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(Article begins on next page)

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The Nature of the Steady State in Models of Optimal Growth Under Uncertainty*

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Summary. We study a one-sector stochastic optimal growth model with a representative agent. Utility is logarithmic and the production function is of the Cobb-Douglas form with capital exponent α. Production is affected by a multiplicative shock taking one of two values with positive probabilities p and $1-p$. It is well known that for this economy, optimal paths converge to a unique steady state, which is an invariant distribution. We are concerned with properties of this distribution. By using the theory of Iterated Function Systems, we are able to characterize such a distribution in terms of singularity versus absolute continuity as parameters α and p change. We establish mutual singularity of the invariant distributions as p varies between 0 and 1 whenever $\alpha < 1/2$. More delicate is the case $\alpha > 1/2$. Singularity with respect to Lebesgue measure also appears for values α, p such that $\alpha < p^p (1-p)^{(1-p)}$. For $\alpha > p^p (1-p)^{(1-p)}$ and $1/3 \leq p \leq 2/3$, Peres and Solomyak (1998) have shown that the distribution is *a.e.* absolutely continuous. Characterization of the invariant distribution in the remaining cases is still an open question. The entire analysis is summarized through a bifurcation diagram, drawn in terms of pairs (α, p).

Key words Stochastic optimal growth, Iterated Function System, singular and absolutely continuous invariant distributions

JEL Classification Numbers C61, O41

1 Introduction

In deterministic optimal growth models, it is customary to characterize a (stable) steady state, and also to examine how the steady state changes as parameters of the model change.

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In contrast to this, relatively little is known about the nature of the steady state of stochastic optimal growth models. A stochastic steady state is identified as an invariant distribution, and, since the seminal papers by Lucas-Prescott [41] and Brock-Mirman [11], a large part of the literature on the subject has focused on the existence, uniqueness and global stability of this invariant distribution. See, for example, [44], [45], [10], [16], [28] and, more recently, [34], [6] and [57].

However, the problem of *characterizing* such a distribution appears to have received far less attention even for the simplest examples arising in such models. Consequently, our state of knowledge, regarding how the stochastic steady state changes with the parameters of the model, is extremely limited. An exception is the paper by Mirman and Zilcha [46] where, for some examples with specific parameter values, the limiting distribution function can be directly computed.

The aim of the present work is to examine *one* example of an optimal growth model under uncertainty, in as much detail as possible, and address specifically two questions regarding the stochastic steady state of the model:

1. What is the nature of the invariant distribution representing the stochastic steady-state? Specifically, is it *absolutely continuous*, so that it can be represented by a density, which could possibly be estimated in terms of a few parameters? Or, is it *singular*, so that, essentially, we do not have a convenient way of representing the function without actually stating the value of the function for every point in its domain?
2. How does the nature of the invariant distribution (specifically, its absolute continuity or singularity) change with parameters of the optimal growth model generating it? Specifically, is it possible to represent this information in the form of a convenient "bifurcation diagram" involving the key parameters of the optimal growth model?

These questions appear to be quite basic. Yet, as we will see, answering them, even in the context of a simple example of an optimal growth model, is surprisingly difficult, and the whole family of invariant measures obtained for a large set of parameters values exhibits an extremely rich and complicated structure.

The particular example of optimal growth under uncertainty that we discuss is a familiar one in which the utility function is logarithmic and the production function is of the Cobb-Douglas form with capital exponent α. Uncertainty is captured in terms of a multiplicative shock to the production function taking one of two values, with positive probabilities p and $1-p$, which are fixed over time. [That is, the process of exogenous shocks is independent and identically distributed with marginal probability distribution given by these specified probabilities].

It is known that this optimal growth problem can be solved, using dynamic programming methods, to obtain an optimal policy function. This, in turn, yields a non-linear stochastic process, governing the evolution of the state variable (the capital stock of the economy). Using the specific structure of the example, it is possible, through an appropriate log-transformation, to convert this into an *iterated function system* (IFS) involving *affine* functions (or *similarities*). A straightforward application of the theory of iterated function systems leads to the functional equation governing the invariant distribution[1] of the (transformed) random dynamical system.

We note that since our random dynamical system is generated by a finite number of affine maps which are contractions, the existence, uniqueness and global stability of the invariant distribution (starting from an arbitrary initial distribution), which the literature is principally concerned with, are guaranteed by a straightforward application of the standard theory of iterated function systems, developed by Karlin [38], Norman [48], Hutchinson [35], Barnsley and Demko

[1] Whenever the IFS involves similarities, its invariant distribution is also called a *self-similar measure*.

[4], and others. [For ease of exposition, a brief review of the theory of IFS is presented in Section 2].

Our objective is to be able to say something useful about the stochastic steady state beyond the above mentioned properties[2]. To this end, we provide a heuristic discussion in Section 3 regarding the features of the stochastic steady state that one may expect to obtain. In particular, we observe that the support of the invariant measure should be "full" or "thin" depending on whether or not the images of the maps of our dynamical system "overlap". In addition, we relate the invariant measure of our IFS to the distribution of a random variable, defined in terms of an infinite series, and known as an Erdös series.

This link allows us to exploit the mathematical literature on the distribution of the Erdös series. The literature on this topic has developed over about sixty years (see Peres, Schlag and Solomyak [50]), and much of it is inaccessible to the non-specialist. The principal task of this paper is to provide a systematic review of the part of this literature that is directly relevant for the economic problem that we are studying. All the mathematical results that we review are known, except for the result on mutual singularity in Section 4. To make our review useful to a wider audience, we have tried to provide (especially in Sections 4 and 5) elementary treatments of the relevant results (on singular distributions), and self-contained expositions of the techniques involved.

We start our formal analysis of the nature of the stochastic steady state in Section 4. In the first subsection we confine our attention to "low" values of our technological parameter, α, which in our case turns out to be the range $\alpha \in (0, 1/2)$. For $\alpha \in (0, 1/2)$, the images of the maps of our IFS do not overlap, so that the support of the invariant distribution is seen to be a Cantor-like set of Lebesgue measure zero. The invariant distribution is clearly singular in this case, and its graph is the well-known "Devil's Staircase". It is worthwhile to note that the actual magnitude of the probability of picking each map of the IFS plays no role in obtaining this conclusion.

In contrast, for the borderline case of $\alpha = 1/2$, the magnitude of the probability of picking each map of the IFS plays a crucial role in determining the nature of the stochastic steady state. Thus, in the equal probability case, it is easy to see that the invariant distribution is uniform and so absolutely continuous. On the other hand, for the unequal probability case, the invariant distribution is always singular, leading to a continuous, strictly increasing distribution function on the unit interval, whose derivative is zero (Lebesgue) almost everywhere. [Construction of a function with these features figures prominently in Lebesgue's theory of integration].

To conclude Section 4, we are able to establish an even stronger property of the invariant distributions for $\alpha \in (0, 1/2]$. We prove that they are all mutually singular; that is, they concentrate over subsets of Cantor-like sets that have empty intersection. This result is in line with the work started by Montrucchio and Privileggi [47], where they found that the support of the invariant distribution is a fractal set (for $\alpha \in (0, 1/2)$). Thus rather strange dynamics can be optimal in standard neo-classical stochastic growth models. The "mutual singularity" property shows that the "thin support" of the invariant distribution is itself highly sensitive to small changes in probabilities.

Section 5 provides a further refinement of the singularity result by showing that for the combination of parameter values (α, p) for which

$$1/2 < \alpha < p^p (1-p)^{(1-p)}$$

[2] The idea of characterizing complexity of measures, usually, but not necessarily, defined over fractal supports, is due to Mandelbrot [42], who called this approach *multifractal analysis*. Far from pursuing a completely exhaustive multifractal study, here we confine our research on singularity versus absolute continuity properties of the invariant distribution.

the invariant distribution is singular with respect to Lebesgue measure. This means that the limiting distribution can be singular for (a set of positive Lebesgue measure of) values $\alpha > 1/2$ as well, provided that probabilities p and $1-p$ are asymmetric enough. Coupled with singularity for $\alpha \in (0, 1/2)$, all p, and for $\alpha = 1/2$, $p \neq 1/2$, this is a well known result in the mathematical literature (see the discussion in Section 5). We provide a simplified proof based on the concept of Hausdorff dimension, which we discuss briefly to keep our exposition self-contained.

In Section 6, we turn our attention to the remaining cases in which the technological parameter, α, is relatively "high" and probabilities p and $1-p$ are not too asymmetric, that is, pairs of parameters (α, p) such that

$$p^p(1-p)^{(1-p)} < \alpha < 1. \tag{1}$$

This turns out to be the most difficult situation to analyze. Our investigation of this case is incomplete; we report an important result, recently established by Peres and Solomyak [52], which establishes almost everywhere absolute continuity of the invariant distribution for values of the probability p in the interval $[1/3, 2/3]$ and pairs (α, p) satisfying (1).

It is interesting to find that the function $p^p(1-p)^{(1-p)}$, at least for $1/3 \leq p \leq 2/3$, turns out to be the borderline between singularity and (generic) absolute continuity of the invariant distribution. In particular, for a given p in $[1/3, 2/3]$, as α increases from $1/2$ to 1 (which means that the degree of "overlap" of the images of the maps of the IFS increases), the distribution turns from a singular one to an absolutely continuous one around the value $\alpha = p^p(1-p)^{(1-p)}$. That is, the turning point from singularity to absolute continuity of the invariant distribution is determined precisely by the entropy, $[-p \ln p - (1-p)\ln(1-p)]$, of the Bernoulli process underlying the exogenous shocks in our growth model. Furthermore, invariant distributions exhibit increasing "smoothness" above the entropy curve: for almost every $p^p(1-p)^{(1-p)} < \alpha \leq p^2 + (1-p)^2$, the distributions have densities in L^γ, where $1 < \gamma \leq 2$, and γ increases as α increases.

The whole discussion leads to Section 7, where a partial "bifurcation diagram" in terms of the parameters (α, p) on the unit square, indicates how the nature of the stochastic steady state varies with changes in these parameters (see Figure 5). By analyzing another result by Peres and Solomyak [52], which establishes a necessary smoothness condition for the invariant probability, we discuss further developments, and we sketch a conjectured "complete" bifurcation diagram (see Figure 6).

2 The Framework

In this section, we will describe the framework for our subsequent analysis. The concepts and basic results of *iterated function systems* (IFS) are essential to this study, and we provide a simplified exposition in Section 2.1. This exposition is based on Stark and Bressloff [58], and Lasota and Mackey [39]. In Section 2.2, we introduce the familiar model of optimal growth under uncertainty, and show, in the context of a particular example, how the analysis of optimal dynamics in this model reduces to the study of an IFS.

2.1 Iterated Function Systems

2.1.1 Definitions Let (X, d) be a complete metric space. An IFS (f, p) consists of a finite number of mappings $\{f_0, \ldots, f_{m-1}\}$ from X to X, and a set of associated probabilities $\{p_0, \ldots, p_{m-1}\}$ (so that $p_z > 0$ for $z = 0, \ldots, m-1$ and $p_0 + \cdots + p_{m-1} = 1$). Such an IFS gives a (random) dynamical system by defining a (random) orbit of f to be a sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} = f_{z_t}(x_t)$ where $\{z_t\}_{t=0}^{\infty}$ are i.i.d. over $\{0, \ldots, m-1\}$ with distribution $\{p_0, \ldots, p_{m-1}\}$.

We are interested in IFS with *contracting* maps. Hence, the f_zs will be assumed to be contractions; that is, for each $z \in \{0, \ldots, m-1\}$ there is a γ_z such that $0 \leq \gamma_z < 1$ and for all $x, y \in X$

$$d(f_z(x), f_z(y)) \leq \gamma_z d(x, y). \tag{2}$$

We shall denote by γ the maximum of the γ_z. Clearly $\gamma < 1$.

The maps $\{f_0, \ldots, f_{m-1}\}$ induce an operator T on X, called the *Barnsley operator*, defined by:

$$T(A) = f_0(A) \cup \cdots \cup f_{m-1}(A), \qquad A \subset X, \tag{3}$$

where $f_z(A)$ denotes the image of the set A through f_z. Operator T allows for a definition of an invariant set of an IFS. A set $A_f \subset X$ is said to be an *invariant set* of f if it is *compact* and satisfies

$$T(A_f) = A_f. \tag{4}$$

One can also define a *Markov operator* M which describes the evolution of probabilities under the process $x_{t+1} = f_{z_t}(x_t)$. Thus let $\mathcal{B}(X)$ be the σ-algebra of Borel measurable subsets of X and $\mathcal{P}(X)$ the space of probability measures on $\mathcal{B}(X)$**.** Then if $\mu \in \mathcal{P}(X)$ is an initial probability distribution we want $M(\mu)$ to be the distribution after one iteration of f. M is then given by

$$M\mu(B) = \sum_{z=0}^{m-1} p_z \mu\left(f_z^{-1}(B)\right), \qquad \text{for all } B \in \mathcal{B}(X)$$

where $f_z^{-1}(B)$ denotes the set $\{x \in X : f_z(x) \in B\}$. Note that M is a linear operator from $\mathcal{P}(X)$ to itself.

2.1.2 The Invariant Measure of an IFS We shall be interested in the asymptotic behavior of measures under repeated application of M; in other words, in the long run behavior of distributions as they evolve under f. Since the maps f_z of the IFS we are focusing on are contractions, it turns out that there is a unique *invariant distribution* μ_f, that is

$$\mu_f = M\left(\mu_f\right) \tag{5}$$

and the iterates $M^t(\mu)$s *converge weakly* to μ_f as $t \to \infty$ for all initial distributions $\mu \in \mathcal{P}(X)$.

The simplest proof of the existence and uniqueness of μ_f uses the *Hutchinson* [35] *metric* L defined over a subset $\mathcal{L}(X)$ of $\mathcal{P}(X)$. For all pairs $\mu, \upsilon \in \mathcal{P}(X)$, define a pseudo-metric[3] by

$$L(\mu, \upsilon) = \sup\left\{\left|\int_X \phi d\mu - \int_X \phi d\upsilon\right| : \mathrm{Lip}(\phi) \leq 1\right\} \tag{6}$$

where the supremum is taken over all functions $\phi : X \to \mathbb{R}$ that are bounded and with Lipschitz constant not larger than 1. Fix any $a \in X$, and define[4]

$$\mathcal{L}(X) = \left\{\mu \in \mathcal{P}(X) : \int d(x, a)\, d\mu < \infty\right\}.$$

L defined in (6) satisfies the triangle inequality, thus $\mathcal{L}(X)$ does not depend on a.

[3] This is not a metric: L is clearly non-negative, symmetric and satisfies the triangle inequality, but it could be plus infinity for some pair $\mu, \upsilon \in \mathcal{P}(X)$; also $L(\mu, \upsilon) = 0$ does not necessarily imply $\mu = \upsilon$. Therefore, we need to restrict our probability space to a subset $\mathcal{L}(X)$.

[4] Note that the space $\mathcal{L}(X)$ is more general than the space of probability measures with bounded support actually used by Hutchinson [35].

When f satisfies (2), the operator $M : \mathcal{L}(X) \to \mathcal{L}(X)$ turns out to be a contraction with respect to L. In fact we have $L(M(\mu), M(\upsilon)) \leq \gamma L(\mu, \upsilon)$. Then through a contraction mapping argument it can be shown that μ_f is the unique attracting fixed point of M. Since the L-metric convergence implies weak convergence over $\mathcal{L}(X)$ (see, Proposition 5 in the Appendix), it follows that $M^t(\mu)$ converges weakly[5] to μ_f.

Since the measure μ_f is the unique invariant measure for M, it must be ergodic. However a much stronger property holds: μ_f is in fact a Sinai-Ruelle-Bowen (SRB) measure. This means that for all $x_0 \in X$ and almost every sequence $\{f_{z_t}\}$ of maps the visiting frequency of the orbit $\{x_t\}$ to a set $B \in \mathcal{B}(X)$ is simply $\mu_f(B)$, that is

$$\lim_{n\to\infty} \frac{\#\{x_t \in B : 1 \leq t \leq n\}}{n} = \mu_f(B) \tag{7}$$

This was first proved by Norman [48] and later independently by Elton [21].

Equation (7) provides the simplest method of estimating μ_f.

2.1.3 The Attractor of an IFS The existence of a (unique) invariant set, A_f, of f can be seen as follows. Consider the space $\mathcal{K}(X)$, the set of all non-empty compact sets in X. This is itself a complete metric space when endowed with the *Hausdorff metric* d_H, which is defined (see Falconer [25]) as follows. Let

$$A_\delta = \{x \in X : d(x - a) < \delta \text{ for some } a \in A\} \tag{8}$$

denote the *δ-parallel body* of $A \in \mathcal{K}(X)$, i.e., the set of points within distance δ of A. Then the Hausdorff metric is defined for $A, B \in \mathcal{K}(X)$ as

$$d_H(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\}.$$

It is easily seen that $d_H(T(A), T(B)) \leq \gamma d_H(A, B)$ (see, for example [25] or [39]), so that T is a contraction on $\mathcal{K}(X)$. Thus, T has a unique fixed point, call it A_f. Then A_f is the unique invariant set of f. Further, we have $T^t(A_0) \to A_f$ as $t \to \infty$ for any initial $A_0 \in \mathcal{K}(X)$; and, given any $A_0 \in \mathcal{K}(X)$ such that $T(A_0) \subset A_0$, we have $A_f \subset T^{t+1}(A_0) \subset T^t(A_0)$ for $t = 0, 1, 2, \ldots$ (see Proposition 12.8.3, p. 439, of Lasota and Mackey [39]).

The support of the invariant distribution, μ_f, is called the *attractor* of f. It is useful to note that this attractor is the same as the unique invariant set, A_f, obtained (above) by purely non-probabilistic methods. This can be seen as follows: given any $\mu \in \mathcal{P}(X)$, define $\mu_t = M^t(\mu)$ for $t = 0, 1, 2, \ldots$. Then, if the support of μ is a compact set, we have (by Proposition 12.8.2, p. 435 of Lasota and Mackey [39]) (support of μ_t) $= T^t$ (support of μ). Applying this result to μ_f, we have $\big($support of $\mu_f\big) = T\big($support of $\mu_f\big)$, so that (since A_f is the unique fixed point of T), $\big($support of $\mu_f\big) = A_f$.

[5] Hutchinson's [35] construction works for the special case of an IFS with contracting maps. However, refinements of the theory are possible for more general classes of IFS. For instance, Karlin [38] proves existence of the invariant measure for the category of affine maps, while allowing the probabilities to depend on the state and be Markovian. For Lipschitz maps, which are not necessarily contractive, but are "contracting on average", the existence and uniqueness of an invariant distribution μ_f to which the iterates $M^t(\mu)$ converge weakly, can be established. For a recent survey of results along these lines see, for example, Diaconis and Freedman [15].

2.2 One Sector Log-Cobb-Douglas Optimal Growth

Let us recall the formulation of the one-sector growth model with a Cobb-Douglas production function $G(x) = x^\alpha$, $0 < \alpha < 1$, with a representative decision maker's utility given by $u(c) = \ln c$. Suppose that an exogenous perturbation may reduce[6] production by some parameter $0 < k < 1$ with probability $p > 0$ (the same for all $t = 0, 1, \ldots$). This independent and identically distributed random shock enters multiplicatively into the production process so that output is given by $G_r(x) = rx^\alpha$ where $r \in \{k, 1\}$. The dynamic optimization problem can be explicitly written as follows:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

where $0 < \beta < 1$ is the discount factor, and the maximization is over all consumption plans $c = c_0, c_1, \ldots$ such that for $t = 0, 1, 2, \ldots$

$$c_t = r_t x_t^\alpha - x_{t+1}, \qquad c_t \geq 0, \qquad x_t \geq 0$$

and x_0, r_0 are given.

It is well known that the optimal policy for the concave problem just described is $g(x, z) = \alpha\beta r x^\alpha$ (see [59]); i.e. the plan x_t generated recursively by

$$x_{t+1} = g(x_t, r_t) = \alpha\beta r_t x_t^\alpha$$

is optimal.

Consider now the random dynamical system obtained by the following logarithmic transformation of x_t:

$$y_t = -\frac{1-\alpha}{\ln k} \ln x_t + 1 + \frac{\ln \alpha + \ln \beta}{\ln k}.$$

The new variable y_t, associated with x_t, evolves according to a linear policy, so that

$$y_{t+1} = \alpha y_t + (1-\alpha)\left(1 - \frac{\ln r_t}{\ln k}\right),$$

which can be rewritten as

$$\begin{cases} y_{t+1} = \alpha y_t & \text{with probability } p \\ y_{t+1} = \alpha y_t + (1-\alpha) & \text{with probability } 1-p \end{cases}$$

Define the maps f_0, f_1 from $[0, 1]$ to $[0, 1]$ by:

$$\begin{cases} f_0(y) = \alpha y \\ f_1(y) = \alpha y + (1-\alpha). \end{cases} \tag{9}$$

It is useful to note here that the map f_0 corresponds to the case where the shock, r, takes the value k; and the map f_1 corresponds to the case where the shock, r, takes the value 1. Denote $(p, 1-p)$ by (p_0, p_1). Then (f_0, f_1), together with (p_0, p_1) is an iterated function system over the interval $[0, 1]$. The maps f_z, for $z \in \{0, 1\}$, are clearly affine (or *similarities*, as they are one-dimensional) and since $0 < \alpha < 1$, they are contractions on $[0, 1]$. Thus, we can apply the methods of Section 2.1 directly to our IFS, and obtain the following proposition, which summarizes for future reference the main results regarding the invariant distribution associated with our IFS.

Proposition 1 *Consider the IFS described by* $[0, 1], (f_0, f_1), (p_0, p_1)$.

[6] The same framework works for a perturbation that "increases" production: $k > 1$.

i) There is a unique compact set $A_f \subset [0,1]$, such that $f_0(A_f) \cup f_1(A_f) = A_f$.
ii) For any compact set A, denoting $A_t = T^t(A)$ for $t = 0, 1, 2, \ldots$, we have $A_0 \supset A_1 \supset A_2 \supset \ldots \supset A_f$ whenever $T(A) \subset A$.
iii) There is a unique probability distribution μ_f on $([0,1], \mathcal{B}([0,1]))$ satisfying the functional equation (5), that is,

$$\mu_f(A) = p_0 \mu_f\left(f_0^{-1}(A)\right) + p_1 \mu_f\left(f_1^{-1}(A)\right) \qquad \text{for all } A \in \mathcal{B}([0,1]).$$

Further, A_f is the support of μ_f.
iv) For $\mu \in \mathcal{L}([0,1])$, denoting $\mu_t = M^t(\mu)$ for $t = 0, 1, 2, \ldots$, $L(\mu_t, \mu_f) \to 0$ as $t \to \infty$, and further μ_t converges weakly to μ_f.

3 Nature of the Stochastic Steady State: Discussion

In this section, we provide a preliminary discussion on the nature of the stochastic steady state (invariant distribution) obtained in Section 2. This heuristic material helps us to proceed with the formal presentation of the main results of the paper in the next three sections.

Our discussion pertains to three sub-topics. First, we comment on the support of the invariant distribution, and how this is related to "overlaps", or lack of it, of the IFS. Second, we relate the study of our iterated function system to the analysis of a certain infinite series, which we refer to as the Erdös series. Third, we summarize what is known in the mathematics literature regarding the behavior of the Erdös series (equivalently the iterated function system).

3.1 The Support of the Invariant Distribution

The graphs of the functions given by (9), show that for $0 < \alpha < 1/2$, the image sets of the two functions f_0 and f_1 are disjoint, a situation which can be described as the "non-overlapping" case (see Figure 1(a)). In this case, the "gap" between the two image sets (in the unit interval) will "spread" through the unit interval by successive applications of the maps (9). Thus, one would expect the support of the invariant distribution to be "thin" (with zero Lebesgue measure) and have features of the usual Cantor ternary set; in fact, for $\alpha = 1/3$, the support is precisely the Cantor ternary set.

On the other hand, for $1/2 \leq \alpha < 1$, the image sets of the functions f_0 and f_1 have a non-empty intersection. We can refer to this as the "overlapping" case (see Figure 1(b)). Here, the successive iterations of the overlap can be expected to "fill up" the unit interval, so the invariant distribution should have full support.

The above heuristics are actually seen to be valid as we demonstrate in the following sections.

It is important to remark that this construction does not depend on the magnitude of the discount factor β nor on the amplitude of the shock k, but only on the technological parameter α. The discount factor β only shifts the support of the invariant distribution of the original model over the real line, while the exogenous shock k affects its amplitude.

3.2 Erdös Series

Let Ω be the space of all infinite sequences of zeros and ones; that is, an element of Ω is $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ where $\omega_t \in \{0,1\}$ for $t = 0, 1, \ldots$. Let $\mathcal{F}$ be the smallest σ-algebra containing all the finite n-cylinders $(\omega_0, \omega_1, \ldots, \omega_{n-1}) \times \{0,1\} \times \{0,1\} \times \cdots$ and P the product measure over $\mathcal{F}$ generated by the probability over $\{0,1\}$ such that $\Pr(0) = p$ and $\Pr(1) = 1 - p$. Then we

have a probability space $(\Omega, \mathcal{F}, P)$ which we refer to as a *Bernoulli space*. Define the projection $\varepsilon_t(\omega) = \omega_t$. Then $\{\omega_t\}$ is a sequence of independent and identically distributed random variables on $(\Omega, \mathcal{F}, P)$.

Using the sequence $\{\omega_t\}$, we can write our iterated function system as:

$$y_{t+1}(\omega) = \alpha y_t(\omega) + (1-\alpha)\omega_t. \tag{10}$$

By Proposition 1, there is a unique limiting distribution μ_f to which the random variable $y_t(\omega)$ converges weakly, starting from any initial probability over $y_0(\omega)$. For simplicity, consider the Dirac measure that concentrates the whole probability over $y_0(\omega) = 0$. Iteration of (10) leads to

$$y_t(\omega) = (1-\alpha)\sum_{s=0}^{t-1} \omega_s \alpha^{t-s-1}.$$

Since ω_s are independent random variables, the probability distribution of $y_t(\omega)$ is the same as that of the random variable

$$(1-\alpha)\sum_{s=0}^{t-1} \omega_s \alpha^{s},$$

which converges almost everywhere to the series

$$\Pi_\alpha(\omega) = (1-\alpha)\sum_{s=0}^{\infty} \omega_s \alpha^{s} \tag{11}$$

which is called Erdös series. Since almost everywhere convergence implies weak convergence (see, e.g., Theorem 25.2, p. 284 in [9]), the distribution of (11) is the same as the invariant distribution, μ_f, of the system (10).

Note that this construction allows us to interpret μ_f as the image probability of P through the map $\Pi_\alpha : \Omega \to [0,1]$.

The Erdös series (11) is usually written in a slightly different format:

$$\sum_{s=0}^{\infty} \pm\alpha^{s}, \tag{12}$$

it being understood that the minus sign is taken with probability p and the plus sign with probability $(1-p)$. Note that (12) is the same as (11) where random variables $\omega'_t \in \{-1, 1\}$ are used in place of $\omega_t \in \{0, 1\}$ and the image set of the corresponding IFS is the translation of interval $[0,1]$ to the interval $[-1/(1-\alpha), 1/(1-\alpha)]$. Thus, the nature of the distribution μ_f of (11) can be studied by analyzing the distribution of the random variable (12).

3.3 Brief Review of the Mathematical Literature

Until a few years ago, the mathematical literature focused on the problem of analyzing the nature of the distribution of the random variable given in (12) exclusively in the special case where $p = 1/2$. The topic is known as the study of "symmetric infinite Bernoulli convolutions". Only very recently some important results have been extended to the asymmetric case $p \neq 1/2$. All this stream of research has been striving around the fundamental question on deciding for what values of α, μ_f is absolutely continuous, and for what values of α, μ_f is singular. For an exhaustive mathematical survey on the whole history of Bernoulli convolutions, see [50].

It is known, in the symmetric case, that the distribution function is "pure"; that is, it is either absolutely continuous or it is singular (Jessen and Wintner [36]). Further, Kershner and Wintner [37] have shown that if $0 < \alpha < 1/2$, the support of the distribution function is a Cantor set of Lebesgue measure zero, and therefore the distribution function is singular. For $\alpha = 1/2$, one gets the uniform distribution, which is *not* singular.

For the symmetric case $p = 1/2$, denote by $S_\perp$ the set of $\alpha \in (1/2, 1)$ such that μ_f is singular. It was often conjectured that the distribution function should be absolutely continuous with respect to Lebesgue measure when $1/2 < \alpha < 1$. Wintner [62] showed that if α is of the form $(1/2)^{1/k}$ where $k \in \{1, 2, 3, \ldots\}$, then the distribution function is absolutely continuous. However, in the other direction, Erdös [22] showed that when α is the positive solution of the equation $\alpha^2 + \alpha - 1 = 0$, so that $\alpha = (\sqrt{5} - 1)/2$, then $\alpha \in S_\perp$. More generally, the results of Erdös [22] may be described as follows. Recall that an *algebraic number* is the solution of a polynomial, all of whose coefficients are integers, and whose leading coefficient is either 1 or -1. An algebraic number is called a *Pisot-Vijayaraghavan number* when all its Galois conjugates are less than one in absolute value. Erdös [22] showed that when α is the reciprocal of a Pisot-Vijayaraghavan number, then $\alpha \in S_\perp$. It is an open question whether these numbers, which form a closed countable set, constitute all of $S_\perp$.

Erdös [23] also showed that $S_\perp \cap (\xi, 1)$ has zero Lebesgue measure for some $\xi < 1$, so that absolute continuity of the invariant distribution obtains for (almost every) α sufficiently close to 1. A conjecture that emerged from these findings is that the set $S_\perp$ itself should have Lebesgue measure zero.

Since the contributions of Erdös, there were a few additional results for the case $1/2 < \alpha < 1$, which are surveyed in Garsia [29]. In their brief discussion of this problem, Dubins and Freedman [17] state that deciding whether the distribution is singular or absolutely continuous for $\alpha > 1/2$ is a "famous open question". And Edgar [19], briefly surveying this topic, refers to it as an "Erdös problem".

Solomyak [56] made a real breakthrough when he showed that $S_\perp$ has zero Lebesgue measure. More precisely, he established that for almost every $\alpha \in (1/2, 1)$, the distribution has density in $L^2(\mathbb{R})$ and for almost every $\alpha \in \left(2^{-1/2}, 1\right)$ the density is bounded and continuous. A simpler proof of the same result was subsequently found by Peres and Solomyak [51]. A stronger result has been very lately achieved by Peres and Schlag [49], where they prove that the Hausdorff dimension (see Section 5.1 in the sequel) of the set $S_\perp \cap (\alpha, 1)$ is less than one for all $1/2 < \alpha < 1$. Note, however, that this still does not settle Garsia's [29] conjecture that the distribution is absolutely continuous for all but a countable number of values of α in $(1/2, 1)$.

Very recent contributions to this literature deal with the asymmetric case $p \neq 1/2$. Mauldin and Simon [43] extended Jessen and Wintner's [36] finding on "purity" of the distribution function also for $p \neq 1/2$. Peres, Schlag and Solomyak [50] further pushed the argument to finite families of contracting similitudes (namely, IFS with $m \geq 2$ affine maps) by showing that if the limiting distribution ν is not singular, then it is absolutely continuous with respect to Lebesgue measure and the restriction of Lebesgue measure to the support of ν is absolutely continuous with respect to ν.

Especially useful for our purposes, is the contribution by Peres and Solomyak [52], established for a generic family of contracting similitudes. We restate here their result in terms of our IFS given by (9). First, they found that μ_f is singular below the "entropy" curve, that is for values of parameters (α, p) such that $0 < \alpha < p^p(1-p)^{(1-p)}$, while it is absolutely continuous for almost every $p^p(1-p)^{(1-p)} < \alpha < 1$ whenever $1/3 \leq p \leq 2/3$. Moreover, for $1/3 \leq p \leq 2/3$, letting $1 < \gamma \leq 2$, for almost every $\left[p^\gamma + (1-p)^\gamma\right]^{1/(\gamma-1)} \leq \alpha < 1$, μ_f has density in L^γ. Finally, this property is further refined by the following necessary smoothness condition: for all p and all

$\gamma > 1$, if μ_f is absolutely continuous with density in L^γ, then $\alpha \geq [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$. We shall discuss in more detail these last findings in Section 6.

4 Singular Invariant Distributions

In this section, we provide an analysis of the invariant distributions when the technological parameter, α, is "low", which in our case translates to $\alpha \in (0, 1/2]$. Thus, our results may be viewed as extensions of the mathematical literature reviewed in Section 3.3 to the case where p is not necessarily equal to $1/2$.

In Section 4.1, we briefly review the case where $\alpha \in (0, 1/2)$. Here, the theory for $p \neq 1/2$ is entirely analogous to the theory for $p = 1/2$, since, as we have noted before, the support of the invariant distribution is a Cantor set of Lebesgue measure zero, and this support is independent of the magnitude of p.

In Section 4.2, we consider the "borderline" case where $\alpha = 1/2$. Here the theory for $p \neq 1/2$ is entirely unlike the theory for $p = 1/2$. While $p = 1/2$ generates the uniform invariant distribution (which is, of course, absolutely continuous), $p \neq 1/2$ always generates a singular invariant distribution. However, unlike in Section 4.1, this singular distribution has full support.

Section 4.3 is devoted to an improvement of the existing theory. By using the relationship with Erdös series discussed in Section 3.2, we show that, for the whole range $\alpha \in (0, 1/2]$, not only are the invariant measures singular with respect to Lebesgue measure, but they are also mutually singular. This, of course, indicates a high sensitivity of the limiting distribution with respect to changes in the parameter p. When $\alpha \in (0, 1/2)$, not only does the support (attractor) exhibit fractal features, but also measures with the same support concentrate over sets with empty intersection for different probabilities p and p'.

4.1 Singular Distributions of the Cantor Type

Consider the case where the parameter, α, has a value in $(0, 1/2)$. Recall the Barnsley operator (3) and define

$$A_0 = [0, 1], \qquad \text{and } A_{n+1} = T(A_n) \qquad \text{for } n = 0, 1, \ldots$$

Then, we have

$$\begin{aligned} A_1 = T(A_0) &= [0, \alpha] \cup [1-\alpha, 1], \\ A_2 = T(A_1) &= [0, \alpha^2] \cup [(1-\alpha), (1-\alpha)+\alpha^2] \\ &\quad \cup [\alpha(1-\alpha), \alpha] \cup [(1-\alpha)+\alpha(1-\alpha), (1-\alpha)+\alpha]. \end{aligned}$$

Note that the Lebesgue measure of A_0 is 1, that of A_1 is 2α, and that of A_2 is $2^2\alpha^2$. By induction, one can verify that $A_n = T^n([0, 1])$ consists of 2^n intervals, each of Lebesgue measure α^n, so that $\lambda(A_n) = 2^n\alpha^n \to 0$ as $n \to \infty$. Since, by Proposition 1 (*ii*), $A_0 \supset A_1 \supset \cdots \supset A_f$, where A_f is the attractor of the IFS, we have $\lambda(A_f) = 0$. Since A_f is also the support of the invariant distribution, μ_f, we clearly have $\mu_f(A_f) = 1$. Thus, μ_f is singular with respect to Lebesgue measure.

The distribution function corresponding to μ_f, when $\alpha = 1/3$, is usually called a *Cantor function* (since its support is precisely the Cantor ternary set) or a *Lebesgue function* (since it was used by Lebesgue to demonstrate the need for absolute continuity in the fundamental theorem of calculus). The graph of the distribution function, corresponding to μ_f, for $\alpha \in (0, 1/2)$ is referred to as the "Devil's Staircase" (see Figure 2).

4.2 Singular Invariant Distributions of the Hellinger Type

Consider now the case where the technological parameter $\alpha = 1/2$. In this case, the invariant distribution function, F, is the unique[7] distribution function over $[0,1]$ satisfying the functional equation:

$$F(x) = \begin{cases} pF(2x) & \text{for } x \in [0,1/2) \\ p + (1-p)F(2x-1) & \text{for } x \in [1/2,1] \end{cases} \tag{13}$$

If $p = 1/2$, then it is easy to see that $F(x) = x$ for all $x \in [0,1]$ satisfies (13), and so the invariant distribution function, F, must be the uniform distribution. In the optimal growth literature, this particular case is discussed in [46].

For $p \neq 1/2$, the function, F, that is generated by (13) has been extensively discussed in the mathematics literature as an example of a *continuous, strictly increasing function*, for which the derivative (which exists Lebesgue almost everywhere) vanishes almost everywhere. This necessarily means that F represents a measure that is singular with respect to the Lebesgue measure.

As Brown [12] points out, this type of function is properly attributed to Hellinger [32] in whose thesis it first appears. The function, F, is obtained as the limit of a sequence of piecewise linear functions, and the relevant properties of F are then verified from the properties of the converging sequence of functions. Hellinger's construction, which was subsequently generalized by Salem [55], is discussed in detail in Riesz and Sz-Nagy [53], Hewitt and Stromberg [33] and Asplund and Bungart [2]. It is also discussed, from the probabilistic viewpoint, in Billingsley [9].

De Rham [14] deduced the properties of F directly from the functional equation (13) governing F, but his presentation is quite terse, as is quite clear from the more comprehensive account of it in Yamaguti, Hata and Kigami [63].

Takacs [60] developed a method consisting in writing down a function, F, in explicit form, as an infinite series, and verifying that it is continuous, strictly increasing and singular with respect to Lebesgue measure. Given this result, one can show that the solution of (13) is singular by simply verifying that his F satisfies (13).

Given any $x \in [0,1]$, we can write it uniquely as

$$x = \sum_{r=0}^{\infty} 2^{-a_r} \tag{14}$$

where $a_0 < a_1 < \cdots < a_r < \cdots$ are positive integers[8]. Define $\delta = (1/p) - 1$, so that $\delta > 0$, and $\delta \neq 1$ (since $p \neq 1/2$). Now define for x given by (14),

$$F(x) = \sum_{r=0}^{\infty} \delta^r (1+\delta)^{-a_r} \tag{15}$$

and $F(0) = 0$. Then, by the theorem of Takacs [60], F is *strictly increasing, continuous, and its derivative is zero Lebesgue almost everywhere in the interval* $[0,1]$. It remains to verify that F satisfies (13).

For $x = 0$, we have $F(x) = pF(2x)$, as required in (13). Next, pick any $x \in (0,1/2]$. Then, $a_0 \geq 2$ in (14), and

$$2x = \sum_{r=0}^{\infty} 2^{-(a_r - 1)} \tag{16}$$

[7] Uniqueness can be shown through the standard IFS argument. As a matter of fact, the graph of F is the invariant set of two linear contraction maps from $[0,1]^2$ to itself, defined by $(x,y) \longmapsto [(1/2)x, py]$ and $(x,y) \longmapsto [(1/2)x + 1/2, (1-p)y + p]$.

[8] Note that (14) corresponds to the dyadic expansion of x, *i.e.*, the representation in base-2 of x, where all multiple representations (such that they are all zero after a certain point) are eliminated.

with $a_0 - 1 < a_1 - 1 < \cdots < a_r - 1 < \cdots$ positive integers. Thus, by definition (15), we have $F(2x) = \sum_{r=0}^{\infty} \delta^r (1+\delta)^{-(a_r-1)} = (1+\delta)F(x) = (1/p)F(x)$ which verifies (13). Finally, pick any $x \in (1/2, 1]$. Then $a_0 = 1$, $a_1 \geq 2$ in (14), and so by (16),

$$(2x-1) = \sum_{r=0}^{\infty} 2^{-(a_{r+1}-1)}$$

with $a_1 - 1 < a_2 - 1 < \cdots < a_r - 1 < \cdots$ positive integers. Thus, by definition (15), we have

$$p + (1-p)F(2x-1) = \frac{1}{1+\delta} + \sum_{r=0}^{\infty} \delta^{r+1}(1+\delta)^{-a_{r+1}} = \sum_{r=0}^{\infty} \delta^r (1+\delta)^{-a_r} = F(x)$$

which verifies (13).

A graph of F is indicated in Figure 3.

4.3 Mutually Singular Distributions

In Section 3.2 we have seen that the invariant probability measure of our IFS can be constructed as the image probability of the Bernoulli space $(\Omega, \mathcal{F}, P)$ through the map $\Pi_\alpha : \Omega \to [0,1]$ defined in (11) as $\Pi_\alpha(\omega) = (1-\alpha)\sum_{t=0}^{\infty} \omega_t \alpha^t$. We now use this fact to prove that for all $\alpha \in (0, 1/2]$, two invariant measures are mutually singular whenever different probabilities p and p' are adopted in the IFS. (For the standard definition of mutual singularity, see [7], p. 374). The idea is to "transfer" this property from the space $(\Omega, \mathcal{F}, P)$ (on which it is well-known to be true) to the image measure of Π_α.

For each $\omega \in \Omega$ and each $n \in \mathbb{N}$, define

$$N(\omega, n) = \#\{0 \leq t \leq n-1 : \omega_t = 1\}.$$

Let $q = (1-p)$ and consider the set $S(q) \subset \Omega$ of sequences such that the frequency of 1's is q:

$$S(q) = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{N(\omega, n)}{n} = q\right\}. \tag{17}$$

Of course, $S(q) \cap S(q') = \varnothing$ for $q \neq q'$. By the strong law of large numbers (see [9], p.70), it follows that for each $p \in (0,1)$, $P(S(1-p)) = 1$. As a consequence, each pair of Bernoulli probabilities P and P' over $(\Omega, \mathcal{F})$ generated by marginal probabilities $(\Pr(0) = p, \Pr(1) = 1-p)$ and $(\Pr(0) = p', \Pr(1) = 1-p')$ respectively, with $p \neq p'$, are mutually singular.

To stress dependence of the invariant measure on parameter p, let us denote the image measure of Π_α (our invariant distribution) by μ_f^p.

For each $\alpha \in (0,1)$, we can introduce a metric, ρ_α, on the space Ω, given by

$$\rho_\alpha(\omega, \omega') = \alpha^{|\omega \wedge \omega'|} \tag{18}$$

where $|\omega \wedge \omega'| = \min\{t : \omega_t \neq \omega'_t\}$.

Since, for all $\alpha \in (0,1)$ and all pairs $\omega, \omega' \in \Omega$,

$$|\Pi_\alpha(\omega) - \Pi_\alpha(\omega')| \leq (1-\alpha)\alpha^{|\omega\wedge\omega'|} \sum_{t=0}^{\infty} \alpha^t = \rho_\alpha(\omega, \omega'),$$

we note that Π_α is a Lipschitz continuous map from the metric space (Ω, ρ_α) to the metric space$([0,1], |\cdot|)$, with Lipschitz constant 1. Moreover, if we restrict our attention only to $\alpha \in (0, 1/2)$, a stronger property holds:

$$|\Pi_\alpha(\omega) - \Pi_\alpha(\omega')| \geq (1-\alpha)\alpha^{|\omega \wedge \omega'|} \left(1 - \sum_{t=1}^{\infty} \alpha^t\right) = (1-2\alpha)\, \rho_\alpha(\omega, \omega'),$$

that is, Π_α is *bi-Lipschitz* for $\alpha < 1/2$. This implies that Π_α is one-to-one, and thus invertible, for $\alpha \in (0, 1/2)$.

A special situation occurs when $\alpha = 1/2$: in this case Π_α is "almost" one-to-one. To see this, note that when $\alpha = 1/2$, Π_α is the base-2 representation of some real number $x \in [0,1]$:

$$x = \sum_{r=0}^{\infty} 2^{-a_r} \approx 0.\omega_0\omega_1\omega_2 \ldots$$

where $a_0 < a_1 < \cdots < a_r < \cdots$ are positive integers [corresponding to values of ω_t in (11) equal to one]. By dropping double representations, we get a (restricted) one-to-one map. For our purposes, it is useful to note that the inverse is not continuous but certainly measurable.

The discussion above is enough to prove the main result of this section.

Proposition 2 *For each $\alpha \in (0, 1/2]$, the invariant distributions $\left\{\mu_f^p : p \in (0,1)\right\}$ are mutually singular.*

Proof We have seen that for $\alpha \in (0, 1/2]$, the map Π_α is invertible. If $\alpha < 1/2$, it is a consequence of bi-Lipschitz property. If $\alpha = 1/2$, we can suitably restrict the domain by dropping a countable set of P-measure zero. The inverse map Π_α^{-1} is measurable. Take two Bernoulli probabilities over Ω, P and P' generated by p and p', with $p \neq p'$. By the discussion at the beginning of this section, P and P' are mutually singular; that is, two sets Ω_1, Ω_2 exist so that $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \varnothing$ and $P(\Omega_1) = P'(\Omega_2) = 0$. Consider the images of these two sets through Π_α: clearly, $\Pi_\alpha(\Omega_1)$ and $\Pi_\alpha(\Omega_2)$ are measurable, disjoint and cover $\Pi_\alpha(\Omega)$. Since $\mu_f^p(\Pi_\alpha(\Omega_1)) = P(\Omega_1) = 0$ and $\mu_f^{p'}(\Pi_\alpha(\Omega_2)) = 0$, the two image measures are mutually singular.

Note that, through a different argument, for $\alpha = 1/2$ we have established singularity of the Hellinger distribution functions discussed in Section 4.2. As a matter of fact, since for $p = 1/2$ the (uniform) invariant distribution is itself the Lebesgue measure on $[0,1]$, by Proposition 2, for all $p \neq 1/2$ the invariant distribution must be singular with respect to Lebesgue measure.

5 A Class of Singular Distributions for High Values of α

We now carry on our analysis by investigating the behavior of the invariant distribution for values $\alpha > 1/2$. From Figure 1(b), it is clear that, due to the overlapping of images of the IFS, the study of the invariant distribution becomes more difficult.

This section is confined to a partial extension of the results on singularity developed before. It turns out that for the combination of parameter values (α, p) satisfying

$$1/2 < \alpha < p^p(1-p)^{(1-p)} \tag{19}$$

the invariant distribution is necessarily singular. Coupled with singularity results already obtained in the previous sections for $0 < \alpha < 1/2$ (plus case $\alpha = 1/2$ and $p \neq 1/2$), we can thus assert that for all pairs (α, p) such that

$$0 < \alpha < p^p(1-p)^{(1-p)}$$

the invariant distribution is singular with respect to Lebesgue measure.

This result leads to a general observation on existence of singular invariant distributions in our context. *When the entropy of the Bernoulli process,* $-(p \ln p + (1-p)\ln(1-p))$ (see [8])*, is relatively "low", which in our context occurs when it is less than* $\ln(1/\alpha)$*, then the invariant distribution is singular.* Since the highest value of the entropy is $\ln 2$, this condition is automatically satisfied when $\alpha < 1/2$. In the special case $\alpha = 1/2$, the entropy is smaller than $\ln(1/\alpha)$ for all p except $p = 1/2$, and so the condition is satisfied for all $p \neq 1/2$. Thus, the results of Sections 4.1 and 4.2 can be viewed as particular cases of this general observation.

It is important to remark that singularity below the entropy curve is a well known property in the mathematical literature, linking entropy and dimension theory. For our case of two possibilities, with probabilities p and $(1-p)$, it follows from the analysis of Besicovitch [5]. Eggleston [20] generalized the result of Besicovitch to the case of any finite number of possibilities. A general version of the statement above can be established, using the Shannon-McMillan-Breiman Theorem. This approach is followed in Billingsley [7].

Here we provide a simplified proof without resorting directly to properties of the entropy. We make use of only the notion of Hausdorff dimension, which we briefly review in Section 5.1. In Section 5.2, we return to the sequence space introduced in Section 3.2, and show how the Hausdorff dimension of a "probability 1 set" can be computed to be less than one. This, in turn, means that the Lebesgue measure of this set is zero, and leads to the singularity of the invariant distribution.

5.1 Hausdorff Dimension

Let (X, d) be a metric space. For $F \subset X$, and $s \in \mathbb{R}_+$, the *s-dimensional Hausdorff outer measure* is denoted by $\mathcal{H}^s(F; d)$ and is defined as follows. A collection $\mathcal{A}$ of subsets of X is called a *countable cover* of F if

$$F \subset \bigcup_{A \in \mathcal{A}} A$$

and $\mathcal{A}$ is a countable family of sets. Given any $\delta > 0$, a countable cover $\mathcal{A}$ of F is a *δ-cover* if $|A|_d \leq \delta$ for all $A \in \mathcal{A}$, where $|A|_d$ denotes the diameter of A:

$$|A|_d = \sup_{x,y \in A} d(x, y)$$

We define

$$\mathcal{H}^s_\delta(F; d) = \inf \sum_{A \in \mathcal{A}} (|A|_d)^s$$

where the infimum is taken over all countable δ-covers $\mathcal{A}$ of the set F. $\mathcal{H}^s_\delta(F; d)$ is non-increasing in δ, and one can define:

$$\mathcal{H}^s(F; d) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F; d)$$

as the *s-dimensional Hausdorff outer measure of* F (given the metric d).

A useful result on Hausdorff outer measures is that for $\mathbb{R}$ (with the usual distance), the one-dimensional Hausdorff outer measure of any set coincides with the Lebesgue outer measure of that set (see, for example, [9], p. 208, or [18], p. 148-149).

Suppose $\delta < 1$, then, given any δ-cover $\mathcal{A}$, for any pair $s < t$,

$$\sum_{A\in\mathcal{A}} (|A|_d)^t = \sum_{A\in\mathcal{A}} (|A|_d)^s (|A|_d)^{t-s} \leq \delta^{t-s} \sum_{A\in\mathcal{A}} (|A|_d)^s ,$$

and so $\mathcal{H}^t(F;d) \leq \delta^{t-s}\mathcal{H}^s(F;d)$. Hence, $\mathcal{H}^t(F;d) > 0$ implies $\mathcal{H}^s(F;d) = \infty$ and $\mathcal{H}^s(F;d) < \infty$ implies $\mathcal{H}^t(F;d) = 0$. This means that $\mathcal{H}^s(F;d)$ is decreasing in s and is everywhere $+\infty$ or 0 but, at most, at a single point $s_0 \geq 0$, where it may satisfy $0 < \mathcal{H}^{s_0}(F;d) < \infty$. This argument leads to a useful definition of dimension. The *Hausdorff-Besicovitch dimension* of F is given by

$$\dim(F;d) = \inf\{s : \mathcal{H}^s(F;d) = 0\} = \sup\{s : \mathcal{H}^s(F;d) = \infty\}$$

Clearly, if $s_0 \equiv \dim(F;d) < \infty$, then for all $s > s_0$, $\mathcal{H}^s(F;d) = 0$; and if $s_0 \equiv \dim(F;d) > 0$, then for all $0 \leq s < s_0$, $\mathcal{H}^s(F;d) = \infty$.

The relationship between s-dimensional Hausdorff outer measure and Hausdorff dimension of a set can be depicted as in Figure 4.

A useful result (see [25] or [18]) on Hausdorff dimension is the following. Let (X,d) and (Y,d') be metric spaces, and f a Lipschitz continuous map from X to Y. Then, for every $F \subset X$,

$$\dim(f(F);d') \leq \dim(F;d).$$

5.2 Singularity for Low Entropy

Recall the probability space $(\Omega,\mathcal{F},P)$ discussed in Section 3.2, the metric ρ_α defined in (18) and the map Π_α defined in (11). We will be concerned, in particular, with ρ_α for $\alpha \in (1/2,1)$, and with $\rho_{1/2}$, which (for convenience) we will denote by ρ.

We have (deliberately) included the metric being used in our definition (and notation) of the Hausdorff dimension, since this dimension clearly varies with the metric. From this perspective, our choice of metric ρ_α for the space Ω is not accidental. Since ρ_α depends on α [that is, on the "degree of overlapping" of the images of the maps f_0 and f_1 in (9)], it turns out that the dimension of subsets of Ω depends on the contraction ratio of our IFS. This will play a crucial role in the main result of this section. The following relation indicates how the dimension varies as we employ different metrics ρ_α, that is, different contraction coefficients α.

Let $F \subset \Omega$, and $0 < \alpha < 1$ be given. Then we have the relation

$$\dim(F;\rho_\alpha) = \left[\frac{\ln(1/2)}{\ln\alpha}\right] \dim(F;\rho). \tag{20}$$

To see this, denote $[\ln(1/2)/\ln\alpha]$ by b, and note that for $s \in \mathbb{R}_+$, and $\mathcal{A}$ a δ-cover of F,

$$\mathcal{H}^{sb}_\delta(F;\rho_\alpha) = \inf \sum_{A\in\mathcal{A}} \left(|A|_{\rho_\alpha}\right)^{sb} = \inf \sum_{A\in\mathcal{A}} \left(|A|_{\rho}\right)^{s} = \mathcal{H}^s_\delta(F;\rho)$$

Thus, letting $\delta \to 0$, we get

$$\mathcal{H}^{sb}(F;\rho_\alpha) = \mathcal{H}^s(F;\rho)$$

and (20) follows consequently.

Recall from Section 4.3 that Π_α is Lipschitz with Lipschitz constant 1. Thus, we have for every $F \subset \Omega$,

$$\dim(\Pi_\alpha(F); |\cdot|) \leq \dim(F; \rho_\alpha). \tag{21}$$

We will now use again the set $S(q) \subset \Omega$ defined in (17). We will show that (by the law of large numbers) $S(q)$ is a set of P-measure 1 and of Hausdorff dimension less than 1 whenever condition (19) holds.

If we consider the set $S(q)$ in the metric space (Ω, ρ) [recall that $\rho_{1/2}$ is denoted by ρ], then we have the Besicovitch-Eggleston-Good result[9] that

$$\dim(S(q); \rho) = \frac{(p \ln p + q \ln q)}{\ln(1/2)} \tag{22}$$

Using (20) and (22) we have

$$\dim(S(q); \rho_\alpha) = \frac{(p \ln p + q \ln q)}{\ln(\alpha)}.$$

It follows that, under (19),

$$\dim(S(q); \rho_\alpha) < 1. \tag{23}$$

Using (21) and (23), we obtain

$$\dim(\Pi_\alpha(S(q)), |\cdot|) < 1. \tag{24}$$

By definition of Hausdorff dimension, (24) implies that

$$\mathcal{H}^1(\Pi_\alpha(S(q)), |\cdot|) = 0$$

and since $\mathcal{H}^1$ coincides with Lebesgue outer measure, λ^*, we have

$$\lambda^* (\Pi_\alpha(S(q))) = 0.$$

In particular then $\Pi_\alpha(S(q))$ is a Lebesgue measurable set (in the sense of Caratheodory); see [54], p. 58. There is then a Borel set, $B \subset [0,1]$, such that $\Pi_\alpha(S(q)) \subset B$, and $\lambda^*(B) = 0$; see Proposition 5, p. 58 in [54]. Thus $\lambda(B) = 0$, where λ is Lebesgue measure on $\mathbb{R}$.

Now, we will show that the Borel set B obtained above has $\mu_f(B) = 1$, where μ_f is the invariant distribution of our IFS. To see this, note that, by the strong law of large numbers, $P(S(q)) = 1$. By continuity of Π_α on (Ω, ρ_α), $\Pi_\alpha^{-1}(B)$ is a Borel set of Ω, and $\Pi_\alpha^{-1}(B) \supset S(q)$. Thus,

$$\mu_f(B) = P(\Pi_\alpha^{-1}(B)) \geq P(S(q)) = 1$$

Since B is a Borel set in $[0,1]$ with $\lambda(B) = 0$ and $\mu_f(B) = 1$, μ_f is singular with respect to Lebesgue measure. We have thus demonstrated the following.

Proposition 3 *For $1/2 < \alpha < p^p (1-p)^{1-p}$, μ_f is singular with respect to Lebesgue measure.*

[9] This important result was established for the case of two possibilities by Besicovitch [5]. For an arbitrary finite number of possibilities, it was conjectured by Good [30], and proved by Eggleston [20]. For alternative approaches, see Billingsley ([7], [8]), and Edgar [18]. Note that $(p \ln p + q \ln q)$ measures the negative of the *metric entropy* of a Bernoulli process with probabilities p and q. Therefore, the ratio on the right side of (22) is the metric entropy relative to its maximum value $\ln 2$.

6 Absolutely Continuous Invariant Distributions

It remains to examine properties of the invariant distribution for pairs (α, p) satisfying (1):

$$p^p (1-p)^{1-p} < \alpha < 1.$$

As we have already indicated in Section 3.3, for $p = 1/2$ and for Lebesgue almost every $\alpha \in (1/2, 1)$, Solomyak [56] showed that the distribution has density in L^2. His original proof was based on *Fourier transforms* and the problem of estimating multiple zeros of certain *power series*. The method of Fourier transform applies naturally to the symmetric case $p = 1/2$ and proves useful in developing a theory in L^2.

Subsequently, Peres and Solomyak [51] developed an alternative method, based on differentiation of measures instead of Fourier transforms, to prove the same theorem as in [56]. The latter approach (by-passing L^2 theory) actually opened a way for some extensions to the asymmetric case $p \neq 1/2$, where L^2 theory fails to capture the main features of the invariant distribution in some circumstances.

The main results along this direction are worked out by Peres and Solomyak [52] and may be summarized as follows. By confining the range of values for the probability p to the interval $[1/3, 2/3]$, the authors established that the invariant distribution μ_f is generically absolutely continuous above the entropy curve, namely for almost every $p^p (1-p)^{1-p} < \alpha < 1$. Furthermore, again for $1/3 \leq p \leq 2/3$, μ_f turns out to exhibit gradually increasing smoothness as the technological parameter α rises above the entropy curve, more specifically, for $1 < \gamma \leq 2$ and for almost every $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} \leq \alpha < 1$, μ_f has density in L^γ. Note that a theory only in L^2 would be far too stringent to investigate this last behavior.

This seems to be the present state of art, at least pertaining to a characterization in terms of (generic) absolute continuity above the entropy curve. More work is needed to extend the above statements to values of p outside the interval $[1/3, 2/3]$; perhaps, as indicated by Peres and Solomyak, a new approach may be needed.

However, another result in [52], stated in terms of necessary smoothness condition, sheds some more light on this "gradually increasing smoothness" of μ_f for higher values of α. The authors, indeed, state that for all p and all $\gamma > 1$, if μ_f is absolutely continuous with density in L^γ, then necessarily $\alpha \geq [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$.

In this section, we confine ourselves to a brief description of the basic ingredients necessary to deal with our specific problem. We provide some material on Fourier Transforms in Section 6.1, while Section 6.2 shows how this material works in Solomyak's [56] construction for the symmetric case and how some method of estimating multiple zeros of power series can be used. Finally, Section 6.3 discusses the extensions to the case $p \neq 1/2$ obtained in [52].

6.1 Fourier Transforms of the Erdös Series

Instead of looking at the invariant distribution μ_f, of our IFS, or equivalently the invariant distribution of the Erdös series (11) discussed in Section 3.2, we find it convenient to work with a class of series that includes (12). We describe this (class of) Erdös series as follows.

Let $J \subset \mathbb{Z}_+$ and ε_j (for $j \in J$) are independent and identically distributed random variables taking values on the set $D = \{-1, +1\}$, with $P(\varepsilon_j = -1) = p$ and $P(\varepsilon_j = +1) = q$. Suppose n is a positive integer such that $j \in J$ if and only if $j + n \in J$. Define $\ell = \#(J \cap [0, n-1])$; then J is a union of ℓ arithmetic progressions with difference equal to n. For example, with $J = \mathbb{Z}_+$, we have $n = 1$ and $\ell = 1$; with $J = \{0, 1, 3, 4, 6, 7, 9, \ldots\}$, every third integer is omitted, so $n = 3$ and $\ell = 2$.

We define the random variable

$$Y_\alpha^J = \sum_{j \in J} \varepsilon_j \alpha^j \tag{25}$$

Note that this is the same as the random variable defined in (12), when $J = \mathbb{Z}_+$. However, we need also to consider "rarefied" versions of the Erdös series here, which is the reason for defining the *class* of Erdös series given by (25). For $J = \mathbb{Z}_+$, we denote Y_α^J by Y_α.

Given J and α, let us denote the distribution of Y_α^J by η. Its *Fourier transform* is given (as the function of the real variable, s) by:

$$\phi_\eta(s) = \int e^{isx} \eta(dx) = \mathbb{E}\left[e^{isY_\alpha^J}\right] = \mathbb{E}\left[\prod_{j \in J} e^{is\varepsilon_j \alpha^j}\right]$$

By independence of ε_j, it is easy to obtain

$$\phi_\eta(s) = \prod_{j \in J} \mathbb{E}\left[e^{is\varepsilon_j \alpha^j}\right] = \prod_{j \in J} \left[pe^{-i\alpha^j s} + qe^{i\alpha^j s}\right]. \tag{26}$$

Working with the Fourier transform of η rather than η itself is primarily motivated, in the current context, by the "inversion formula". In particular, if

$$\int_{-\infty}^{\infty} \left|\phi_\eta(s)\right| ds < \infty \tag{27}$$

then the distribution η has a continuous density, given by

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} e^{-isx} \phi_\eta(s) ds.$$

See Billingsley [9], p. 296-304, for the basic properties of Fourier transforms of probability measures (known as *characteristic functions*) on the real line.

Since $\left|\phi_\eta(s)\right| \leq 1$ for all $s \in \mathbb{R}$, if (27) holds ($\phi_\eta \in L^1$), we also have

$$\int_{-\infty}^{\infty} \left|\phi_\eta(s)\right|^2 ds < \infty \tag{28}$$

so that $\phi_\eta \in L^2$. Thus, a natural question is whether (28) itself can ensure the absolute continuity of η. This is indeed possible, as shown by the "L^2-theory" of Fourier transforms. [See, for example, Chung [13], p. 159]. Briefly, if ϕ_η satisfies (28), then by Plancherel's theorem (see, for example, Feller [27], p. 599-601), there exists $f \in L^2$ such that:

$$\int_0^x f(u) du = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(\frac{e^{isx} - 1}{-is}\right) \phi_\eta(s) ds$$

Now, one can use the inversion formula to show that

$$F(x) - F(0) = \frac{1}{\sqrt{2\pi}} \int_0^x f(u) du$$

where F is the distribution function, corresponding to the distribution η.

In view of this, if one can show that for every $\varepsilon > 0$

$$\int_{a+\varepsilon}^{b-\varepsilon} \left[\int_{-\infty}^{\infty} \left|\phi_\eta(s)\right|^2 ds\right] d\alpha < \infty \tag{29}$$

where $a < b$, then it would follow that for Lebesgue almost every $\alpha \in (a, b)$, (28) holds, and so the distribution η is absolutely continuous, with L^2 density.

The results of Solomyak [56], which we discuss in Section 6.2, provides sufficient conditions under which (29) is satisfied, for certain specifications of a and b.

6.2 The Result of Solomyak in the Symmetric case $p = 1/2$

The result of Solomyak [56] that solves the Erdös problem in the symmetric case is the following.

Theorem 1 (Solomyak) *For $p = 1/2$, the distribution of Y_α has a density in $L^2(\mathbb{R})$ for Lebesgue almost every $\alpha \in (1/2, 1)$.*

The proof of this theorem rests upon a more general result, which establishes a sufficient condition for (L^2) absolute continuity based on the existence of a certain type of power series.

Proposition 4 *For $p = 1/2$, the distribution of Y_α^J has $L^2(\mathbb{R})$ density for a.e. $\alpha \in \left((1/2)^{\ell/n}, \tau\right)$ provided*

$$(1/2)^{\ell/n} < \tau$$

where $\tau \in (0,1)$ is the smallest number for which there exists a power series $g(x) \not\equiv 0$,

$$g(x) = \sum_{j \in J} g_j x^j, \qquad \text{where } g_j \in \{-1, 0, 1\},$$

with $g(\tau) = g'(\tau) = 0$.

The specifications required at the end of the last section in order to apply Fourier transforms theory as in (29), are indeed $a = (1/2)^{\ell/n}$ and $b = \tau$. To be more precise, existence of such power series is a sufficient condition for the following *transversality condition* to hold on the interval $(a, b) \subset (0, 1)$. Say that the transversality condition holds on (a, b) for a given pair $\omega, \omega' \in \Omega$ if

$$\alpha \longmapsto \Pi_\alpha(\omega) - \Pi_\alpha(\omega') \qquad \text{has no double zero on } (a, b) \text{ for } \omega \neq \omega'.$$

Here a "double zero" for the difference $f(\alpha) = \Pi_\alpha(\omega) - \Pi_\alpha(\omega')$ means α_0 such that $f(\alpha_0) = f'(\alpha_0) = 0$. The graphic interpretation is that the graphs of functions $\alpha \longmapsto \Pi_\alpha(\omega)$ and $\alpha \longmapsto \Pi_\alpha(\omega')$ restricted to the interval (a, b), intersect transversally (if at all).

In order to apply Proposition 4 to obtain Theorem 1, one has to have an efficient method of isolating "multiple zeros" of power series. Solomyak himself calculates some lower bounds for such zeros that we shall now exploit to see how Proposition 4 and the use of Fourier transforms yield Theorem 1.

The first step in proving Theorem 1 is to find an interval $(a, b) \subset (0, 1)$ such that the transversality condition holds. We shall see that such an interval is $(a, b) = \left((1/2)^{\ell/n}, \tau\right) = \left(1/2, 2^{-1/2}\right)$; then Proposition 4 states that for *a.e.* $\alpha \in \left(1/2, 2^{-1/2}\right)$, the distribution μ_f is absolutely continuous with L^2 density. In order to fill the whole interval $(1/2, 1)$, one must rely on convolutions through Fourier transform as in (26). Let us see this procedure in some detail.

Take $J = \mathbb{Z}_+$, so that $\ell = n = 1$ and $(\ell/n) = 1$ in Proposition 4. Thus, the first step tackles directly our Erdös series. Solomyak finds a lower estimation $\tau > 0.649$ for this case. Hence, in view of Proposition 4, the distribution μ_f of Y_α has density in $L^2(\mathbb{R})$ for almost every $\alpha \in (1/2, 0.649)$.

Next, consider the "rarefied" Erdös series so that every third integer from Z_+ is omitted, namely, let $J = \{0, 1, 3, 4, 6, 7, 9, \ldots\}$, so that $n = 3$ and $\ell = 2$. By Solomyak's calculations, the number τ for this rarefied series satisfies $\tau > 0.713$. Since $2^{-2/3} \approx 0.630 < 0.713$, Proposition 4 (for $\ell/n = 2/3$) is nonempty and thus for *a.e.* $\alpha \in \left(2^{-2/3}, 0.713\right)$ the distribution η of Y_α^J has L^2 density. In this case, its Fourier transform $\phi_\eta \in L^2$ and since $\left|\phi_{\mu_f}\right| \leq \left|\phi_\eta\right|$, we also have $\phi_{\mu_f} \in L^2$. That is, we have found that for *a.e.* $\alpha \in (1/2, 0.649) \cup \left(2^{-2/3}, 0.713\right)$, μ_f has L^2 density. Note, however, that these two open intervals overlap, as $2^{-2/3} \approx 0.630 < 0.649$.

Moreover, $2^{-1/2} \approx 0.707 < 0.713$. Therefore, we conclude that for *a.e.* $\alpha \in \left(1/2, 2^{-1/2}\right)$, μ_f has L^2 density.

The second step consists in extending this finding to *a.e.* $\alpha \in \left(2^{-1/k}, 2^{-1/2k}\right)$ for $k = 2, 3, \ldots$. Then, since $2^{-1/k} \to 1$ as $k \to \infty$, it follows that μ_f has L^2 density for *a.e.* $\alpha \in (1/2, 1)$. We demonstrate this for $k = 2$, the rest following by induction. For *a.e.* $\alpha \in (2^{-1/2}, 2^{-1/4})$, we have the Fourier transform of η given by (26), which yields

$$\phi_\eta^\alpha(s) = \phi_\eta^{\alpha^2}(s)\, \phi_\eta^{\alpha^2}(\alpha s). \tag{30}$$

Since $\alpha \in (2^{-1/2}, 2^{-1/4})$ implies $\alpha^2 \in (1/2, 2^{-1/2})$, we have each of the two Fourier transforms on the right hand side of (30) in L^2, and so by Hölder's inequality ϕ_η^α is in L^1. Since ϕ_η^α is also in L^∞, we have $\phi_\eta^\alpha \in L^1(R) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$.

Finally, by induction, one can check that for *a.e.* $\alpha \in \bigcup_{k=0}^{\infty} \left(2^{-1/k}, 2^{-1/2k}\right)$, $\phi_\eta^\alpha \in L^2(\mathbb{R})$, and so η has L^2 density. Thus, μ_f is absolutely continuous with L^2 density for *a.e.* $\alpha \in (1/2, 1)$.

6.3 An Extension of Solomyak Theorem to the Asymmetric Case $p \neq 1/2$

The main findings in [52] for the asymmetric case pertaining to our IFS are the following.

Theorem 2 (Peres - Solomyak)

i) If $1/3 \leq p \leq 2/3$, *then for almost every* $p^p (1-p)^{1-p} < \alpha < 1$ μ_f *is absolutely continuous;*

ii) If $1/3 \leq p \leq 2/3$ *and* $1 < \gamma \leq 2$, *then for almost every* $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} \leq \alpha < 1$ μ_f *has density in* L^γ*;*

iii) for all $0 < p < 1$ *and all* $\gamma > 1$, *if* μ_f *is absolutely continuous with density in* L^γ, *then* $\alpha \geq [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$.

The meaningful innovation in the proof of both parts (i) and (ii) with respect to the technique adopted in the proof of Theorem 1, is the use of the Radon-Nykodym derivative of the invariant measure, $\partial \mu_f / \partial x$, which actually is its density and exists whenever μ_f is absolutely continuous. The importance of this method lies in by-passing L^2 theory[10] which relies on (29). As a matter of fact, in place of (29) the authors show that

$$\int_a^b \left[\int_{-\infty}^{\infty} \left(\underline{D}\left(\mu_f, x\right) \right)^{\gamma-1} d\mu_f(x) \right] d\alpha < \infty,$$

where

$$\underline{D}\left(\mu_f, x\right) = \liminf_{r \to 0^+} (2r)^{-1} \mu_f\left[B_r(x)\right]$$

and $B_r(x)$ denotes the ball of center x and radius r, holds for any interval $(a, b) \subset (0, 1)$ such that $a > [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$ and the transversality condition holds. As for the upper extremum of interval (a, b), they use the same estimation as in [56]: $b = 0.649 < \tau$, which clearly works since, for $1/3 \leq p \leq 2/3$ and $\gamma \leq 2$,

$$[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} \leq p^2 + (1-p)^2 \leq (1/3)^2 + (2/3)^2 = 5/9 < 0.649.$$

[10] Note that this approach includes the L^2 case for $\gamma \geq 2$, since $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} = p^2 + (1-p)^2$ for $\gamma = 2$. However, a theory purely in L^2, like the one pursued in [56], would have missed absolute continuity of μ_f (with density in L^γ for $1 < \gamma < 2$) between $p^p (1-p)^{1-p}$ and $p^2 + (1-p)^2$.

Hence, part (i) of Theorem 2 is true for $p^p(1-p)^{1-p} < \alpha < 0.649$, and the extension to the whole interval $\left(p^p(1-p)^{1-p}, 1\right)$ [and to interval $\left[[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}, 1\right)$ in part (ii)] is established through convolution techniques similar to that discussed in Section 6.2.

Actually slightly more can be said: since the transversality condition holds for all pairs (α, p) such that $\alpha \leq 0.649$, part (i) of Theorem 2 it is immediately extended for values of p in the larger interval $(0.156, 0.844)$ [where the extrema are the solutions of the equation $p^p(1-p)^{1-p} = 0.649$] and for $p^p(1-p)^{1-p} < \alpha < 0.649$. A similar argument also applies to part (ii) of Theorem 2, therefore yielding the following corollary.

Corollary 1

i) If $0.156 < p < 0.844$, *then for almost every* $p^p(1-p)^{1-p} < \alpha < 0.649$ μ_f *is absolutely continuous;*

ii) If $0.156 < p < 0.844$ *and* $1 < \gamma \leq 2$ *is such that* $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} < 0.649$, *then for almost every* $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} \leq \alpha < 0.649$ μ_f *has density in* L^γ.

The difficulty in extending the last result for values of $p \in (0.156, 0.844)$ outside the interval $[1/3, 2/3]$ and for values of α larger than 0.649 lies in the use of convolutions techniques, which do not work in this case.

Necessary smoothness condition (iii) in Theorem 2 is derived from a theorem of Hardy-Littlewood [31].

7 A Bifurcation Diagram

It is clear that in the equal probability case, we are able to describe the nature of the stochastic steady-state for (Lebesgue) almost every value of the technological parameter, α. However, in the asymmetric case ($p \neq 1/2$), the results of Sections 4, 5 and 6 are unable to completely characterize the invariant distribution, especially for values of technological parameter α above the entropy curve and values of probability p outside the interval $[1/3, 2/3]$.

In view of all known results reviewed in the previous sections, we are able to sketch only a partial "bifurcation diagram" that includes the combinations of parameter values covered by our analysis. It appears as in figure 5.

In view of necessary smoothness condition expressed in part (iii) of Theorem 2, it is natural to formulate a conjecture for the missing values of α above the entropy curve and p outside interval $[1/3, 2/3]$. In (iii) is excluded the possibility that μ_f is too smooth below curve $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$, namely, given that μ_f has density in L^γ, then $\alpha \geq [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$, for any $\gamma > 1$. Note that $\lim_{\gamma \to 1^+} [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} = p^p(1-p)^{(1-p)}$ and the graph of function $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$ moves upward away from the graph of the entropy curve $p^p(1-p)^{1-p}$ for increasing values of γ, until converging to the graph of function $\max(p, 1-p)$, that is, $\lim_{\gamma \to +\infty} [p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} = \max(p, 1-p)$.

The conjecture, thus, is that for all $0 < p < 1$, as one moves upward above the entropy curve, μ_f is generically absolutely continuous with density becoming gradually better shaped, starting from having density in L^1 (close to the entropy curve), then having density in L^2 for $\gamma \geq 2$ [this is already known to be true by part (ii) of Theorem 2 at least for $p \in [1/3, 2/3]$] and eventually having an essentially bounded density for $\gamma \to +\infty$, that is, for almost every $\max(p, 1-p) \leq \alpha < 1$. This situation is depicted in figure 6.

However, since very few cases of singular measures above the entropy curve are known (actually the only known examples are those given by Erdös for $p = 1/2$, see [22]), even if the portrait

in figure 6 were true, there is still room for investigating Garsia's [29] much stronger conjecture that the invariant measure is absolutely continuous for all $p^p(1-p)^{(1-p)} < \alpha < 1$ but a countable set of points.

8 Appendix

In this final section we present a result connecting L-metric convergence to weak convergence of measures, which we used in our exposition in Section 2.1.2.

Proposition 5 *Convergence in the L-metric defined in (6) implies weak convergence over $\mathcal{L}(X)$; that is, given any sequence μ_t in $\mathcal{L}(X)$ and a measure μ_f in $\mathcal{L}(X)$, $L\left(\mu_t, \mu_f\right) \to 0$ implies $\mu_t \Longrightarrow \mu_f$.*

To prove this result we need the following lemma which is a variant of Uryshon Lemma.

Lemma 1 *Let (X, d) be a metric space and $A, B \subset X$ non-empty closed disjoint sets. If $d(A, B) > 0$, then there exists a Lipschitz function $g : X \to \mathbb{R}$ such that $g(a) = 0$ for all $a \in A$, $g(b) = 1$ for all $b \in B$ and $0 < g(x) < 1$ for all $x \notin A \cup B$.*

Proof Consider

$$g(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

Since $d(A, B) > 0$, there is δ such that $d(x, A) + d(x, B) > \delta > 0$. Hence, g satisfies all requirements but Lipschitz property. To see that g is Lipschitz, for any $x, y \in X$ and $\varepsilon > 0$, find $a \in A$ so that $d(a, y) \leq d(y, A) + \varepsilon$. Then

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \leq d(x, y) + d(y, A) + \varepsilon$$

and

$$d(x, A) - d(y, A) \leq d(x, y) + \varepsilon$$

Since ε is arbitrary, and by exchanging x with y, we get

$$|d(x, A) - d(y, A)| \leq d(x, y)$$

from which follows

$$|g(x) - g(y)| \leq \delta^{-1} d(x, y)$$

Proof (Proposition 5) It suffices to show that

$$\limsup_{t \to \infty} \mu_t(C) \leq \mu_f(C) \tag{31}$$

for all closed sets $C \subset X$.

Note that

$$L\left(\mu_t, \mu_f\right) \to 0 \Longrightarrow \int g d\mu_t \to \int g d\mu_f$$

for any Lipschitz function g. Take any closed (proper) subset $C \subset X$ and consider the sequence of open sets $\left\{C_{1/n}\right\}$, where C_δ denotes the δ-parallel body of C defined in (8). Clearly $C \subset C_{1/n}$ for all n and the complement $C^c_{1/n}$ is non-empty for n large enough. By Lemma 1 there exists a Lipschitz function g_n such that $g_n(x) = 1$ for $x \in C$ and $g_n(x) = 0$ for $x \in C^c_{1/n}$. Therefore

$$\mu_t(C) \leq \int g_n d\mu_t \leq \mu_t\left(C_{1/n}\right)$$

and

$$\limsup_{t \to \infty} \mu_t(C) \leq \int g_n d\mu_f \leq \mu_f\left(C_{1/n}\right),$$

which, for $n \to \infty$, gives (31).

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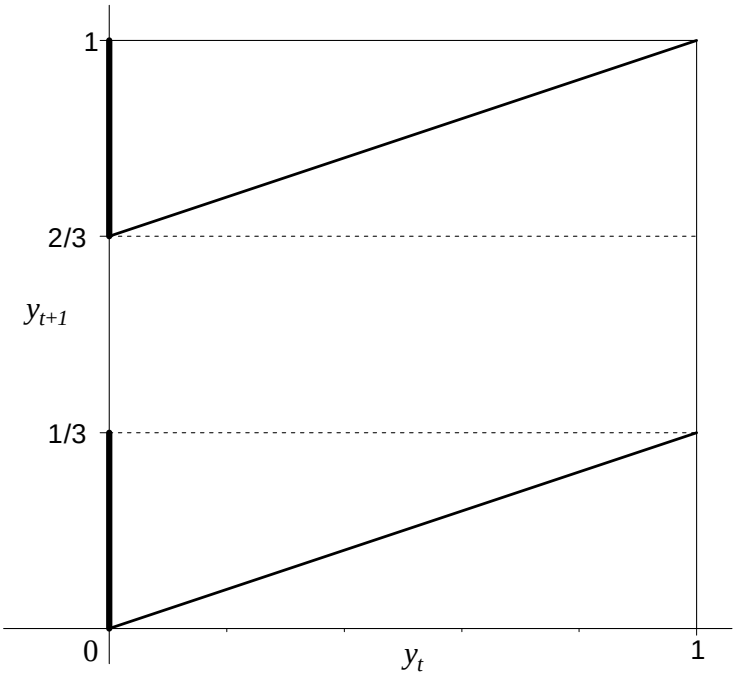
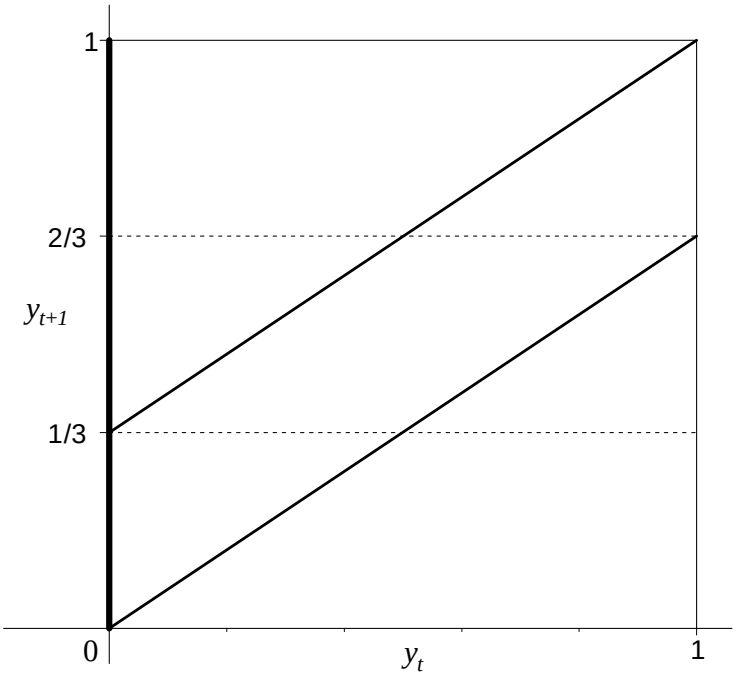
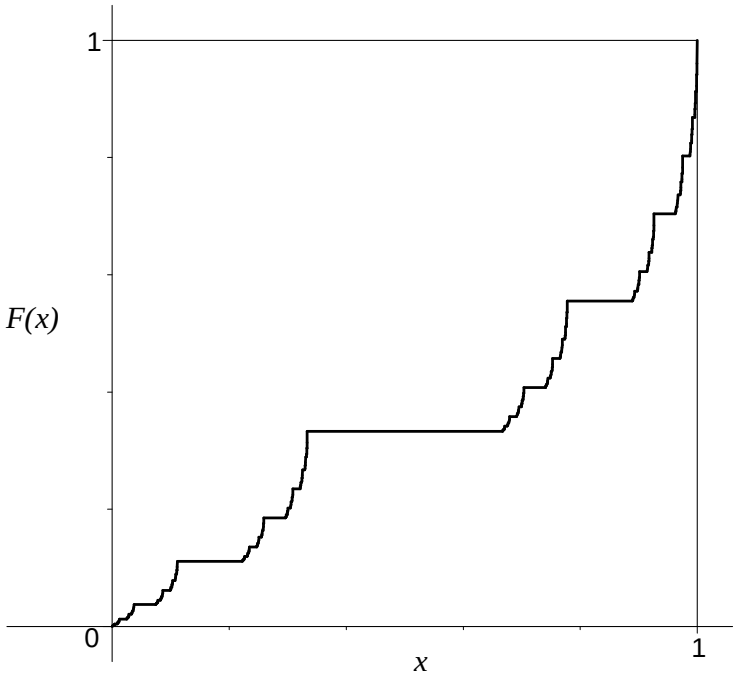


a)

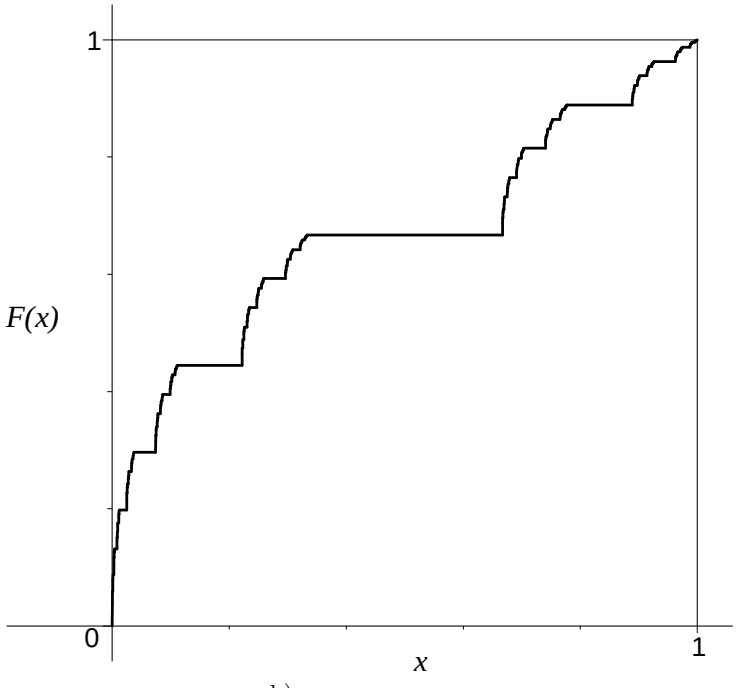


b)

Fig. 1 Plot of $y_{t+1} = f_z(y_t)$, $z \in \{0,1\}$: a) $\alpha = 1/3$, non overlapping images; b) $\alpha = 2/3$, overlapping images.

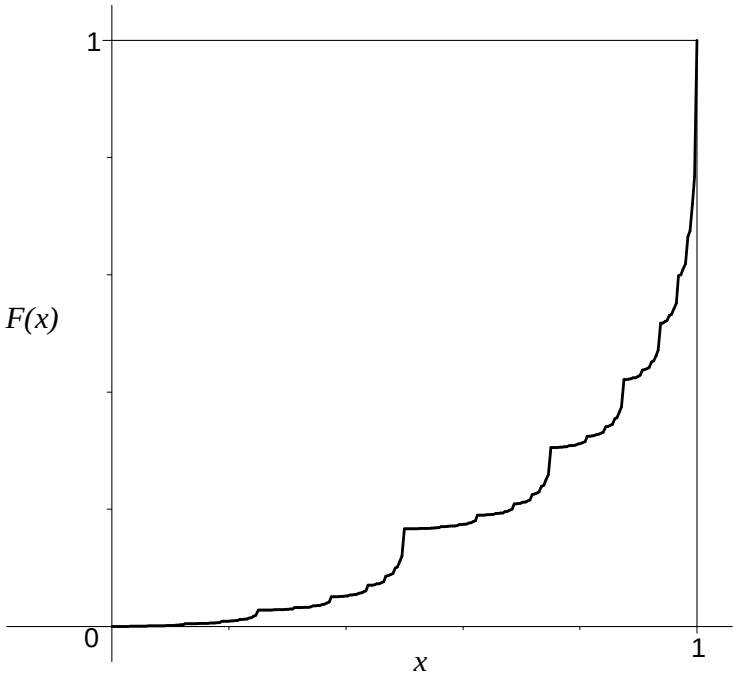


a)

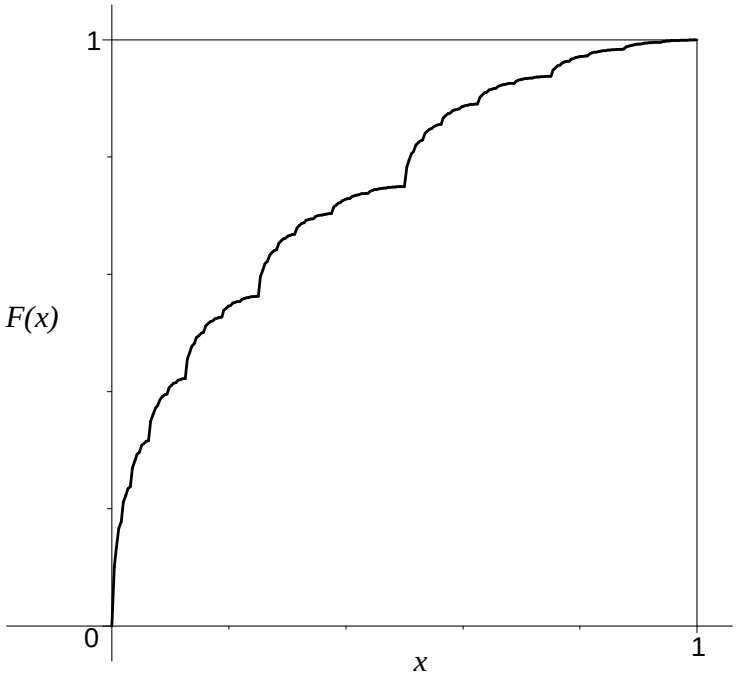


b)

Fig. 2 Devil's Staircase: a) $\alpha = 1/3$ and $p = 1/3$; b) $\alpha = 1/3$ and $p = 2/3$.



a)



b)

Fig. 3 Hellinger distributions: a) $\alpha = 1/2$ and $p = 1/6$; b) $\alpha = 1/2$ and $p = 3/4$.

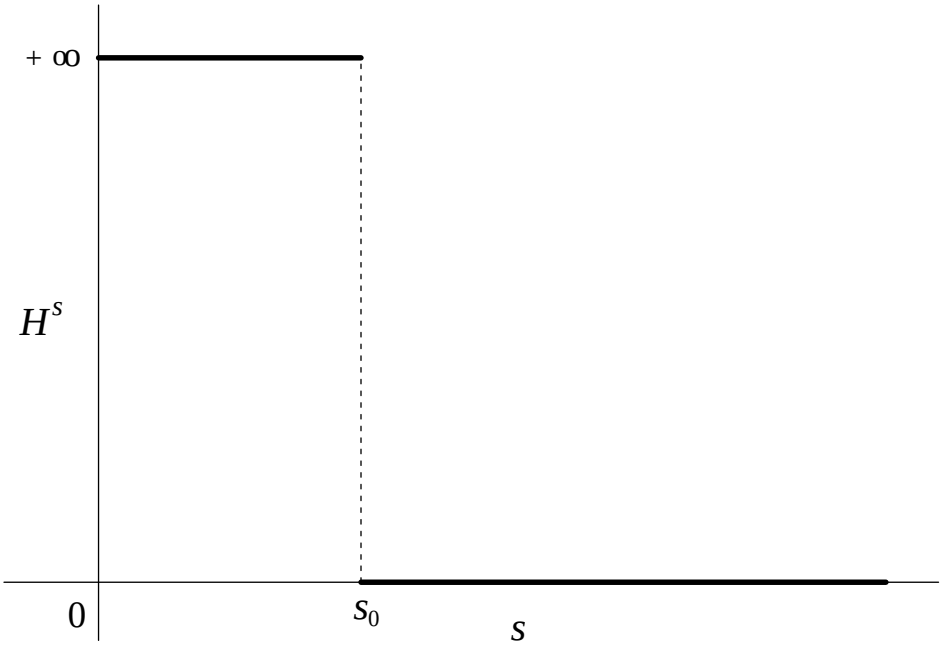


Fig. 4 Plot of $\mathcal{H}^s(F;d)$ where $s_0 \equiv \dim(F;d)$.

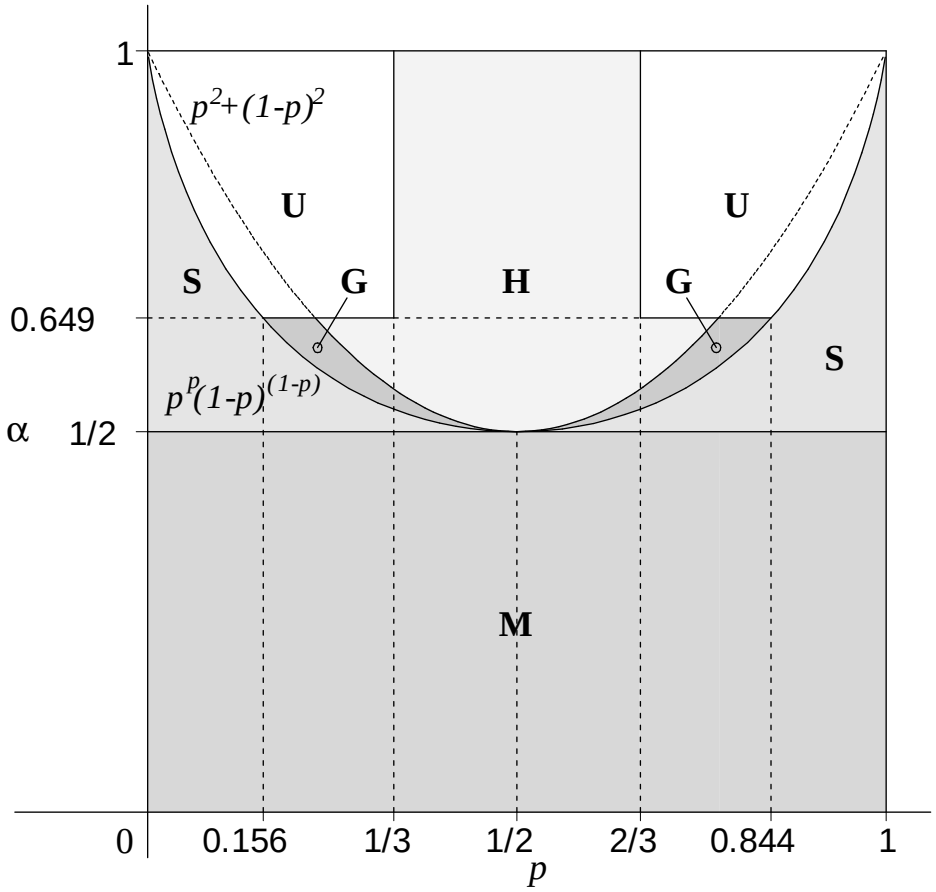


Fig. 5 **M**: mutually singular measures over Cantor sets; **S**: measures singular with respect to Lebesgue measure; **H**: *a.e.* absolutely continuous measures with density in L^2; **G**: *a.e.* absolutely continuous measures with density in L^γ, with $1 < \gamma \leq 2$ [recall that $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)} = p^2 + (1-p)^2$ when $\gamma = 2$]; **U**: unknown area.

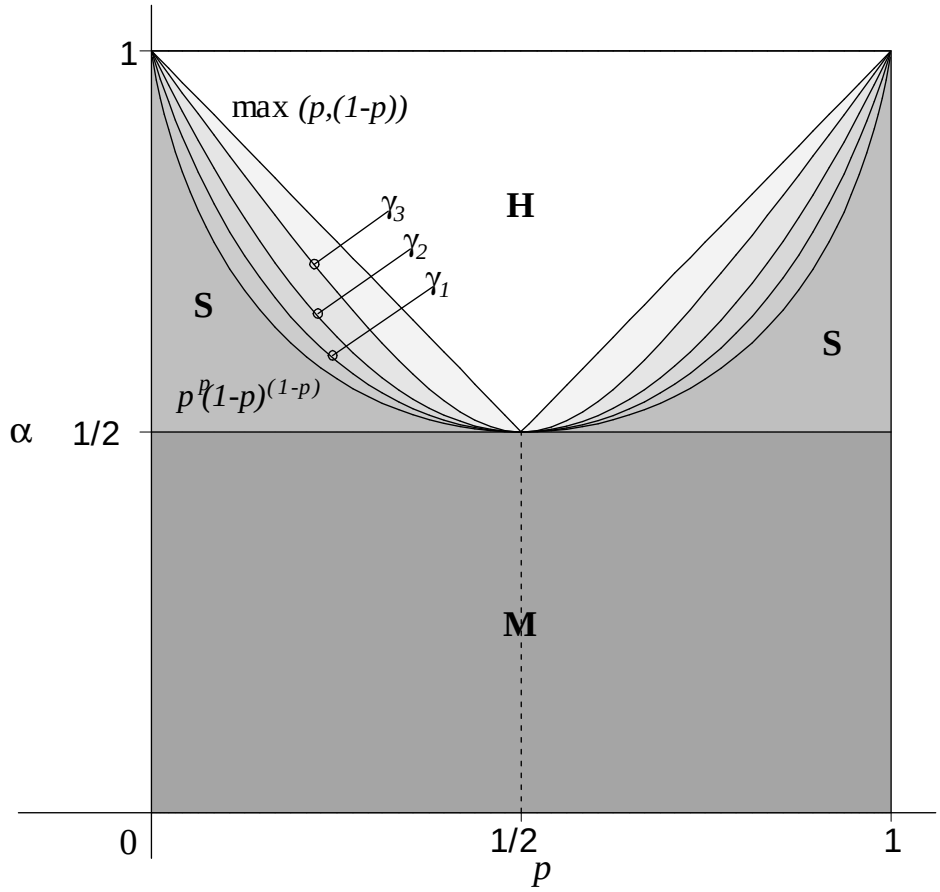


Fig. 6 **M**: mutually singular measures over Cantor sets; **S**: measures singular with respect to Lebesgue measure; **H**: *a.e.* absolutely continuous measures with bounded density. In the area between **S** and **H,** the invariant measure has density in L^γ, $\gamma > 1$, depending on $[p^\gamma + (1-p)^\gamma]^{1/(\gamma-1)}$. Three increasingly smooth cases are plotted between the entropy curve and $\max(p, 1-p)$: $\gamma_1 = 11/8$, $\gamma_2 = 2$ and $\gamma_3 = 4$.