Transition Dynamics in Endogenous Recombinant Growth Models by means of Projection Methods

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Transition Dynamics in Endogenous Recombinant Growth Models by means of Projection Methods

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Abstract

This paper provides a step further in the computation of the transition path of a continuous time endogenous growth model discussed by Privileggi (2010) – based on the setting first introduced by Tsur and Zemel (2007) – in which knowledge evolves according to the Weitzman (1998) recombinant process. A projection method, based on the least squares of the residual function corresponding to the ODE defining the optimal policy of the 'detrended' model, allows for the numeric approximation of such policy for a positive Lebesgue measure range of values of the efficiency parameter characterizing the probability function of the recombinant process. Although the projection method's performance rapidly degenerates as one departs from a benchmark value for the efficiency parameter, we are able to numerically compute time-path trajectories which are sufficiently regular to allow for sensitivity analysis under changes in parameters' values.

JEL Classification Numbers: C61, C63, O31, O41.

Key words: Knowledge Production, Endogenous Recombinant Growth, Transition Dynamics, Projection Methods, Least Squares.

1 Introduction

The main contribution of this work is to provide a numeric algorithm capable of approximating the complete transition path of some parametrization of the two-sector continuous time endogenous growth model introduced by Tsur and Zemel [8] in which knowledge evolves according to the Weitzman [9] recombinant process. Privileggi [7] provides functional forms suitable to 'detrend' the model and thus obtain a closed form for the ODE defining the optimal policy which depends on a number of parameters. The author, however, through a finite-difference, fourth-order, Runge-Kutta method is able to compute a numeric approximation of the policy only for zero Lebesgue-measure sets of parameters' values; specifically, for a given set of parameters describing the production process, consumers' preferences, etc., the finite-difference method turns out to be reliable only for one single value for the – most interesting – efficiency parameter that characterizes the recombinant process of knowledge generation. Here a projection method based on the least squares of the residual function associated to the ODE defining the optimal policy is developed. Given a set of parameters' values for the economy, such method provides a meaningful numeric approximation of the policy for a positive Lebesgue-measure range of values of the efficiency parameter defining the probability of successful knowledge production, thus providing a substantial improvement upon the result obtained in [7].

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The economy is described by an endogenous growth model in which the stock of knowledge evolves according to Weitzman's [9] recombinant expansion process and is used, together with physical capital, as input factor in the competitive sector in order to produce a unique physical good. A 'regulator' maximizes the discounted utility of a representative consumer over an infinite time horizon and directly finances new knowledge production through a tax levied on the consumers; at each instant the public good 'knowledge' is produced by an independent R&D sector which is controlled by the regulator. Tsur and Zemel [8] provide conditions under which this economy exhibits sustained balanced growth in the long-run and characterize the asymptotic optimal tax rate and the common growth rate of all variables, thus generalizing Weitzman's [9] Solow-type model in which investment in knowledge production was assumed to be constant and exogenous.

Privileggi [7] provides a suitable (hyperbolic) form for the probability of success in new knowledge production through the Weitzman's recombinant process of matching existing ideas which, coupled with a Cobb-Douglas production function for the physical good and CIES preferences of the representative consumer, allows for a closed-form ODE defining the optimal policy. Such policy defines the instantaneous optimal consumption in terms of 'detrended' state-like and control-like variables when the conditions for sustained long-run growth are met. Once a reliable numeric approximation of the policy is available, the optimal time-path trajectories of the stock of knowledge, capital, output and consumption, as well as their transition growth rates, for the original model are easily computed by means of a finite-difference, Runge-Kutta method, thus allowing for comparative dynamics analysis along the transition path.

In this work a *projection method* (see, *e.g.*, Chapter 11 in Judd [2], Chapter 6 in Heer & Maussner [1], or Paragraph 5.5.2 in Novales, Fernández & Ruíz [6]) which computes the *least squares* of the *residual function* associated to the ODE defining the optimal policy as obtained in [7] is pursued. We are able to approximate the solution of such ODE – *i.e.*, the optimal policy – for a full range of values for the parameter related to the efficiency of the recombinant process in the production of new useful knowledge around the benchmark value $\beta = 0.0124$ considered in [7].

Although, due to the analytical complexity of the ODE under study, specific tests available for projection methods show that our method's performance rapidly degenerates as one departs from the benchmark value $\beta = 0.0124$, we are able to compute time-path trajectories which are sufficiently regular to allow for comparative dynamics analysis under changes in parameters' values. Specifically, even if our method exhibits an accuracy only of order around 10^{-2}, qualitative behavior of the resulting optimal time-paths trajectories (transition dynamics) appears to be sufficiently neat for a direct comparison among different trajectories, provided that they are not too far from the benchmark one. We thus draw the broader conclusion that projection methods are overall superior to finite-difference methods in numerically solving particularly awkward ODEs.

Section 2 recalls the endogenous recombinant growth model first introduced by Weitzman [9] and then further developed by Tsur and Zemel [8], while Section 3 reports its parameterization as in Privileggi [7]. Section 4 briefly describes projection methods, which are then applied in Section 5 to approximate the optimal policies for three parametrizations of the model through a least square algorithm. Finally, in Section 6 the optimal policies just obtained are used to trace out the optimal time-path trajectories of all relevant variables and a qualitative discussion of the transition dynamics is carried out, while Section 7 reports some concluding remarks and topics for future research.

2 The Model

Weitzman's [9] knowledge generation device assumes that unprocessed (*seed*) ideas are matched with other ideas available in order to yield new *hybrid* seed ideas. As a matter of fact, not all hybrid ideas turn out to be successful, only a fraction of them is, which will again be recombined with other existent (successful) ideas to produce yet new hybrids. Let us assume that m ideas are matched together. If $A(t)$ is the stock of knowledge at time t (measured as the total number of successful ideas), let $C_m[A(t)]$ denote the number of different combinations of m elements (hybrids) of $A(t)$; *i.e.*: $C_m[A(t)] = A(t)!/\{m![A(t)-m]!\}$ [*e.g.*, $C_2(A) = A(A-1)/2$]. Then, at time t the number of hybrid seed ideas is given by $H(t) = C_m[A(t)] - C_m[A(t-1)]$.

In continuous time $A(t)$, $C_m[A(t)]$ and $H(t)$ become flows and

$$H(t) = C'_m[A(t)]\,\dot{A}(t), \tag{1}$$

defining the flow of hybrid seed ideas as a function of the rate of change of knowledge stock. Let π denote the probability of obtaining a successful idea from each matching, which is assumed to depend on the ratio $J(t)/H(t)$, where $J(t)$ is a measure of the physical resources devoted to R&D (matching ideas) at instant t. Hence, this R&D expenditure produces a flow of successful new ideas that accrue the existing stock of knowledge according to

$$\dot{A}(t) = H(t)\,\pi[J(t)/H(t)]. \tag{2}$$

In our specification of the model we shall assume that only pairs of ideas can be matched and that the probability of success is described by a hyperbolic function.

A. 1 $m = 2$ *and the success probability function is given by*[1]

$$\pi(x) = \beta x/(\beta x + 1), \qquad \beta > 0. \tag{3}$$

Parameter β provides a measure of the '*degree of efficiency*' of the Weitzman matching process: the larger β the higher probability of obtaining a new successful idea out of each (pairwise) matching of seed ideas.

Combining (1) and (2) the law of motion for the stock of knowledge $A(t)$ is:

$$\dot{A}(t) = J(t)/\varphi[A(t)], \tag{4}$$

where $\varphi(A) = C'_m(A)\,\pi^{-1}[1/C'_m(A)]$ is the *expected unit cost of knowledge production*. As, for $m = 2$, $C'_2(A) = (2A-1)/2$, and from (3) we get $\pi^{-1}(x) = x/[\beta(1-x)]$, the unit cost of knowledge production can be written as

$$\varphi(A) = (2A-1)/[\beta(2A-3)] = (1/\beta)[1 + 2/(2A-3)], \tag{5}$$

which is defined for $A > 3/2$, is decreasing in A, and $\lim_{A\to\infty}\varphi(A) = 1/\pi'(0) = 1/\beta > 0$.

A 'regulator' chooses the optimal amount J to be employed in production of new knowledge in order to maximize the discounted utility of a representative consumer over an infinite time horizon. The exact amount J is levied as a tax on the representative consumer and is used to generate new useful knowledge according to (4), which is immediately and freely passed to the output producing firms operating in a competitive market.

With no loss of generality, we assume that labour is constant and normalized to one: $L \equiv 1$.

A. 2 *Output is produced according to a Cobb-Douglas technology:*

$$y(t) = \theta[k(t)]^{\alpha}[A(t)]^{1-\alpha} = \theta A(t)[k(t)/A(t)]^{\alpha}, \qquad \theta > 0,\ 0 < \alpha < 1, \tag{6}$$

depending on aggregate capital and knowledge-augmented labour, $A(t)L$ *for* $L = 1$.

[1] Note that π is independent of time and satisfies Weitzman's assumptions (p. 345 in [9]): $\pi' > 0$, $\pi'' < 0$, $\pi(0) = 0$ and $\pi(\infty) \leq 1$; moreover, $\pi'(0) = \beta < +\infty$.

Each output producing firm i maximizes instantaneous profit by renting capital k_i and hiring labour $L_i \leq 1$ from the households, taking as given the capital rental rate r, the labour wage w and the stock of knowledge A. As all firms use the same technology and operate in a competitive market, and all households are the same, the subscript i can be dropped; moreover, as firms act competitively, in equilibrium their profit is zero, that is, households earn $y = \theta A \left(k/A\right)^{\alpha} = rk + w$, moreover, the amount of capital demanded, k, satisfies

$$\theta\alpha \left(k/A\right)^{\alpha-1} = r. \tag{7}$$

Given that, at each instant t, a fraction $J(t)$ of the whole endowment of the economy, $k(t) + y(t)$, is being employed to finance R&D firms, and a fraction $c(t)$ is being consumed, capital evolves through time according to

$$\dot{k}(t) = y(t) - J(t) - c(t), \tag{8}$$

where it is assumed that capital does not depreciate. Since the upper bound for $J(t)$ and $c(t)$ is jointly given by $J(t) + c(t) \leq k(t) + y(t)$, $\dot{k}(t)$ in (8) may be negative.

A. 3 *All households enjoy an instantaneous CIES utility,*

$$u(c) = \left(c^{1-\sigma} - 1\right) / (1-\sigma), \qquad \sigma \geq 1, \tag{9}$$

and have a common discount rate, $\rho > 0$*.*

Tsur and Zemel [8] showed that three curves on the space (A, k) are useful for characterizing the solutions of the social planner problem in our regulated economy.

1. The locus $\theta\alpha \left(k/A\right)^{\alpha-1} - \theta(1-\alpha)\left(k/A\right)^{\alpha}/\varphi(A) = 0$ defines the curve on the space (A, k) on which the marginal product of capital equals that of knowledge per unit cost. Under Assumptions A.1, A.2 and using (5) it can be rewritten as a function of the only variable A:

$$\tilde{k}(A) = \left[\alpha/(1-\alpha)\right]\varphi(A)\,A = \left\{\alpha/\left[\beta(1-\alpha)\right]\right\}\left[1 + 2/(2A-3)\right]A. \tag{10}$$

   We call $\tilde{k}(A)$ in (10) the *(transitory) turnpike*.

2. The function $\tilde{k}(A)$ in (10) for large A becomes affine, defining the curve

$$\tilde{k}_{\infty}(A) = \left\{\alpha/\left[\beta(1-\alpha)\right]\right\}(A+1). \tag{11}$$

   Note that $\tilde{k}(A)$ lies above $\tilde{k}_{\infty}(A)$ for all $A < \infty$, approaching $\tilde{k}_{\infty}(A)$ as A increases. We call $\tilde{k}_{\infty}(A)$ in (11) the *asymptotic turnpike*.

3. Finally, on the locus $\theta\alpha\left(k/A\right)^{\alpha-1} = \rho$ the marginal product of capital equals the individual discount rate, which, by (7), implies $r = \rho$. It can be written as a linear function of A:

$$\hat{k}(A) = \left(\theta\alpha/\rho\right)^{1/(1-\alpha)} A. \tag{12}$$

   We call $\hat{k}(A)$ in (12) the *stagnation line*.

Proposition 1 (Tsur and Zemel [8])

i) *A necessary condition for the economy to sustain long-run growth requires the stagnation line to lie above the asymptotic turnpike for sufficiently large A, that is, $(\theta\alpha/\rho)^{1/(1-\alpha)} > \alpha/\left[\beta(1-\alpha)\right]$ must hold, which can be written in terms of an upper bound for the discount rate,*

$$\rho < r_\infty, \tag{13}$$

where $r_\infty = \theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}$ defines the long-run capital rental rate. *Conversely, if $\rho \geq r_\infty$ the economy eventually reaches a steady (stagnation) point on the line $\hat{k}(A)$ in (12) corresponding to zero growth.*

ii) *Under (13), for any given initial knowledge stock A_0 there is a corresponding threshold capital stock $k^{sk}(A_0) \geq 0$ such that whenever $k_0 \geq k^{sk}(A_0)$ the economy – possibly after an initial transition outside the turnpike – first reaches the turnpike $\tilde{k}(A)$ in a finite time, and then continues to grow along it as time elapses until the asymptotic turnpike $\tilde{k}_\infty(A)$ is reached in the long-run. Along $\tilde{k}_\infty(A)$ the economy follows a balanced growth path characterized by a common constant growth rate of output, knowledge, capital and consumption given by*

$$\gamma = (r_\infty - \rho)/\sigma. \tag{14}$$

Moreover, $J(t) > 0$ for all t, while, as $t \to \infty$, $J(t) < y(t)$ and the income shares devoted to investments in knowledge and capital are constant and given respectively by

$$s_\infty = (1-\alpha)\gamma/r_\infty \qquad \text{and} \qquad s_\infty^k = \alpha\gamma/r_\infty. \tag{15}$$

If $k_0 < k^{sk}(A_0)$ the economy eventually stagnates.

Proposition 1, whose proof can be found in [8], establishes that if (13) holds – *i.e.*, either households are patient enough or, given the impatience of households, economies are more productive (larger θ) and/or with more efficient R&D (larger β) – and k_0 is sufficiently high with respect to the initial knowledge stock, A_0, the economy grows along a turnpike path which, in the long-run, converges to a balanced growth path with knowledge and capital growing at the same constant rate and with constant saving rate. For a Cobb-Douglas economy the long-run income shares allocated to R&D and saving turn out to be proportional to the knowledge and capital shares in the production function.

Two optimal regimes are possible:

1. zero R&D, $J \equiv 0$, which, if maintained forever, eventually leads the economy to some steady state (stagnation point) on the line $\hat{k}(A)$, and

2. a path along the turnpike $\tilde{k}(A)$ – maybe started after a finite period of transition outside the turnpike – envisaging growth as time elapses and, if maintained forever, eventually leading to a balanced growth path along the asymptotic turnpike $\tilde{k}_\infty(A)$.

Under (13) and if $k_0 \geq k^{sk}(A_0)$ it can be shown that the turnpike $\tilde{k}(A)$ is 'trapping', *i.e.*, the economy keeps growing along it after it is reached. Hence, there are two types of transitions: one driving the system toward the turnpike starting from outside it, and another characterizing the optimal path along $\tilde{k}(A)$ after it has been entered. We shall focus on the latter; specifically, we shall assume that (13) holds and, for simplicity, that the economy starts *exactly* on the turnpike: $k_0 = \tilde{k}(A_0)$. In this scenario $k_0 \geq k^{sk}(A_0)$ is certainly satisfied, as $\tilde{k}(A)$ is trapping.

3 Optimal Dynamics Along the Turnpike and 'Detrended' Policy

In what follows we recall without proofs the main steps involved in the construction pursued by Privileggi [7] in order to build the ODE that defines the optimal policy for our economy. We refer the reader to [7] for all details.

As, under condition (13) and assuming that $k_0 = \tilde{k}(A_0) = \{\alpha/[\beta(1-\alpha)]\}[1 + 2/(2A_0 - 3)]A_0$, the relevant variables are bound to move along the turnpike $\tilde{k}(A)$, the social planner problem can be reduced to a infinite-horizon, continuous-time optimization problem in only two variables – the stock of knowledge, A, and consumption, c, which are the state and control variables respectively – and one dynamic constraint:

$$\max_{\{c\}} \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \tag{16}$$

$$\text{subject to} \begin{cases} \dot{A} = [\tilde{y}(A) - c] / \left[\tilde{k}'(A) + \varphi(A)\right] \\ 0 \leq c \leq \tilde{k}(A) + \tilde{y}(A) \\ A(0) = A_0 > 0, \end{cases}$$

where the time argument has been dropped for simplicity, $\tilde{y}(A) = \theta A \left[\tilde{k}(A)/A\right]^\alpha$ is the output as a function of the sole variable A on the turnpike $\tilde{k}(A)$ as defined in (10), $\tilde{k}'(A) = \partial\tilde{k}(A)/\partial A$, and $\varphi(A)$ is given by (5).

Necessary conditions on the current-value Hamiltonian for problem (16) yield the following system of ODEs defining the optimal dynamics for A and c along the turnpike under Assumption A.3:

$$\begin{cases} \dot{A} = \left\{\theta A \left[\tilde{k}(A)/A\right]^\alpha - c\right\} / \left[\tilde{k}'(A) + \varphi(A)\right] \\ \dot{c} = c \left\{\theta\alpha \left[\tilde{k}(A)/A\right]^{\alpha-1} - \rho\right\} / \sigma. \end{cases} \tag{17}$$

As the stock of knowledge A cannot be depleted and, by Proposition 1 (ii), the optimal investment in R&D must be positive along the turnpike, A must grow: $\dot{A}(t) > 0$ for all $t \geq 0$. It can be shown that the graph of $\tilde{k}(A)$ is a U-shaped curve on $(3/2, +\infty)$, reaching its unique minimum on $\underline{A} = 3/2 + \sqrt{6}/2 > 3/2$. This implies that capital $\tilde{k}(t)$ decreases when t is small and increases for larger t, envisaging that in early times it is optimal to take away some physical capital from the output-producing sector and invest it in R&D, so that the stock of knowledge A can take-off.

Moreover, the denominator on the RHS of the first equation in (17), $\tilde{k}'(A) + \varphi(A)$, vanishes on the unique point

$$A^s = 1 + (1/2)\left(\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right), \tag{18}$$

which is larger than $3/2$, and $\tilde{k}'(A) + \varphi(A) < 0$ for $3/2 < A < A^s$, while $\tilde{k}'(A) + \varphi(A) > 0$ for $A > A^s$. As $\dot{A}(t) > 0$ for all $t \geq 0$, the whole ratio on the RHS of the first equation in (17) must be positive for all $t \geq 0$, that is, the numerator, $\theta A \left[\tilde{k}(A)/A\right]^\alpha - c$, must have the same sign of the denominator, $\tilde{k}'(A) + \varphi(A)$, and must vanish on A^s as well. In other words, as $\theta A \left[\tilde{k}(A)/A\right]^\alpha = \tilde{y}(A)$, the optimal consumption c must satisfy $c > \tilde{y}(A)$ for $3/2 < A < A^s$, $c < \tilde{y}(A)$ for $A > A^s$, and $c = \tilde{y}(A)$ for $A = A^s$, where A^s is defined in (18) (see Proposition 3 in [7]). We thus conclude that in early times it is optimal to take away physical capital from the output-producing sector both for investment in R&D and consumption.

Clearly, system (17) diverges in the long-run. Thus, we transform the state variable A and the control c in a state-like variable, μ, and a control-like variable, χ, defined respectively by

$$\mu = \tilde{k}(A)/A = [\alpha/(1-\alpha)]\,\varphi(A) = \{\alpha/[\beta(1-\alpha)]\}[1+2/(2A-3)] \tag{19}$$
$$\chi = c/A, \tag{20}$$

where in (19) we used (10) and (5). Hence, A is related to μ as follows:

$$A = \alpha/[\beta(1-\alpha)\mu - \alpha] + 3/2. \tag{21}$$

Following the steps in [7], under Assumptions A.1-A.3, we obtain the following system of ODEs describing the transition optimal dynamics in the detrended variables μ (state) and χ (control):

$$\begin{cases} \dot{\mu} = [1 - 2\beta(1-\alpha)\mu/Q(\mu)](\theta\mu^{\alpha} - \chi) \\ \dot{\chi} = [(\theta\alpha\mu^{\alpha-1} - \rho)/\sigma - 2\alpha\beta(1-\alpha)(\theta\mu^{\alpha} - \chi)/Q(\mu)]\,\chi, \end{cases} \tag{22}$$

where

$$Q(\mu) = -3\beta^2(1-\alpha)^2\mu^2 + 2\beta(1-\alpha)(1+2\alpha)\mu - \alpha^2. \tag{23}$$

Since $A > 3/2$, from (19) one immediately obtains the range $[\alpha/(\beta(1-\alpha)), +\infty)$ for the state-like variable μ, with endpoints corresponding to $A \to +\infty$ and $A \to 3/2$ respectively. System (22) has *three steady states* in the (μ, χ) *phase diagram* (see [7] for details).

1. The steady value
$$\mu^* = \alpha/[\beta(1-\alpha)] \tag{24}$$
for variable μ corresponds to *long-run capital/knowledge ratio* along the asymptotic turnpike $\tilde{k}_{\infty}(A)$ [μ^* is the slope of $\tilde{k}_{\infty}(A)$ in (11)]. To this value corresponds the *long-run consumption/knowledge ratio* defined by
$$\chi^* = \theta\{\alpha/[\beta(1-\alpha)]\}^{\alpha}(1-1/\sigma) + \rho/[\beta\sigma(1-\alpha)], \tag{25}$$
which is the asymptotic slope of the optimal policy $c(A)$, when consumption steadily grows at the constant rate γ defined in (14). The point (μ^*, χ^*), with coordinates defined in (24) and (25), is *saddle-path stable*, with the stable arm converging to it from north-east whenever the initial values $(\mu(t_0), \chi(t_0))$ are suitably chosen.

2. The point $(\hat{\mu}, \hat{\chi})$, with
$$\hat{\mu} = (\theta\alpha/\rho)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \hat{\chi} = \theta(\theta\alpha/\rho)^{\frac{\alpha}{1-\alpha}}, \tag{26}$$
is an unstable clockwise-rotating spiral which is irrelevant for our analysis, as the optimal trajectory keeps well apart from it; $\hat{\mu}$ in (26) corresponds to $\hat{A} = \alpha/\left[\beta(1-\alpha)(\theta\alpha/\rho)^{\frac{1}{1-\alpha}} - \alpha\right] + 3/2$, point at which the turnpike $\tilde{k}(A)$ intersects the stagnation line $\hat{k}(A)$.

3. The point (μ^s, χ^s) defined by
$$\mu^s = \left(1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right)/[3\beta(1-\alpha)] \qquad \text{and} \qquad \chi^s = \theta(\mu^s)^{\alpha}, \tag{27}$$
is a 'supersingular' steady state whose Jacobian contains elements diverging to infinity, so that its stability/instability properties cannot be classified analytically. It is crossed by the stable arm of the saddle-path at low values of the stock of knowledge A, that is, in early times (in

proximity of the very beginning of the economy's dynamics). The coordinate μ^s, as defined in (27), is larger than μ^* in (24) and is the largest (and only admissible) root of the function $Q(\mu)$ defined in (23), with $Q(\mu) > 0$ for $\mu^* \leq \mu < \mu^s$ and $Q(\mu) < 0$ for $\mu > \mu^s$. The point (μ^s, χ^s) corresponds to the critical point $(A^s, c(A^s))$, with $c(A^s) = \tilde{y}(A^s)$, defined by (18) in the space (A, c).

While the singular point (μ^s, χ^s) lies north-east of the long-run steady state (μ^*, χ^*) for all admissible parameters' values, the position of $(\hat{\mu}, \hat{\chi})$ depends on the magnitude of the discount factor ρ with respect to parameters α, θ and β. Following [7], we shall assume that

$$\theta\alpha (\mu^s)^{\alpha-1} < \rho < \theta\alpha (\mu^*)^{\alpha-1}, \tag{28}$$

envisaging a phase diagram in which $(\hat{\mu}, \hat{\chi})$ lies north-east of (μ^*, χ^*) and south-west of (μ^s, χ^s) (see Proposition 4 and Figure 1 in [7], where all loci are drawn and stability/instability properties of steady states are illustrated). Note that the RHS in (28) equals the necessary condition (13) for long-run growth.

We have seen in Section 2 that $\tilde{k}(A) > \tilde{k}_\infty(A)$ for all A (and thus for all t); this is consistent with $\mu(t) > \mu^*$ for all t. Moreover, it can be shown that the stable arm $\chi(\mu)$, which is the *optimal policy expressed in terms of state-like and control-like variables*, approaches (μ^*, χ^*) from north-east; consequently, along the turnpike both ratios $\tilde{k}[A(t)]/A(t)$ [$= \mu(t)$] and $c(t)/A(t)$ [$= \chi(t)$] decline in time when they are approaching the asymptotic turnpike $\tilde{k}_\infty(A)$ corresponding to (μ^*, χ^*).

In order to study the policy function $\chi(\mu)$ – which is the conjugate of $c(A)$ in the original model – we apply the technique developed by Mulligan and Sala-i-Martin [4] and tackle the unique ODE given by the ratio between the equations in (22):

$$\chi'(\mu) = \frac{[(\alpha\theta\mu^{\alpha-1} - \rho)/\sigma] Q(\mu) - 2\alpha\beta(1-\alpha)[\theta\mu^\alpha - \chi(\mu)]}{[Q(\mu) - 2\beta(1-\alpha)\mu][\theta\mu^\alpha - \chi(\mu)]} \chi(\mu), \tag{29}$$

where $Q(\mu)$ is defined in (23). We plan to apply a *Projection method* to approximate (29).

4 Projection Methods

Here we briefly recall the main features of Projection methods for approximating solutions of functional equations. For more details we refer the reader to, among others, Chapter 11 in [2], [3], Chapter 6 in [1], or Paragraph 5.5.2 in [6].

Consider the following functional equation:

$$\mathcal{N}(f) = 0, \tag{30}$$

where $\mathcal{N} : B_1 \to B_2$ and B_1, B_2 are vector spaces of functions $f : X \subset \mathbb{R}^m \to \mathbb{R}^l$.

To approximate the solution f satisfying (30), we choose a degree of approximation, n, and make the assumption that the approximation $\bar{f}$ is a linear combination of $n+1$ simple functions, $\psi_i(x)$:

$$\bar{f}(x, a) = \sum_{i=0}^{n} a_i \psi_i(x), \qquad x \in X \subset \mathbb{R}^m,\ a = (a_0, a_1, \ldots, a_n). \tag{31}$$

If B_1 is endowed with an inner product, then the functions $\psi_0(x), \psi_1(x), \ldots, \psi_n(x)$, are usually chosen to be the first $n+1$ elements of an infinite orthogonal family of polynomials, so that they are numerically more distinguishable from each other. The choices of the family $\{\psi_i(x)\}_{i=0}^n$ and of n

lay down the structure and flexibility of the approximation respectively. As the only unknown is the vector a, the original infinite-dimensional problem has been reduced to a finite-dimensional one.

Specifically, we shall look for a vector of coefficients $a = (a_0, a_1, \ldots, a_n)$ so that $\mathcal{N}\left(\bar{f}\right)$ turns out to be as close as possible to the zero function in B_2, where 'closeness' is measured in terms of a norm in B_2, itself defined through some inner product $\langle \cdot, \cdot \rangle_2$ over B_2. In order to check proximity of $\mathcal{N}\left(\bar{f}\right)$ to the zero function in B_2, we must compute $\mathcal{N}\left(\bar{f}\right)$; this often requires a numeric approximation, $\bar{\mathcal{N}}$, of the map $\mathcal{N}$ itself, as will be the case in our simulation.

Define the *residual function* as the (approximated) functional equation (30) evaluated at the approximate solution $\bar{f}$ defined in (31):

$$R(x, a) = \bar{\mathcal{N}}\left[\bar{f}(x, a)\right]. \tag{32}$$

For each given $a = (a_0, a_1, \ldots, a_n)$, a set of *weight functions* $\{g_i(x)\}_{i=0}^n$ in B_2 together with $R(x, a)$ in (32) define an inner product inducing a norm on B_2 through $n+1$ *projections*:

$$P_i(a) = \langle R(x, a), g_i(x) \rangle_2 = \int_X R(x, a)\, g_i(x)\, dx, \qquad i = 0, 1, \ldots, n. \tag{33}$$

Given the set of weight functions $\{g_i(x)\}_{i=0}^n$, the goal is to choose $a = (a_0, a_1, \ldots, a_n)$ so that the projections P_i in (33) are close to zero:

$$\int_X R(x, a)\, g_i(x)\, dx = 0 \qquad \text{for all } i = 0, 1, \ldots, n. \tag{34}$$

Projection methods are also called "weighted residuals methods" (see [3]) as (34) get the residual R in (32) close to zero in a weighted integral sense.

Methods differ according to the form of the weight functions g_is; the following are the three most widely used methods.

1. **Least squares method** assumes that $g_i(x) = \partial R(x, a) / \partial a_i$, which are derived by calculating the FOC for the following minimization problem:

$$\min_a \int_X [R(x, a)]^2\, dx. \tag{35}$$

   In other words, it computes the L^2 norm of $R(x, a)$ and chooses a that solves (35).

2. **Galerkin method** assumes that $g_i(x) = \psi_i(x)$, thus forcing $R(x, a)$ to be orthogonal to each of the basis polynomials. Therefore, a is chosen to solve the following system of $n+1$ equations:

$$\langle R(x, a), \psi_i(x) \rangle_2 = \int_X R(x, a)\, \psi_i(x)\, dx = 0, \qquad i = 0, 1, \ldots, n. \tag{36}$$

3. **Orthogonal collocation method** assumes that $g_i(x) = \delta(x - x_i)$, where $\delta(\cdot)$ is the Dirac delta function and x_i are n *collocation points* corresponding to the zeros of the largest-degree basis polynomial, $\psi_n(x)$. That is, $a = (a_0, a_1, \ldots, a_n)$ is chosen to solve a system of $n+1$ equations in which the first n are

$$\langle R(x, a), \delta(x - x_i) \rangle_2 = R(x_i, a) = 0, \qquad i = 1, \ldots, n, \tag{37}$$

   while the $n + 1^{th}$ is usually given by some condition provided by the problem under study.[2] Note that, unlike the previous two, this method has the advantage of avoiding the call for a numeric approximation of integrals.

[2]An equation independent of the projection method is required as in most commonly used families of orthogonal polinomials the first element is $\psi_0(x) \equiv 1$, having no zeros. For example, when (30) is an ODE, the condition may be the boundary value provided by the Cauchy problem associated to it.

Four *accuracy tests* can be performed on the approximate solution $\bar{f}(x,a)$ in (31) for the vector of coefficients a^* obtained by solving any of the systems above. First, one can evaluate the residual function $R(x,a^*)$ over a grid of points $x \in X$ and check whether it actually remains sufficiently close to the zero function; this provides an estimate of the error made when using the approximation $\bar{f}(x,a^*)$ in place of the true solution of (30). Second, if a plot of the residual function $R(x,a^*)$ is available, it should be checked that $R(x,a^*)$ exhibits oscillating behavior around zero, so to be sure that all polynomials ψ_i in the basis give a fair contribution in the approximation $\bar{f}(x,a^*)$. Third, for special basis of polynomials $\{\psi_i(x)\}_{i=0}^n$ used in (31), a good approximation requires the coefficients a_i to decrease rapidly and that a_n is small; for example, this property must hold when Chebyshev polynomials are used, as will be the case in our simulation (see Theorem 6.4.2 on p. 209 in [2]). Finally, a qualitative analysis of the approximated solution based on the expected behavior of the model under scrutiny clearly provides another accuracy test.

5 Simulations of the Optimal Policy

We fix the same values for parameters α, ρ, σ and θ as assumed in [7] and are common in the macroeconomic literature (see, *e.g.*, [5]):

$$\alpha = 0.5, \qquad \rho = 0.04, \qquad \sigma = \theta = 1. \tag{38}$$

Note that $\sigma = 1$ implies logarithmic instantaneous utility.

Our goal is to perform comparative dynamics analysis among different transition trajectories characterized by the same parameters' values as in (38) and starting from the same initial stock of knowledge, A_0, for different values of parameter β measuring the degree of efficiency of the recombinant process of matching ideas. In other words, we aim at comparing the whole time-path trajectory of economies which are equal in all respects but for the technological parameter β determining the speed of knowledge evolution. The following three values will be considered for parameter β:

$$\beta_L = 0.0108, \qquad \beta_M = 0.0124 \qquad \text{and} \qquad \beta_U = 0.0146. \tag{39}$$

For the parameters' values in (39) – all satisfying the necessary growth condition (13), which for the other parameters' values in (38) turns out to be $\beta > 0.0064$ – we are able to produce satisfactory approximations of the optimal trajectories for our economy through the projection method. The middle value, $\beta_M = 0.0124$, corresponding to the unique value for which Privileggi [7] managed to simulate the optimal policy for the same model, will be considered as benchmark. Such value allows for a comparison between the approximate solution here delivered by a projection method and the approximate solution produced by the finite-difference method used in [7] for the same policy.

The three economies characterized by β_L, β_M and β_U as in (39) will be labeled L, M and U respectively; accordingly, all variables belonging to each of the three economies will have the same letter as subscript. For $h \in \{L, M, U\}$, we shall exploit informations provided by the steady states (μ_h^*, χ_h^*) and (μ_h^s, χ_h^s) [see (24), (25) and (27)] of the three economies in order to build the projection method algorithm. Table 1 reports the coordinates of such points.

Economy	μ_h^*	χ_h^*	μ_h^s	χ_h^s
L	92.5926	7.4074	234.7392	15.3212
M	80.6452	6.4516	204.4503	14.2986
U	68.4932	5.4795	173.6427	13.1774

TABLE 1: coordinates of the relevant steady states in the three economies.

It is easily seen that all three pairs (μ_h^*, μ_h^s) in Table 1 satisfy condition (28) for the parameters' values in (38). In order to establish the initial stock of knowledge, A_0, common to the three economies we take a μ-value larger than the largest value for μ_h^s in Table 1, corresponding to economy L; specifically, we set $\mu_{\max} = \mu_L^s + 30 = 264.7392$. Using (21), we get the common stock of knowledge at time $t = 0$ given by

$$A_0 = \alpha / [\beta (1 - \alpha) \mu_{\max} - \alpha] + 3/2 = 2.0379. \tag{40}$$

Initial stocks of capital are obtained by computing the turnpike value at $A_0 = 2.0379$ for each economy using (10): $(k_0)_h = \tilde{k}_h (A_0)$; *i.e.*,

$$(k_0)_L = 539.5043, \qquad (k_0)_M = 469.8908 \qquad \text{and} \qquad (k_0)_U = 399.0853. \tag{41}$$

Figure 1(a) shows the (transitory) turnpike curves in the (A, k) space corresponding to each economy under study: *light grey*, *dark grey* and *black* curves correspond to turnpikes in the L, M and U economies respectively. We shall identify with these colors all relevant curves related to each economy throughout the paper. Each curve converges to its own linear asymptotic turnpike $\tilde{k}_\infty (A)$ defined in (11), corresponding to long-run balanced growth with (constant) growth rates given by (14):

$$\gamma_L = 0.0120, \qquad \gamma_M = 0.0157 \qquad \text{and} \qquad \gamma_U = 0.0204. \tag{42}$$

More efficient recombinant processes (larger β) require less capital, k, for a given stock of knowledge A along the turnpike; moreover, economies with larger β clearly grow faster in the long-run.

Figure 1(b) draws loci and steady states of all three economies in the (μ, χ) space; the three pairs of steady states useful for our analysis – (μ_h^*, χ_h^*) and (μ_h^s, χ_h^s) for $h \in \{L, M, U\}$ – are the balls colored light grey to black. The intersection of the four curves south west of (μ_U^s, χ_U^s) corresponds to the steady state $(\hat{\mu}, \hat{\chi})$ defined in (26) – irrelevant in our study – which is the same for all economies. For details on the loci's analytic forms see [7].

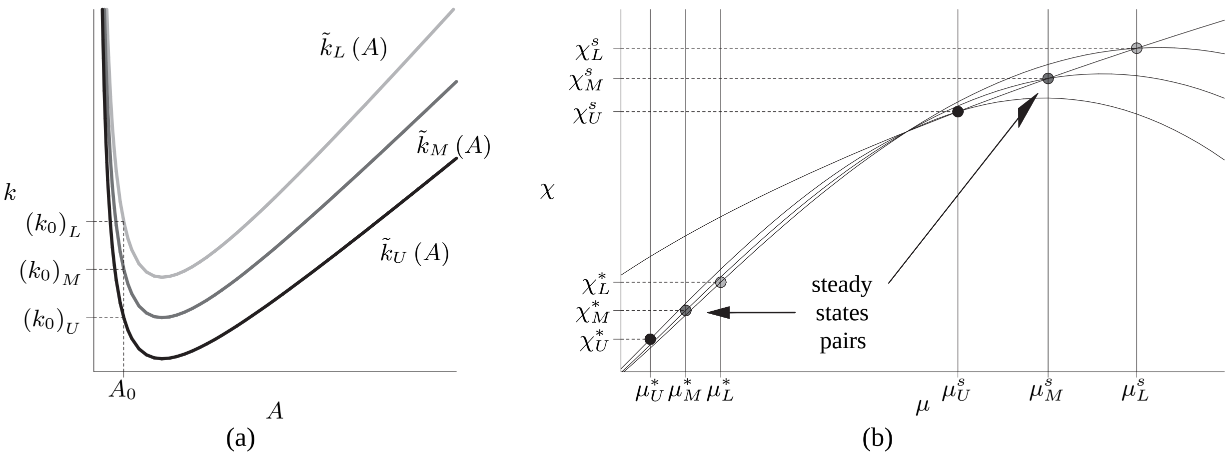

FIGURE 1: (a) turnpikes and (b) loci and steady states of the three economies.

For each of the three economies we now apply a projection method as described in Section 4 to the ODE defining the optimal policy according to (29). To approximate the true policy function, $\chi(\mu)$, solving the functional equation (29) we build the approximate function in (31) as a linear combination of **Chebyshev polynomials**, $T_i(x)$, which are mutually orthogonal with respect to the *weighting function* $w(x) = 1/\sqrt{1 - x^2}$ on the (closed) interval $[-1, 1]$. They are defined either directly as $T_i(x) = \cos[i \arccos(x)]$ or recursively by $T_0(x) \equiv 1$, $T_1(x) = x$ and $T_{i+1}(x) =$

$2xT_i(x) - T_{i-1}(x)$. After several tests, we chose $\boldsymbol{n} = \mathbf{7}$ as **degree of approximation**, because larger n would not add any improvement upon the simulation, as will be clarified later.

For each $h \in \{L, M, U\}$, an interval $\left[\underline{\mu}_h, \overline{\mu}_h\right]$ containing the abscissae of the two relevant steady states, μ_h^* and μ_h^s, is chosen as (compact) state space for the approximation. Hence, the argument of each polynomial in the linear combination must be translated from $\left[\underline{\mu}_h, \overline{\mu}_h\right]$ to $[-1, 1]$; accordingly, for $h \in \{L, M, U\}$, the *approximate functions* in (31) are

$$\bar{\chi}_h(\mu, a) = \sum_{i=0}^{7} a_i T_i \left(\frac{2\mu - \underline{\mu}_h - \overline{\mu}_h}{\overline{\mu}_h - \underline{\mu}_h} \right), \qquad \mu \in \left[\underline{\mu}_h, \overline{\mu}_h\right], \; a = (a_0, a_1, \ldots, a_7). \tag{43}$$

Using (43) in (29), the *residual functions* in (32) turn out to be

$$\begin{aligned} R_h(\mu, a) = {} & \left[\frac{\partial}{\partial \mu} \bar{\chi}_h(\mu, a)\right] \left[Q(\mu) - 2\beta(1-\alpha)\mu\right] \left[\theta\mu^\alpha - \bar{\chi}_h(\mu, a)\right] \\ & - \left\{ \frac{\alpha\theta\mu^{\alpha-1} - \rho}{\sigma} Q(\mu) - 2\alpha\beta(1-\alpha)\left[\theta\mu^\alpha - \bar{\chi}_h(\mu, a)\right] \right\} \bar{\chi}_h(\mu, a), \end{aligned} \tag{44}$$

with $Q(\mu)$ defined in (23) and $h \in \{L, M, U\}$.

Under the assumption that all three economies start with the same stock of knowledge, $A_0 = 2.0379$, at $t = 0$ as computed in (40), and that the initial amounts of capital are given by the turnpike values on A_0 as in (41), the starting point of each trajectory in the (μ, χ) space, corresponding to the the upper endpoint of the interval $\left[\underline{\mu}_h, \overline{\mu}_h\right]$, is given by $\overline{\mu}_h = (k_0)_h / A_0$. As $t \to \infty$, each economy's trajectory ends up on its own (unique) saddle-path stable steady state, (μ_h^*, χ_h^*). As a matter of fact, we let the lower endpoints be slightly smaller than μ_h^*;[3] specifically, we set $\underline{\mu}_h = \mu_h^* - 0.01$ for all $h \in \{L, M, U\}$. Thus, each projection procedure is run on the following state spaces:

$$\begin{aligned} \left[\underline{\mu}_L, \overline{\mu}_L\right] &= [92.5826, 264.7392] \\ \left[\underline{\mu}_M, \overline{\mu}_M\right] &= [80.6352, 230.5793] \\ \left[\underline{\mu}_U, \overline{\mu}_U\right] &= [68.4832, 195.8345], \end{aligned}$$

while the whole range of the analysis will be $[\mu_{\min}, \mu_{\max}] = \left[\underline{\mu}_U, \overline{\mu}_L\right] = [68.4832, 264.7392]$.

After trying all three methods recalled in Section 4 (least squares, Galerkin and orthogonal collocation), we opted for **least squares**, as in this case it outperforms the other two in all respects. More precisely, for each $h \in \{L, M, U\}$ we solve

$$\min_a F_h(a) = \min_a \int_{\left[\underline{\mu}_h, \overline{\mu}_h\right]} \left[R_h(\mu, a)\right]^2 d\mu, \tag{45}$$

with respect to $a = (a_0, a_1, \ldots, a_7) \in \mathbb{R}^8$, where the residual function $R_h(\mu, a)$ is defined in (44) and the integral in the objective function is approximated through **Gauss-Chebyshev quadrature** over

[3]As the steady state value (μ_h^*, χ_h^*) is explicitly used in the projection algorithm, in order to avoid singularities it is convenient to treat it as an interior point of the state space.

57 nodes[4] on $\left[\underline{\mu}_h, \overline{\mu}_h\right]$; that is, we set (see eqn. 7.2.7 on p. 260 in [2])

$$F_h(a) \approx \frac{\pi\left(\overline{\mu}_h - \underline{\mu}_h\right)}{2 \times 57} \sum_{i=1}^{57} \left\{ R_h \left[\frac{(x_i + 1)\left(\overline{\mu}_h - \underline{\mu}_h\right)}{2} + \underline{\mu}_h, a \right] \right\}^2 \sqrt{1 - x_i^2}, \tag{46}$$

where x_i are the zeros of the 57-order Chebyshev polynomial, $T_{57}(x)$, over $[-1, 1]$.

Problem (45) is stated as unconstrained optimization problem. However, since our final goal is to use the approximate solutions (43) of the optimal policies, $\chi_h(\mu)$, in the computation of time-path trajectories, we must handle with circumspection the steady states (μ_h^s, χ_h^s) and (μ_h^*, χ_h^*), especially the former, which, as discussed in Section 2, is a 'supersingular' point. As (μ_h^s, χ_h^s) in the (μ, χ) space corresponds to $\left(A^s, \tilde{k}_h(A^s)\right)$ in the (A, k) space, where A^s is defined in (18) as a point on which both the numerator and the denominator of the law of motion of the state variable, A, vanish [see the first equation in (17)], our approximate trajectories $\bar{\chi}_h$ must keep as close as possible to (μ_h^s, χ_h^s) so to prevent the subsequent algorithm computing time-path trajectories from diverging to infinity on A^s. Similarly, $\bar{\chi}_h$ must be sufficiently close to (μ_h^*, χ_h^*) so to guarantee convergence of the transitory turnpikes $\tilde{k}_h(A)$ to their own asymptotic turnpikes as defined in (11) in the (A, k) space. In principle, any numeric unconstrained optimization algorithm cannot assure that the approximate solution is sufficiently close to some point. Therefore, we decided to apply a *constrained optimization routine* by adding to (45) the two equality constraints $\bar{\chi}_h(\mu_h^s, a) = \chi_h^s$ and $\bar{\chi}_h(\mu_h^*, a) = \chi_h^*$. By applying *Maple 13 nonlinear programming (NLP) solver* with the *sequential quadratic programming* (sqp) method[5] to the constrained version of (45) with objective approximated by (46), we have been able to keep the distance between the approximated trajectory $\bar{\chi}_h$ and the two critical points less to 10^{-5}, enough to prevent the algorithm approximating time-path trajectories to break apart.

As in all optimization algorithms, the **initial guess** a^0 from which the Maple 13 solver starts its search is critical. Learning from the (only) approximate solution computed in [7], we know that the graph of the approximate solution $\bar{\chi}_h$ does not lie too far from the segment joining the two steady states (μ_h^*, χ_h^*) and (μ_h^s, χ_h^s) (see Figure 3 on p. 270 in [7]). Therefore, we build the initial vector a^0 through a *Chebyshev regression of order* 7 on such segment; to this purpose, we implement Algorithm 6.2 on p. 223 in [2] with $m = n + 1 = 8$ on the line crossing the two steady states.

The approximate optimal policies $\bar{\chi}_h(\mu)$ obtained through our procedure for the three economies, together with loci and steady states, are plotted in Figure 2 in light grey, dark grey and black. The complete Maple 13 code of the whole procedure is available from the author upon request.[6]

Applying the *accuracy tests* described at the end of Section 4 to the approximate policies $\bar{\chi}_h(\mu)$ obtained in the last paragraph it turns out that our results exhibit a maximum error of 3×10^{-4} only for the benchmark case M – corresponding to $\beta_M = 0.0124$ – already solved in [7] through finite-difference methods.[7] For $h = L$ and $h = U$ the maximum errors rapidly worsen, getting around

[4]To avoid collinearity with the approximate function, the number of nodes must be different than n, the number of elements of the basis used in $\bar{\chi}_h$ defined by (43). Also, the larger the nodes the better the approximation in (46). We observed no differences in the results with nodes between 10 and 100, while larger nodes start slowing down the algorithm with no improvement. Hence, we opted for an intermediate value around $100/2 = 50$, to which we add the degree of approximation, $n = 7$.

[5]See pp.125-126 in [2].

[6]We also tried a minimization algorithm in (45) based on purely *Newton iterations* with *line search* and a quadratic *penalty function* for the two constraints [see equation (4.6.7) on p. 123 in [2]], but at best we were able to exactly replicate the solution provided by the NLP solver in the 'Optimization' package of Maple 13, adding no improvement over it. Furthermore, Galerkin and orthogonal collocation methods have been tried through the 'fsolve' algorithm of Maple 13 applied to systems (36) and (37) respectively: both yielded definitely worse results from all points of view.

[7]For this special case the results from finite-difference and least squares projection methods coincide.

3×10^{-2}. These errors correspond to the maximum of the (absolute value of the) residual function $R_h(\mu, a)$ – evaluated at a solving (45) – over the interval $\left[\underline{\mu}_h, \overline{\mu}_h\right]$, as shown in Figures 3(a) and 3(b) [Figure 3(b) amplifies the values of R_M in Figure 3(a)], which provide a measure of the maximum distance between the approximate $\bar{\chi}_h(\mu)$ and the true $\chi_h(\mu)$. Moreover, it is clear from the figures that none of them really oscillate around zero.

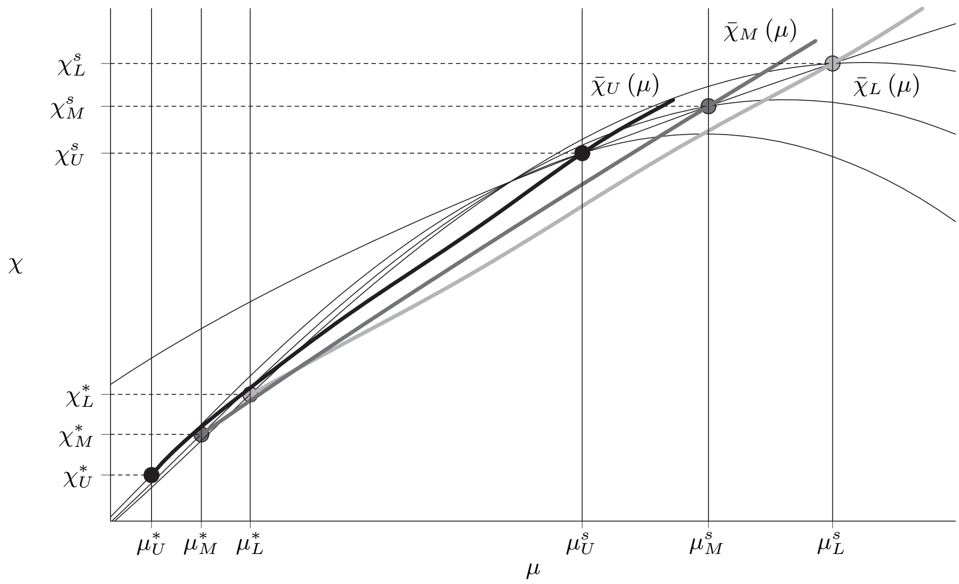

FIGURE 2: approximate detrended policies in the three economies $h \in \{L, M, U\}$.

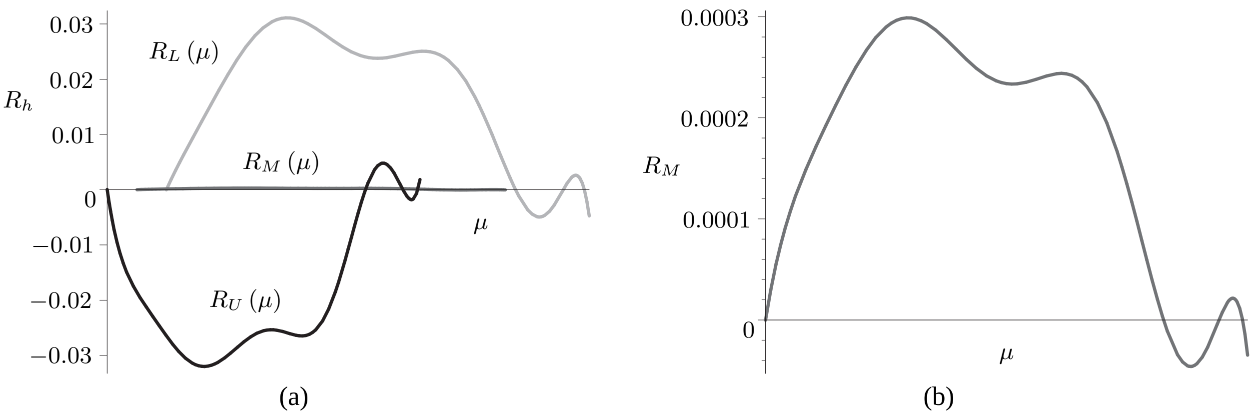

FIGURE 3: (a) residual functions for the three economies; (b) magnification of residuals for the benchmark economy $h = M$.

Finally, Table 2 lists the coefficients a_is of the approximate policies $\bar{\chi}_h(\mu, a)$ in (43) obtained in the last paragraph for $h \in \{L, M, U\}$. It is immediately seen that when $h = M$ they exhibit a decreasing pattern only up to a_6 – at values around 10^{-4} – and they start to oscillate thereafter, while when $h = L$ or $h = U$ they start to oscillate already after a_3 – at values around 3×10^{-2}.

Coefficients	Economy L	Economy M	Economy U
a_0	12.205573	11.236049	10.216761
a_1	4.867376	4.693670	4.435077
a_2	0.038693	-0.080298	-0.185641
a_3	-0.025176	0.009586	0.039838
a_4	0.037159	-0.001159	-0.059263
a_5	0.037148	0.000507	-0.009536
a_6	-0.011437	-0.000069	-0.0075921
a_7	-0.016252	-0.000168	0.020616

TABLE 2: approximate policies' coefficients in the three economies.

6 Qualitative Analysis of the Transition Dynamics

To get the approximated time-path trajectories $\mu(t)$ we substitute the approximate optimal policies $\bar{\chi}_h(\mu)$ computed before into the first equation of (22), yielding the following ODE in t,

$$\dot{\mu}(t) = \{1 - 2\beta(1-\alpha)\mu(t)/Q[\mu(t)]\}\{\theta[\mu(t)]^\alpha - \bar{\chi}_h[\mu(t)]\}, \tag{47}$$

with $Q(\cdot)$ defined in (23). ODE (47) can be numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method available in Maple 13. The time-path trajectory $\chi(t)$ is then computed by letting $\chi(t) = \bar{\chi}_h[\mu(t)]$ with $\mu(t)$ just obtained. Figures 4(a) and 4(b) report the approximate trajectories $\mu_h(t)$ and $\chi_h(t)$ for $0 \le t \le 400$ and $h \in \{L, M, U\}$.

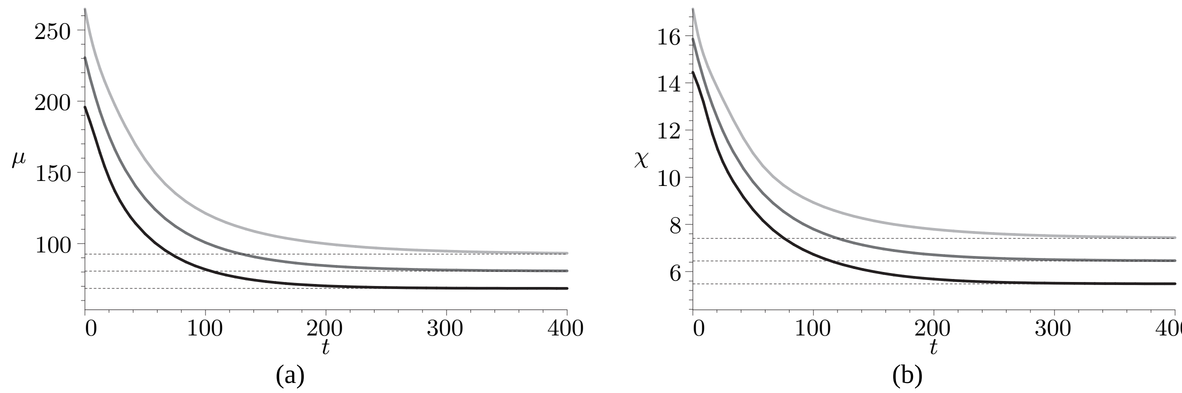

FIGURE 4: time-path trajectories (a) $\mu_h(t)$ and (b) $\chi_h(t)$ for $h = L$ (light grey), $h = M$ (dark grey) and $h = U$ (black).

With $\mu(t)$ and $\chi(t)$ at hand, we can compute the time-path trajectories of the stock of knowledge, $A(t)$, and capital, $k(t)$, again by using $\mu(t)$ in (21) and then computing $k(t) = \tilde{k}[A(t)]$ from the definition of turnpike in (10). Similarly, the time-path trajectory of output is given by $y(t) = \theta A(t)\left\{\tilde{k}[A(t)]/A(t)\right\}^\alpha$, while the time-path trajectory of the optimal consumption $c(t)$ is obtained by using trajectories $\chi(t)$ and $A(t)$ in (20). Figure 5(a) draws the $A_h(t)$ trajectories all starting from $A_0 = 2.0379$ in $t = 0$, where it is seen that the stock of knowledge in economy U overcome that in the other economies after a short time. Figures 5(b) shows that the same occurs for the stock of capital, $k_h(t)$, but after a longer period of time. Similar patterns are shown in Figures 5(c) and 5(d) for the output, $y_h(t)$, and consumption, $c_h(t)$, trajectories.

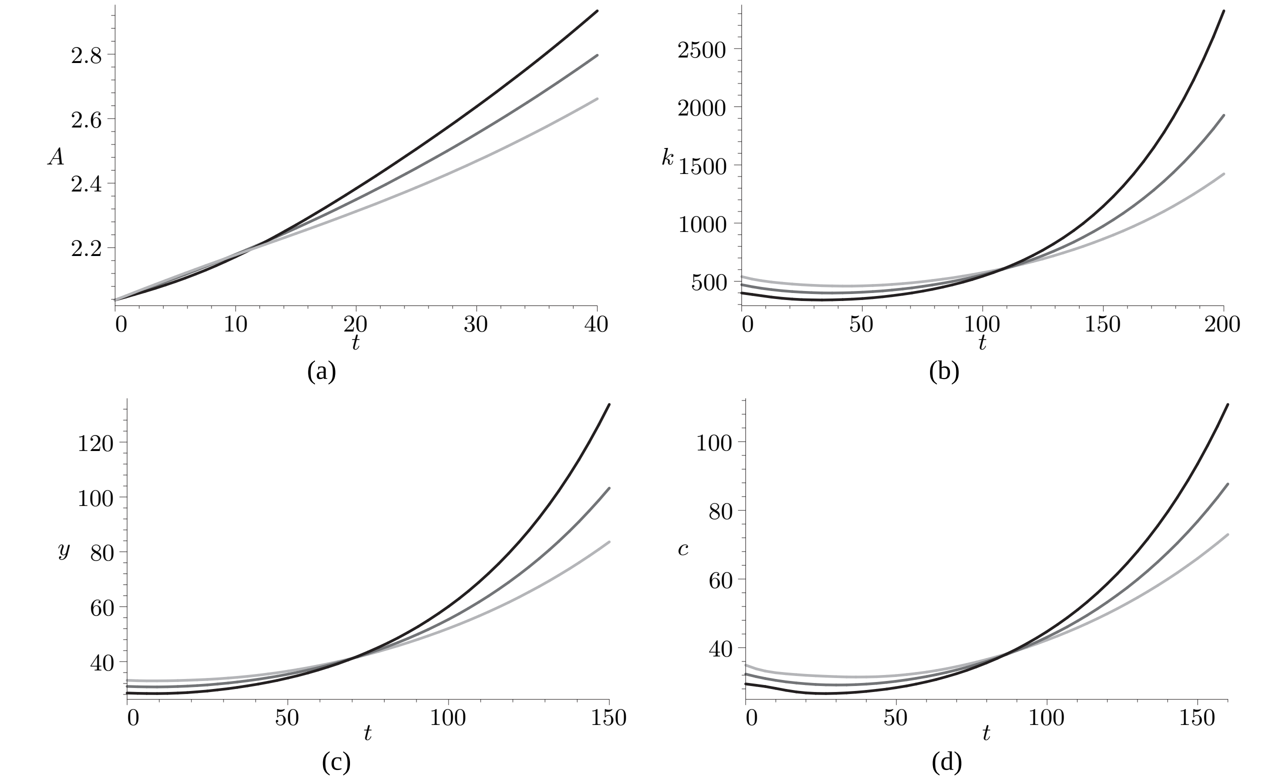

FIGURE 5: time-path trajectories (a) $A_h(t)$, (b) $k_h(t)$, (c) $y_h(t)$ and (d) $c_h(t)$ for $h = L$ (light grey), $h = M$ (dark grey) and $h = U$ (black).

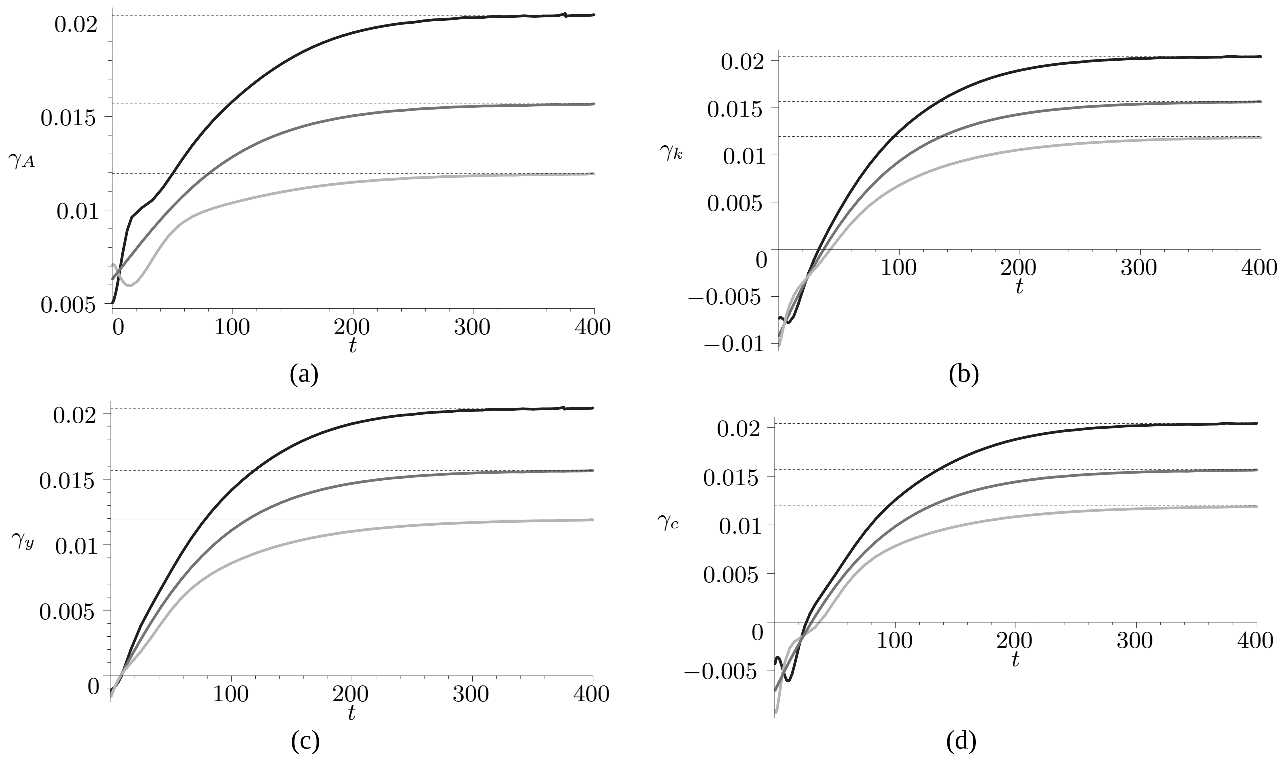

FIGURE 6: growth rate time-path trajectories of (a) $A_h(t)$, (b) $k_h(t)$, (c) $y_h(t)$ and (d) $c_h(t)$ for $h = L$ (light grey), $h = M$ (dark grey) and $h = U$ (black).

All these figures provide a *qualitative test* for the approximation technique developed in former sections: they show that our approximate trajectories exhibit a quite reasonable behavior. Only Figure 6, reporting transitory growth rates for the four variables, A, k, y and c, all converging to the asymptotic constant growth rates computed in (42) for each economy, actually detects some blurry behavior of such variables in early times for the economies $h = L$ and $h = U$, thus emphasizing larger inaccuracies of the approximations $\bar{\chi}_h$ close to the upper endpoints $\overline{\mu}_h$ of the state spaces $\left[\underline{\mu}_h, \overline{\mu}_h\right]$ in these two scenarios.

7 Conclusions

In this paper a projection method based on computing the least squares of the residual function built upon an approximate solution for the ODE (29) defining the optimal policy of the detrended version of the recombinant growth model discussed by Privileggi [7] is proposed. The same parameters' values for the discount rate, the CIES instantaneous utility and the Cobb-Douglas production for the output sector as in [7] has been assumed, while we left the efficiency parameter defining the (hyperbolic) probability of success (3) in matching pairs of ideas, β, free to change. Although our results quickly degenerates as one departs from the benchmark value $\beta_M = 0.0124$ used in [7], we succeeded in approximating time-path trajectories – that is, transition dynamics – which appear to be sufficiently regular for β-values in a neighborhood of 0.0124. This is a substantial step forward with respect to the result obtained by Privileggi [7].

After trying finite-difference and projection methods, however, we must conclude that ODE (29) is quite tough to tackle numerically. We tested our code for all projection methods described in Section 4 – least squares, Galerkin and Orthogonal collocation – on more standard Ramsey-like model and we found that it works reasonably well. Vice versa, when applied to the recombinant growth model, the same code does not work satisfactorily in case of Galerkin and Orthogonal collocation, while in the least square algorithm it works only up to a degree of approximation $n = 7$, as larger values of n do not reduce residuals and the coefficients a_i of the approximate $\bar{\chi}_h(\mu, a)$ in (43) keep oscillating for $i > 7$. Nonetheless, we believe that there is room for applying the projection technique to study the transition dynamics in a similar model in which the production of knowledge is monopolized through *intellectual property rights*, which will be the topic of future research.

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