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1 Introduction

In the production of complex Manufacturing-to-Order products, uncertainty may stem from a number of possible sources, both internal and external, affecting the execution of the scheduled activities. A disrupted schedule incurs high costs due to missed due dates, resource idleness, or higher work-in-process inventory. Robust scheduling approaches aim at being able to provide a balanced compromise between expected performance and the protection against rare but extremely unfavourable events. Tackling this problem entails the need of estimating the probability distribution associated with a scheduling objective function. One approach, described in (Urgo 2014), is to use phase type distributions to approximate generally distributed activity durations.

The aim of this work is to show that an efficient manner to deal with activity networks in which durations are described by phase type distributions is provided by Kronecker algebra. This algebra alleviates the state space explosion problem, which is typical when phase type distributions are used in models where several activities are executed in parallel, and opens a way to an efficient and modular analysis methodology. The proposed approach is applied to a simple test case demonstrating how a proper design of the PH approximation provides a tool to asses the completion time distribution with a good accuracy.

2 Markov Activity Networks

In Markov Activity Networks (MAN) (Kulkarni & Adlakha 1986), given that (i) the durations of the activities are mutually independent and (ii) exponentially distributed, the execution of the activity network can be represented through a *continuous time Markov chain* (CTMC). Modelling the execution of a network of activities with a Markov chain provides the capability of exploiting the wide set of tools and approaches available for Markov models to address stochastic scheduling problems. As an example we consider the activity network in Figure 1, representing the execution of a set of activities $1, 2, ..., 5$ and their precedence relations. The durations of the activities are exponentially distributed with parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and λ_5,, respectively.

The CTMC describing the execution of the set of activities is depicted in Figure 2 where the on-going activities of a state are indicated inside the circle. The initial state is the state in which no activities have yet been executed. Assuming that this state is the first state of the CTMC, the initial probability vector is given as $\pi(0) = |1, 0, ..., 0|$. Denoting by Q the infinitesimal generator of the CTMC, the transient probabilities of the states at time t is given by the vector $\pi(t) = \pi(0) \exp(tQ)$ where $\exp(\bullet)$ is the matrix exponential function. Assuming that the last state of the CTMC corresponds to the situation when all activities are executed, the last entry of $\pi(t)$ gives the probability that the execution of all activities took less than t time units. I.e., the last entry of $\pi(t)$ provides the cumulative distribution function of the makespan of the underlying stochastic scheduling problem. Note that the
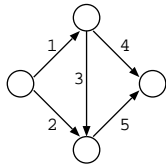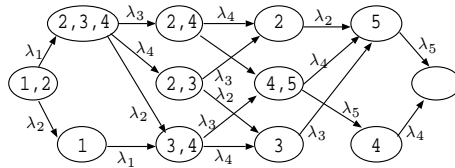
Fig. 1. An activity network.

Fig. 2. CTMC of the activity network in Figure 1.

last state of the Markov chain is absorbing and the chain is acyclic. However, the restriction to exponentially distributed activity durations represents a limiting hypothesis, since the exponential distribution is quite seldom applicable to real industrial processes.

3   Phase Type Distributions

In the field of Markov models, phase type (PH) distributions are widely used to provide an approximation of a general distribution. Basically, a set of inter-related exponential delays are put together to form a distribution to approximate a general one. Formally, a continuous time PH distribution is the distribution of the time to absorption of a CTMC and the order of the PH distribution is given by the number of transient states of the chain. Consequently, the PH distribution is determined by a vector, β, which gives the initial probabilities of the transient states and a matrix, T, which contains the intensities of the transitions among the transient state. The cumulative distribution function, the probability density function and the moments of a PH distribution are given by

$$F(x) = P\{X \leq x\} = 1 - \beta e^{Tx}\mathbb{I}, \quad f(x) = \beta e^{Tx}(-T)\mathbb{I}, \quad m_i = i!\beta(-T)^{-i}\mathbb{I}$$

where $\mathbb{I}$ is a column vector of ones.

The use of PH distributions is popular because these distributions can be easily used as building blocks of more complex models. Indeed, if we are given a system in which all sojourn times are according to PH distributions and the next state distribution is Markovian then the overall system behavior can be described by a Markov chain.

The class of PH distributions is dense in the field of positive valued distributions, i.e., any positive valued distribution can be approximated by PH distributions with any accuracy. This fact does not provide however directly a practical method to fit distributions by PH distributions. Several authors proposed fitting methods and most of these fall into two categories: maximum likelihood (ML) based estimation of the parameters and moment matching techniques. One of the first works on ML estimation considered acyclic PH distributions (Bobbio & Cumani 1992) while an approach for the whole family, based on the expectation-maximization method, is proposed in (Asmussen, Nerman & Olsson 1996). Since these early papers, many new methods and improvements have been suggested for the whole PH family and for its sub-classes. For what concerns moment matching methods the following results are available. For low order (≤ 3) PH distributions, moment bounds and moment matching formulas are either known in an explicit manner or there exist iterative numerical methods to check if given moments can be captured (Telek & Heindl 2002, Horváth & Telek 2007). For higher order there exist matching algorithms, but these often result in improper density functions and the validity check is a non-trivial problem. In (Bobbio, Horváth & Telek 2005) a simple method is provided that constructs a minimal order acyclic PH distribution given three moments. Tool support is available for the construction of PH and ME distributions. Specifically, ML based fitting is implemented in PhFit (Horváth & Telek 2002) and a set of moment matching functions is provided in BuTools.

4 Analysis of Activity Networks with Phase Type Distributed Activity Durations by Kronecker Algebra

We assume now that the time to carry out an activity is distributed according to a PH distribution. We will denote the vector-matrix pair that describes the PH distributions associated with activity i by (β_i, T_i). Then infinitesimal generator of the overall system can be constructed by blocks. The diagonal blocks describe the parallel execution of a number of activities and the diagonal block associated with state j is given by

$$Q_{j,j} = \bigoplus_{i \in \mathcal{A}(j)} T_i$$

where $\bigoplus$ denotes the Kronecker sum operator and $\mathcal{A}(j)$ is the set of on-going activities in state j. An off diagonal block describes the finishing of an activity, the initialization of one or more activities and must maintain the phase of those activities that remain active. Accordingly, the block that describes the transition from state j to state k (with $j \neq k$) is given as

$$Q_{j,k} = \bigotimes_{i \in \mathcal{A}} R_i \quad \text{with} \quad R_i = \begin{cases} (-T_i)\mathbb{1} & \text{if } i \in \mathcal{A}(j) \text{ and } i \notin \mathcal{A}(k) \\ \beta_i & \text{if } i \notin \mathcal{A}(j) \text{ and } i \in \mathcal{A}(k) \\ I_i & \text{if } i \in \mathcal{A}(j) \text{ and } i \in \mathcal{A}(k) \\ 1 & \text{if } i \notin \mathcal{A}(j) \text{ and } i \notin \mathcal{A}(k) \end{cases}$$

where $\bigotimes$ denotes the Kronecker product operator and I_i is an identity matrix whose size is equal to the order of the PH distribution associated with activity i. The four cases in the above equation correspond to the cases: activity i finishes (described by the vector $(-T_i)\mathbb{1}$ which contains the "finishing" intensities of the associated PH distribution), activity i starts (described by β_i), activity i remain active (described by I_i), activity i is neither active in state j nor in state k (scalar 1 in the Kronecker product). The initial probability vector of the overall system is composed of the initial probabilities of the starting activities: $\pi(0) = |\bigotimes_{i \in \mathcal{A}(1)} \beta_i, 0, ..., 0|$.

The infinitesimal generator for the model depicted in Figure 1 is as follows

$$Q = \begin{vmatrix} T_1 \oplus T_2 & t_1 \otimes I_2 \otimes \beta_3 \otimes \beta_4 & 0 & 0 & I_1 \otimes t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_2 \oplus T_3 \oplus T_4 & I_2 \otimes I_3 \otimes t_4 & 0 & 0 & I_2 \otimes t_3 \otimes I_4 & t_2 \otimes I_3 \otimes I_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_2 \oplus T_3 & I_2 \otimes t_3 & 0 & 0 & 0 & t_2 \otimes I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & t_2 \otimes \beta_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_1 & 0 & t_1 \otimes \beta_3 \otimes \beta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \otimes t_4 & 0 & T_2 \oplus T_4 & 0 & 0 & 0 & t_2 \otimes I_4 \otimes \beta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_3 \oplus T_4 & I_3 \otimes t_4 & 0 & t_3 \otimes I_4 \otimes \beta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_3 & t_3 \otimes \beta_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_5 & 0 & 0 & t_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_4 \otimes I_5 & T_4 \oplus T_5 & I_4 \otimes t_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4 & t_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

where t_i is the finishing vector of the PH distribution (β_i, T_i) (i.e.,$t_x = -T_x \mathbb{1}$). Each block of rows corresponds to a set of currently ongoing activities and these sets are ordered as $\{1,2\}, \{2,3,4\}, \{2,3\}, \{2\}, \{1\}, \{2,4\}, \{3,4\}, \{3\}, \{5\}, \{4,5\}, \{4\}$ and the empty set, $\emptyset$.

As a numerical example, we assume that activity 1 follows a log-normal distribution with mean equals to one and with pdf $f(x) = 1/(b\sqrt{2\pi}x) \, e^{-\frac{(\log(x)-a)^2}{2b^2}}$ and parameters $a = -1.62, b = 1.8$ while the duration of the other activities are given by order 4 Erlang distribution with mean equal to one (note that Erlang distributions are in the PH family). We constructed four different fitting distributions for the log-normal distribution of activity 1. The first one is an order 8 ML estimation by PhFit (Horváth & Telek 2002). The second and the third one are order 8 ML estimations extended with 2 and 4 phases, respectively, to fit the tail (obtained with PhFit). The last one matches three moments of the log-normal distribution (this can be done with an order 2 PH distribution by (Bobbio et al. 2005)). The pdf of the log-normal and the fitting PH distributions are depicted in Figure 3. The pure ML estimation fails to catch the tail behaviour and the moment matching PH distribution fails to capture the shape of the log-normal distribution.
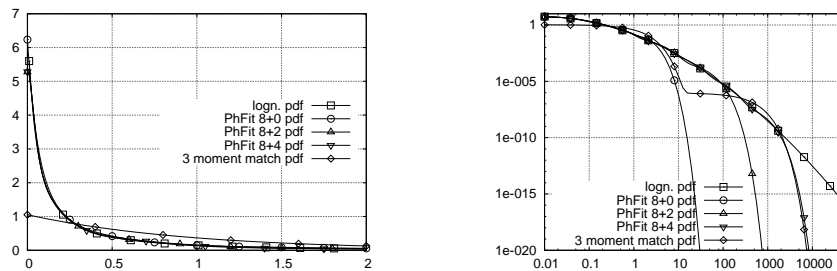
Fig. 3. Main part (left) and tail (right) of the pdf of the PH distributions fitting the log-normal distribution.
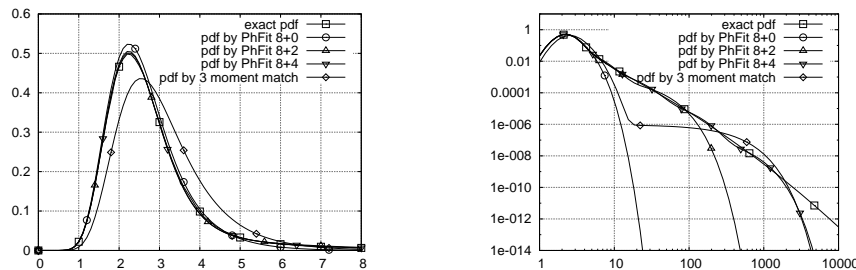

Fig. 4. Main part (left) and tail (right) of the pdf of the makespan of the activity network.

In Figure 4 we depicted the pdf of the makespan of the activity network both with the original log-normal distribution and with the approximating PH distributions. It can be seen that the best approximation is achieved by applying the third PH distributions that captures both the main part and the tail of the pdf.

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