-\alpha^{46}-b\alpha^{56},\ d\alpha^6=0$

$\mathfrak{g}_{6.28}^{-2}$	$d\alpha^1 = -\alpha^{23} - 2\alpha^{16},\ d\alpha^2 = -\alpha^{26},\ d\alpha^3 = -\alpha^{26} - \alpha^{36},$ $d\alpha^4 = 2\alpha^{46},\ d\alpha^5 = -\alpha^{46} + 2\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.29}^{-2b,b,\varepsilon}$ $b\varepsilon = 0$	$d\alpha^1 = -\alpha^{23} + b\alpha^{16} - \varepsilon\alpha^{56},\ d\alpha^2 = 2b\alpha^{26},\ d\alpha^3 = -b\alpha^{36},$ $d\alpha^4 = -\alpha^{36} - b\alpha^{46},\ d\alpha^5 = -\alpha^{46} - b\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.32}^{a,-6a-h,h,\varepsilon}$ $a > -\frac{1}{4}h, \varepsilon h = 0$	$d\alpha^1 = -\alpha^{23} - 2a\alpha^{16} - \varepsilon\alpha^{46},\ d\alpha^2 = -a\alpha^{26} + \alpha^{36},$ $d\alpha^3 = -\alpha^{26} - a\alpha^{36},\ d\alpha^4 = -(2a+h)\alpha^{46},$ $d\alpha^5 = (6a+h)\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.33}^{a,-6a}$ $a \geq 0$	$d\alpha^1 = -\alpha^{23} - 2a\alpha^{16} - \alpha^{56},\ d\alpha^2 = -a\alpha^{26} + \alpha^{36},$ $d\alpha^3 = -\alpha^{26} - a\alpha^{36},\ d\alpha^4 = 6a\alpha^{46},$ $d\alpha^5 = -2a\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.34}^{a,-4a,\varepsilon}$ $\varepsilon a = 0$	$d\alpha^1 = -\alpha^{23} - 2a\alpha^{16} - \varepsilon\alpha^{56},\ d\alpha^2 = -a\alpha^{26} + \alpha^{36},$ $d\alpha^3 = -\alpha^{26} - a\alpha^{36},\ d\alpha^4 = 2a\alpha^{46},\ d\alpha^5 = 2a\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.35}^{a,b,c}$ $a+b+c=0,\ a^2+b^2 \neq 0$	$d\alpha^1 = -\alpha^{23} - (a+b)\alpha^{16},\ d\alpha^2 = -a\alpha^{26},\ d\alpha^3 = -b\alpha^{36},$ $d\alpha^4 = -c\alpha^{46} + \alpha^{56},\ d\alpha^5 = -\alpha^{46} - c\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.36}^{a,-2a}$	$d\alpha^1 = -\alpha^{23} - 2a\alpha^{16},\ d\alpha^2 = -a\alpha^{26},\ d\alpha^3 = -\alpha^{26} - a\alpha^{36},$ $d\alpha^4 = 2a\alpha^{46} + \alpha^{56},\ d\alpha^5 = -\alpha^{46} + 2a\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.37}^{a,-2a,s}$ $s \neq 0$	$d\alpha^1 = -\alpha^{23} - 2a\alpha^{16},\ d\alpha^2 = -a\alpha^{26} + \alpha^{36},\ d\alpha^3 = -\alpha^{26} - a\alpha^{36},$ $d\alpha^4 = 2a\alpha^{46} + s\alpha^{56},\ d\alpha^5 = -s\alpha^{46} + 2a\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.38}^{0}$	$d\alpha^1 = -\alpha^{23},\ d\alpha^2 = \alpha^{36},\ d\alpha^3 = -\alpha^{26},$ $d\alpha^4 = -\alpha^{26} + \alpha^{56},\ d\alpha^5 = -\alpha^{36} - \alpha^{46},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.39}^{-4-3h,h}$ $h \neq -\frac{4}{3}$	$d\alpha^1 = -\alpha^{45} - (1+h)\alpha^{16},\ d\alpha^2 = -\alpha^{15} - (2+h)\alpha^{26},$ $d\alpha^3 = (4+3h)\alpha^{36},\ d\alpha^4 = -h\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.40}^{-\frac{3}{2}}$	$d\alpha^1 = -\alpha^{45} + \frac{1}{2}\alpha^{16},\ d\alpha^2 = -\alpha^{15} - \frac{1}{2}\alpha^{26} - \alpha^{36},$ $d\alpha^3 = -\frac{1}{2}\alpha^{36},\ d\alpha^4 = \frac{3}{2}\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.41}^{-1}$	$d\alpha^1 = -\alpha^{45},\ d\alpha^2 = -\alpha^{15} - \alpha^{26},\ d\alpha^3 = \alpha^{36} - \alpha^{46},$ $d\alpha^4 = \alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.42}^{-\frac{5}{3}}$	$d\alpha^1 = -\alpha^{45} + \frac{2}{3}\alpha^{16},\ d\alpha^2 = -\alpha^{15} - \frac{1}{3}\alpha^{26},\ d\alpha^3 = \alpha^{36} - \alpha^{56},$ $d\alpha^4 = \frac{5}{3}\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.44}^{-7}$	$d\alpha^1 = -\alpha^{45} - 2\alpha^{16},\ d\alpha^2 = -\alpha^{15} - 3\alpha^{26},\ d\alpha^3 = 7\alpha^{36},$ $d\alpha^4 = -\alpha^{46} - \alpha^{56},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.47}^{-3,\varepsilon}$ $\varepsilon = 0, \pm 1$	$d\alpha^1 = -\alpha^{45} - \alpha^{16},\ d\alpha^2 = -\alpha^{15} - \alpha^{26} - \varepsilon\alpha^{46},$ $d\alpha^3 = 3\alpha^{36},\ d\alpha^4 = -\alpha^{46},\ d\alpha^5 = 0,\ d\alpha^6 = 0$
$\mathfrak{g}_{6.54}^{2(1+l),l}$	$d\alpha^1 = -\alpha^{35} - \alpha^{16},\ d\alpha^2 = -\alpha^{45} - l\alpha^{26},\ d\alpha^3 = (1+2l)\alpha^{36},$ $d\alpha^4 = (2+l)\alpha^{46},\ d\alpha^5 = -2(1+l)\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.55}^{-4}$	$d\alpha^1 = -\alpha^{35} - \alpha^{16} - \alpha^{46},\ d\alpha^2 = -\alpha^{45} + 3\alpha^{26},$ $d\alpha^3 = -5\alpha^{36},\ d\alpha^4 = -\alpha^{46},\ d\alpha^5 = 4\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.56}^{\frac{4}{3}}$	$d\alpha^1 = -\alpha^{35} - \alpha^{16},\ d\alpha^2 = -\alpha^{45} + \frac{1}{3}\alpha^{26} - \alpha^{36},$

	$d\alpha^3 = \frac{1}{3}\alpha^{36},\ d\alpha^4 = \frac{5}{3}\alpha^{46},\ d\alpha^5 = -\frac{4}{3}\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.57}^{-\frac{2}{3}}$	$d\alpha^1 = -\alpha^{35} - \alpha^{16},\ d\alpha^2 = -\alpha^{45} + \frac{4}{3}\alpha^{26},$ $d\alpha^3 = -\frac{5}{3}\alpha^{36},\ d\alpha^4 = \frac{2}{3}\alpha^{46} - \alpha^{56},\ d\alpha^5 = \frac{2}{3}\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.61}^{-\frac{3}{4}}$	$d\alpha^1 = -\alpha^{35} - 2\alpha^{16},\ d\alpha^2 = -\alpha^{45} + \frac{3}{2}\alpha^{26},$ $d\alpha^3 = -\alpha^{36} - \alpha^{56},\ d\alpha^4 = \frac{5}{2}\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.63}^{-1}$	$d\alpha^1 = -\alpha^{35} - \alpha^{16},\ d\alpha^2 = -\alpha^{45} + \alpha^{26} - \alpha^{46},$ $d\alpha^3 = -\alpha^{36},\ d\alpha^4 = \alpha^{46},\ d\alpha^5 = 0,\ d\alpha^6 = 0$
$\mathfrak{g}_{6.65}^{4l,l}$	$d\alpha^1 = -\alpha^{35} - l\alpha^{16},\ d\alpha^2 = -\alpha^{45} - \alpha^{16} - l\alpha^{26},$ $d\alpha^3 = 3l\alpha^{36},\ d\alpha^4 = -\alpha^{36} + 3l\alpha^{46},\ d\alpha^5 = -4l\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.70}^{4p,p}$	$d\alpha^1 = -\alpha^{35} - p\alpha^{16} + \alpha^{26},\ d\alpha^2 = -\alpha^{45} - \alpha^{16} - p\alpha^{26},$ $d\alpha^3 = 3p\alpha^{36} + \alpha^{46},\ d\alpha^4 = -\alpha^{36} + 3p\alpha^{46},\ d\alpha^5 = -4p\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.71}^{-\frac{7}{4}}$	$d\alpha^1 = -\alpha^{25} - \frac{5}{4}\alpha^{16},\ d\alpha^2 = -\alpha^{35} - \frac{1}{4}\alpha^{26},$ $d\alpha^3 = -\alpha^{45} + \frac{3}{4}\alpha^{36},\ d\alpha^4 = \frac{7}{4}\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.76}^{-1}$	$d\alpha^1 = -\alpha^{25} + \alpha^{16},\ d\alpha^2 = -\alpha^{45},$ $d\alpha^3 = -\alpha^{24} - \alpha^{36},\ d\alpha^4 = -\alpha^{46},\ d\alpha^5 = \alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.78}$	$d\alpha^1 = -\alpha^{25} + \alpha^{16},\ d\alpha^2 = -\alpha^{45},\ d\alpha^3 = -\alpha^{24} - \alpha^{36} - \alpha^{46},$ $d\alpha^4 = -\alpha^{46},\ d\alpha^5 = \alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.83}^{0,l}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -l\alpha^{26} - \alpha^{36},\ d\alpha^3 = -l\alpha^{36},$ $d\alpha^4 = l\alpha^{46},\ d\alpha^5 = \alpha^{46} + l\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.84}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -\alpha^{26},\ d\alpha^3 = -\alpha^{56},$ $d\alpha^4 = \alpha^{46},\ d\alpha^5 = 0,\ d\alpha^6 = 0$
$\mathfrak{g}_{6.88}^{0,\mu,\nu}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -\mu\alpha^{26} + \nu\alpha^{36},\ d\alpha^3 = -\nu\alpha^{26} - \mu\alpha^{36},$ $d\alpha^4 = \mu\alpha^{46} + \nu\alpha^{56},\ d\alpha^5 = -\nu\alpha^{46} + \mu\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.89}^{0,\nu,s}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -s\alpha^{26},\ d\alpha^3 = \nu\alpha^{56},$ $d\alpha^4 = s\alpha^{46},\ d\alpha^5 = -\nu\alpha^{36},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.90}^{0,\nu}$ $\nu \neq 1$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -\alpha^{46},\ d\alpha^3 = \nu\alpha^{56},$ $d\alpha^4 = -\alpha^{26},\ d\alpha^5 = -\nu\alpha^{36},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.91}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -\alpha^{46},\ d\alpha^3 = \alpha^{56},$ $d\alpha^4 = -\alpha^{26},\ d\alpha^5 = -\alpha^{36},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.92}^{0,\mu,\nu}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = \mu\alpha^{36},\ d\alpha^3 = -\nu\alpha^{26},$ $d\alpha^4 = \nu\alpha^{56},\ d\alpha^5 = -\mu\alpha^{46},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.92^*}^{0}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = \alpha^{46},\ d\alpha^3 = \alpha^{56},$ $d\alpha^4 = -\alpha^{26},\ d\alpha^5 = -\alpha^{36},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.93}^{0,\nu}$	$d\alpha^1 = -\alpha^{24} - \alpha^{35},\ d\alpha^2 = -\alpha^{46} + \nu\alpha^{56},\ d\alpha^3 = \nu\alpha^{46},$ $d\alpha^4 = -\alpha^{26} - \nu\alpha^{36},\ d\alpha^5 = -\nu\alpha^{26},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.94}^{-2}$	$d\alpha^1 = -\alpha^{25} - \alpha^{34},\ d\alpha^2 = -\alpha^{35} + \alpha^{26},\ d\alpha^3 = 2\alpha^{36},$ $d\alpha^4 = -2\alpha^{46},\ d\alpha^5 = -\alpha^{56},\ d\alpha^6 = 0$
$\mathfrak{g}_{6.101}^{a,b,c,e}$	$d\alpha^1 = a\alpha^{15} + b\alpha^{16},\ d\alpha^2 = c\alpha^{25} + e\alpha^{26},$

$a+c=-1,\ b+e=-1,$ $ab\neq 0,\ c^2+e^2\neq 0$	$d\alpha^3=\alpha^{36},\ d\alpha^4=\alpha^{45},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.102}^{-1,b,-2-b}$	$d\alpha^1=-\alpha^{15}+b\alpha^{16},\ d\alpha^2=\alpha^{25}-(2+b)\alpha^{26},$ $d\alpha^3=\alpha^{36},\ d\alpha^4=\alpha^{35}+\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.105}^{-2,-1}$	$d\alpha^1=-2\alpha^{15}-\alpha^{16},\ d\alpha^2=\alpha^{26},\ d\alpha^3=\alpha^{35},$ $d\alpha^4=\alpha^{35}+\alpha^{45},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.107}^{-1,0}$	$d\alpha^1=-\alpha^{15}-\alpha^{26},\ d\alpha^2=-\alpha^{25}-\alpha^{16},\ d\alpha^3=\alpha^{35},$ $d\alpha^4=\alpha^{35}+\alpha^{45},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.113}^{a,b,-a,c}$ $a^2+b^2\neq 0,\ a^2+c^2\neq 0,$ $b+c=-2$	$d\alpha^1=a\alpha^{15}+b\alpha^{16},\ d\alpha^2=-a\alpha^{25}+c\alpha^{26},\ d\alpha^3=\alpha^{36},$ $d\alpha^4=\alpha^{35}+\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.114}^{a,-1,-\frac{a}{2}}$ $a\neq 0,$	$d\alpha^1=a\alpha^{15}-\alpha^{16},\ d\alpha^2=\alpha^{26},\ d\alpha^3=-\frac{a}{2}\alpha^{35}-\alpha^{45},$ $d\alpha^4=\alpha^{35}-\frac{a}{2}\alpha^{45},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.115}^{-1,b,c,-c}$ $b\neq 0,$	$d\alpha^1=\alpha^{15}+c\alpha^{16}-\alpha^{26},\ d\alpha^2=\alpha^{25}+\alpha^{16}+c\alpha^{26},$ $d\alpha^3=-\alpha^{35}-b\alpha^{45}-c\alpha^{36},\ d\alpha^4=b\alpha^{35}-\alpha^{45}-c\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.116}^{0,-1}$	$d\alpha^1=\alpha^{16},\ d\alpha^2=\alpha^{15}+\alpha^{26},\ d\alpha^3=-\alpha^{45}-\alpha^{36},$ $d\alpha^4=\alpha^{35}-\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.118}^{0,b,-1}$	$d\alpha^1=-\alpha^{25}+\alpha^{16},\ d\alpha^2=\alpha^{15}+\alpha^{26},\ d\alpha^3=-b\alpha^{45}-\alpha^{36},$ $d\alpha^4=b\alpha^{35}-\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.120}^{-1,-1}$	$d\alpha^1=-\alpha^{56},\ d\alpha^2=-\alpha^{25}-\alpha^{26},\ d\alpha^3=\alpha^{36},$ $d\alpha^4=\alpha^{45},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.125}^{0,-2}$	$d\alpha^1=-\alpha^{56},\ d\alpha^2=-2\alpha^{26},\ d\alpha^3=-\alpha^{45}+\alpha^{36},$ $d\alpha^4=\alpha^{35}+\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.129}^{-2,-2}$	$d\alpha^1=-\alpha^{23}+\alpha^{15}+\alpha^{16},\ d\alpha^2=\alpha^{25},\ d\alpha^3=\alpha^{36},$ $d\alpha^4=-2\alpha^{45}-2\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{g}_{6.135}^{0,-4}$	$d\alpha^1=-\alpha^{23}+2\alpha^{16},\ d\alpha^2=\alpha^{26},\ d\alpha^3=\alpha^{25}+\alpha^{36},$ $d\alpha^4=-4\alpha^{46},\ d\alpha^5=0,\ d\alpha^6=0$
$\mathfrak{n}_{6.83}^{0}$	$d\alpha^1=-\alpha^{45},\ d\alpha^2=-\alpha^{15}-\alpha^{36},\ d\alpha^3=-\alpha^{14}+\alpha^{26},$ $d\alpha^4=\alpha^{56},\ d\alpha^5=-\alpha^{46},\ d\alpha^6=0$
$\mathfrak{n}_{6.84}^{\varepsilon}$ $\varepsilon=\pm 1$	$d\alpha^1=-\alpha^{45},\ d\alpha^2=-\alpha^{15}-\alpha^{36},\ d\alpha^3=-\alpha^{14}+\alpha^{26}-\varepsilon\alpha^{56},$ $d\alpha^4=\alpha^{56},\ d\alpha^5=-\alpha^{46},\ d\alpha^6=0$
$\mathfrak{n}_{6.96}^{b}$	$d\alpha^1=-\alpha^{24}-\alpha^{35},\ d\alpha^2=-\alpha^{46},\ d\alpha^3=-b\alpha^{56},$ $d\alpha^4=\alpha^{26},\ d\alpha^5=-b\alpha^{36},\ d\alpha^6=0$

$\mathfrak{g}_{5.7}^{p,q,r}$ $-1\leq r\leq q\leq p\leq 1,$ $pqr\neq 0,\ p+q+r=-1$	$d\alpha^1=-\alpha^{15},\ d\alpha^2=-p\alpha^{25},\ d\alpha^3=-q\alpha^{35},$ $d\alpha^4=-r\alpha^{45},\ d\alpha^5=0$
$\mathfrak{g}_{5.8}^{-1}$	$d\alpha^1=-\alpha^{25},\ d\alpha^2=0,\ d\alpha^3=-\alpha^{35},\ d\alpha^4=\alpha^{45},\ d\alpha^5=0$
$\mathfrak{g}_{5.9}^{p,-2-p}$	$d\alpha^1=-\alpha^{15}-\alpha^{25},\ d\alpha^2=-\alpha^{25},\ d\alpha^3=-p\alpha^{35},$

$p \geq -1$,	$d\alpha^4 = -(p+2)\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.11}^{-3}$	$d\alpha^1 = -\alpha^{15} - \alpha^{25},\ d\alpha^2 = -\alpha^{25} - \alpha^{35},\ d\alpha^3 = -\alpha^{35},$ $d\alpha^4 = 3\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$ $-1 \leq q \leq 0,$ $q \neq -\frac{1}{2},\ r \neq 0$	$d\alpha^1 = -\alpha^{15},\ d\alpha^2 = (1+2q)\alpha^{25},\ d\alpha^3 = -q\alpha^{35} - r\alpha^{45},$ $d\alpha^4 = r\alpha^{35} - q\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.14}^{0}$	$d\alpha^1 = -\alpha^{25},\ d\alpha^2 = 0,\ d\alpha^3 = -\alpha^{45},\ d\alpha^4 = \alpha^{35},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.15}^{-1}$	$d\alpha^1 = -\alpha^{15} - \alpha^{25},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = \alpha^{35} - \alpha^{45},$ $d\alpha^4 = \alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.16}^{-1,q}$ $q \neq 0$	$d\alpha^1 = -\alpha^{15} - \alpha^{25},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = \alpha^{35} - q\alpha^{45},$ $d\alpha^4 = q\alpha^{35} + \alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.17}^{p,-p,r}$ $r \neq 0$	$d\alpha^1 = -p\alpha^{15} - \alpha^{25},\ d\alpha^2 = \alpha^{15} - p\alpha^{25},\ d\alpha^3 = p\alpha^{35} - r\alpha^{45},$ $d\alpha^4 = r\alpha^{35} + p\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.18}^{0}$	$d\alpha^1 = -\alpha^{25} - \alpha^{35},\ d\alpha^2 = \alpha^{15} - \alpha^{45},\ d\alpha^3 = -\alpha^{45},$ $d\alpha^4 = \alpha^{35},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.19}^{p,-2p-2}$ $p \neq -1$	$d\alpha^1 = -\alpha^{23} - (p+1)\alpha^{15},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = -p\alpha^{35},$ $d\alpha^4 = (2p+2)\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.20}^{-1}$	$d\alpha^1 = -\alpha^{23} - \alpha^{45},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = \alpha^{35},\ d\alpha^4 = 0,\ d\alpha^5 = 0$
$\mathfrak{g}_{5.23}^{-4}$	$d\alpha^1 = -\alpha^{23} - 2\alpha^{15},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = -\alpha^{25} - \alpha^{35},$ $d\alpha^4 = 4\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.25}^{p,4p}$ $p \neq 0$	$d\alpha^1 = -\alpha^{23} - 2p\alpha^{15},\ d\alpha^2 = -p\alpha^{25} + \alpha^{35},\ d\alpha^3 = -\alpha^{25} - p\alpha^{35},$ $d\alpha^4 = 4p\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.26}^{0,\varepsilon}$ $\varepsilon = \pm 1$	$d\alpha^1 = -\alpha^{23} - \varepsilon\alpha^{45},\ d\alpha^2 = \alpha^{35},\ d\alpha^3 = -\alpha^{25},$ $d\alpha^4 = 0,\ d\alpha^5 = 0$
$\mathfrak{g}_{5.28}^{-\frac{3}{2}}$	$d\alpha^1 = -\alpha^{23} + \frac{1}{2}\alpha^{15},\ d\alpha^2 = \frac{3}{2}\alpha^{25},\ d\alpha^3 = -\alpha^{35},$ $d\alpha^4 = -\alpha^{35} - \alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.30}^{-\frac{4}{3}}$	$d\alpha^1 = -\alpha^{24} - \frac{2}{3}\alpha^{15},\ d\alpha^2 = -\alpha^{34} + \frac{1}{3}\alpha^{25},\ d\alpha^3 = \frac{4}{3}\alpha^{35},$ $d\alpha^4 = -\alpha^{45},\ d\alpha^5 = 0$
$\mathfrak{g}_{5.33}^{-1,-1}$	$d\alpha^1 = -\alpha^{14},\ d\alpha^2 = -\alpha^{25},\ d\alpha^3 = \alpha^{34} + \alpha^{35},$ $d\alpha^4 = 0,\ d\alpha^5 = 0$
$\mathfrak{g}_{5.35}^{-2,0}$	$d\alpha^1 = 2\alpha^{14},\ d\alpha^2 = -\alpha^{24} - \alpha^{35},\ d\alpha^3 = -\alpha^{34} + \alpha^{25},$ $d\alpha^4 = 0,\ d\alpha^5 = 0$

$\mathfrak{g}_{4.2}^{-2}$	$d\alpha^1 = 2\alpha^{14},\ d\alpha^2 = -\alpha^{24} - \alpha^{34},$ $d\alpha^3 = -\alpha^{34},\ d\alpha^4 = 0$
$\mathfrak{g}_{4.5}^{p,-p-1}$ $-1 \leq p \leq -\frac{1}{2},$	$d\alpha^1 = -\alpha^{14},\ d\alpha^2 = -p\alpha^{24},$ $d\alpha^3 = (p+1)\alpha^{34},\ d\alpha^4 = 0$
$\mathfrak{g}_{4.6}^{-2p,p}$ $p > 0$	$d\alpha^1 = 2p\alpha^{14},\ d\alpha^2 = -p\alpha^{24} - \alpha^{34},$ $d\alpha^3 = \alpha^{24} - p\alpha^{34},\ d\alpha^4 = 0$

$\mathfrak{g}_{4.8}^{-1}$	$d\alpha^1 = -\alpha^{23},\ d\alpha^2 = -\alpha^{24},\ d\alpha^3 = \alpha^{34},\ d\alpha^4 = 0$
$\mathfrak{g}_{4.9}^{0}$	$d\alpha^1 = -\alpha^{23},\ d\alpha^2 = -\alpha^{34},\ d\alpha^3 = \alpha^{24},\ d\alpha^4 = 0$

$\mathfrak{g}_{3.1}$ nilpotent	$d\alpha^1 = -\alpha^{23},\ d\alpha^2 = 0,\ d\alpha^3 = 0,$
$\mathfrak{g}_{3.4}^{-1}$	$d\alpha^1 = -\alpha^{13},\ d\alpha^2 = \alpha^{23},\ d\alpha^3 = 0,$
$\mathfrak{g}_{3.5}^{0}$	$d\alpha^1 = -\alpha^{23},\ d\alpha^2 = \alpha^{13},\ d\alpha^3 = 0,$

Appendix B

Betti numbers of 6 dimensional unimodular, solvable, non-nilpotent Lie algebras [1]

$\mathfrak{g}$	b_1	b_2	b_3
$\mathfrak{g}_{6.1}$	1	0 if $a \neq -1,\ b \neq -1,\ b \neq -a,$ $c \neq -a,\ c+b \neq -1,\ c \neq -b,$ $a+b \neq -1,\ a+c \neq -1$ 1 if $a=-1$, or if $b=-a$, or if $b=-c$, or if $a+b \neq -1$ 2 if $b=-1$, or if $c=-a$ or if $c=-1-a$, or if $a=-1$ and $b=1$, or if $a=-1$ and $b+c=-1$, or if $b=c=-a$, or if $b=-c=\pm a$, or if $b=-c=\pm(1+a)$, or if $b=c=-1-a$ 3 if $a=-\frac{1}{2}$ and $b=-c=\pm\frac{1}{2}$ or if $a=b=c-\frac{1}{2}$	0 if $a \neq -1,\ b \neq -1,\ b \neq -a,$ $c \neq -a,\ c+b \neq -1,\ c \neq -b,$ $a+b \neq -1,\ a+c \neq -1$ 2 if $a=-1$, or if $b=-a$, or if $b=-c$, or if $a+b \neq -1$ 4 if $b=-1$, or if $c=-a$ or if $c=-1-a$, or if $a=-1$ and $b=1$, or if $a=-1$ and $b+c=-1$, or if $b=c=-a$, or if $b=-c=\pm a$, or if $b=-c=\pm(1+a)$, or if $b=c=-1-a$ 6 if $a=-\frac{1}{2}$ and $b=-c=\pm\frac{1}{2}$ or if $a=b=c-\frac{1}{2}$

[1] In Table 1 we impose conditions which become at every step more restrictive. It is therefore implicit that the previous conditions hold only when the more restrictive ones are not satisfied.

		4 if $a=-b=c=\frac{1}{2}$	5 if $a=-b=c=\frac{1}{2}$
$\mathfrak{g}_{6.2}$	1 if $a\neq 0$ 2 if $a=0$	0 if $a\neq 0,\ c\neq -1,\ e\neq -c,$ $c-e\neq \pm 1,\ c+e\neq 1,$ 1 if $c=-1,$ or if $e=-c$, or if $e=c+1,$ or if $e=\pm(1-c),$ 2 if $a=0$ or if $c=-e=\pm\frac{1}{2}$ or if $e=-1,$ or if $c=-e=-1$	0 if $a\neq 0,\ c\neq -1,\ e\neq -c,$ $c-e\neq \pm 1,\ c+e\neq 1,$ 2 if $a=0$ or if $c=-1$ or if $e=-c$, or if $e=c+1,$ or if $e=\pm(1-c),$ 4 if $c=-e=\pm\frac{1}{2}$ or if $e=-1,$ or if $c=-e=-1$
$\mathfrak{g}_{6.3}$	1 if $a\neq -1$ 2 if $a=-1$	0 if $a\neq -1,\frac{1}{2}$ 1 if $a=\frac{1}{2}$ 3 if $a=-1$	0 if $a\neq -1,\frac{1}{2}$ 2 if $a=\frac{1}{2}$ 4 if $a=-1$
$\mathfrak{g}_{6.4}$	1	0	0
$\mathfrak{g}_{6.6}$	1 if $a\neq -\frac{1}{2}$ 2 if $a=-\frac{1}{2}$	0 if $a\neq -1,-\frac{1}{2}$ 1 if $a=-1$ 2 if $a=-\frac{1}{2}$	0 if $a\neq -1,-\frac{1}{2}$ 2 if $a=-1$ 2 if $a=-\frac{1}{2}$
$\mathfrak{g}_{6.7}$	1	0	0
$\mathfrak{g}_{6.8}$	1	0 if $a+b\neq 0,\ a+c\neq 0,$ $b+c\neq 0,\ p\neq 0$ 1 if $a+b=0$, or if $b+c=0,$ or if $p=0$ 2 if $a=-b=c,$ or if $a+c=0$	0 if $a+b\neq 0,\ a+c\neq 0,$ $b+c\neq 0,\ p\neq 0$ 2 if $a+b=0,$ or if $b+c=0,$ or if $p=0$ 4 if $a=-b=c$, or if $a+c=0$
$\mathfrak{g}_{6.9}$	1 if $b\neq 0$ 2 if $b=0$	0 if $bp\neq 0,\ a+b\neq 0$ 1 if $p=0$ or if $a+b=0$ 2 if $b=0$	0 if $bp\neq 0,\ a+b\neq 0$ 2 if $bp=0$ or if $a+b=0$
$\mathfrak{g}_{6.10}$	1 if $a\neq 0$ 2 if $a=0$	0 if $a\neq 0$ 3 if $a=0$	0 if $a\neq 0$ 4 if $a=0$
$\mathfrak{g}_{6.11}$	1	0 if $pq\neq 0$ 1 if $pq=0$	0 if $pq\neq 0$ 2 if $pq=0$
$\mathfrak{g}_{6.12}$	1	0	0
$\mathfrak{g}_{6.13}$	1 if $bh\neq 0$ 2 if $b=0$ or if $h=0$	0 if $a\neq -1,\ b\neq -1,$ $a+b\neq 0,\ 2a+b\neq 0,$ $a+2b\neq 0,\ a+2b+1\neq 0,$ $b+2a+1\neq 0$ 1 if $a=-1$ or if $b=-1$ or if $a+b=0$ or if $a+2b=0,-1$	0 if $a\neq -1,\ b\neq -1,$ $a+b\neq 0,\ 2a+b\neq 0,$ $a+2b\neq 0,\ a+2b+1\neq 0,$ $b+2a+1\neq 0$ 2 if $a=-1$ or if $b=-1$ or if $a+b=0$ or if $a+2b=0,-1$

		or if $b+2a=0,-1$ 2 if $a=-1$ and $b=2$ or if $b=-1$ and $a=2$ or if $a=\frac{1}{3}$ and $b=-\frac{2}{3}$ or if $a=-\frac{2}{3}$ and $b=\frac{1}{3}$ or if $a=b=-1,-\frac{1}{3}$ 3 if $a=\frac{1}{2}$ and $b=-1$ or if $a=-1$ and $b=0,\frac{1}{2}$ or if $a=-b=\pm 1$	or if $b+2a=0,-1$ 4 if $a=-1$ and $b=2$ or if $b=-1$ and $a=2$ or if $a=\frac{1}{3}$ and $b=-\frac{2}{3}$ or if $a=-\frac{2}{3}$ and $b=\frac{1}{3}$ or if $a=b=-1,-\frac{1}{3}$ 4 if $a=\frac{1}{2}$ and $b=-1$ or if $a=-1$ and $b=0,\frac{1}{2}$ 6 if $a=-b=\pm 1$
$\mathfrak{g}_{6.14}$	1 if $a\neq-\frac{1}{3}$ 2 if $a=-\frac{1}{3}$	0 if $a\neq-1,-\frac{2}{3},-\frac{1}{3},\frac{1}{3},\frac{2}{3}$ 1 if $a=-1,-\frac{2}{3},-\frac{1}{3},\frac{1}{3},\frac{2}{3}$	0 if $a\neq-1,-\frac{2}{3},\frac{1}{3},\frac{2}{3}$ 2 if $a\neq-1,-\frac{2}{3},\frac{1}{3},\frac{2}{3}$
$\mathfrak{g}_{6.15}$	1	2	4
$\mathfrak{g}_{6.17}$	2	2	1
$\mathfrak{g}_{6.18}$	1 if $a\neq 0$ 2 if $a=0$	0 if $a\neq 0,-\frac{1}{2},-1,-2,-3$ 1 if $a=0,-\frac{1}{2},-2,-3$ 2 if $a=-1$	0 if $a\neq-\frac{1}{2},-1,-2,-3$ 2 if $a=-\frac{1}{2},-2,-3$ 4 if $a=-1$
$\mathfrak{g}_{6.19}$	1	0	0
$\mathfrak{g}_{6.20}$	2	1	0
$\mathfrak{g}_{6.21}$	1 if $a\neq 0$ 2 if $a=0$	0 if $a\neq 0,-\frac{1}{3},-1$ 1 if $a=-\frac{1}{3},-1$ 3 if $a=0$	0 if $a\neq 0,-\frac{1}{3},-1$ 2 if $a=-\frac{1}{3},-1$ 4 if $a=0$
$\mathfrak{g}_{6.22}$	1	0	0
$\mathfrak{g}_{6.23}$	1 if $a\neq 0$ 3 if $a=0$	0 if $a\neq 0$ 5 if $a=0$	0 if $a\neq 0$ 6 if $a=0$
$\mathfrak{g}_{6.25}$	1 if $b\neq 0,-1$ 2 if $b=0,-1$	0 if $b\neq 0,-1,-\frac{1}{2},1$ 1 if $b=-\frac{1}{2},1$ 2 if $b=0,-1$	0 if $b\neq 0,-1,-\frac{1}{2},1$ 2 if $b=0,-1,-\frac{1}{2},1$
$\mathfrak{g}_{6.26}$	2	2	2
$\mathfrak{g}_{6.27}$	1	1	2
$\mathfrak{g}_{6.28}$	1	0	0
$\mathfrak{g}_{6.29}$	1 if $b\neq 0$ 3 if $b=0$	2 if $b\neq 0$ 5 if $b=0$ and $\varepsilon\neq 0$ 6 if $b=0$ and $\varepsilon=0$	2 if $b\neq 0$ 6 if $b=0$ and $\varepsilon\neq 0$ 8 if $b=0$ and $\varepsilon=0$
$\mathfrak{g}_{6.32}$	1 if $a\neq-\frac{h}{2},-\frac{h}{6}$ 2 if $a=-\frac{h}{2},-\frac{h}{6}$	0 if $a\neq 0,-\frac{h}{2},-\frac{h}{6}$ 1 if $a=0,-\frac{h}{2},-\frac{h}{6}$	0 if $a\neq 0$ 2 if $a=0$
$\mathfrak{g}_{6.33}$	1 if $a\neq 0$ 3 if $a=0$	0 if $a\neq 0$ 3 if $a=0$	0 if $a\neq 0$ 1 if $a=0$
$\mathfrak{g}_{6.34}$	1 if $a\neq 0$	0 if $a\neq 0$	0 if $a\neq 0$

	3 if $a = 0$	3 if $a = 0$	1 if $a = 0$
$\mathfrak{g}_{6.35}$	1 if $ab \neq 0$ 2 if $a = 0$ or $b = 0$	0 if $c \neq 0, a \neq 0, -2b,$ $b \neq 0, -2a$ 1 if $a = 0$ or if $b = 0$ or if $c = 0$ or if $a = -2b$ or if $b = -2a$	0 if $c \neq 0, a \neq -2b,$ $b \neq -2a$ 2 if $c = 0$ or if $a = -2b$ or if $b = -2a$
$\mathfrak{g}_{6.36}$	1 if $a \neq 0$ 2 if $a = 0$	0 if $a \neq 0$ 3 if $a = 0$	0 if $a \neq 0$ 4 if $a = 0$
$\mathfrak{g}_{6.37}$	1	0 if $a \neq 0$ 1 if $a = 0$	0 if $a \neq 0$ 2 if $a = 0$
$\mathfrak{g}_{6.38}$	1	2	4
$\mathfrak{g}_{6.39}$	1 if $h \neq 0$ 2 if $h = 0$	0 if $h \neq 0, -\frac{1}{2}, -1, -2, -3$ 1 if $h = 0, -\frac{1}{2}, -1, -2, -3$	0 if $h \neq -\frac{1}{2}, -1, -2, -3$ 2 if $h = -\frac{1}{2}, -1, -2, -3$
$\mathfrak{g}_{6.40}$	1	0	0
$\mathfrak{g}_{6.41}$	1	1	2
$\mathfrak{g}_{6.42}$	1	0	0
$\mathfrak{g}_{6.44}$	1	0	0
$\mathfrak{g}_{6.47}$	2	1	0
$\mathfrak{g}_{6.54}$	1 if $l \neq -\frac{1}{2},$ $-1, -2$ 2 if $l = -\frac{1}{2},$ $-1, -2$	0 if $l \neq 0, -\frac{1}{2}, -1, -2, -\frac{3}{2}, -\frac{2}{3}$ 1 if $l = 0, -\frac{1}{2}, -2, -\frac{3}{2}, -\frac{2}{3}$ 3 if $l = -1$	0 if $l \neq 0, -1, -\frac{3}{2}, -\frac{2}{3}$ 2 if $l = 0, -\frac{3}{2}, -\frac{2}{3}$ 4 if $l = -1$
$\mathfrak{g}_{6.55}$	1	0	0
$\mathfrak{g}_{6.56}$	1	0	0
$\mathfrak{g}_{6.57}$	1	0	0
$\mathfrak{g}_{6.61}$	1	0	0
$\mathfrak{g}_{6.63}$	2	2	2
$\mathfrak{g}_{6.65}$	1 if $l \neq 0$ 3 if $l = 0$	0 if $l \neq 0$ 5 if $l = 0$	0 if $l \neq 0$ 6 if $l = 0$
$\mathfrak{g}_{6.70}$	1 if $p \neq 0$ 2 if $p = 0$	0 if $p \neq 0$ 3 if $p = 0$	0 if $p \neq 0$ 4 if $p = 0$
$\mathfrak{g}_{6.71}$	1	0	0
$\mathfrak{g}_{6.76}$	1	1	1
$\mathfrak{g}_{6.78}$	1	1	2
$\mathfrak{g}_{6.83}$	1 if $l \neq 0$ 3 if $l = 0$	1 if $l \neq 0$ 5 if $l = 0$	2 if $l \neq 0$ 6 if $l = 0$
$\mathfrak{g}_{6.84}$	2	2	2
$\mathfrak{g}_{6.88}$	1 if $\mu \neq 0$ or $\nu \neq 0$	1 if $\mu\nu \neq 0$ 3 if $\mu \neq 0$ and $\nu = 0$	2 $\mu\nu \neq 0$ 6 if $\mu \neq 0$ and $\nu = 0$

	5 if $\mu = 0$ and $\nu = 0$	or if $\mu = 0$ and $\nu \neq 0$ 9 if $\mu = 0$ and $\nu = 0$	or if $\mu = 0$ and $\nu \neq 0$ 10 if $\mu = 0$ and $\nu = 0$
$\mathfrak{g}_{6.89}$	1 if $s\nu \neq 0$ 2 if $s \neq 0$ or $\nu \neq 0$ 5 if $\nu = 0$ and $s = 0$	1 if $s\nu \neq 0$ 3 if $s \neq 0$ and $\nu = 0$ or if $s = 0$ and $\nu \neq 0$ 9 if $s = 0$ and $\nu = 0$	2 if $s\nu \neq 0$ or if $s \neq 0$ and $\nu = 0$ or if $s = 0$ and $\nu \neq 0$ 10 if $s = 0$ and $\nu = 0$
$\mathfrak{g}_{6.90}$	1 if $\nu \neq 0$ 3 if $\nu = 0$	1 if $\nu \neq 0$ 3 if $\nu = 0$	2
$\mathfrak{g}_{6.91}$	1	1	2
$\mathfrak{g}_{6.92}$	1 if $\mu\nu \neq 0$ 2 if $\mu \neq 0$ or if $\nu \neq 0$ 5 if $\nu = 0$ and $\mu = 0$	3 if $\mu\nu \neq 0$ 5 if $\mu \neq 0$ and $\nu = 0$ or if $\mu = 0$ and $\nu \neq 0$ 9 if $\mu = 0$ and $\nu = 0$	4 if $\mu\nu \neq 0$ 6 if $\mu \neq 0$ and $\nu = 0$ or if $\mu = 0$ and $\nu \neq 0$ 10 if $\mu = 0$ and $\nu = 0$
$\mathfrak{g}^*_{6.92}$	1	3	6
$\mathfrak{g}_{6.93}$	1 if $\nu \neq 0$ 3 if $\nu = 0$	1 if $\nu \neq 0$ 3 if $\nu = 0$	2 if $\nu \neq 0$ 2 if $\nu = 0$
$\mathfrak{g}_{6.94}$	1	1	2
$\mathfrak{g}_{6.101}$	2	1 if $a \neq -2$ or $b \neq -1$ 2 if $a = -2$ and $b = -1$	0 if $a \neq -2$ or $b \neq -1$ 1 if $a = -2$ and $b = -1$
$\mathfrak{g}_{6.102}$	2	1	0
$\mathfrak{g}_{6.105}$	2	1	0
$\mathfrak{g}_{6.107}$	2	1	0
$\mathfrak{g}_{6.113}$	2	1 if $a \neq 0$ or $b \neq -1$ 3 if $a = 0$ and $b = -1$	0 if $a \neq 0$ or $b \neq -1$ 2 if $a = 0$ and $b = -1$
$\mathfrak{g}_{6.114}$	2	2 if $a \neq \pm 2$ 3 if $a = \pm 2$	2 if $a \neq \pm 2$ 3 if $a = \pm 2$
$\mathfrak{g}_{6.115}$	2	1	0
$\mathfrak{g}_{6.116}$	2	1	0
$\mathfrak{g}_{6.118}$	2	1 if $b \neq \pm 1$ 3 if $b = \pm 1$	0 if $b \neq \pm 1$ 4 if $b = \pm 1$
$\mathfrak{g}_{6.120}$	2	2	2
$\mathfrak{g}_{6.121}$	2	2	2
$\mathfrak{g}_{6.129}$	2	1	0
$\mathfrak{g}_{6.135}$	2	1	0
$\mathfrak{n}_{6.83}$	1	1	2
$\mathfrak{n}_{6.84}$	1	1	1

$\mathfrak{n}_{6.96}$	1 if $b \neq 0$ 3 if $b = 0$	1 if $b \neq 0$ 3 if $b = 0$	2
$\mathfrak{g}_{5.7} \oplus \mathbb{R}$	2	1 if $r \neq -1$ 3 if $r = -1$ and $q \neq -1$ 5 if $r = -1$ and $q = -1$	0 if $r \neq -1$ 4 if $r = -1$ and $q \neq -1$ 8 if $r = -1$ and $q = -1$
$\mathfrak{g}_{5.8} \oplus \mathbb{R}$	3	5	6
$\mathfrak{g}_{5.9} \oplus \mathbb{R}$	2 if $p \neq 0$ 3 if $p = 0$	1 if $p \neq 0, -1$ 3 if $p = 0, -1$	0 if $p \neq 0, -1$ 2 if $p = 0$ 4 if $p = -1$
$\mathfrak{g}_{5.11} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.13} \oplus \mathbb{R}$	2	1 if $q \neq 0$ 3 if $q = 0$	0 if $q \neq 0$ 4 if $q = 0$
$\mathfrak{g}_{5.14} \oplus \mathbb{R}$	2	5	6
$\mathfrak{g}_{5.15} \oplus \mathbb{R}$	2	3	4
$\mathfrak{g}_{5.16} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.17} \oplus \mathbb{R}$	2	1 if $p \neq 0$ and $r \neq \pm 1$ 3 if $p = 0$ and $r \neq \pm 1$ or if $p \neq 0$ and $r = \pm 1$ 5 if $p = 0$ and $r = \pm 1$	0 if $p \neq 0$ and $r \neq \pm 1$ 4 if $p = 0$ and $r \neq \pm 1$ or if $p \neq 0$ and $r = \pm 1$ 8 if $p = 0$ and $r = \pm 1$
$\mathfrak{g}_{5.18} \oplus \mathbb{R}$	2	3	4
$\mathfrak{g}_{5.19} \oplus \mathbb{R}$	2 if $p \neq 0$ 3 if $p = 0$	1 if $p \neq 0, -\frac{1}{2}, -2$ 3 if $p = 0, -\frac{1}{2}, -2$	0 if $p \neq 0, -\frac{1}{2}, -2$ 2 if $p = 0$ 4 if $p = -\frac{1}{2}, -2$
$\mathfrak{g}_{5.20} \oplus \mathbb{R}$	3	3	3
$\mathfrak{g}_{5.23} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.25} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.26} \oplus \mathbb{R}$	3	3	2
$\mathfrak{g}_{5.28} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.30} \oplus \mathbb{R}$	2	1	0
$\mathfrak{g}_{5.33} \oplus \mathbb{R}$	3	3	2
$\mathfrak{g}_{5.35} \oplus \mathbb{R}$	3	3	1
$\mathfrak{g}_{4.2} \oplus 2\mathbb{R}$	3	3	2
$\mathfrak{g}_{4.5} \oplus 2\mathbb{R}$	3	3	2
$\mathfrak{g}_{4.6} \oplus 2\mathbb{R}$	3	3	2
$\mathfrak{g}_{4.8} \oplus 2\mathbb{R}$	3	3	2
$\mathfrak{g}_{4.9} \oplus 2\mathbb{R}$	3	3	2
$\mathfrak{g}_{3.4} \oplus 3\mathbb{R}$	4	7	8

$\mathfrak{g}_{3.5} \oplus 3\mathbb{R}$	4	7	8
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}$	3	5	6
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.5}$	3	5	6
$\mathfrak{g}_{3.4} \oplus \mathfrak{g}_{3.4}$	2	3	4
$\mathfrak{g}_{3.4} \oplus \mathfrak{g}_{3.5}$	2	3	4
$\mathfrak{g}_{3.5} \oplus \mathfrak{g}_{3.5}$	2	3	4

Appendix C

Symplectic structures on 6-dimensional solvable unimodular Lie algebras

Lie algebra	Symplectic form	Conditions on $\omega_{i,j}$
$\mathfrak{g}_{6.3}^{0,-1}$	$\omega=\omega_{1,6}\alpha^{16}+\omega_{2,3}\alpha^{23}+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+$ $+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\omega_{2,3}\omega_{4,5}\neq 0$
$\mathfrak{g}_{6.10}^{0,0}$	$\omega=\omega_{1,6}\alpha^{16}+\omega_{2,3}\alpha^{23}+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+$ $+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\omega_{2,3}\omega_{4,5}\neq 0$
$\mathfrak{g}_{6.13}^{\frac{1}{2},-1,0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{2,3}(-\frac{1}{2}\alpha^{16}+\alpha^{23})+\omega_{2,6}\alpha^{26}+$ $+\omega_{3,4}\alpha^{34}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{6.13}^{-1,\frac{1}{2},0}$	$\omega=\omega_{1,3}\alpha^{13}+\omega_{2,3}(-\frac{1}{2}\alpha^{16}+\alpha^{23})+\omega_{2,4}\alpha^{24}+$ $+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,3}\omega_{2,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{6.15}^{-1}$	$\omega=\omega_{1,6}\alpha^{16}+(\frac{\omega_{1,6}+\omega_{3,4}}{2})\alpha^{25}+\omega_{3,4}\alpha^{34}+$ $+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\omega_{3,4}\neq 0$, $\omega_{1,6}\neq -\omega_{3,4}$
$\mathfrak{g}_{6.18}^{-1,-1}$	$\omega=\omega_{1,6}(\alpha^{16}+\alpha^{24})+\omega_{2,6}\alpha^{26}+\omega_{3,5}\alpha^{35}+$ $+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\omega_{3,5}\neq 0$
$\mathfrak{g}_{6.21}^{0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{2,3}\alpha^{23}+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+$ $+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,6}\omega_{4,5}\neq 0$
$\mathfrak{g}_{6.23}^{0,0,\varepsilon}$	$\omega=\omega_{1,2}(\alpha^{12}+\varepsilon\alpha^{35})+\omega_{1,6}(\alpha^{16}+\alpha^{24})+\omega_{2,3}\alpha^{23}+$ $+\omega_{2,5}\alpha^{25}+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\neq 0$, $\omega_{1,6}^2+\omega_{1,2}\omega_{4,6}\neq 0$
$\mathfrak{g}_{6.29}^{0,0,\varepsilon\neq 0}$	$\omega=\omega_{1,3}(\alpha^{13}+\varepsilon\alpha^{45})+\omega_{1,6}(\alpha^{16}+\alpha^{24})+\omega_{2,3}\alpha^{23}+$	$\omega_{1,3}\neq 0$,

	$+\omega_{2,6}\alpha^{26}+\omega_{3,4}\alpha^{34}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$(\omega_{1,6},\omega_{2,6})\neq(0,0)$, $\omega_{5,6}\omega_{1,6}\neq\varepsilon(\omega_{2,3}\omega_{1,6}-$ $\omega_{2,6}\omega_{1,3})$
$\mathfrak{g}_{6.29}^{0,0,0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,3}\alpha^{13}+\omega_{1,6}(\alpha^{16}+\alpha^{24})+\omega_{2,3}\alpha^{23}+$ $+\omega_{2,6}\alpha^{26}+\omega_{3,4}\alpha^{34}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{5,6}\neq 0$, $\omega_{1,3}\omega_{1,6}-\omega_{1,2}\omega_{3,4}\neq 0$
$\mathfrak{g}_{6.36}^{0,0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{2,3}\alpha^{23}+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+$ $+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,6}\omega_{4,5}\neq 0$
$\mathfrak{g}_{6.38}^{0}$	$\omega=\omega_{1,6}(2\alpha^{16}+\alpha^{25}-\alpha^{34})+\omega_{2,3}\alpha^{23}++\omega_{2,4}(\alpha^{24}+$ $+\alpha^{35})+\omega_{2,6}\alpha^{26}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\neq 0$
$\mathfrak{g}_{6.54}^{0,-1}$	$\omega=\omega_{1,4}(\alpha^{14}+\alpha^{23})+\omega_{1,6}(\alpha^{16}+\alpha^{35})+\omega_{2,6}(\alpha^{26}+$ $-\alpha^{45})+\omega_{3,4}\alpha^{34}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{6.70}^{0,0}$	$\omega=\omega_{1,3}(\alpha^{13}+\alpha^{24})+\omega_{1,6}(\alpha^{16}+\alpha^{45})+\omega_{3,4}\alpha^{34}+$ $+\omega_{3,5}\alpha^{35}+\omega_{3,6}\alpha^{36}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,3}\omega_{1,6}\omega_{3,5}\omega_{5,6}\neq 0$, $\omega_{1,3}\omega_{5,6}+\omega_{1,6}\omega_{3,5}\neq 0$
$\mathfrak{g}_{6.78}$	$\omega=\omega_{1,4}(\alpha^{14}+\alpha^{35}-\alpha^{26})+\omega_{1,6}(\alpha^{16}-\alpha^{25})+$ $+\omega_{2,4}(\alpha^{24}+\alpha^{36})+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\neq 0$
$\mathfrak{g}_{6.118}^{0,\pm1,-1}$	$\omega=\omega_{1,3}(\alpha^{13}\pm\alpha^{24})+\omega_{1,4}(\alpha^{14}\mp\alpha^{23})+\omega_{1,5}(\alpha^{15}+$ $\alpha^{26})+\omega_{1,6}(\alpha^{16}+\alpha^{25})+\omega_{3,5}(\pm\alpha^{35}-\alpha^{46})+$ $+\omega_{3,6}(\alpha^{36}\pm\alpha^{45})+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\omega_{1,6}\omega_{5,6}\neq 0$, $2\omega_{1,4}\omega_{1,6}\omega_{3,6}+\omega_{1,3}\omega_{3,5}\pm$ $\omega_{1,3}^2\omega_{5,6}\pm\omega_{1,4}^2\omega_{5,6}\neq 0$
$\mathfrak{n}_{6.84}$	$\omega=\omega_{1,4}(\alpha^{14}-\alpha^{26})+\omega_{1,5}(\alpha^{15}+\alpha^{36})+$ $+\omega_{1,6}(-\varepsilon\alpha^{16}+\alpha^{25}+\alpha^{34})+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+$ $\omega_{5,6}\alpha^{56}$	$\omega_{1,6}\neq 0$
$\mathfrak{g}_{5.7}^{p,-p,-1}\oplus\mathbb{R}$	$\omega=\omega_{1,4}\alpha^{14}+\omega_{1,5}\alpha^{15}+\omega_{2,3}\alpha^{23}+\omega_{2,5}\alpha^{25}+$ $+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\omega_{2,3}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.7}^{1,-1,-1}\oplus\mathbb{R}$	$\omega=\omega_{1,3}\alpha^{13}+\omega_{1,4}\alpha^{14}+\omega_{1,5}\alpha^{15}+\omega_{2,3}\alpha^{23}+$ $+\omega_{2,4}\alpha^{24}+\omega_{2,5}\alpha^{25}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\omega_{2,3}-\omega_{1,3}\omega_{2,4}\neq 0$, $\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.8}^{-1}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{2,6}\alpha^{26}+$ $+\omega_{3,4}\alpha^{34}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{3,4}\neq 0$, $\omega_{1,2}\omega_{5,6}-\omega_{1,5}\omega_{2,6}\neq 0$
$\mathfrak{g}_{5.13}^{-1,0,r}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{3,4}\alpha^{34}+$ $+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.14}^{0}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{2,6}\alpha^{26}+$ $+\omega_{3,4}\alpha^{34}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,5}\omega_{2,6}\omega_{3,4}\omega_{5,6}\neq 0$, $\omega_{1,2}\omega_{5,6}-\omega_{1,5}\omega_{2,6}\neq 0$
$\mathfrak{g}_{5.15}^{-1}\oplus\mathbb{R}$	$\omega=\omega_{1,4}(\alpha^{14}-\alpha^{23})+\omega_{1,5}\alpha^{15}+\omega_{2,4}\alpha^{24}+$ $+\omega_{2,5}\alpha^{25}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.17}^{0,0,r}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{3,4}\alpha^{34}+$ $+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.17}^{p,-p,\pm1}\oplus\mathbb{R}$	$\omega=\omega_{1,3}(\pm\alpha^{13}+\alpha^{24})+\omega_{1,4}(\mp\alpha^{14}+\alpha^{23})+$ $+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,3}\mp\omega_{1,4}\neq 0,\ \omega_{5,6}\neq 0$
$\mathfrak{g}_{5.17}^{0,0,\pm1}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,3}(\pm\alpha^{13}+\alpha^{24})+\omega_{1,4}(\mp\alpha^{14}+\alpha^{23})$ $+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+\omega_{3,4}\alpha^{34}+\omega_{3,5}\alpha^{35}+$	$\omega_{1,2}\omega_{3,4}-\omega_{1,3}^2-\omega_{1,4}^2\neq 0$, $\omega_{3,4}\omega_{5,6}\neq 0$

	$+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	
$\mathfrak{g}_{5.18}^{0}\oplus\mathbb{R}$	$\omega=\omega_{1,3}(\alpha^{13}+\alpha^{24})+\omega_{1,5}\alpha^{15}+\omega_{2,5}\alpha^{25}+$ $+\omega_{3,4}\alpha^{34}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,3}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.19}^{-2,2}\oplus\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,5}(\alpha^{15}-\alpha^{23})+\omega_{2,5}\alpha^{25}+$ $+\omega_{3,4}\alpha^{34}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{5.19}^{-\frac{1}{2},-1}\oplus\mathbb{R}$	$\omega=\omega_{1,3}\alpha^{13}+\omega_{1,5}(\alpha^{15}-\alpha^{23})+\omega_{2,4}\alpha^{24}+$ $+\omega_{2,5}\alpha^{25}+\omega_{3,5}\alpha^{35}+\omega_{4,5}\alpha^{45}+\omega_{5,6}\alpha^{56}$	$\omega_{1,3}\omega_{2,4}\omega_{5,6}\neq 0$
$\mathfrak{g}_{3.4}^{-1}\oplus 3\mathbb{R}$ $\mathfrak{g}_{3.5}^{0}\oplus 3\mathbb{R}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,3}\alpha^{13}+\omega_{2,3}\alpha^{23}+\omega_{3,4}\alpha^{34}+$ $+\omega_{3,5}\alpha^{35}+\omega_{3,6}\alpha^{36}+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{5,6}\omega_{3,4}-\omega_{1,2}\omega_{4,6}\omega_{3,5}+$ $\omega_{1,2}\omega_{4,5}\omega_{3,6}\neq 0$
$\mathfrak{g}_{3.1}\oplus\mathfrak{g}_{3.4}^{-1}$ $\mathfrak{g}_{3.1}\oplus\mathfrak{g}_{3.5}^{0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,3}\alpha^{13}+\omega_{2,3}\alpha^{23}+\omega_{2,6}\alpha^{26}+$ $+\omega_{3,6}\alpha^{36}+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{4,5}\neq 0$, $\omega_{3,6}\omega_{1,2}-\omega_{2,6}\omega_{1,3}\neq 0$
$\mathfrak{g}_{3.4}^{-1}\oplus\mathfrak{g}_{3.4}^{-1}$ $\mathfrak{g}_{3.4}^{-1}\oplus\mathfrak{g}_{3.5}^{0}$ $\mathfrak{g}_{3.5}^{0}\oplus\mathfrak{g}_{3.5}^{0}$	$\omega=\omega_{1,2}\alpha^{12}+\omega_{1,3}\alpha^{13}+\omega_{2,3}\alpha^{23}+\omega_{3,6}\alpha^{36}+$ $+\omega_{4,5}\alpha^{45}+\omega_{4,6}\alpha^{46}+\omega_{5,6}\alpha^{56}$	$\omega_{1,2}\omega_{3,6}\omega_{4,5}\neq 0$

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