Teaching practices with the derivative concept. A problematic meeting between Algebra and Calculus in secondary school.

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TEACHING PRACTICES WITH THE DERIVATIVE CONCEPT

A problematic meeting between Algebra and Calculus in secondary school

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Abstract

The derivative is one of the crucial notions related to functions in secondary teaching. Some rooted algebraic practices intervene in the development of practices that are proper to Calculus, such as the use of limit. Hence, the introduction of the derivative is a delicate moment for both students and teachers.

In this thesis, the problematics of the transition from Algebra to Calculus is approached through an historical and epistemological analysis. In entering the Calculus domain, the work on functions becomes increasingly grounded on local properties, which are valid in the neighbourhood of a point. This is the case of the derivative introduction that requires the activation of a local perspective on functions and poses an additional difficulty, since from a global perspective the derivative of a function is a function itself.

This thesis investigates the intervention of the local perspective in the secondary teaching of the derivative, within the Italian context. Framing our research in the Anthropological Theory of the Didactic (ATD), we study the didactic transposition of the derivative notion, when it is considered both as a tool for studying a function and as a function itself. In our analysis, we network three different theoretical elements: we focus on two types of task and the related mathematical and didactic praxeologies; we identify the perspectives activated on the involved functions (i.e., pointwise, global and local); we analyse the employed semiotic resources (e.g., speech, gestures, symbols, drawings) to convey these perspectives and to construct such praxeologies. The didactic transposition of the derivative concept is presented through the analysis of the intended curriculum and the implemented curriculum, with insights into the attained curriculum. At each stage, we specifically concentrate on the presence and the role given to the local perspective. In particular, the core of the thesis is the analysis of three case studies involving three teachers who introduce the derivative to their grade 13 students of scientific high school. After an interview, we have observed each teacher in her classroom and finally we have proposed two activities to the students.

One of the main results is the identification of two different derivative-related praxeologies. They are based on different concept images of the tangent line: on the one hand, the limit of secant lines, and on the other hand, the best linear approximation. We discuss them by distinguishing the different levels of intervention of the local perspective in the work on functions.

Contents

Introduction

Problematics

In the Italian school, Algebra and Calculus are introduced in two successive moments. Algebra is presented as literal calculus at the end of lower secondary school and it acquires a great importance in the first two years of upper secondary school, when it is identified with the study of number systems and of their properties, with the resolution of equations, inequalities and systems. Calculus dominates the final year of secondary school with a systematic study of the functions of a real variable, limits, differential and integral calculus. Between these two stages, there is some preliminary work on elementary functions, which is essentially algebraic and graphical. The Calculus studied in the secondary school is strongly based on an algebraic work. Is this enough to make students suitably master the fundamental concepts of Calculus? Is a student well prepared to deal with the Analysis he/she will study at university?
This dissertation is not a study of the so-called "secondary-tertiary transition" (from secondary school to university), to which many researches in Mathematics Education are devoted (e.g., Robert, 1998; Bloch & Ghedamsi, 2005; Artigue, Batanero & Kent, 2007; Gueudet, 2008; Vandebrouck, 2011b). However, the same concern has fostered us to investigate further the connection of Algebra and Calculus domains.

From an historical point of view, Calculus was introduced in mathematics as infinitesimal calculus, that is algebraic calculus on infinitesimal quantities. Leibniz, Newton, and the contemporary mathematicians began to find anomalies in algebraic procedures when they involved infinitely small quantities. These anomalies could not be explained by the universally accepted paradigm (Kuhn, 1970). It seems to us that the introduction of the "infinitely small" has represented a paradigm shift in mathematics. It led to the adoption of a local perspective - or way to regard - on curves and to the idea of best approximation which replaced somehow the concept of equality. An historical and epistemological analysis leads us to recognize that the transition from Algebra to Calculus is significantly marked by the introduction of a local perspective on functions.

Research problem, focus and aim

We intend, therefore, to study the didactic transposition (Chevallard, 1985) of this complex transition from Algebra to Calculus in the context of secondary school. In particular, we consider the derivative notion. Indeed, this is one of the first concepts that make the local perspective intervene in the study of a function. Within this landscape, our research problem consists in investigating the didactic transposition of the derivative concept in the secondary school context. The first phase of the transposition moves from the "scholar" knowledge to the knowledge to be taught: curricula, textbooks, and types of task that the students are expected to be able to solve at the end of their schooling. The second transposition phase, which is the core of this dissertation, transforms the knowledge to be taught into the taught knowledge. The main focus are thus the teachers' practices in classroom and their effects (even though partial) on students. At each stage of the derivative concept transposition, we are mainly concerned in identifying the intervention of the local perspective on the involved functions and in analysing how it intervenes. For instance, we wonder if it is implicitly or explicitly present, spontaneously introduced or forced, which are the triggering practices or ideas and what is their implementation.

Theoretical framework

By adopting the research practice of networking theories (Bikner-Ahsbahs & Prediger, 2014), we coordinate three elements coming from three different theoretical approaches in Mathematics Education. The first theoretical tool is the praxeology (Chevallard, 1999), which is a fundamental notion of Chevallard's Anthropological Theory of the Didactic (ATD). This lens allows us to identify the types of task involved, the techniques to solve them and the related justifications, along with the theoretical elements that support the justifying arguments. However, it provides us with a quite static and general picture of the derivative-related practices. In other words, it does not help us to account for the deep differences that may characterize the underlying dynamics of praxeologies. Which perspectives on functions (i.e., pointwise, global or local) are activated (Vandebrouck, 2011)? What and how are the semiotic resources (e.g., speech, gestures, symbols, drawing) employed (Arzarello, 2006)? We introduce and coordinate in the analysis these two further tools: indeed, it is important not only that a certain pointwise, global or local property is identified on a function, but also the way in which such a property is claimed.

Research questions

Within our theoretical framework, we pose the following research question:

(RQ) How does the local perspective intervene in the development of the derivative-related praxeologies in the secondary school?

In particular, we focus on two types of task, with their related praxeologies. Specifically, they consist of determining the equation of the tangent to a generic function at a point, and representing the derivative function. The indications for teaching the derivative and especially the approach to these two types of task are analysed at different levels: within the intended curriculum (national guidelines, textbooks and final examination); within the implemented curriculum (teachers' praxeologies in classroom) with possible effects on the attained curriculum (students' praxeologies). RQ is articulated in the following sub-questions, which guide the analysis:

(RQ.1) What role is given to the local perspective on functions in the secondary teaching of the derivative?

(RQ.2) How do teachers construct the derivative-related praxeologies with and for their students?

(RQ.1+2) What role is given by the teachers to the local perspective on functions in the construction of such praxeologies?

(RQ.3) In which ways different praxeologies developed in classroom can affect the students' praxeologies, in terms of local perspective?

We try to answer such research questions, also considering some possible implications on teaching.

Methodology and data collection

We enter the study of the derivative concept transposition at different levels. We adopt at each stage the methodology we consider the most appropriate one.
As far as the intended curriculum is concerned, we adopt a global view on the Italian National guidelines for high schools (*Indicazioni Nazionali per i Licei*). In particular, we consider the indications involving the derivative devoted to scientific high schools. Afterwards, we opt for a more restricted point of view on the Italian textbooks. More precisely, we examine the approach to the two selected types of task within both the theory and the exercises of only two textbooks. However, they are among the most widespread ones in scientific high schools in Piedmont. Finally, we analyse the tasks that make the derivative intervene in the mathematics final examination proposed to experimental courses in scientific high schools in 2013.
With regard to the implemented curriculum, we rely upon a methodology based on case studies. The three teachers joining the project follow the analysed textbooks and their prexeologies are rather different on both didactic and mathematical side. In particular, our data consist of audio-recorded preliminary interviews and videotaped lessons for each teacher.
As a small insight also into the attained curriculum, we consider the results of two activities properly designed on the derivative. We propose them to the students working in small homogeneous groups, in each of the three observed classrooms. In this phase, the collected data encompass the written productions of all the groups and the videos of some of them.
Therefore, the data we dispose of do not form a statistic sample from which deducing quantitative evidences. On the contrary, our analysis, results, discussion and conclusions will be of qualitative kind.

Overview of the dissertation

The dissertation is subdivided into two parts. Part I is devoted to the problematics and the theoretical framework, whereas Part II involves data analysis.

In Part I, Chapter 1 presents the problematics from which the study originates: the transition from Algebra to Calculus. Firstly, this transition is examined from an historical and epistemological point of view. In particular, the perspectives on functions are introduced as an important epistemological component of the work on functions. Secondly, it is justified the choice of the derivative concept as an significant notion to investigate the transition. The definitions of differentiable function and derivative are given and discussed within the "scholarly" mathematics. Thirdly, it is provided an overview of the literature in Mathematics Education about the derivative and the related concepts. The chapter is closed by a first outline of the research problem, set in the secondary school context.

Chapter 2, then, introduces the theoretical framework of the research. It frames our study in the Anthropological Theory of the Didactic (ATD), with particular regard to the notion of didactic transposition. Further, the theoretical tools coordinated in the analysis are presented. It is stressed that the focus is especially on teachers, and the theory of the meta-didactic transposition is considered, as a natural extension of the ATD to teachers' practices in a research context. Finally, our research questions take shape.

In Part II, Chapter 3 is devoted to the analysis of the intended curriculum involving the derivative. In particular, the analysis concerns part of the Italian National guidelines, two textbooks and two problems given in one recent final examination.

Chapter 4 presents the case-study analysis of three teachers, while introducing the derivative concept and the derivative function in their classrooms.

Chapter 5 provides a brief analysis of the students working in small groups on two activities involving the derivative. The description, and the a priori and a posteriori analysis are given for both the activities proposed.

Finally, Chapter 6 draws the conclusions, starting from a discussion of the results obtained through the analysis. The analysis tool is evaluated as well. Some implications on teaching are detected, and possible further developments of the research are identified.

Part I

Problematics and theoretical framework

Chapter 1

The problematics

This chapter aims to introduce the problematics from which the study originates: the transition from Algebra to Calculus[1]. An historical and epistemological analysis of the articulation of these two mathematical domains will support us in explaining why it reveals "a problematic meeting". We have chosen the derivative concept as a significant notion which lives between Algebra and Calculus. The different ways to define the derivative in mathematics are presented and discussed in detail, in order to have insight into the fundamental mathematical objects, conceptions and definitions involved. Then, we will move on to an overlook of the existing literature in Mathematics Education about the derivative notion and the elements it involves. With the contextualization of our study in the secondary school, and in particular by focusing on the teaching of the derivative concept, we will finally come to outline the research problem from which this thesis arises.

1.1 The transition from Algebra to Calculus

> "And whatever the common Analysis [that is, algebra] performs by Means of Equations of a finite number of Terms (provided that can be done) this new method can always perform the same by Means of infinite Equations. So that I have not made any Question of giving this the Name of *Analysis* likewise. For the Reasonings in this are no less certain than in the other; nor the Equations less exact; albeit we Mortals whose reasoning Powers are confined within narrow Limits, can neither express, nor so conceive all the Terms of these Equations as to know exactly from thence the Quantities we want. To conclude, we may justly reckon that to belong to the *Analytic Art*, by the help of which the Areas and Lengths, etc. of Curves may be exactly and geometrically determined." (Newton, *De Analysi*, in Boyer & Merzbach, 2011, p.362)

[1]In several European countries Calculus is called Elementary Analysis. It can be considered as the study of functions, of their properties and of the operations on them. Sometimes, we will use the term "Analysis" to intend the broader mathematical domain studied at university courses.

With these words, Newton opens one of his greatest masterpieces, *De Analysi per aequationes numero terminorum infinitas*. This sentence reflects the sensation of the time with respect to the mathematical practices related to Algebra, when they were applied to a more general set of objects. In the case of mathematical equations involving an infinite number of terms, the common practices perform in the same way. As we will outline in the next paragraph, many other mathematicians also before Newton tried to apply the current rules of Algebra to infinite or infinitesimal entities or quantities. In the history of mathematics, a sort of transition between Algebra and Calculus can be perceived. Obviously it is something we can say in retrospect because we have a global view of the historical development of concepts which were just rising and were so ineffable at the time. Perhaps this sensation was not made explicit but it was inherently present in how mathematicians employed algebraic rules, trying them beyond the known mathematical boundaries.

1.1.1 From an historical viewpoint

We have not the pretension of giving a complete history of the infinitesimal calculus and its origins. Our intention is to find some clues of the transition between Algebra and Calculus in the history of mathematics, taking into particular account the 16th-17th centuries, when the infinitesimal calculus arose.
Calculus indeed has been introduced in mathematics as infinitesimal calculus. And the mathematics of the time was mainly based on arithmetical, algebraic and geometrical works. Therefore, it is natural that the mathematicians, who started working with embryonic forms of those that later would be the fundamental concepts of Calculus, handled the material at their disposal. In particular, we search for evidences of algebraic practices employed in a non-standard or extended way. Perhaps they were non-rigorous modes of application for the time but we find them interesting examples of how Algebra intervened in the construction of the practices of Calculus.

In his history of Calculus, Boyer (1949) points out that in the development of the algorithm of Calculus an essential role was played, in 16th century, by the systematic introduction of symbols for the quantities involved in algebraic relations. The introduction in algebra of this literal symbolism, largely due to the French mathematician François Viète, allowed the rapid progress of analytic geometry and permitted the concepts of variability and functionality to enter into algebraic thought. Thanks to the improved notation, in the century preceding Newton's and Leibniz's works, new methods were developed. They were modifications of the ancient geometrical procedures known at the time (such as the exhaustion method largely used by Euclid) but easier to apply. It was a real cognitive jump that allowed these methods to arise and they were finally recognized as forming a "new Analysis": the Calculus.
In particular, we search for evidences of this evolution in some examples taken from the work of Stevin, Kepler, Cavalieri, Fermat, Wallis and, obviously, Newton and Leibniz.

The Dutch **Stevin** (1548-1620) in some of the propositions on fluid pressure, accompanied the geometric proof with a "demonstration by numbers" in which a sequence of numbers tended to a limiting value. In his *Hypomnemata Mathematica*, he supplemented the proof that the average pressure on a vertical square wall of a vessel full of water corresponds to the pressure at its mid-point with a numerical example. Subdividing the square into four horizontal strips he remarked that the force on each strip is grater than $0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}$ units and smaller than $\frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}$ units respectively. Thus, the resulting force is included between $\frac{6}{16}$ and $\frac{10}{16}$. If the wall were subdivided into ten strips, the total force would be included between $\frac{45}{100}$ and $\frac{55}{100}$ units. If the wall were subdivided into one thousand strips, it would be included between $\frac{499500}{100000}$ and $\frac{500500}{100000}$ units. He concluded that by increasing the number of the strips it is possible to approach the ratio $\frac{1}{2}$ as closely as desired. Boyer remarks that "the procedures substituted by Stevin for the method of exhaustion constituted a marked step towards the limit concept" (Boyer, 1949, p.104). This example is based more on arithmetic rather than on algebra. However, Stevin must have based his reasoning on the algebraic relation between the number of strips n and the bounds a and b. Indeed, we can not think that he has arithmetically counted till $\frac{499500}{1000000}$ and $\frac{500500}{1000000}$. It is more plausible that he has exploited the algebraic rules for finding $a = \frac{(n-1)n/2}{n^2}$ and $b = \frac{n(n+1)/2}{n^2}$. In this sense, Stevin's demonstration can be considered as an example of using algebraic formulas to develop new infinitesimal methods.

This was the nature of the proofs of another great mathematician and astronomer of the time, the German **Kepler** (1571-1630). His first two laws of astronomy announced in *Astronomia nova* in 1609 are universally known: (1) the planets move about the sun in elliptical orbits with the sun at one focus, and (2) the radius vector joining a planet to the sun sweeps out equal areas in equal times.
In working on problems of that kind, Kepler thought of the area as made up of infinitely small triangles with one vertex at the sun and the other two vertices at points infinitely close together along the orbit. This idea is illustrated by Kepler in *Nova stereometria* of 1615 to determine the area of a circle. He considered the circle as a regular polygon with an infinite number of sides. He noticed that the heights of the infinitely thin triangles are equal to the radius. The area of each triangle is given by the semi-product of the infinitely small base, lying along the circumference, and the height, that is r. The total area of the circle, that corresponds to the sum of all these areas, is thus half the product of the apothem and the perimeter (see Kepler's figure 1.1.1). It gives $\frac{1}{2}rC$. This infinitesimal procedure hides an algebraic sum of an infinite sequence: $C = b_1 + b_2 + b_3 + ... + b_n + ...$

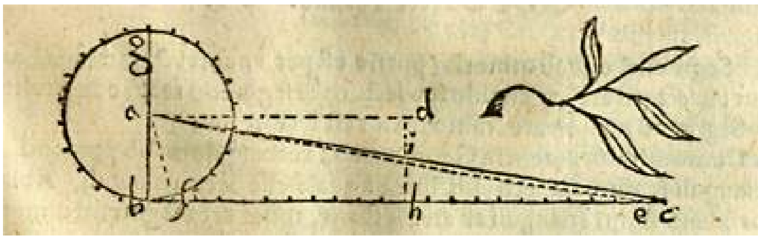

Figure 1.1.1 - Kepler's figure to illustrate the problem of determining the area of the circle (on www.matematicasvisuales.com).

In 1635, the Keplerian ideas were systematically expanded in the work of **Cavalieri** (1598-1647), disciple of Galileo, in his *Geometria Indivisibilibus*. In particular, we consider here the proposition known as "Cavalieri's Theorem" and the algebraic idea that is at its core. Proposition 24 of Book II states as follows: "Given any parallelogram and drawn one of its diagonals, all the squares of the parallelogram will be the triple of all the squares of either of the triangles formed by the diagonal, set as common reference one of the sides of the parallelogram"[2] (Lombardo-Radice, 1966, p.265). Cavalieri considered the lines in a parallelogram (see Figure 1.1.2). First of all he proved that the diagonal divides a parallelogram into two equal triangles by showing that for every indivisible of one of the triangles (e.g., BM) there exists an indivisible that is equal in the other triangle (e.g., HE). Cavalieri could conclude that the sum of the lines in one of the constituent triangles is half of the sum of the lines in the parallelogram.

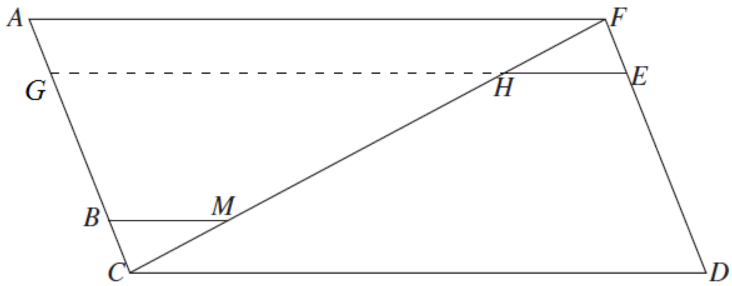

Figure 1.1.2 - Cavalieri's idea shown on the parallelogram AFDC (in Boyer & Merzbach, 2011, p.304; with our addition of the segment GH).

This result, written in our terms, corresponds to

$$\int_0^a x \mathrm{d}x = \frac{a^2}{2}.$$

[2] Our English translation of the passage "*Dato un parallelogramma qualunque e condotta in esso una diagonale, tutti i quadrati del parallelogramma saranno il triplo di tutti i quadrati di uno qualsiasi dei triangoli formati ad opera di detto diametro, posto come riferimento comune uno dei lati del parallelogramma*".

He generalized this outcome, by comparing powers of the lines parallel to the base in a parallelogram with the corresponding powers of the lines in either of the two triangles. Thus, Cavalieri found that the sum of the squares of the lines in the triangle is $\frac{1}{3}$ the sum of the squares of the lines in the parallelogram. For the cubes of the lines, the ratio was $\frac{1}{4}$. In his *Exercitationes geometricae sex*, he then concluded by analogy that the ratio between the sum of n times the lines in the triangle and the sum of the n times the lines in the parallelogram is 1 to $n+1$. In our notation, he found

$$\int_0^a x^n \mathrm{d}x = \frac{a^{n+1}}{n+1}.$$

The method employed by Cavalieri, as Boyer (1949) remarked, was based on several lemmas which are equivalent to special cases of the binomial theorem. For example, for the ratio of the cubes of the lines, he began with $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Then, his procedure is somehow the following. Calling $AF = c$, $GH = a$ and $HE = b$ in Fig. 1.1.2, we have $\Sigma c^3 = \Sigma a^3 + 3\Sigma a^2 b + 3\Sigma ab^2 + \Sigma b^3$ where the sums Σ are taken over the lines of the parallelogram and the triangles. Because of the symmetrical situation, it becomes

$$\Sigma c^3 = 2\Sigma a^3 + 6\Sigma a^2 b. \tag{1.1}$$

Now, $\Sigma c^3 = c\Sigma c^2 = c\Sigma(a+b)^2 = c(\Sigma a^2 + 2\Sigma ab + \Sigma b^2)$. At this point, the previous Cavalieri's proposition on the squares gives $\Sigma a^2 = \Sigma b^2 = \frac{1}{3}\Sigma c^2$. Thus,

$$\begin{aligned}
\Sigma c^3 =& c\Big(\frac{1}{3}\Sigma c^2 + 2\Sigma ab + \frac{1}{3}\Sigma c^2\Big) \\
=& \frac{2}{3}c\Sigma c^2 + 2c\Sigma ab \\
=& \frac{2}{3}\Sigma c^3 + 2(a+b)\Sigma ab \\
=& \frac{2}{3}\Sigma c^3 + 2\Sigma a^2 b + 2\Sigma ab^2 \\
=& \frac{2}{3}\Sigma c^3 + 4\Sigma a^2 b
\end{aligned}$$

Hence, $4\Sigma a^2 b = \frac{1}{3}\Sigma c^3$, that is $\Sigma a^2 b = \frac{1}{12}\Sigma c^3$. Replacing this result in (1.1) one obtains

$$\Sigma c^3 = 2\Sigma a^3 + 6\frac{1}{12}\Sigma c^3$$

and so

$$\frac{1}{2}\Sigma c^3 = 2\Sigma a^3$$

that gives the ratio of the cubes:

$$\Sigma a^3 = \frac{1}{4}\Sigma c^3.$$

Applying the known algebraic procedures, such as that illustrated by Boyer (1949), Cavalieri could reach an embryonic formulation of one important integration theorem. However, the sums imagined by Cavalieri are not the "classical" sums of finite terms, but they represent the collection of an infinite number of lines.

Cavalieri's proposition, which appeared in 1639, was also independently worked out by many other mathematicians between 1635 and 1655, such as Torricelli, Roberval, Pascal, Fermat and Wallis. This theorem played a significant role in the development of infinitesimal methods. Boyer writes that "it was perhaps the first theorem in infinitesimal analysis to point toward the possibility of a more general algebraic rule of procedure, such as that which, formulated a generation after by Newton and Leibniz, became basic in the integral calculus" (Boyer, 1949, p.121).
Nonetheless, all the anticipations of methods of the Calculus developed in those years were related to geometry and the role of algebra was still feeble. Moreover, infinitesimal lines, surfaces and solids appeared but not infinitesimal numbers. The algebra and the analytic geometry of 16th-17th century were an essential step to go in this direction, namely to uproot the Aristotelian idea that in arithmetic no number can be smaller than one. Fermat and Descartes developed and exploited an analytic geometry, by considering symbols entering in an equation as indeterminate constants. To these symbols line segments could be associated. Thus, there was the tacit assumption that to every segment corresponds a number. Boyer notices: "To such a view there was nothing incongruous with the idea of infinitesimal constants or numbers, since they would correspond to the geometrical indivisibles which were being used so successfully." (Boyer, 1949, p.155).

To make an example, let us show the ingenious method of the French **Fermat** (1601-1665) for determining maxima and minima, first appeared in 1637. We consider the first problem he solved in *Methodus ad disquirendam maximam et minimam*. The problem consists in dividing a segment of length b in two parts a and $b-a$ in such a way that their product $a(b-a)$ is maximum. Fermat's argument develops as follows: let the first segment be $a+E$, the second segment thus will be $b-a-E$ and the product of the new parts will be $(a+E)(b-a-E) = ba - a^2 + bE - 2aE - E^2$. The arithmetic-algebraic method of Fermat is based on an observation made by Pappus, which corresponds to our property: a function is stationary in the neighbourhood of a value that makes it maximum or minimum. Then, Fermat compared the two obtained expressions and he set a pseudo-equality (*adaequalitatem*) between them. For convenience, we will use our symbol $\sim$ where Fermat writes in words "*adaeguabitur*": $ba-a^2+bE-2aE-E^2 \sim ab-a^2$ which gives $bE \sim 2aE+E^2$. The smaller is the interval E, the nearer the pseudo-equality becomes a true equation. Consequently Fermat divided by E and set $E=0$, obtaining the result $a = \frac{b}{2}$ (he writes "*aequabitur*"). In our terms, his procedure can be summarized as follows:

$$\frac{a(b-a)-(a+E)(b-a-E)}{E} \xrightarrow{E\to 0} 0.$$

Now we will write $E \to 0$, but Fermat seems to have interpreted the vanishing of E in

the sense that E is actually zero.
Fermat employed his method also for determining tangents to curves. Here are the steps he followed to find the tangent to the parabola in Fig. 1.1.3.

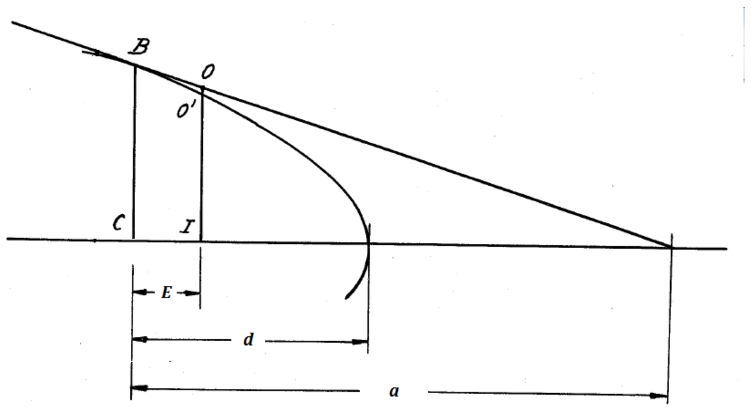

FIGURE 1.1.3 - FIGURE SUPPORTING FERMAT'S DETERMINATION OF THE TANGENT TO A PARABOLA (IN DUPONT, 1975, P.15).

From what he called the "specific property" of a curve (today we would say its equation) and the similarity of the two right triangles in figure, we shall have

$$\frac{d}{d-E} = \frac{(CB)^2}{(IO')^2} > \frac{(CB)^2}{(IO)^2} = \frac{a^2}{(a-E)^2}$$
$$\frac{d}{d-E} > \frac{a^2}{(a-E)^2}$$
$$dE^2 - 2daE > -a^2E.$$

The smaller we take E, the more the point O can be considered as if it was on the curve. Therefore, the inequality becomes a pseudo-equality:

$$dE^2 - 2daE \sim -a^2E.$$

Dividing by E and then setting $E = 0$ we find the equality:

$$\begin{aligned} dE + a^2 &\sim 2ad \\ a^2 &= 2ad \\ a &= 2d. \end{aligned}$$

Fermat thought in terms of equations and the infinitely small. In his method, algebra is the primary ingredient. However, from the following criticisms, we start noticing a first sensation of unease in using algebraic rules on quantities like E. The greatest incongruity criticized in this procedure was the fact that all the algebraic operations made with the quantity E, included the division by E, as if it were an ordinary positive quantity, lose their sense if in the end E is set equal to zero.

In England, the mathematician and theologian **Wallis** (1616-1703) conducted similar studies, by employing the concepts of infinity and infinitesimals, although no rigorous definition was established. He was largely inspired by the reading of Cavalieri's work on indivisibles. The approach of the latter was geometrical, whereas Wallis proceeded mainly arithmetically. He used in his own demonstrations the arithmetical symbolism in order to give them brevity and clarity. As Cavalieri, Wallis considered a plane figure as made up by an infinite number of parallel lines. Nevertheless, he preferred to describe them as parallelograms having height $\frac{1}{\infty}$ or an infinitely small part of the height of the figure. The symbol ∞ for infinity appears here for the first time. Wallis said that $\frac{1}{\infty}$ represents an infinitely small quantity or *non-quanta*. By using this approach, in his *De Sectionibus Conicis* appeared in 1655, he proved that the area of a triangle is the product of the base and half the height. He supposed the triangle be divided into an infinite number of parallelograms having height $\frac{1}{\infty}$, taken as lines parallel to the base. The areas of these parallelograms, starting from the vertex to the base, form an arithmetic progression which begins with zero. Wallis used a well-known algebraic rule that the sum of such an arithmetic progression is the product of the last term and half the number of terms. In applying this formula, Wallis made no distinction between finite of infinite numbers. Thus, calling A the base of the triangle and B its height, the last term of the progression is $\frac{1}{\infty}AB$ and the total area is given by

$$\frac{1}{\infty}AB\frac{\infty}{2} = \frac{1}{2}AB.$$

Most of his results base on crude manipulations of the symbols ∞, such as his demonstration of the proposition that we can write as

$$\int_0^a x^n \mathrm{d}x = \frac{a^{n+1}}{n+1}.$$

In *Arithmetica Infinitorum*, he proceeded arithmetically by showing that

$$\frac{0+1}{1+1} = \frac{1}{2}; \quad \frac{0+1+2}{2+2+2} = \frac{1}{2}; \quad \frac{0+1+2+3}{3+3+3+3} = \frac{1}{2}; \ldots$$

So the ratio of any finite number of terms is always $\frac{1}{2}$ and he concluded it is $\frac{1}{2}$ also for an infinite number of terms. For the squares he showed

$$\frac{0+1}{1+1} = \frac{1}{3} + \frac{1}{6}; \quad \frac{0+1+4}{4+4+4} = \frac{1}{3} + \frac{1}{12}; \quad \frac{0+1+4+9}{9+9+9+9} = \frac{1}{3} + \frac{1}{18}; \ldots$$

He then observed that the more the number of terms increases, the more the difference between the result and $\frac{1}{3}$ decreases. For an infinite number of terms, he concluded, this

difference is about to vanish completely and, as a consequence, the ratio is $\frac{1}{3}$.
Wallis's work declared the independence of arithmetic by geometry. His use of the infinitesimals however was not clearer than what it was in Fermat's work. Indeed, if Fermat set his vanishing E equal to 0, also Wallis wrote his infinitely small height as $\frac{1}{\infty} = 0$. Nonetheless, thirty years later Wallis made an interesting remark about the infinitesimal nature of $\frac{1}{\infty}$. He said: "We may observe a great difference between the proportion of *Infinite* to *Finite*, and, of *Finite* to *Nothing*. For $\frac{1}{\infty}$ that which is a part infinitely small may, by infinite Multiplication, equal the whole. But $\frac{0}{1}$, that which is Nothing, can by no Multiplication become equal to Something" (Wallis, *Defense of the angle of contact*. In Scott, 1981, p.21).

Let us come to Newton and Leibniz who are considered as the fathers of infinitesimal calculus. A quick analysis of the methods they developed and disseminated in the second half of the 17th century. The solved problems can start from or be given in a geometrical situation, but their resolution is almost completely independent from geometry, leaving a greater space to algebra. In the spirit of this paragraph, it is not our intention to present all the problems solved by Newton and Leibniz. With the shown examples we intend to give an idea of the algebraic procedures on which their different calculus methods were grounded.

Newton (1642-1727) worked on power series expansion. One of his most known results is the so-called "Newton's binomial", but we can say that Newton went farther extending the formula

$$(a+b)^n = \sum_{h=0}^{n} \binom{n}{h} a^{n-h} b^h,$$

for any integer n, to the more general series expansion[3]

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n,$$

for any integer or rational r. Newton reached results like this one through several methods such as division, root extraction, etc. For instance, the power series expansion of $\frac{a^2}{b+x}$ is obtained dividing a^2 by $b+x$.
In *De analysi per aequationes numero terminorum infinitas*, written in 1669 but published later in 1711, we can appreciate Newton's method of integration intended as antiderivation. We can illustrate it as follows. Let z represent the area of ABD where AB is x, BD is y and AD is a portion of curve. Given to x a positive and finite increment o (which recalls Fermat's E), $B\beta$ in Fig. 1.1.4, the area z will be increased of the area of

[3]The used notation is that of today.

$B\beta\delta D$. Consider the equivalent rectangle $B\beta HK$ which has o as base and v as height. Then the area z is incremented of ov. Thus, notice that Newton introduced two different increments: one for the independent variable and another one for the dependent variable. It allowed him to avoid speaking of pseudo-equalities.

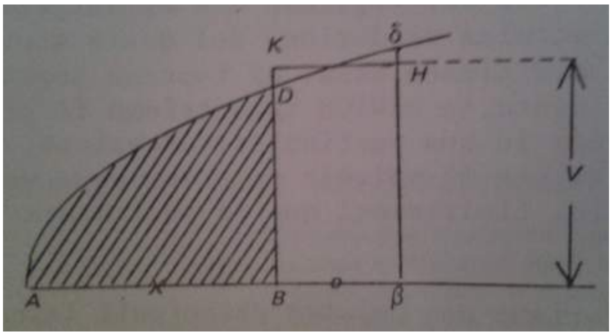

FIGURE 1.1.4 - FIGURE SUPPORTING NEWTON'S ARGUMENT (IN DUPONT, 1980A, P.374).

Newton shows his method on a particular example:

$$z = \frac{2}{3}x^{3/2}$$

which he rewrites as

$$z^2 = \frac{4}{9}x^3.$$

To the increment o of x corresponds an increment ov of the area z so

$$(z + ov)^2 = \frac{4}{9}(x + o)^3.$$

Expanding the powers and knowing that $z^2 = \frac{4}{9}x^3$, he obtains

$$\begin{aligned} z^2 + 2ovz + o^2v^2 &= \frac{4}{9}x^3 + \frac{4}{9}3x^2o + \frac{4}{9}3xo^2 + \frac{4}{9}o^3 \\ 2ovz + o^2v^2 &= \frac{4}{3}x^2o + \frac{4}{3}xo^2 + \frac{4}{9}o^3, \end{aligned}$$

and dividing by o

$$2vz + ov^2 = \frac{4}{3}x^2 + \frac{4}{3}xo + \frac{4}{9}o^2. \tag{1.2}$$

Then he writes, with reference to Fig. 1.1.4: "Now if we suppose $B\beta$ to be diminished infinitely and to vanish, or o to be nothing, v and y, in that Case will be equal, and the Terms which are multiplied by o will vanish" (English translation of 1964; in Dupont, 1981a, p.376). Here, Newton makes explicitly o tend to zero, noticing that for $o = 0$ the equality (1.2) "tends" to (1.3):

$$2yz = \frac{4}{3}x^2 \tag{1.3}$$

that is

$$2y\frac{2}{3}x^{3/2} = \frac{4}{3}x^2$$
$$y = x^{1/2}.$$

At this point, Newton erroneously thought that also the converse his true, namely that if $y = x^{1/2}$ the area will be $z = \dfrac{2}{3}x^{3/2}$. Now we know that it lacks the integration constant, and that Newton actually found one particular primitive.
In the *Methodus Fluxionum*, written in 1671 but published only after his death, Newton called *fluents* the magnitudes x and y which depend on time (an artificial time) and he considered their *fluxions* in time $\dot{x}$ and $\dot{y}$. Considering a little increment of the time o, the fluents x and y will become respectively $x + \dot{x}o$ and $y + \dot{y}o$. He stated clearly the fundamental problem of the Calculus: given the relation of the fluents, determining the relation of their fluxions; and conversely. Newton's method bases on the algebraic substitution of x, y, z, ... with the incremented quantities $x + \dot{x}o$, $y + \dot{y}o$, $z + \dot{z}o$, ... Then, he expands and calculates, deletes the terms which do not contain o (which satisfy the equality given for hypothesis) and divides by o. Moreover, since o is supposed to be infinitely small, the terms which contain it can be considered as zero with respect to the quantities that do not depend on it, and are to be neglected.
Using this method, Newton found also the inversion theorem (fundamental theorem of Calculus): if z is an area generated by the flowing of y, then he writes

$$\dot{z} : \dot{x} = y : 1.$$

Hence, $\dfrac{\dot{z}}{\dot{x}} = y$.
The method of the fluents and fluxions is only one of the two methods Newton worked out in his infinitesimal studies. The other one is the method of prime and ultimate ratios and limits exposed in the *Principia* published in 1687. Newton presented the "ultimate ratio in which quantities vanish" as "limits to which the ratios of these quantities decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminished indefinitely" (Newton, *Opera omnia.* In Boyer, 1949, p.198). With this method, Newton overcame the conception of the indivisibles and substituted it with an intuitive idea of limit.

In the same period, in Germany, **Leibniz** (1646-1716) published in 1684 on the *Acta Eruditorum* the work *Nova methodus pro maximis et minimis, itemque tangentibus quae nec fractas nec irrationales quantitates moratur et singulare pro illis calculi genus.* His method recalls that of Fermat, with the introduction of the differential instead of E. For Leibniz, the increment of the variable x is $\mathrm{d}x$, defined as an arbitrary segment. Given a curve, Leibniz considered the corresponding function $v(x)$ and introduced the differential of v, $\mathrm{d}v$, which is a segment that is to $\mathrm{d}x$ as v is to the sub-tangent (see Fig. 1.1.5). He also said that $\mathrm{d}v$ is the difference $v(x+\mathrm{d}x)-v(x)$, however we know that it does not occur for any increment $\mathrm{d}v$ but the two definitions coincide if the increment is infinitesimal.

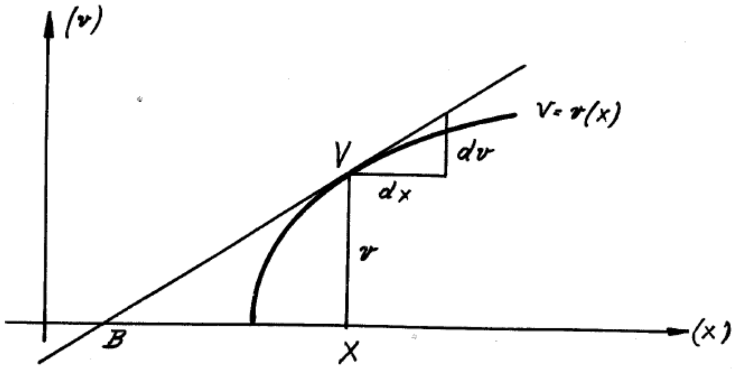

FIGURE 1.1.5 - FIGURE SUPPORTING LEIBNIZ'S ARGUMENT (IN DUPONT, 1975, P.82).

As Dupont observes in his *Storia del concetto di derivata* (1975), with the introduction of the differential and the algebraic operations on it, algebra had not to be considered as calculus of the finite magnitudes anymore. It became an infinitesimal algebra and the relations involving infinitely small differences had to be written in new proper symbols. It is in line with Leibniz's theory of the *characteristica universalis*, according to which the symbols must have the inherent capability of expressing a notion. It is on these bases that Leibniz grounded his infinitesimal calculus. Let $\mathrm{d}v, \mathrm{d}w, \mathrm{d}y, \mathrm{d}z$ be the differentials of the functions $v(x), w(x), y(x), z(x)$. Leibniz stated that

- if a is a constant, then $\mathrm{d}a = 0$ and $\mathrm{d}ax = a\mathrm{d}x$
- if $y = v$, then $\mathrm{d}y = \mathrm{d}v$
- $\mathrm{d}(z - y + w + x) = \mathrm{d}z - \mathrm{d}y + \mathrm{d}w + \mathrm{d}x$
- $\mathrm{d}vx = x\mathrm{d}v + v\mathrm{d}x$
- $\mathrm{d}\frac{v}{y} = \dfrac{\pm y\mathrm{d}v \mp v\mathrm{d}y}{y^2}$.

These rules were given without proof in the *Nova Methodus*. The differential $\mathrm{d}x$ remained an arbitrary segment. No allusion was made to an infinitesimal quantity. If we retrace the story of the differential $\mathrm{d}x$ in Leibniz's mathematical work, it has to be related to the difference operator d he used to calculate the finite difference between two consecutive terms of a sequence. In the previous Leibniz's manuscripts (around 1675), we can see that the differential calculus arose as "extrapolation" (to use Bos's term in Bos, 1974) of the calculus on numerical sequences, in a sort of passage from discrete to continuous. In these manuscripts we can find the proof of the algebraic rules given later in *Nova Methodus* and in them we can clearly notice that Leibniz actually considered the differential as an infinitesimal difference between two successive values of the variable. For instance, let us consider the proof of the product formula $\mathrm{d}xy = x\mathrm{d}y + y\mathrm{d}x$. Leibniz calculated it as difference between the incremented product, obtained replacing x with $x+\mathrm{d}x$ and y with $y+\mathrm{d}y$, and xy. Thus

$$\mathrm{d}xy = (x+\mathrm{d}x)\cdot(y+\mathrm{d}y) - xy = x\mathrm{d}y + y\mathrm{d}x + \mathrm{d}x\mathrm{d}y.$$

Then, Leibniz omitted the term $\mathrm{d}x\mathrm{d}y$ and here we see that the differentials he was considering were infinitesimal. Indeed, the product of two infinitesimal quantities $\mathrm{d}x\mathrm{d}y$ is negligible with respect to all the other terms in the sum. This was one of the most discussed points in Leibnizian differential calculus. Notice that algebra alone could not explain how $\mathrm{d}xy$ could be equal to $x\mathrm{d}y+y\mathrm{d}x+\mathrm{d}x\mathrm{d}y$ and to $x\mathrm{d}y+y\mathrm{d}x$ at the same time. As far as the integration is concerned, Dupont (1981b) follows Bos (1974) affirming that Leibniz stressed the difference between his own method and the indivisibles one. Leibniz highlighted that, in his calculus, he obtained quadratures as sums of area-differentials (i.e., areas of rectangles having dimensions y and $\mathrm{d}x$) rather than sums of lines in a figure. As Dupont (1981b) observes, it occurred in an advanced process of algebraisation of the infinitesimal calculus which detached from both the finite Cartesian algebra and the pure Cavallieri's geometry. Leibniz algebraised Cavalieri's geometry by introducing the bidimensional differentials $y\mathrm{d}x$. It was Leibniz that introduced the symbol *omn* (from the Latin "omnia") and then $\int$ for the sum. For instance, he found the result $\frac{1}{2}(\int \mathrm{d}y)^2 = \int(\int \mathrm{d}y)\mathrm{d}y$, which we have written in our symbols, but maintaining the structure and the meaning given by Leibniz to the single parts.

Leibnizian "Nova Methodus" resulted very difficult to read also for the greatest mathematicians of the time. As Dupont (1981b) remarks, this work contained uncertainties, enigmas, contradictions and fundamental rules given without any proof. Perhaps, this was due to the fact that Leibniz desired avoiding possible critics related to the infinitely small.

It is exactly this perception of having find great results in calculus, but not being able to fully explain them through rigorous algebraic methods, that we tried to highlight in this first paragraph. We saw how the role of algebra became more and more important in the construction of calculus practices. Sometimes algebra perfectly explained, more quickly than geometry, certain results; sometimes, instead, mathematicians had to do great efforts to justify their theorems or proofs to the mathematical community or they tried not to mention what may attract criticisms. These ambiguities are largely due to the fact that they were doing algebraic calculations with infinite or infinitesimal quantities. Something in the way of thinking mathematics was changing and certain algebraic practices seemed not to fit perfectly the new ideas and conceptions.

This situation recalls us the concept of **paradigm shift** introduced by Kuhn in *The structure of scientific revolutions* (1970). In the next paragraph we try to develop this idea and we propose an approach through paradigms for studying the transition from Algebra to Calculus.

1.1.2 An approach through paradigms

A deep historical research of a given specialized science reveals the presence of "a set of recurrent and quasi-standard illustrations of various theories in their conceptual, observational and instrumental applications" (Kuhn, 1970, p.43). In this way, the American physicist, historian, and philosopher of science Kuhn describes the **paradigms** of a scientific community. He distinguishes the paradigms from the rules. "Paradigms may be prior to, more binding, and more complete than any set of rules that could be unequivocally abstracted from them" (Kuhn, 1970, p.46). "Normal science", that is the research grounded on one or more stable results, may proceed without any rule only as long as the scientific community accepts and does not discuss the particular solutions already achieved. The rules become relevant whenever the paradigms are perceived as unsure. The pre-paradigmatic period is marked by frequent and deep debates about the validity of certain methods, problems or models of solutions. These periods "when paradigms are first under attack and then subject to change" (Kuhn, 1970, p.48) are, according to Kuhn, characteristic of **scientific revolutions**.
In line with Kuhn's thought, we share the definition proposed by Kuzniak in his research on the paradigms in geometry. Kuzniak (2011) intends the paradigm as the set of the beliefs, the techniques and the principles shared by a scientific group. A **paradigm shift**, described by Kuhn as a change in procedures and expectations, is then for us a change in the way of thinking and in the way of working on a mathematical object or a set of objects.
We can read under this lens the examples shown in the previous paragraph. They all have in common the attempt to approach the problems of the normal science with the standard techniques of the time. In particular, the rules of what Newton called "common Analysis", that is Algebra, were applied to usual problems of quadrature, area and volume, maxima and minima. Nevertheless, in all the cases taken into account we can find insights of a new way of thinking of the figures, the curves, and the involved variables. A new way of working on these mathematical objects was emerging. Symptomatic were the reference to the indivisibles, the lines or the infinitesimal rectangles composing the figure, and the introduction of infinitesimal heights, segments or increments.
Moreover, as Kuhn states, the discovery of a new paradigm is marked by the perception of some anomalies, something that breaks the expectations given by the paradigms of the normal science. The researcher's reaction to these anomalies is characterized by a first attempt of adapting the paradigmatic theory so that what appeared anomalous seems to be expected. It is exactly what occurred in all the attempts, presented in the previous paragraph, of using algebraic rules on the introduced infinitesimal and infinite quantities. Sometimes this fact led to anomalies that algebra alone could not explain. We refer, for instance, to the use of the finite and positive increment E that turns out to be equal to zero in Fermat; or to the infinitesimal height of the rectangles in the figures of Wallis, written as $\frac{1}{\infty} = 0$; or to the negligible $\mathrm{d}x\mathrm{d}y$ in Leibniz and o in Newton. They all were anomalies for common algebra, but also the only way to get the solution. Thus, they appeared as a necessary change in procedures and expectations.

Adopting an approach through paradigms, we can describe this historical shift to infinitesimal calculus as a **paradigm change**.
More precisely, let us refer to a study in progress on paradigms in the Calculus domain, carried out by Kuzniak, Vivier, Estrella and Montoya-Delgadillo. This research (Estrella *et al.*, 2014)[4] interprets the mathematical work in the Calculus domain through three paradigms.

- AG: *Analytic-Geometric* which allows interpretations arisen from geometry or the real world. The geometrical intuition guides the work with several implicits.
- AC: *Analytic-Calculative* where the rules of calculus are defined, more or less explicitly, and are applied independently from the reflection upon the existence and validity of the introduced objects.
- AI: *Analytic-Infinitesimal* is characterized by a work of approximation, with a loss of information in order to solve a problem: maxima and minima, the entry into a work on neighbourhoods (or a more topological entry): "searching for ϵ", "the negligible".

Therefore, following this characterization, we can read the transition from Algebra to Calculus as a shift from the analytic-geometric and the analytic-calculative paradigms to the analytic-infinitesimal one.
Then, the incredible mathematical production of the 18th century and the following foundation research contributed to shape the Analysis domain as we study it nowadays. The appearance of the primitive concept of function (with Leibniz in 1692) and its following development has certainly been a crucial step towards a clear formulation of the limit notion. A turning point has then been the definition of real numbers in the second half of the 19th century. Indeed, the well-formed concept of real number has allowed to formally define the limit notion and has fostered the process of arithmetisation of Analysis. The scholarly knowledge[5], as it is presented in our university textbooks and lectures, is strongly grounded on the analytic-infinitesimal paradigm. From an epistemological point of view, this paradigm has introduced a particular way of regarding functions, curves and involved variables: the **local perspective**. The term "perspective" is used with reference to the French "*perspective*", introduced in literature by Vandebrouck (2011a). Besides information in a point (pointwise properties) and in an interval (global properties), the local perspective gives access to information on the function in the neighbourhood of a point.
In the next paragraph, we introduce the perspectives on functions (Rogalski, 2008; Maschietto, 2002, 2008; Vandebrouck, 2011a, 2011b) as an important epistemological component of the transition from Algebra to Calculus. As we will see in the following, this epistemological aspect has shaped our research questions and has become a fundamental lens for investigating Calculus practices in classroom.

[4] Research conducted within the international project ECOS-CONICYT C13H03 involving France and Chile.

[5] We use this expression in the sense of Chevallard (1992), to indicate mathematical knowledge as it is produced by mathematicians or other producers

1.1.3 Different perspectives on functions

Calculus, or as it is called in other European countries Elementary Analysis, can be considered as the study of functions, of their properties and of the operations on them. This mathematical domain is nowadays grounded on differential and integral calculus. It is strictly related to problems of variation, calculation of areas, volumes and determination of the tangent line to a curve. In the Italian curriculum, the last year of secondary school (grade 13) is almost all devoted to the teaching of the fundamental concepts of Calculus: limits, differentiation and integration. Nevertheless, the teaching and learning of functions begins before and permeates all the upper secondary teaching (grade 9-13). The functions are introduced to study particular problems in algebra and geometry. Their learning is related to the solution of equations and inequalities and it develops with the study of analytic geometry. Therefore, the teaching and learning of functions cannot be separated from the learning of the other mathematical domains, especially that of algebra. With the introduction of limits, derivatives and integrals, and so with the real entry into Calculus work, a transition from Algebra to Calculus has to be operated. The functions continue to be the object of the mathematical work, but their study acquires different features.
In Mathematics Education, several studies have been conducted on the notion of function and on its conceptualization. Some researches have focused on the multiple existing representations of functions (e.g., Duval, 1993). Another aspect that has been deepened is the idea of covariance at the base of the notion of function (e.g., Slavit, 1997).
In this thesis, we concentrate on another important epistemological component: the **perspectives** from which a function can be studied (Rogalski, 2008; Maschietto, 2002, 2008; Vandebrouck, 2011a, 2011b). Indeed, they can help us in describing the transition from Algebra to Calculus in the teaching context.

In the previous paragraph, we claimed that the entrance into Calculus is marked by the introduction of an analytic-infinitesimal paradigm. Its great novelty stands in the change of regard on functions, which we can formalise as a change in the adopted **perspective** on them. According to the properties that are taken into account on a given function, the adopted perspective can be **pointwise**, **global** or **local**. Following Vandebrouck (2011a, 2011b), let us define pointwise, global and local properties on a given real function f of one real variable.
A property of f at a given point x_0 is **pointwise** when it depends only on the value assumed by f at x_0. For instance, $f(x_0) = 3$ is a pointwise property which tells us nothing about $f(x_1)$ if $x_1 \neq x_0$.
Further, we have **global** properties of f which are valid on intervals: parity, periodicity, sign, variation, etc. For instance, f is increasing in the interval (a, b) is a global property.
Finally, a property of f is **local** at the point x_0 if it depends on the values of f in a neighbourhood of x_0. For instance, f has limit l in x_0, f is continuous or differentiable in x_0, f is negligible with respect to another function in a neighbourhood of x_0, etc.
The distinction between the global and the local perspectives requires further development. Indeed, the former is based on properties that are valid in an interval (a, b); the

latter takes into account a neighbourhood of the point x_0, that is an open interval containing x_0, for instance in the form $(x_0-\delta, x_0+\delta)$. Thus, the notion of interval intervenes in the definition of both global and local properties. Nevertheless, the two resulting perspectives are very different. Let us distinguish in details the two situations.
From a global perspective, we can consider the whole function over all its domain or we can fix a specific interval (a, b) on which the function has a particular (global) property. So, by saying "f is even" or "f is defined as $f(x) = x^2$" we are considering the function over all its (explicit or implicit) domain. On the other hand, if we say "f is increasing in $(0, 1)$", we are detecting a specific interval in which the function f has the property of increasing.
From a local perspective, by observing for instance that f has a corner point in $x = 1$, we cannot fix any particular interval containing 1, but an infinite family of intervals (e.g., $(1-\delta, 1+\delta)$), more or less narrow, can satisfy the (local) property.

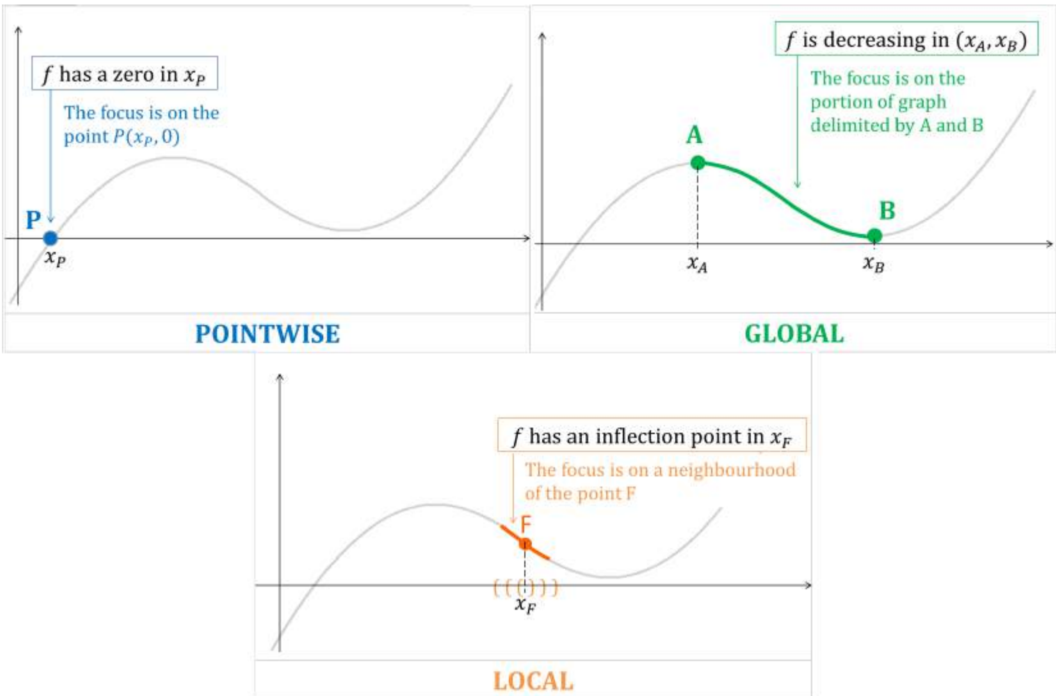

FIGURE 1.1.6 - EXAMPLES OF POINTWISE, GLOBAL AND LOCAL PERSPECTIVES ACTIVATED ON THE GRAPH OF A FUNCTION.

Other researches in Mathematics Education particularly focus on the global/local game on functions (Maschietto, 2002, 2008; Rogalski, 2008). Maschietto discusses it at a secondary school level, centring her research on the property of "local straightness" of a graph as a cognitive root (Tall, 2000) for the property of "local linearity" of the corresponding function. Her teaching experiments base on the use of the zoom in a technological environment, in order to introduce micro-straightness. She shows how local or micro-straightness, playing the role of invariant in the zoom processes, support the initiation of the global/local game. She stresses: "Linearity was previously seen as a global

phenomenon, associated with a particular class of functions defined on intervals (or on $\mathbb{R}$). It has then to assume local features. This change also demands the reconstruction of the relationship with the notion of tangent line to a curve at a point" (Maschietto, 2008, p.209).
Rogalski analyses this game local/global at university, claiming that one of the fundamental activities in Analysis consists in the distinction between these two points of view. Specifically, he stresses the epistemological difficulty related to quantifiers. The formal definitions in mathematics make intervene a subtle *mélange* of existential quantifiers ("it exists y for which we have...") and universal quantifiers ("for all x we have..."). They appear since the first definitions involving continuity and differentiability and carry on with the notion of uniformity. Emblematic is the case of continuity and uniform continuity.
A function f is uniformly continuous in I if:

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, x_0 \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \tag{1.4}$$

A function f is continuous in a point $x_0 \in I$ if:

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \tag{1.5}$$

Notice that the δ, which limits the variation of the abscissa, in (1.4) depends only on ϵ, which limits the variation of the ordinate, and not from the chosen points in I. Instead, in (1.5) δ depends on both ϵ and the chosen point $x_0 \in I$. Therefore, the uniform continuity is a global property of the function, on the contrary of the simple continuity which is a local property. Now, let us express the global property that the function f is continuous in I.

$$\forall x_0 \in I, \forall \epsilon > 0, \exists \delta > 0 : \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \tag{1.6}$$

This property is now global, in the sense that it is verified for all $x_0 \in I$ (Rogalski would define it a "universal local" property). Let us compare (1.6) and (1.4) at the level of quantifiers. The phrase "$\forall x_0 \in I$" appears after the existential quantifier for δ in (1.4) and before it in (1.6). This apparently "little" change in the formal definition entails a great difference at the level of meaning. A function which is uniformly continuous on I is always continuous in I, but not conversely. And it is formally due to a slight difference in the order of quantifiers.

Rogalski extends local to **universal local** properties. Similarly Vandebrouck considers the extension of pointwise to **universal pointwise** properties, which are pointwise properties verified for all the values x in the domain or in a given interval. Thus, universal pointwise properties are global properties on I because they consist in pointwise properties verified for all x in I. For instance,

$$\begin{array}{lcl} f \text{ is even} & \Leftrightarrow & \text{its definition domain } D \text{ is symmetric} \\ & & \text{and } \forall x \in D \text{ it is } f(-x) = f(x). \end{array}$$

Another slightly different example is the definition of increasing function on an interval:

$$f \text{ is increasing on } I \quad \Leftrightarrow \quad \forall x_1, x_2 \in I,\ x_1 \leq x_2 \text{ it is } f(x_1) \leq f(x_2).$$

In this case, the interest is on the values taken by f in two different points of the interval I, for every couple of points (x_1, x_2). The property is universal pointwise on the set of the couples of points of I.
Notice that every universal pointwise property is expressed through a universal quantifier on the domain or a subset of it.
Therefore, mathematically verifying a global property of the function f on an interval I means verifying a universal pointwise property for all x in I. Thus, mathematically, universal pointwise properties and global properties coincide.
However, cognitively to verify a global property one can be interested in what the function does point by point in a certain interval or one can look at the function over the whole interval. Then we distinguish between

- a global perspective acquired as **universal pointwise perspective** on the function f, when the interest is on the values of f point by point.
- a global perspective on the function f as such, when the interest is on f over a whole interval (without distinguishing the different points), or on the whole f on its domain.

In the classical logic, if one proves by contradiction that $\exists x \in I \quad | \quad f(x) = k$, such an x has not been constructed. It can be everywhere in the interval I. So, the perspective on f is global as such, without any distinction of the single values taken in the different points of I. The classical logic does not distinguish if the existence of an x in I for which $f(x) = k$ has been proven by contradiction or by construction. Thus, in this field, the two perspectives distinguished above can coincide. This may not occur for other kind of logic, for example in the constructivist logic.

The global and the local works on functions lead us to consider two fundamental ideas: that of "generic" and that of "approximation"[6]. They both are often source of difficulties in the learning of Calculus.

From a global perspective, the mathematical sign to express "for each", "for every" and "for all" is the same, namely the universal quantifier "$\forall$". The natural language makes a distinction between the terms "every" and "all". They can both be used to talk about things in general, but "every" has the distributive meaning of "all in each of its parts". So, for example, if we say "Every Italian citizen has a name and a surname" we want to underline the fact that "if taken one by one", to all Italian citizens has been given a name and a surname. But we would prefer to say "All Italian citizens older than 18 has the right to vote" in order to stress that the totality of Italian people can vote, without inner distinctions. In mathematics, instead, the expressions "for every" and "for all" are completely equivalent from a logical point of view. This universal quantifier is

[6] Panero (2013) and Panero *et al.* (2014) provide a starting point of the study of these two ideas within the transition from Algebra to Calculus in secondary school.

often used together with the idea of generic. We find this fundamental idea already in Euclid (see his "generic" proof of the infinity of prime numbers) and in some speculations of philosophers (e.g., Locke: see below), but it was particularly exploited within algebraic geometry at the turning of 19th century. The geometers introduced explicitly the notion of "generic point", and the related practices in their discipline. However, while generic points were widely used, one cannot find easily their definition. For that we can refer to Van der Waerden:

> "Indeed, by generic point of a variety, one usually means, even if this is not always clearly explained, a point which satisfies no special equation, except those equations which are met at every point." (Van der Waerden, 1926, p.197)

and to Enriques and Chisini:

> "The notion of a generic 'point' or 'element' of a variety, i.e., the distinction between properties that pertain in general to the points of a variety and properties that only pertain to exceptional points, now takes on a precise meaning for all algebraic varieties. A property is said to pertain in general to the points of a variety V_n, of dimension n, if the points of V_n not satisfying it form – inside V_n – a variety of less than n dimensions." (Enriques & Chisini, 1915, p.139)

An interesting point of view is given by Speranza (1996), who analysed the idea of "general triangle" through the words of the great philosopher Locke (1690): "[The general triangle] must be neither oblique nor rectangle, neither equilateral, equicrural, nor scalenon: but all and none of these at once"[7].
The idea of generic occurs in the mathematical practices leading to the research of a "generic stereotype" which represents all the basic features desired without any added specific singularity.
In Calculus, it occurs when we know how a function behaves for some values of x and we shift our reasoning on a "generic abscissa x", which must belong to the function domain, but has no particular added characteristic.
The same happens in Algebra with the work on generic examples. For instance, let us imagine that we have to decide if the sum of two even numbers is even or odd. In Algebra, we can proceed empirically, testing several cases (e.g., $2+4=6, 4+8=12$, and so on) and, then, inducing the general property: the sum is an even number. This is an example of generalization, that means inducing the general case, from a sequence of particular cases. From an epistemological point of view, generalization is an empiric induction, which entails an empiric, but not real proof. However, we can follow another way in order to decide if the sum of two even numbers is even or odd: we can reason on a particular example, giving emphasis to a general feature that characterizes all the examples similar to the proposed one.

[7]Original quotation from the English translation we found in Speranza, 1996, p.15.

$$\begin{array}{rcrcr} 14 & = & 7 & + & 7 \\ 22 & = & 11 & + & 11 \\ \hline 36 & = & 18 & + & 18 \end{array}$$

As a consequence, it becomes useless to provide other examples: the given one can be conceived as generic. This is a "generic proof", that is "a proof carried out on a generic example" (Leron & Zaslavsky, 2013, p.24).

From a local perspective, at the base of fundamental concepts and reasoning processes of Calculus are the ideas of approximation and convergence involving infinite processes. Then, the mathematical work can be very different from that students are familiar with, in approaching the study of Calculus. Let us recall Legrand's reflections on the procedures of Analysis, which he distinguishes from those of Algebra (Legrand, 1993). He specifically makes the example of the notion of equality, which changes moving from Algebra to Analysis. He states that for proving that an unknown quantity A is equal to a known quantity B, and so for writing $A = B$, in Algebra, one passes through a finite number of intermediaries: $A = C, C = D, \ldots F = B$ whereas in Analysis, one proves that the distance between A and B is less than any positive number, for an appropriate distance. Thus, the meaning of equality is different: in Algebra, we have an equality when two expressions are equivalent; in Analysis, we have an equality when two quantities are arbitrary close. Therefore, also the concept of equality between functions changes. In Algebra, the fact that the function f is equal to the function g implies a coincidence point by point (within a universal pointwise perspective)

$$\begin{aligned} f = g \quad \Leftrightarrow \quad & Domf = Domg = D \\ & \text{and } f(x) = g(x) \quad \forall x \in D. \end{aligned}$$

Instead, in Analysis, the idea of equality of functions can also take a **local dimension**. This is the idea of approximation at the base of series expansions in x_0.

$$f \simeq g \text{ in } x_0 \quad \Rightarrow \quad f(x) - g(x) \to 0 \quad \text{for} \quad x \to x_0.$$

Roughly speaking, in the most part of Analysis processes, it is sufficient that two functions are coincident in a neighbourhood of the point x_0, and it is not necessary that this happens for all the points x in their domain. We make the example of "germs" of functions. These are classes of equivalence of functions defined as follows:

> Def. of germs of functions - A germ of a real function f at $x_0 \in I$ is the equivalence class of f under the equivalence relation:
>
> $$f_1 \sim f_2 \Leftrightarrow f_1(x) = f_2(x) \quad \forall x \in J, \quad \text{for some neighborhood} \quad J \subset I \quad \text{of} \quad x.$$
>
> The collection of germs at a point x_0 can be denoted by $C^\infty(x_0)$.

Two functions representing the same germ of f at x_0 have the same derivative on neighbourhoods of x_0, thus in such neighbourhoods it is not possible to distinguish one from the other.

Also within the local perspective, we want to point out some interesting issues which are related to the natural language. As Cornu (1991) stresses, investigations on the limit concept, such as those of Schwarzenberger & Tall (1978) or Robert (1982), have shown that the verbal expressions "tends to" and "limit" have a previous meaning for students before any lessons begin. The students continue to rely on these meanings after they have been given a formal definition. So the words "tending towards" continue to mean "approaching", "becoming", "being closer and closer to", ... They assume a dynamic connotation. Cornu observes that the idea of approximation is usually first encountered through a dynamic notion of limit and Sierpinska (1985) notices in this a "physical" (epistemological) obstacle. She describes it as "The obstacle which consists of associating the passage to the limit with a physical movement, with an approaching: 'one indefinitely approaches' or 'one approaches more and more', whilst the limit notion within the formal theory is conceived in a 'static' way."[8] (Sierpinska, 1985, p.40). Sierpinska is referring to our modern formal theory, mainly due to the work of Weierstrass, which is based on the concept of real number and on the $\epsilon - \delta$ definition of limit. Moreover, the idea of approaching makes another epistemological question arise, which has been at the center of historical debates: the limit is attained or not? With regards to it, Boyer observes that "In the light of the precision of Weierstrassian theory of limits, however, the question is seen to be entirely inapposite. The limit concept does not involve the idea of *approaching*, but only a static state of affairs. The single question amounts really to two: first, does the variable $f(x)$ have a limit L for the value a of x. Secondly, is this limit L the value of the function for the value a of x. If $f(a) = L$, then one can say that the limit of the variable [i.e., $f(x)$] for the value x in question [i.e., a] *is* the value of the variable [i.e., $f(x)$] for this value of x, but not that $f(x)$ *reaches* $f(a)$ or L, for this latter statement has no meaning" (Boyer, 1949, pp.287-288).

1.2 The derivative concept

In order to study the transition from Algebra to Calculus, we are interested in the concepts that make the algebraic practices, whose recall or introduction is necessary when teaching Calculus, a bridge or a rupture.
Since, within a PhD study, it was not possible to investigate all the various facets of the problematic articulation of Algebra and Calculus, we chose to focus on a specific Calculus concept: the derivative.

1.2.1 Why the derivative concept?

The derivative is an important mathematical and didactic node. It is one of the first and fundamental concepts of Calculus which evokes and calls into question competences, notions and registers which are proper to the algebraic or the geometrical domain. In

[8]Our English translation of the passage: "*L'obstacle qui consiste à associer le passage à la limite à un mouvement physique, à un rapprochement: 'on s'approche indéfinement' ou 'on s'approche de plus en plus', alors que la notion de limite dans la théorie formelle est conçue de façon 'statique'.*"

particular, it involves functions and their properties, limits and also geometrical objects such as the tangent line. At the same time, the derivative permits to solve several problems belonging to the Calculus domain, such as optimization problems, approximation methods for the zeros, primitives of functions, and many others. Therefore, the derivative seems a suitable theme within which facing algebraic practices (also coming from analytic geometry) which are involved in the construction of Calculus ones. Moreover, the derivative treatment permits to work with various registers of representation for functions and their possible coordination is rich. On the calculative (more algebraic) side, we cannot ignore the huge importance that the differential calculus has not only in mathematics but also in all applied sciences. On the functional (more analytic) side, the derivative concept is both complex and fascinating, especially because the derivative of a function is a function itself. Hence, as far as the perspectives are concerned, not only the derivative allows a local kind of work on functions, but also the derivative itself can be studied as a function from all the introduced perspectives. In relating a function with its derivative, for instance, all the perspectives can be activated and have to be coordinated. Imagine that you want to establish a relationship between the stationary points of a function f and the zeros of its derivative f'. Then, you need to adopt a local perspective on f and, at the same time, a pointwise perspective on f'. In the process for carrying out such a relationship an interesting dialectic of perspectives occurs. Moreover, imagine that you want to link f' to f''. The function f' which before was considered as the derivative of f, now has to be seen as the starting function whose differentiation leads to f''. Thus, working on derivatives may produce also an engaging exchange of roles. Within this context, the properties and the relations of functions (and the perspectives with them) can be worked at a really deep stage.

From the teaching point of view, the derivative notion is considered as a cornerstone in the mathematics curriculum of the last year of secondary school, with relevant applications to physics. However, if learning how to differentiate a function is rather simple in terms of computation, conceptualizing the object derivative, and especially the derivative function, may not reveal that simple.

1.2.2 The derivative concept in the scholarly mathematics

At the level of scholarly mathematics, various approaches lead to the definition of derivative, and consequently to different conceptualizations of the notion.

The first approach consists of the classical limit of the incremental ratio of a function f. To make an example, we quote the definition given by Bramanti *et al.* (2000, p.171).

DEF. 1 - Let $f : (a,b) \to \mathbb{R}$; we say that f has derivative in $x_0 \in (a,b)$ if $\lim\limits_{h \to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exists and is finite. This limit is called first derivative (or simply, derivative) of f in x_0 and it is denoted with one of the following symbols:

$$f'(x_0) \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} \qquad Df(x_0) \qquad \dot{f}(x_0)$$

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) \tag{1.7}$$

The straight line of equation $y = f(x_0) + f'(x_0)(x - x_0)$ is called tangent line to the graph of f at the point $(x_0, f(x_0))$.

The derivative is presented as the gradient of the tangent line, defined as the limit position of a secant to the function (see Fig. 1.2.1). We owe to d'Alembert this definition. He introduced the "*nombre dérivé*"[9] as the limit value of the growth rate of the function f at the point x_0. Moreover, he claimed that the secant line becomes the tangent line when the two points of intersection with the curve become only one and then the tangent is the limit of the secant line (d'Alembert, *Mélanges de littérature, d'histoire et de philosophie.* In Sierpinska, 1985, p.20).

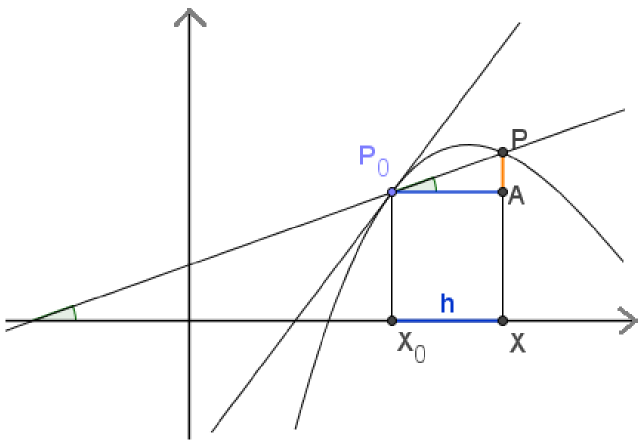

FIGURE 1.2.1 - DEFINITION 1 COMPARES THE INFINITESIMALS PA AND XX_0: THE LIMIT OF THEIR RATIO IS $f'(x_0)$ (SEE DEF. 1).

Another definition we find on university textbooks is due to Fréchet and it is based on the idea of approximation of functions through power series expansion. We can read it on Geymonat's textbook (Geymonat, 1981, p.188), for example.

[9]We want to highlight that often the choice of the names denoting mathematical entities can really influence the conceptions and the images the one can make of them. For example, the expression "*nombre dérivé*" for $f'(x_0)$ makes it explicit and evident that it is a number. In Italian, instead, to refer to $f'(x_0)$ we commonly use the verbal phrase "*derivata della funzione nel punto*" (derivative of the function in the point) which hides the fact that it is a number.

DEF. 2 - Let f be a real function of a real variable defined in the [open] interval I (not reduced to a single point) and let $x_0 \in I$. We say that f is differentiable in x_0 with derivative $\lambda \in \mathbb{R}$ if and only if it is

$$f(x) = f(x_0) + \lambda(x - x_0) + o(x - x_0) \quad \text{as} \quad x \to x_0. \tag{1.8}$$

[...] If f is a real function of a real variable, the straight line through $P_0(x_0, f(x_0))$ of equation

$$y = f(x_0) + \lambda(x - x_0)$$

where λ is given by (1.8) is called tangent line to the graph of f in the point P_0 and denoted with $T_{x_0}f$; (1.8) then means that for x close to x_0 the function f can be approximated with a linear function, i.e., the graph of f is well approximated by the graph of its tangent. Intuitively, we can say that the graph of f in the neighbourhood of P_0 is "almost linear" (or "almost straight").

In DEF. 2, the derivative is presented as a real number λ, which is a coefficient of a series expansion of first order of the function f in x_0. It is shown to be the gradient of the tangent line at x_0, defined as the straight line that best approximates the graph of the function in the point. In other words, by substituting the curve with the straight line, the committed error is an infinitesimal of higher order with respect to the increment $x - x_0$ as $x - x_0 \to 0$ (see Fig. 1.2.2). As a consequence, the function turns out to be "almost straight".

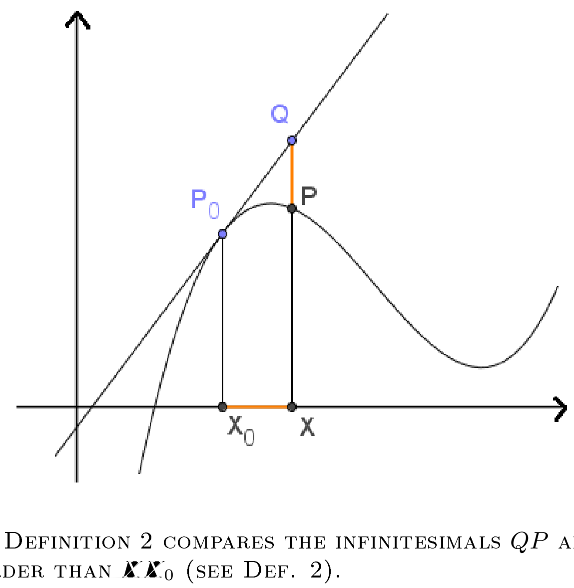

FIGURE 1.2.2 - DEFINITION 2 COMPARES THE INFINITESIMALS QP AND XX_0: QP MUST BE OF HIGHER ORDER THAN XX_0 (SEE DEF. 2).

Let us prove that DEF. 2 is equivalent to DEF. 1[10].

Proof. Saying that for f it exists and it is finite the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

where $h = x - x_0$, means that

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) = 0$$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} = 0.$$

It is equivalent to say that

$$f(x)-f(x_0)-f'(x_0)(x-x_0) = o(x-x_0) \quad \text{as} \quad x \to x_0$$

so

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+o(x-x_0) \quad \text{as} \quad x \to x_0$$

that is (1.8) with $\lambda = f'(x_0)$. □

Another approach bases again on the idea of substituting the function with a straight line in a neighbourhood of the point x_0. Indeed, the series expansion (1.8) of f can be also read as

$$f(x)-f(x_0) = \lambda(x-x_0)+o(x-x_0) \quad \text{as} \quad x \to x_0.$$

Hence, for $\lambda = 0$ we obtain the straight line $y = f(x_0)$, whereas for $\lambda \neq 0$ we obtain the following definition.

DEF. 2' - Let f be a real function of a real variable defined in the open interval I and let $x_0 \in I$. We say that f is differentiable in x_0 with derivative $\lambda \in \mathbb{R}$ if and only if it is

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{\lambda(x-x_0)} = 1. \tag{1.9}$$

This approach considers a straight line passing through the point $(x_0, f(x_0))$ which is asymptotically equivalent to the function f in a neighbourhood of x_0 (see Fig. 1.2.3). Such a straight line is the tangent line at x_0. From DEF. 2', we find immediately the value of the derivative λ. Indeed, in (1.9) λ does not depend on x and so it can be pull out of the limit sign:

$$\frac{1}{\lambda} \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 1$$

[10] Notice that this equivalence is valid in dimension 1. However, in dimension $n > 1$, DEF. 2 implies DEF. 1 but not conversely.

and since $\lambda \neq 0$ we obtain

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lambda.$$

Setting then $h = x - x_0$ the latter limit is exactly that of (1.7) with $\lambda = f'(x_0)$.

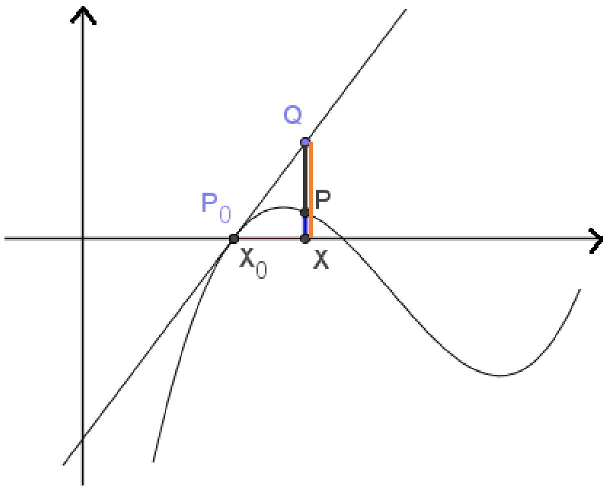

FIGURE 1.2.3 - DEFINITION 2' COMPARES THE INFINITESIMALS QX AND PX: THEY MUST HAVE THE SAME ORDER (SEE DEF. 2').

A third approach is based on the idea of best affine approximation, whose definition is given as follows.

DEF. OF BEST AFFINE APPROXIMATION - We call affine approximation of f in a neighbourhood of x_0 a function r_a, with $a \in \mathbb{R}$, defined by $r_a(x) = f(x_0) + a(x - x_0)$. We say that r_a is the best affine approximation of f if

$$\forall b \in \mathbb{R}, \exists \delta > 0 : \forall x \in I_\delta(x_0) \Rightarrow |f(x) - r_a(x)| \leq |f(x) - r_b(x)|.$$

Briefly, the distance of $f(x)$ from the best affine approximation is smaller than the distance of $f(x)$ from any other affine approximation of f. See Fig. 1.2.4 for a graphical representation.

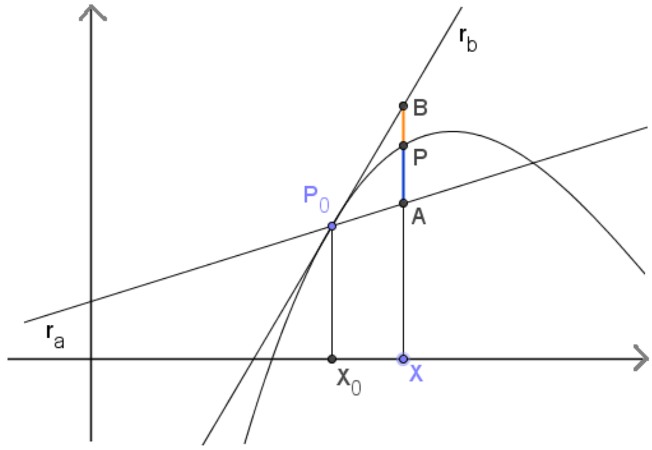

Figure 1.2.4 - Definition 3 compares the distances PB and PA: the former must be the smallest one (see Def. 3).

Therefore, here is Def. 3 statement.

> Def. 3 - Let f be a real function of a real variable defined in the open interval I and let $x_0 \in I$. We say that f is differentiable in x_0 with derivative $f'(x_0)$ if and only if it admits a best affine approximation t. Then, t is unique and $t = r_{f'(x_0)}$.

Def. 3 presents the tangent line of equation $y = f(x_0) + f'(x_0)(x - x_0)$ as the best affine approximation of the function f in a neighbourhood of x_0. Let us prove that this definition is equivalent to Def. 1.

Proof. "$\Leftarrow$" Let us suppose that $r_a(x) = f(x_0) + a(x - x_0)$ is *the* best affine approximation of f in x_0. Thus,

$$\forall b \in \mathbb{R}, \exists \delta > 0 : \forall x, 0 < |x - x_0| < \delta \Rightarrow |f(x) - r_a(x)| \leq |f(x) - r_b(x)|.$$

Dividing by $|x - x_0|$, we obtain

$$\Big|\frac{f(x) - f(x_0) - a(x - x_0)}{x - x_0}\Big| \leq \Big|\frac{f(x) - f(x_0) - b(x - x_0)}{x - x_0}\Big|$$

$$\Big|\frac{f(x) - f(x_0)}{x - x_0} - a\Big| \leq \Big|\frac{f(x) - f(x_0)}{x - x_0} - b\Big|.$$

For $\epsilon > 0$, we can choose $b = a + \epsilon$, and so

$$\Big|\frac{f(x) - f(x_0)}{x - x_0} - a\Big| \leq \Big|\frac{f(x) - f(x_0)}{x - x_0} - a - \epsilon\Big|.$$

But also it can be $b = a - \epsilon$, so

$$\Big|\frac{f(x) - f(x_0)}{x - x_0} - a\Big| \leq \Big|\frac{f(x) - f(x_0)}{x - x_0} - a + \epsilon\Big|.$$

For the same quantity $u(x) = \dfrac{f(x) - f(x_0)}{x - x_0} - a$, we have simultaneously $|u(x)| \leq |u(x) - \epsilon|$ and $|u(x)| \leq |u(x) + \epsilon|$. So, $|u(x)| \leq \epsilon/2$.
Hence, as $x \to x_0$, $u(x) \to 0$, which means that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = a.$$

Therefore, f has derivative in x_0 and it is $f'(x_0) = a$. The best affine approximation of f in x_0 is then $r_{f'(x_0)}$.

"$\Rightarrow$" Let us suppose that for f it exists and it is finite the limit (1.7), which we can rewrite with $h = x - x_0$:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Let us consider any $b \in \mathbb{R}$, $b \neq f'(x_0)$. We have to prove that

$$|f(x) - r_{f'(x_0)}(x)| < |f(x) - r_b(x)|$$

and so that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| < |f(x) - f(x_0) - b(x - x_0)|$$

which, divided by $|x - x_0|$, is

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \left|\frac{f(x) - f(x_0)}{x - x_0} - b\right|.$$

If x tends to x_0, $\dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \to 0$ and $\dfrac{f(x) - f(x_0)}{x - x_0} \to f'(x_0)$ for hypothesis. Thus, our thesis transforms into

$$0 < |f'(x_0) - b|$$

which is always true since for hypothesis $b \neq f'(x_0)$, so it is either $b > f'(x_0)$ or $b < f'(x_0)$. $\square$

Finally, let us present a geometrical approach which bases on the distances in the plane. For the graphical situation see Fig. 1.2.5.

DEF. 4 - Let f be a real function of a real variable defined in the open interval I and let $x_0 \in I$. We say that f is differentiable in x_0 with derivative $\lambda \in \mathbb{R}$ if and only if it exists a non-vertical straight line t with gradient λ which passes through $P_0(x_0, f(x_0))$ and so that

$$\lim_{x \to x_0} \frac{d(P, t)}{||PP_0||} = 0 \tag{1.10}$$

where $d(P, t)$ is the distance from the generic point P on the curve representing the function and the straight line t, and $|| \cdot ||$ denotes the distance between two points in $\mathbb{R}^2$.

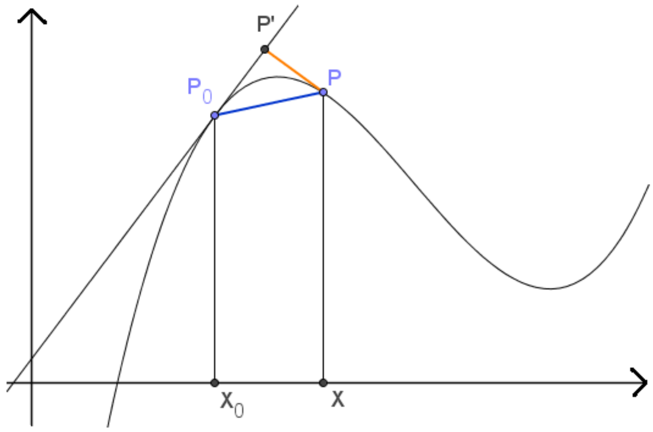

Figure 1.2.5 - Definition 4 compares the infinitesimals PP' and PP_0: PP' must be of higher order than PP_0 (see Def. 4).

In this case, we say that the curve representing the function "flattens" on the straight line t in a neighbourhood of P_0. t is unique and it is the tangent line, whose slope is given by $f'(x_0)$. Let us prove that also Def. 4 is equivalent to Def. 1.

Proof. First of all, let us make explicit the ratio of the distances appearing in (1.10). A non-vertical straight line t passing through P_0 has generic equation $y = f(x_0) + a(x - x_0) \Rightarrow ax - y + f(x_0) - ax_0 = 0$. Let us take $P(x, f(x))$ as a generic point of the curve. Then,

$$d(P,t) = \frac{|ax - f(x) + f(x_0) - ax_0|}{\sqrt{a^2+1}}$$

and

$$||PP_0|| = \sqrt{(x-x_0)^2 + (f(x) - f(x_0))^2}.$$

So the ratio of the distances is

$$\frac{d(P,t)}{||PP_0||} = \frac{|ax - ax_0 - f(x) + f(x_0)|}{\sqrt{a^2+1}\sqrt{(x-x_0)^2 + (f(x)-f(x_0))^2}}$$

which, divided and multiplied by $|x - x_0|$, becomes

$$\frac{d(P,t)}{||PP_0||} = \frac{\left|a - \dfrac{f(x)-f(x_0)}{x-x_0}\right|}{\sqrt{a^2+1}\sqrt{1 + \left(\dfrac{f(x)-f(x_0)}{x-x_0}\right)^2}}.$$

Let us set $\theta(x) = \dfrac{f(x)-f(x_0)}{x-x_0}$. Thus,

$$\frac{d(P,t)}{||PP_0||} = \frac{|a - \theta(x)|}{\sqrt{a^2+1}\sqrt{1+\theta^2(x)}}. \tag{1.11}$$

"$\Rightarrow$" Let us suppose that for f it exists and it is finite the limit (1.7), which we can rewrite with $h = x - x_0$:

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0).$$

Then, $\lim_{x\to x_0} \theta(x) = f'(x_0)$. Let make x tend to x_0 in (1.11). We obtain

$$\lim_{x\to x_0} \frac{d(P,t)}{||PP_0||} = \lim_{x\to x_0} \frac{|a-\theta(x)|}{\sqrt{a^2+1}\sqrt{1+\theta^2(x)}}.$$

Since $\frac{|a-\theta(x)|}{\sqrt{a^2+1}\sqrt{1+\theta^2(x)}} \leq |a-\theta(x)|$, then

$$0 \leq \lim_{x\to x_0} \frac{d(P,t)}{||PP_0||} \leq \lim_{x\to x_0} |a-\theta(x)| = a - f'(x_0).$$

Therefore, it exists a straight line, namely that of gradient $a = f'(x_0)$, for which $\lim_{x\to x_0} \frac{d(P,t)}{||PP_0||} = 0$.

"$\Leftarrow$" Let us suppose $\lim_{x\to x_0} \frac{d(P,t)}{||PP_0||} = 0$ for the straight line $t : y = f(x_0) + a(x - x_0)$. Let us set $u(x) = \sqrt{a^2+1}\frac{d(P,t)}{||PP_0||}$, then also $u(x)$ goes to 0 as $x \to x_0$. Recalling (1.11), we obtain the following equation of second order in θ.

$$u = \frac{|a-\theta|}{\sqrt{1+\theta^2}}$$

$$u^2 = \frac{a^2 - 2a\theta + \theta^2}{1+\theta^2}$$

$$u^2 + u^2\theta^2 = a^2 - 2a\theta + \theta^2$$

$$(1-u^2)\theta^2 - 2a\theta + a^2 - u^2 = 0.$$

Hence, we obtain as solutions

$$\theta_{1,2} = \frac{a \pm \sqrt{a^2 - (1-u^2)(a^2-u^2)}}{1-u^2} = \frac{a \pm u\sqrt{1+a^2-u^2}}{1-u^2}.$$

For $x \to x_0$, $u \to 0$, then $\theta_{1,2} \to a$, which means

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = a.$$

Therefore, f has derivative in x_0 and this derivative is a. $\square$

Even though all the presented definitions entail a local perspective on the involved function, they may lead to a different conceptualization of the derivative and of the tangent line. As a didactic remark, we observe that each of these definitions could be transposed to secondary school teaching, obviously with proper adaptations. Recalling the distinction between **concept image** and **concept definition** made by Tall and Vinner (1981), we claim that the proposed concept definitions of the derivative are generated by different concept images of the tangent line. This strictly depends on the fact that, in each of the different approaches, the derivative is always defined as the gradient of the tangent line to the given function in a point, but the conveyed concept image of the tangent line is not the same for all the given definitions. Such definitions are all

equivalent, as we proved, but the first (Def. 1) is supported by a tangent concept image as "the limit of secant lines", whereas the others (Def. 2,2',3,4) exploit the concept image of the tangent as "the straight line which best approximates the function". As we showed in the accompanying figures, these two images can be graphically and geometrically described as the interdependence of different variables brought into play.
In the next section we better portray the importance of the derivative concept in the field of Mathematics Education, through the lenses of the existing researches concerning the theme. In particular, we go in-depth in the concept image and concept definition research about the tangent and the function, in order to analyse what Tall calls the **cognitive roots** (Tall, 1989, 2000) for these concepts.

1.3 The derivative concept in Mathematics Education

Introducing the derivative concept in the secondary school entails talking about the tangent line to a function in a given point and, more generally, working with functions: the function to be differentiated and the derivative function itself. In Italy, differentiation is part of the curriculum of the last year of secondary school. However, the tangent line is already treated as a geometrical and algebraic object since the first years of the upper secondary school, specifically within the work with the conics. Usually, the tangent to the circle at its point P is defined as the straight line that has only one intersection with the circle. An alternative definition is the straight line which passes through P and is perpendicular to the radius OP, where O is the center of the circle. In the case of the other conics (parabola, ellipse and hyperbola) the first definition is maintained: a straight line is tangent to the conic if and only if it has only one point of intersection with the conic. This means that, when the students face the derivative concept as the gradient of the tangent line to a generic curve in a point, they have already in their mind some concept images of the tangent linked to the previous work with conics. And when the derivative has to be introduced as a function, some function concept images can emerge in the construction of a concept definition for the derivative function.

1.3.1 Concept image and concept definition

Concept image and concept definition are distinguished by Tall and Vinner (1981). They observed indeed that

> "Many concepts we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding a variety of personal mental images when a concept is evoked" (Tall & Vinner, 1981, p.151).

Thus, they highlighted "a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived" (Tall & Vinner, 1981, p.151). They speak about **concept image** referring to the cognitive structure that an individual associate in his mind with a specific concept. Thus, it includes all the mental

pictures and all the properties and processes associated with it. Instead, the **concept definition** is "a form of words used to specify that concept. [...] It may also be a personal reconstruction of a definition made by the student. It is then the form of words that the student uses for his own explanation of his (evoked) concept image" (Tall & Vinner, 1981, p.152). In this way, a personal concept definition can differ from the formal concept definition which is accepted by the mathematical community. As a matter of fact, there can be a factor in the concept image which is potentially in conflict with the formal concept definition. The so-called "potential conflict factors" may obstacle the learning of the formal theory. Indeed, if such a potential conflict factor affects their concept image, there is the risk that students might proceed sure in their own interpretations of the involved notions and simply consider the formal theory as inoperative and superfluous.
Tall (1989) focused then on the secondary school curriculum, in particular the British one, observing that presenting concepts in a simplified context and then build on it in order to get formal concept definitions does not actually reveal effective as didactic methodology. Indeed, he noticed that "When ideas are presented in a restricted context, the concept image may include characteristics that are true in this context but not in general" (Tall, 1989, p.38). He made the example of the tangent to the circle, referring to a study of Vinner (1982). In that restricted context, the developed concept image of the tangent is that it touches the circle at a single point and does not cross it.

Concept image and concept definition of the tangent

Let us discuss some fundamental studies involving the tangent line in the field of Mathematics Education research.

Vinner's study (1982), to which Tall refers (1987, 1989), showed that early experiences in geometry of the tangent to the circle foster the students' belief that the tangent is a line that touches the graph at one point and does not pass across it. The associated concept image causes cognitive conflict, when for example the notion of tangent is met in other more complex contexts, such as the tangent at an inflection point, where it does cross the curve. Also Fischbein (1987), in his researches about intuition, regarded the tangent to a circle as a *paradigmatic model* of the tangent line.

Sierpinska (1985) conducted a deep study of the epistemological obstacles related to the notion of limit. The context of her research was an activity involving a mobile material device properly designed to foster the identification of the tangent as the limit of a variable secant. A pair of students was required to write a tangent definition, only according to their direct experience with the device. Another couple of students received only this written information for tracing a tangent to a graph. Then, both the pairs were given the task of determining the equation of the tangent to the curve $y = \sin x$ at the point $x = 0$. The orchestration of the whole activity was clearly based on the definition of the tangent to a curve in P as the limit position of a secant to the curve in P and in another point, when the points coincide in P. As an unexpected conclusive remark on the experiment, Sierpinska verified that

> "[...] the notion of tangent is a new concept for the students who need to

> free themselves from several obstacles. One cannot count on the fact that the interpretation of the derivative as the angular coefficient of the tangent could approach this notion if it is introduced firstly as the limit of the sequence of differential quotients"[11] (Sierpinska, 1985, p.58).

We can read this comment as a first highlighting that the concept definition of the tangent as "limit of secants" actually could be not as effective as it seems. Indeed, the students showed some difficulties to give meaning to the notion of "limit of secants".

In Tall (1987), the concept image of the tangent has been studied in deep details. The context was a broader experience of introducing the idea of gradient of a graph through the use of computers. An experimental group of students worked on activities specifically designed in a computer-based environment. In particular, they got familiar with the use of computer for magnifying graphs to see if they "looked straight", and to draw a line through two close points on a graph. Five control classes followed traditional lessons in which an intuitive meaning of a tangent was assumed. A graphical test was given to both the experimental and the control group. From a comparison of the results, Tall concluded that

> "[...] the experiences of the experimental group helped them to develop a more coherent concept image, with an enhanced ability to transfer this knowledge to a new context. [...] However, potential conflicts remained, with a significant number of students retaining the notion of a 'generic tangent' which 'touches the graph at a single point', giving difficulties when the tangent coincides with part of the graph." (Tall, 1987, p.75).

Through this remark, Tall stressed the power of the concept image of the tangent to the circle (or more general to the conics) as the straight line that touches the curve at a single point. Such an idea persists in the students mind and it is source of obstacles when later the students have to face the case of a generic curve.

Castela (1995) chose exactly this situation as a significant example of students learning with and against their previous knowledge. The focus of her study indeed was on the role played by old conceptions[12] in the development of new ones. More precisely, Castela investigated the concept images which the students show when they face a task involving the tangent in the graphical frame. In particular, from an analysis of the French secondary curriculum, she detected three different points of view that intervene more or less in the teaching of the tangent to the circle and can be more or less extended by the students to other more generic curves. A point of view is intended by Castela as "a section of the body of mathematical knowledge about a given object, which gather

[11] Our English translation of the original passage: "*[...] la notion de tangente est un concept nouveau pour les élèves qui demandent le franchissement de beaucoup d'obstacles. On ne peut pas compter sur le fait que l'interprétation de la dérivée comme le coefficient angulaire de la tangente puisse approcher cette notion si elle est introduite d'abord comme limite de la suite de quotients différentiels.*"

[12] Castela (1995) uses the term *conception* to account for the coherence perceived in the observable behaviours of students who are facing a group of tasks involving the same mathematical concept.

definitions, theorems, situations and signifiers"[13] (Castela, 1995, p.10). Thus, for the circle, she distinguished the following points of view.

- "Calculus", in which the tangent is characterized as the straight line that best approximates the curve locally. This point of view is not explicitly approached in lower secondary school.

- "Intersection", in which a straight line is tangent according to its position with respect to the curve, neither external nor secant, and to the number of points of intersection with the curve, that must be only one. This point of view can be easily generalized by the students.

- "Perpendicular to the radius", in which the characteristic of the tangent is to be orthogonal to the radius that match the center of the circle with the tangency point on it. This point of view can be hardly generalized by the students, since usually no reference to the osculator circle is made at secondary school level.

Castela designed a questionnaire based on different graphical situations in which a straight line is drawn tangent to a curve in a certain point. She gave it to 6 classes of students who had not been taught derivatives yet, and to 7 classes of students who instead had already learnt derivatives. The context was mainly the scientific high school, from grade 10 to 12. This test allowed the researcher to investigate the presence of the local point of view related to "Calculus" in the students' concept image. For instance, through mixed approaches some students tried to verify the criteria "Intersection" in a local neighbourhood of the marked point on the given graphs. Thus, there have been detected some intermediary stages between the simple generalization of the point of view "Intersection" and the point of view "Calculus". Castela finally observed not only a persistence of the taught knowledge related to the circle and extended to different curves, but even a sort of implicit encouragement to its generalization in the teaching practices. Comparing the results by grade, she found that, in the situation of pre-teaching of derivatives, the students do not have occasion to get in contact with concept images involving the local point of view on the considered curve. So, when they are taught the derivative, they have to partially break with the previous knowledge, quitting the characterization referred to the circle and the conics in general. However, the latter should not be abandoned definitively, but should be restricted to a smaller validity field, for possible future generalization (osculator circle, etc.). At the same time, the students' concept image has to be enriched by the new approximation point of view. Using Harel and Tall's terms (Harel & Tall, 1991) the teaching-learning of the tangent notion cannot consist of a simple *expansive generalization*. Indeed, it is not a matter of just extending the student's existing cognitive structure without requiring changes in his fundamental images. Rather, it entails a *reconstructive generalization* because it requires reconstruction of the existing cognitive structure. In the reconstructive generalization students have to

[13] Our English translation of the original passage: "*un découpage dans le corps des savoirs mathématiques sur un objet donné, rassemblant définitions, théorèmes, situations et signifiants*".

change radically the old concept images so as to be applicable in a broader context. In conclusion of her study, Castela claimed

> "The rupture with a primitive point of view which is partially unsuitable, the emergence of a general fresh point of view, this has led us to the will of describing the involved learning in terms of change, of discontinuity, of old-new opposition. [...] The establishment of the point of view 'Calculus', generally far from being a *coup d'état*, could be anticipated and prepared, perhaps also allowed, through a progressive evolution of the conception [...]"[14] (Castela, 1995, p.39)

A similar characterization of the students' concept images of the tangent has been made by Biza and Zachariades (2010). In addition to other previous works, this research considered both secondary school pupils and undergraduate students and took into account also epistemological issues of the tangent possible concept images. Indeed, Biza and Zachariades stressed the importance that the notion of tangent holds with regard to many Calculus concepts (e.g., limiting processes, geometrical interpretation of derivative, linear approximation of a curve). Moreover, they pointed out the involvement of the tangent line at all instruction levels and in several mathematical contexts, namely Euclidean geometry, analytic geometry, Calculus. Furthermore, they highlighted that the ways to define the tangent are multiple as well as its representations through the use of different semiotic systems, in particular the graphical and the symbolic ones. They discussed then the different perspectives (term used in the same sense as Vandebrouck, 2011a) characterizing the tangent concept image in the Geometry and the Calculus domains. In Calculus, the fact that a curve admits or not a tangent line at a certain point is a local property of the curve, because it involves the curve at the point and at a neighbourhood of it (e.g., the tangent line is the limiting position of secant lines, or the tangent line is the best linear approximation of the curve at that point). However, in Geometry, in the case of the circle, the tangent line is usually defined through global properties referring to the entire curve (e.g., the tangent line to a circle has only one common point with the curve). The transition from the global to the local perspective (the global/local game) has been deeply studied by Maschietto (2002, 2008) and it comes to be an essential issue in the transition from Algebra to Calculus, as we have already underlined. Tasks involving the tangent notion, then, can provide a powerful stage for the investigation of the perspectives and in particular the local one in the teaching of Calculus. In this light, Biza and Zachariades detected three different perspectives on tangent line with respect to the curve. They are

- the *Analytical Local* perspective, which fits the definition and the uses of the tangent line in Calculus domains;

[14] Our English translation of the original passage: "*Rupture avec un point de vue primitif partiellement inadéquat, émergence d'un point de vue général inédit, ceci nous a conduit à vouloir décrire l'apprentissage en jeu en termes de changement, de discontinuité, d'opposition ancien-nouveau. [...] L'installation du point de vue 'Analyse', loin de prendre en général la forme d'un coup d'état, pouvait etre précédée et préparée, peut-etre meme autorisée, par une évolution progressive de la conception [...].*"

- the *Geometrical Global* perspective, in which, in contrast, the tangent preserves geometrical properties applied globally on the entire curve;
- the *Intermediate Local* perspective, which lies between the other two and is characterized by the application of geometrical properties locally at a neighbourhood of the tangency point.

In the Intermediate Local perspective, the students simply try to apply the geometrical properties valid in a global perspective in a neighbourhood of the tangency point. Thus, they achieve an expansive generalization. Within the Analytical Local perspective, the students put into play a reconstructive generalization (Biza & Zachariades, 2010; Harel & Tall, 1991). Biza and Zachariades proposed a questionnaire both to a group of secondary school pupils and a group of undergraduates attending their first year of university. The given tasks required the participants to: describe in their own words the tangent line and its properties; identify or construct the tangent line; provide definitions; and, write and apply the formula in general and specific cases. The results of this study have also provided evidences about the tangent in its symbolic representation, especially with respect to the group of undergraduates. Indeed, they showed that

> "[...] the knowledge of the formula and its applications does not necessarily imply that an undergraduate has a rich and accurate image of the tangent line, its properties and its relationship with the curve. [...] Sometimes they use argumentation based on the concept definition in order to support insufficient concept images they have about tangency" (Biza & Zachariades, 2010, p.20[15]).

For instance, the perfect knowledge of the formal and symbolic definition of the tangent did not prevent many of the involved students from arguing that a curve has not tangent line when it is vertical. Therefore, the only correct formulation of a definition is not enough to reveal the student's understanding about the notion. In conclusion to their research paper, Biza and Zachariades suggested what a teaching practice related to the tangent notion should take into account. We report a significant excerpt from their conclusions:

> "A teaching practice that is informed by the complexity of students' understandings and does not confine itself in the development of algorithmic skills has to engage students with more representations and richer set of examples in order to facilitate students firstly to reveal their understandings and then, if necessary, to reconstruct their images. A system of assessment that looks for evidence of algorithmic skills is not sufficient in ensuring students' understandings being brought from one level of education to another" (Biza & Zachariades, 2010, pp.27-28[16]).

[15]The page is referred to the online version in the Loughborough University Institutional Repository. Metadata Record: https://dspace.lboro.ac.uk/2134/8869

[16]See note n.15

Biza and Zachariades stressed the importance of the didactic implications of their research also in the foresight that the undergraduates involved in the study may become future mathematics teachers. If their concept images have not been properly enriched and some inconsistency persists with their correct formal concept definitions, their actions as teachers could be counter-productive.

Finally, let us consider Páez and Vivier (2013), who conducted and reported on a similar study about teachers' conceptions[17] of tangent line. They detected the following conceptions (see Table 1.1).

	P *Sphere of practice*	R *Set of operators*	L *Representation system*	Σ *Control structure*
C_V (visual)	Vast class of curves Without subset of straight line Problem for inflexion points	Drawing Visual perception	S_G	Visual, even instrumented One single point (C_V^{glob}) or not (C_V^{loc}) Cross the curve or not Uniqueness of tangent?
C_{AG}^{perp} (perpendicular to a radius of a circle)	Only circles	Visual perception, geometrical tools	S_G (right angle) S_A (equation of circles)	Visual, instrumented Tangent is unique
C_A^{der} (derivative)	Large class of curves, $y=f(x)$, possibly $f(x, y)=0$ (cf. implicit theorem) Problem for *vertical* tangent No multiple point	Algebraic and functional including specifically derivation	S_A (with derivatives) S_G	Related to derivation (use of values a, $f(a)$ and $f'(a)$) Including Σ_V in S_G Tangent is unique
C_A^{Des} (from Descartes' algebraic conception)	Algebraic curves (even implicit) Quite large class of curves (cf. secondary teaching)	Algebraic and functional including specific algebra on polynomials	Close to L_A^{der} (without derivatives) Curves equations	Close to Σ_A^{der} (without derivatives) Tangent is not necessary unique (multiple point)

Table 1.1 - Table of the conceptions of tangent line (in Páez & Vivier, 2013, Annex D).

Some activities, properly designed to cause cognitive unbalance playing with these conceptions of the tangent line, were proposed in a teachers' education workshop in México, in 2010. The participants were all upper secondary school teachers and the main aim was to make them reorganize their conceptions, to connect them, to reflect about them. The teachers had to work in teams, which were formed according to their answers in a preliminary diagnostic questionnaire about the tangent. During the working group the teachers could use the Digital Geometrical Environment of GeoGebra. Afterwards, a plenary discussion was orchestrated, followed by an individual autoreflection. As a result, Páez and Vivier have found a very strong global and visual conception linked to the circle (C_V^{glob} in Table 1.1). Furthermore, as Biza and Zachariades (2010) observed, even though the tangent and the curve are two concepts that seem well known for mathematics teachers, after deeper investigation they may be easily destabilized and they may feel the need to reorganize their concept images to fit them with the formal concept definition.

Concept image and concept definition of the derivative function

Since the derivative of a function is a function itself, in the introduction of the derivative function, concept images and concept definitions of the function intervene.
The concept of function and the functional thinking include many aspects. On the one

[17] Páez and Vivier use the term conception in the sense of Balacheff: "A conception is the state of dynamical equilibrium of an action-feedback loop between a subject and a milieu under proscriptive constraints of viability" (Balacheff & Gaudin, 2013, p.213).

hand functional dependencies can be described and detected in several representational systems such as graphs, words, tables or formulas (Duval, 1993; Arzarello, 2006). On the other hand the nature of functional dependencies has different characteristics (Vollrath, 1989; Malle, 2000). Vollrath (1989) distinguished three fundamental aspects of the functional thinking.

1. The *relation aspect*: functional dependencies are regarded as pointwise relations (static view).

2. The *change aspect*: dynamic view of functional dependencies including the aspect of covariation of values as the change of a variable affecting the change of the other.

3. The *object aspect*: functions seen as objects or *as a whole*.

Malle (2000) slightly modified the aspects 1. and 2. of Vollrath in:

1. *Assignment aspect*: to each x it is assigned exactly one $f(x)$.

2. *Covariation aspect*: any change of x produces a certain change of $f(x)$ and vice versa.

Several research in Mathematics Education have focused on the change or covariation aspect, opposing it to the static view of the relation or assignment aspect.

Tall (1996) stated "One purpose of the function is to represent how things change" (Tall, 1996, p.1). Thus Calculus, intended as the study of variations, is necessarily and inherently related to the function concept and the functional thinking. Tall exalted the dynamic feature of a function, seeing it in the algebraic register as an example of *procept* (Gray & Tall, 1994), that is a combination of process and concept. From this point of view, the function, as well as other Calculus notions (e.g., limit, derivative, integral), is firstly perceived as a process and then, through recurring cycles of activity, it is internalized as a concept. Beside the process-concept dialectic, another interesting one is proposed by Douady (1986) concerning the *tool* aspect or the *object* aspect of a mathematical concept. Thus, the function, as well as the limit or the derivative, is a tool "when we focus our attention on its use to solve a problem"[18] (Douady, 1986, p.9). Moreover, it is intended as "the cultural object having its place in the larger building that is the socially recognized scholarly knowledge in a certain moment"[19] (Douady, 1986, p.9).

Slavit (1997) promoted a property-oriented view on functions which enhances the aspects of functional growth and covariation, instead of the relational aspect. His intention was to propose a new pattern for students' *reification* of functions (in the sense of Sfard, 1991). He described such an approach as "the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyse functions by identifying the presence or absence of these growth properties"

[18]Our English translation of the original excerpt: "*lorsque nous focalisons notre intérêt sur l'usage qui en est fait pour résoudre un problème*".

[19]Our English translation of the original excerpt: "*l'objet culturel ayant sa place dans un édifice plus large qui est le savoir savant à un moment donné, reconnu socialement*".

(Slavit, 1997, p.265). Examples of functional growth properties are zeros, concavity, and asymptotes. For acquiring a property-oriented view of functions, a student has to develop the ability of recognizing the equivalence of procedures that are performed in different semiotic systems, and the ability of generalizing procedures across different classes and types of functions. Through this route, Slavit claimed that "A student can reify the notion of function as a mathematical object capable of possessing or not possessing these properties" (Slavit, 1997, p.265).

In the Italian school, the definition of function highlights the relational aspect. Indeed, a function f is defined as follows.

> DEF. OF FUNCTION - Let A and B be two sets. A *function* $f : A \to B$ is a relation that with each x, belonging to A, associates one and only one element of B. The set A is called *domain of* f and all the elements in B which correspond to at least one element of A form a subset of B called *codomain of* f.

This definition, in which we can recognize the Bourbakist influence[20], can already appear in grade 8 (at the end of lower secondary school) and it is recalled at the beginning of the upper secondary school.Several studies have underlined some misconceptions linked to this definition, due to its abstractness and generality. The change or covariation aspect is not made evident by it. Moreover, the general character of the definition risks to be quickly forgotten since students actually create their concept images of the function getting in contact with a few classes of functions (linear, quadratic, polynomial functions, etc.).

Vinner and Dreyfus (1989) pointed out that students' criteria to decide whether a given example is a function or not are based on their concept image and not on the concept definition. Vinner (1983) worked with high school students, testing their conceptions of functions. He found that even students who could give a correct formal definition of function actually used their intuitive concept images in answering questions about functions. Around 40% of them believed that the graph of a function should be necessarily regular, persistent or reasonably increasing. Moreover, he found many students thinking that a function has to be given by a single formula.

The latter is only one of the typical misconceptions of functions that often make the concept image non-consistent with the concept definition. Tall (1996) reported on the following students' specific conceptions of function, discussed in literature: a function is given by a formula, and if y is a function of x, then it must include x in the formula; the graph of a function is expected to have a recognizable shape (e.g., polynomial, trigonometric or exponential) and to have certain "continuous" properties.

Sierpinska (1992) conducted a study on the epistemological obstacles related to the work with functions. In parallel with the other researches, she also found the obstacle

[20] BOURBAKI'S DEF. OF FUNCTION - f is a function from one set to another, say A to B, if f is a subset of the Cartesian product of A (the domain) and B (the range or codomain), such that for every $a \in A$ there is exactly one $b \in B$ with $(a, b) \in f$

given by the belief that only relationships that one can write through an analytic formula are functions. Another typical example of obstacle she noticed is the identification of a function with its representation (graphical, tabular, etc).

Concerning misconceptions of functions that are linked particularly to graphical representation, we can refer to Clement (1985). He highlighted the "graph-as-picture" misconception, which is frequent especially in physics, when students appear to treat the graph as a literal picture of the problem situation. Such a misconception points out the difficulty to interpret the functional dependency in a dynamic way. Moreover, Clement talked about the "height-for-slope" error that students are likely to commit when asked to compare the slope of two points on a graph. In some cases, they reason in a pointwise way, comparing the ordinates and not correctly the slope at these points.

With regard to this latter misconception, we move on to consider difficulties that occur in articulating the relationship between a function and its derivative function. Although it seems that students can grasp it better working on graphs rather than on other representations, the research in Mathematics Education warns us against some possible misconceptions also arising from the graphical work. In particular, we refer to Nemirovsky and Rubin's study (1992) on the students' tendency to assume resemblances between a function and its derivative. They referred to the "height-for-slope" misconception studied by Clement (1985) by showing a typical example of its occurrence (see Fig. 1.3.1). There are two possible sources for this mistake: a representational one (in the Cartesian graph, the comparison occurs at the level of the ordinates, and the slope is not taken into account) and a conceptual one (position and velocity are not adequately distinguished).

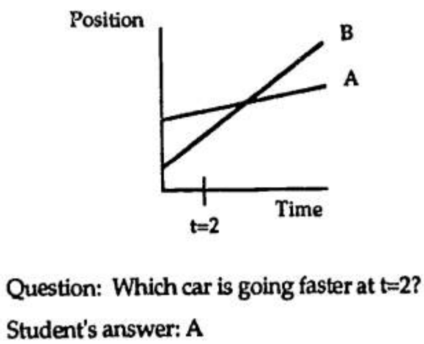

Figure 1.3.1 - A typical problem triggering the "height-for-slope" misconception (in Nemirovsky & Rubin, 1992, p.1).

For their inquiry, Nemirovsky and Rubin provided the participating students with one of three experimental contexts: motion, fluids, or numerical integration. They all were high school students who had not been taught Calculus yet. The researchers started from the assumption that human beings can intuitively relate function and derivative, in the sense that they are able to construct complex bodies of knowledge to make sense of situations involving change. The analysis of the students' interviews revealed a frequent tendency to assume resemblances between the behaviour or appearance of a function and that of its derivative. For "assumptions of resemblance" the authors intend "premises that the graphs of two different functions will be perceived as having common attributes." (Nemirovsky & Rubin, 1992, p.5). They distinguished different types of resemblance:

1. simple replication (the predicted graph is identical to the original graph);
2. same direction of change (e.g., increasing derivatives correspond to increasing functions, and decreasing derivatives correspond to decreasing functions);
3. same shape (e.g., straight lines correspond to straight lines);
4. same sign (graphs above the x-axis generate graphs above the x-axis and vice versa);
5. same geometrical transformation, when students are given two velocity graphs and one position graph and they are asked to infer what the second position graph should be.

Resemblance appears as a tool for the students to make sense of a complex situation. It is not chosen by chance. Nemirovsky and Rubin underlined that this choice emerges from an interplay between expectations and cues, the latter being of syntactic, semantic and linguistic nature. They differently prompt the students to choose specific resemblances. The syntactic cues are based on graphical features and, for instance, they foster the students to resemblances of type 5. Instead, the so-called "isomorphic variation", which occurs when students assume that a function and its derivative change in a similar way (resemblance of type 2.), is due to semantic cues, based on real-world knowledge (for example, the common experience that going faster implies travelling further). Finally, linguistic cues are ambiguities of language that support resemblance (e.g., the uses of "more/less" and "up/down"). So, more (or less) velocity means more (or less) distance, as if they always changed in the same direction (resemblance of type 2.).
It is only after some time and some practice that certain students became aware that one function (the derivative) described the local variation of the other. Some students perceived the "steepness" of a graph, which is a graphical-perceptual feature, or the slope of a curve, as the rate of change of y with respect to the change of x. Thus, gradually from resemblance principles application, they moved on to what Nemirovsky and Rubin define "variational approaches". The authors concluded that

> "[The initial] assumptions of resemblance lead to a particular approach to problems of prediction between a function and its derivative, characterized by forcing a match of global features of the two graphs (e.g., increasing/decreasing, sign) and by focusing on one of them (function or derivative) rather than on their relationship" (Nemirovsky & Rubin, 1992, p.32).

On the contrary, a variational approach focuses on the relationship between a function and its derivative rather than on the global properties of each graph. However, for students it is not simple to develop a variational approach: it takes time and also, we would add, a change in students' perspective on the involved functions, namely from global to local. That of function and its derivative then remains a delicate relationship.

1.3.2 An approach to the derivative based on cognitive roots

Tall (1989; Tall *et al.*, 2000) proposed a curriculum change consisting of new didactic sequences that start not from mathematical foundations, but from what he calls the **cognitive roots**. Here is the definition he gave in 1989:

> "A cognitive root is an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall, 1989, p.40).

Later, from Tall's collaboration with Barnard, the notion of *cognitive unit* has been formulated as "a piece of cognitive structure that can be held in the focus of attention all at one time" together with its immediately available cognitive connections (Barnard and Tall, 1997, p.41). This work fostered Tall to re-elaborate the fundamental idea of cognitive root as a special type of cognitive unit that relates to the fundamental knowledge familiar to the student who is beginning a new conceptual development. In another collaboration, the following more complete definition was given.

> "A cognitive root is a concept that:
>
> (i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,
>
> (ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,
>
> (iii) contains the possibility of long-term meaning in later developments,
>
> (iv) is robust enough to remain useful as more sophisticated understanding develops" (Tall, McGowen & DeMarois, 2000, p.4).

Referring to the learning of mathematical concepts, Tall added: "It is hoped that a firmly based cognitive root will allow the learning sequence to build from meaningful foundations that may be enriched and adjusted whilst maintaining the strength of the entire structure" (Tall, McGowen & DeMarois, 2000, p.4).

For the purpose of our research, centred on the derivative notion, in particular two cognitive roots of Calculus reveal very important.

The notion of "local straightness" has been proposed by Tall (1989) as a cognitive root of the local linearity of functions, and so of their differentiability. Local straightness is identified with the property of looking straight under high magnification (see Fig. 1.3.2). Several researches in Mathematics Education base on this cognitive root for a different approach to Calculus teaching (e.g., Tall, 1989, 2000; Maschietto, 2002, 2008; Giraldo & Carvalho, 2003).

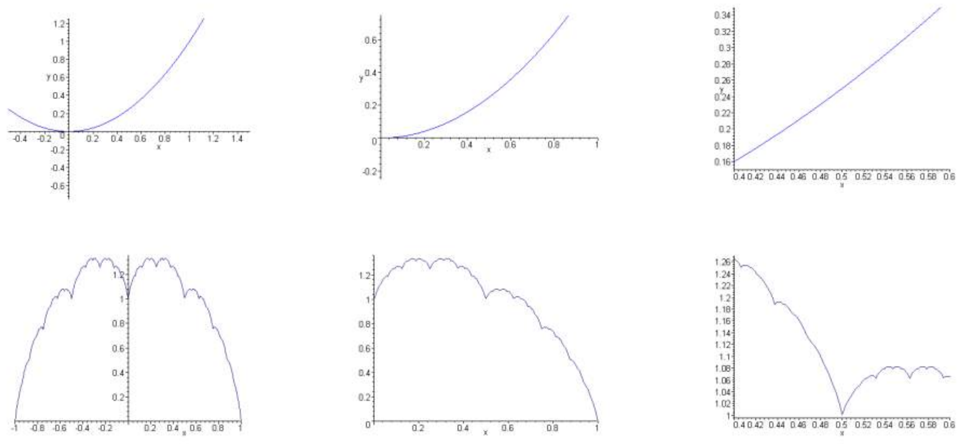

Figure 1.3.2 - The local magnification process for differentiable and non-differentiable curves (in Giraldo & Carvalho, 2003, p.2).

The notion of local straightness naturally leads to talk about "the slope of the graph", that corresponds to the slope of the straight line the function looks like under high magnification. That straight line can be seen as the tangent line to a point of the curve which lies in the zoomed window. Thus, in turn, the intuitive and embodied notion of slope of the graph can be seen as a cognitive root for the derivative concept.

Another cognitive root proposed by Tall is instead related to the concept of function: the "function machine" as an input-output box (see Fig. 1.3.3). Given the complexity of the notion of function, such a cognitive root embodies both its process-object duality and also its multiple representations (Tall, McGowen & DeMarois, 2000).

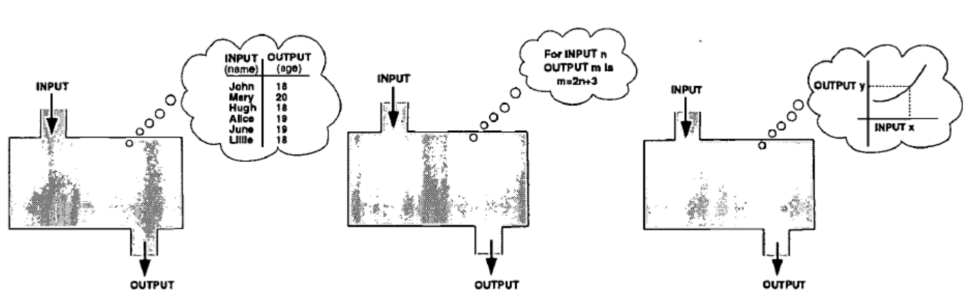

Figure 1.3.3 - The function box as a table, a formula and a graph (in Tall, McGowen & DeMarois, 2000, p.5).

Tall and his colleagues (Tall *et al.*, 2000) specify that a function box is not to be intended, as usually occurs, as a sort of "guess the internal rule/formula" problem. That kind of activity indeed generates often the epistemological obstacle that all functions are given by a formula. The function box may be used in a different and more general way, working with functions given by a procedure rather than a simple formula.

This kind of analysis leads Tall to formulate a didactic project for the secondary teaching of Calculus. He bases on Bruner's three distinct ways in which "the individual translates experience into a model of the world", namely, enactive, iconic and symbolic (Bruner, 1966, p.10). Thus, Tall comes to interpret the cognitive roots in terms of cognitive growth. As they are initially approached, the cognitive roots allow a student to enter the first world of mathematics: the "*conceptual-embodied* world". It "includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery" (Tall, 2004, p.285). Then, a work of reconstruction has to be made on the cognitive root, in order to strengthen the deriving cognitive structure. Thus, the student can access at the "*proceptual-symbolic* world", where the embodied processes become procepts (Gray & Tall, 1994), and finally at the "*formal-axiomatic* world", where the concepts are expressed through formal definitions.
The possible approaches to Calculus that different curricula can propose are very various: from the real-world Calculus, in which intuitions are built through visuospatial representations, through numeric, symbolic and graphic representations in elementary Calculus, and on to the formal approach of definition-theorem-proof-illustration typical of the Analysis. For an effective curriculum, Tall proposes a good balance of work in the three worlds of mathematics, starting from embodied cognitive roots of concepts. In 1989, he criticized curriculum regulations, as the British ones, which suggest a very formal entry to Calculus (that is the one we generally go through in Italian secondary schools nowadays):

1. work on limits in general;
2. fix x to calculate the limit of $\dfrac{f(x+h)-f(x)}{h}$ as h gets small and call the limit $f'(x)$;
3. vary x in $f'(x)$ to get the derivative as a function.

Tall highlighted the cognitive obstacles that the research in Mathematics Education has shown at each stage. In particular, he observed that "the geometric idea of using a secant approaching a tangent is not cognitively intuitive in the sense that it does not occur spontaneously" (Tall, 1989, p.40), as it has been verified also by Sierpinska (1985). Tall and other researchers in Mathematics Education (Tall, 1989, 2000; Maschietto, 2002, 2008; Giraldo & Carvalho, 2003) have proposed innovative approaches to Calculus concepts, and in particular to the derivative, basing on the use of technology and on the notion of local straightness.

In particular, Maschietto (2002, 2008) investigated the implications that the use of zoom-controls has on transformations of the graphical representation of functions. The analysis of the experiments revealed the construction for the students and the teacher of the mathematical meaning of a strongly perceptive phenomenon, that is the "micro-straightness". Thus, related to this phenomenon, a specific language has been formulated and new gestures and specific signs emerge. Maschietto observed that they can be actually used effectively in the processes of constructing mathematical meaning.

1.4 Outline of our research problem

In this chapter, we have highlighted some of the epistemological components that combined together make the introduction to Calculus a delicate didactic moment. It occurs at the end of the schooling path of a student, in his high school studies. In Italy, we refer to the *Indicazioni nazionali degli obiettivi specifici di apprendimento per i licei* (National guidelines[21] of the specific learning objectives for high schools, issued by the Italian Ministerial Decree n.211 of October 7, 2010), which constitute part of the Italian secondary curricula. Let us consider the guidelines specifically devoted to scientific high schools (Attachment F to the M.D.211), in the section titled "*Matematica*" ("Mathematics") (pp.13-17). In the general premise,

> "[...] the elements of algebraic calculus, the elements of the Cartesian analytic geometry, a good knowledge of the elementary functions of Calculus, the elementary notions of the differential and integral calculus" (p.13)

form one of the so-called "groups of concepts and methods to be studied". It is relevant that the algebraic calculus, the analytic geometry and the differential and integral calculus have been gather together.
The theme "*Aritmetica e algebra*" ("Arithmetic and algebra") holds a central position in the first two years of upper secondary school. Usually, it is already introduced as literal calculus and solution of first degree equations at the end of lower secondary school (grade 8). In grade 9 and 10, it acquires an increasing importance, being identified with the study of numeric sets and of their properties, and the resolution of second degree equations, inequalities and systems.
The theme "*Relazioni e funzioni*" ("Relations and functions") develops throughout the five years of upper secondary school, representing at the beginning about one fifth of the curriculum contents and becoming more and more relevant. About three quarters of the last year curriculum (grade 13) is devoted to the systematic study of real functions of one real variable, limits, differential and integral calculus. In Table 1.2 we provide a schematic overlook of the curriculum contents with respect to the themes involving arithmetic and algebra (first row) and relations and functions (second row)[22].

[21] In this dissertation, we translate the Italian term "indicazioni" as "guidelines". Indeed, in the Italian case, they actually consist in a discursive text, where some indications about contents and methods are given to teachers, without any rigid standards definition.

[22] The upper secondary contents involving Algebra and Calculus derive by our elaboration of the National guidelines. The latter, indeed, are entirely made of discursive indications for teachers. We have extrapolated the contents from the explications concerning the themes "*Aritmetica e Algebra*" and "*Relazioni e funzioni*".

	END OF LOWER SECONDARY SCHOOL (grade 8)	1^{ST} – 2^{ND} YEARS OF UPPER SECONDARY SCHOOL (grade 9-10)	3^{RD} – 4^{TH} YEARS OF UPPER SECONDARY SCHOOL (grade 11-12)	LAST YEAR OF UPPER SECONDARY SCHOOL (grade 13)
Arithmetic and algebra	- Identification of significant data and variables within a problem - Setting and calculation of arithmetic expressions - Reading, writing, use and transformation of simple formulas - Simple numerical equations and inequalities of first degree	- Arithmetic calculus to algebraic calculus (integers, fractions, operations properties) - Euclidean algorithm for the GCD - Intuitive introduction to reals - Radicals - Polynomials and operations, factorization, division with remainders - Putting word problems into equation (inequality or system) - General proofs (e.g. in arithmetic) - Vectors, linear dependence and independence, scalar and vector product - Elements of matrix calculus	- In-depth study of real numbers: transcendent π and e - Formalization of real numbers with the introduction of the mathematical infinity - Approximated calculus - Complex numbers	
Relations and functions	- Simple mathematical laws and their Cartesian representation; direct and inverse proportionality, quadratic dependence, etc. - Concepts of relation, correspondence, function, law of composition.	- Functions: set notion of domain, composition, inverse, etc. - Representation of phenomena, mathematical models - Putting word problems into equation and its resolution - $f(x)=ax+b$, $f(x)=ax^2+bx+c$, straight lines and parabolas in the Cartesian plane, graphical and algebraic resolution of equations, inequalities and systems - $f(x)=\|x\|$, $f(x)=a/x$, linear piecewise functions, circular functions - Direct and inverse proportionality - Various registers of representation for functions (numerical, graphical, functional)	- Number of solutions of polynomial equations - Numerical sequences, arithmetic and geometric progressions - Exponential and logarithmic functions - Models of exponential growth/decay, periodical trend - Study of functions, composition and inverse of a function - Concept of variation speed	- Study of fundamental functions of Analysis - Limit of sequence - Limit of function - Continuity, differentiability and integrability - Differentiation (product, quotient, composition) - Integration (integer polynomial functions, elementary functions) - Areas and volumes determination - Differential equations: solutions and properties - Optimization problems

Table 1.2 - From the Italian curriculum.

Notice that, the fundamental concepts and operations of Calculus, including the derivative, are not treated until grade 13. However, in the preceding years (grade 9-12) a preliminary work on elementary functions occurs, whose features are mainly algebraic and graphical.

As far as the perspectives on functions are concerned, we can observe that in a first moment, specifically when algebra plays the main role, a universal pointwise perspective prevails in the mathematical work. Indeed, the object of study are algebraic expressions in one (or more) unknowns. They can be thought as possible equations of certain functions. Since their expressions are intended valid for each x belonging to the existing domain, the perspective on the corresponding functions comes to be universal pointwise. After, when a graphical work intervenes, the global perspective takes shape in addition to the pointwise one. Finally, with the approach to limits a local perspective is introduced. Because of this specific distribution of mathematical topics throughout the secondary teaching, the approach to Calculus is strongly dependent on a prior algebraic work on functions. Is this base enough to make students internalize suitably the fundamental concepts of Calculus? Is a student well-prepared to face the local study of Analysis at university? We do not have the pretension to answer to such great questions, but it is with these concerns and taking into account the evolution of perspectives described above that our research problem arises. We ask:

How do the algebraic practices intervene in the development of Calculus practices, such as those involving the derivative concept, in secondary school?

Our hypothesis is that algebraic formulas, techniques and procedures are certainly an essential base on which constructing Calculus practices. However, they can also contribute to create some obstacles in understanding Calculus concepts. We use the expression "obstacle" in the sense of Brousseau, who writes "[A learned] notion receives certain particularizations, limitations, deformations of language and meaning; if it succeeds well enough and long enough, it takes on a value, a consistency, a meaning, a development that make its modification, re-use, generalization and rejection more and more difficult. For later acquisitions, it becomes both an obstacle and a support" (Brousseau, 1997, p.82). In claiming so, he recalls Bachelard's assumption that "we know against a previous knowing" (Bachelard, 1938, p.13). As we saw in Section 1.3, some algebraic practices can be so rooted in the students' mathematical work that then is really difficult to reinterpret or uproot them. Sometimes they foster in the learners' mind ideas and images which it is not easy to rework. But such a re-elaboration is strictly needed in the case of several algebraic rules in the initiation to Calculus work. For instance, let us consider the common algebraic practice of setting up the existence conditions for a fractional equation and, then, rejecting a solution if not acceptable for the conditions. It leads the students to keep in their mind the idea that the algebraic fraction $\frac{1}{x}$ for $x \in \mathbb{R}$, for example, *is not defined* in $\mathbb{R}$ if $x = 0$. After having employed this practices for years, they find themselves to cope again with the situation when they approach Calculus. Indeed, they learn that the limit $\lim_{x \to 0} \frac{1}{x}$ is ∞. The latter practice and outcome entail a reinterpretation of the old algebraic practice: it is true that the function $f(x) = \frac{1}{x}$ does not exist in $x = 0$, but we can say something more, namely that the function tends to ∞ as x goes to 0. Notice that the two claims reveal a different perspective on the function $\frac{1}{x}$: the former is pointwise, while the latter is local. It is not a chance, since many of the reworkings needed in the Algebra/Calculus transition correspond to a change in perspective, especially towards a local one. Hence, our initial research problem can be reformulated as follows.

What role is given to the local perspective when the Algebra/Calculus transition occurs in the secondary school?

In this chapter, we have shown that the local perspective is strongly present in the scholarly mathematical knowledge concerning functions and their properties, in particular the differentiability. So we wonder about the effects of the didactic transposition process (Chevallard, 1985; see Paragraph 2.1.2) on the local perspective on functions in the secondary school context. Our supposition is that its presence in the mathematical work becomes feebler and feebler. In the Italian curriculum, a great importance is given to the algebraic procedures in teaching Calculus, often with more interest in the techniques rather than in the justification of them. Consequently, the local perspective seems to be not really worked in secondary school. As Vandebrouck highlights "Even though the notion of limits, continuity and differentiability of functions are introduced in secondary school, the local perspective seems necessary only at university level. [...] Problems of continuity or differentiability are introduced in an algebraic way and they consist mainly in calculating limits by algebraic rules. So the algebraisation of tasks erases the pointwise and global perspectives and moreover doesn't allow reaching out to the local perspective" (Vandebrouck, 2011b, p.2096).

This question represents a first outline of the broader problem we intend to study, focusing on the derivative concept. In the next chapter, the introduction of our theoretical framework will enable us to formulate properly our research questions, starting from the research problem we have just posed.

Chapter 2

Theoretical framework

In this chapter we frame our study in the Anthropological Theory of the Didactic (ATD), which has been elaborated and disseminated by Chevallard for the last 30 years. In this research background, we will move on to introduce the theoretical tools through which our analysis is made. The networking of three main lenses will help us in investigating how the derivative concept is transposed in the secondary school context: the praxeologies (Chevallard, 1999), the perspectives (Vandebrouck, 2011) and the semiotic bundle (Arzarello, 2006). The integration of these three lenses will allow us to reformulate our research problem, outlined in Section 1.4, in terms of research questions. Through these combined theoretical tools, the study of the transposition process will develop on three levels: the analysis of the curricular materials (Chapter 3), the analysis of three case studies of teachers' practices in classroom (Chapter 4) and the analysis of two activities proposed to the students (Chapter 5). Closing this chapter, we will underline that our focus is particularly on the teacher's role who, availing of the curricular materials (intended curriculum) and taking into account the effects of her teaching on her students' learning (attained curriculum), has to transpose the concept in classroom (implemented curriculum).

2.1 Anthropological Theory of the Didactic

This thesis is deeply rooted in the theoretical background outlined by Chevallard's research during the last 30 years. It is through a great number of contributions that Chevallard has disseminated the principles of his theory of didactics, named the Anthropological Theory of the Didactic (*Théorie Anthropologique du Didactique*), shorten in ATD from now on. We refer only to some of these papers, workshops or lectures (Chevallard, 1985, 1992, 1999). Moreover, we advert to a paper written by Bosch & Gascón (2006) to present the theory, its application and its development in Mathematics Education research.

2.1.1 Fundamental elements of the theory

As an initial remark, let us give reason for the name of this theory: "Anthropological Theory of the Didactic".
Firstly, the ATD is a theory of *the* didactic ("*du* didactique" in French), that consists of "all that is didactic". Secondly, the fundamental postulate of the ATD, which we report in Chevallard's words, justifies the adjective "anthropological".

> "[...] The ATD situates the mathematical activity, and so the study activity in mathematics, within the set of human activities and social institutions"[1] (Chevallard, 1999, p.223).

In order to explain the structural elements of the ATD, Chevallard (1992) starts exactly from the *institutions*, which are presented as the core of the theory. They are to be intended in a large sense: an institution can be the school, the classroom or the lesson, but the family is an institution as well. With every institution $\mathcal{I}$ a number of *objects* are associated, O_I, which represents the set of institutional objects for $\mathcal{I}$. An object O exists for $\mathcal{I}$ if $\mathcal{I}$ has defined a *relation* with it, denoted with $\mathcal{R}_I(O)$. Institutions and objects are the first primitive terms that constitute the theory. A third one are the *people*: a person $\mathcal{X}$ becomes a *subject* of $\mathcal{I}$ if he/she "enters it". If $\mathcal{X}$ is a person entering the institution $\mathcal{I}$, where the object O is worked, then a *personal relation* is built between the subject $\mathcal{X}$ and the object O, denoted by $\mathcal{R}(\mathcal{X}, O)$. Fig. 2.1.1 illustrates a schematic elaboration where these first fundamental notions are interrelated. To make an example, in the institution C = classroom, we can consider the subjects: s, a student, and t, the teacher. With the object $\mathcal{M}$ = mathematics, each subject will have his/her own personal relation: the teacher's relation with mathematics $\mathcal{R}(t, \mathcal{M})$ and the student's relation with mathematics $\mathcal{R}(s, \mathcal{M})$. Their personal relations with the object mathematics are different first of all because of the different *position* they occupy inside the institution.

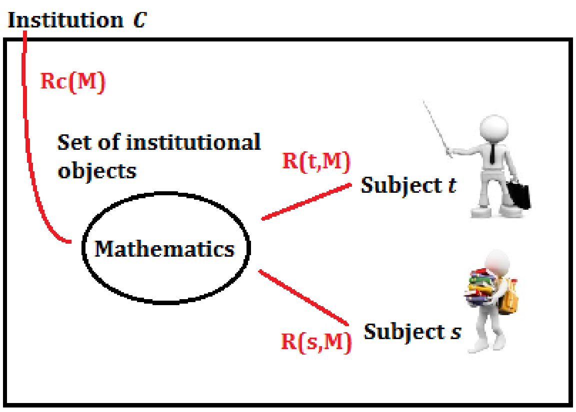

Figure 2.1.1 - Example of relations subject-object within an institution.

[1] Our English translation of the original passage "[...] La TAD situe l'activité mathématique, et donc l'activité d'étude en mathématiques, dans l'ensemble des activités humaines et des institutions sociales."

Chevallard (1992) then states that knowing an object means having a certain relation with it. Thus, in our example, the involved subjects know the object mathematics through their personal relation with it, with respect to the wider institutional relation that the classroom has with is, $R_C(M)$. Chevallard (1992) talks about *learning* whenever the personal relation that a subject has with O, that is $R(X, O)$, changes. In this way, he distinguishes some particular institutions that are the *didactic institutions*, which have the aim of making $R(X, O)$ conform to $R_I(O)$ for every subjects X in the student's position and for all the institutional objects O which are *didactic issues*. There can be some objects O for which $R(X, O)$ is empty: those are institutional objects that the subjects X do not have to know. Chevallard (1992) defines as *institutional instruction* the set of changes operated on the personal relations $R(X, O)$ where O is a didactic issue to be learnt by all the students X. The didactic intention of an institution I is made visible through the creation of *didactic systems*. This entails that one or more subjects of I come to hold the teacher's position and some other subjects come to hold the student's position, developing their relations with one or more didactic issues. For example, the schools are particular didactic systems. There are some "ecological" conditions, as Chevallard (1992) defines them, thanks to which a didactic system can not only exist but also work. These conditions consist of what Brousseau (1997) calls the *didactic contract* and the *milieu*. A didactic system cannot work without these two components and, conversely, the working of a didactic system modifies them. Moreover, behind every didactic system there has to exist a teaching system that shelters it. Thus, a didactic system lives inside a teaching system that in turn is determined by a *noosphere*. The latter is defined as the "sphere of those who think about education" (Bosch & Gascón, 2006, p.52) or more precisely "a plurality of agents [...] including politicians, mathematicians ('scholars') and members of the teaching system (teachers in particular)" (Bosch & Gascón, 2006, p.53).
Thus, after this complex and detailed description of the elements which interplay in the ATD, let us summarize, in Fig. 2.1.2, the institutions and their encapsulation.

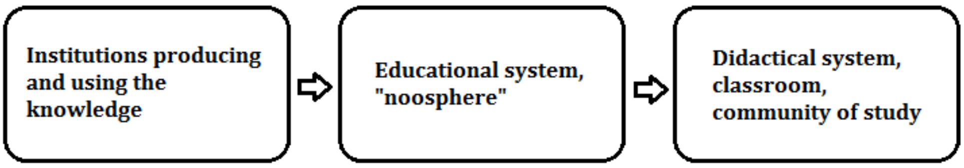

Figure 2.1.2 - Our schematic summary of the institutions related to the school context.

2.1.2 The notion of didactic transposition

Starting from the scheme in Fig. 2.1.2, we focus on the following important issue: how the knowledge relates to the whole structure. A piece of knowledge, as stated by Chevallard (1992), is a particular category of objects which can be learnt, can be taught, can be used, but first of all has to be produced. For each piece of knowledge S (that stands for *savoir*) Chevallard (1992) considers the associated institution $P(S)$ which produces S. Therefore, a certain piece of knowledge initially lives in its natural habitat that is

$P(S)$. Its presence in some other institution I presupposes that a sort of "transport" from $P(S)$ to I has occurred. Chevallard calls it *institutional transposition* of $P(S)$ in I. If we consider the school associated with I, then the process of the school reconstruction of S, starting from what $P(S)$ has produced is called *didactic transposition*. Reflecting upon it, Bosch & Gascón state that

> "[the didactic transposition] formulates the need to consider that what is being taught at school ('contents' or 'knowledge') is, in a certain way, an exogenous production, something generated outside school that is moved — 'transposed' — to school out of a social need of education and diffusion. For this purpose, it needs to go through a series of adapting transformations to be able to 'live' in the new environment that school offers [...] The process of didactic transposition then starts far away from school, in the choice of the bodies of knowledge that have to be transmitted. Then follows a clearly creative type of work — not a mere "transference", adaptation or simplification —, namely a process of de-construction and rebuilding of the different elements of the knowledge, with the aim of making it 'teachable' while keeping its power and functional character." (Bosch & Gascón, 2006, p.53)

Therefore there is "original" or "scholarly" mathematical knowledge as it is produced by mathematicians or other producers. Then, it is transformed in the knowledge "to be taught" as it is officially designed by curricula. The responsible for the first step of the transposition (from scholarly knowledge to knowledge to be taught) are the agents composing the "noosphere", which organize and disseminate the knowledge to be taught through the production of official programmes, textbooks, recommendations to teachers, didactic materials, etc. Afterwards, there is the mathematical knowledge as it is actually taught by teachers in their classrooms and the mathematical knowledge as it is actually learnt by students. Mathematical knowledge in each of these steps is subjected to a transposition, operated firstly by the noosphere, then by the teachers, and finally by the students. We can see in Fig. 2.1.3 a scheme of the process.

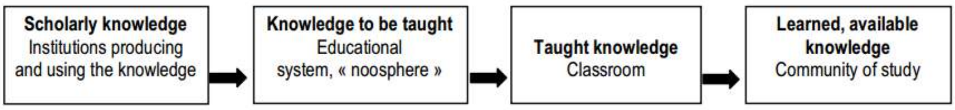

Figure 2.1.3 - The didactic transposition process (in Bosch & Gascón, 2006, p.56).

In Paragraph 1.2.2 we discussed the introduction of the derivative concept in the university mathematical courses, so at a scholar level. In order to study the didactic transposition of the concept, from the productions of the noosphere to the effective teachers' practices in classroom, we refer to the three levels of a national curriculum introduced in Mathematics Education around the 80s. More precisely, a curriculum develops on three interdependent planes:

- the *intended curriculum*;

- the *implemented curriculum*;
- the *attained curriculum*.

They were introduced by the *Second International Mathematics Study* (*SIMS*), which started at the middle of the 70s and was conducted throughout all the 80s. The intended curriculum consists of "the curricular goals and intentions that the country has for its students" (Mullis & Martin, 2007, p.11). Through official documents such as national guidelines, syllabus and textbooks, the education minister conveys the contents to be taught in classroom along with indications. The implemented curriculum pertains what is actually taught at school. The teacher has a central role in implementing the mathematics curriculum, and her beliefs, choices, decisions and approaches shape the taught knowledge in the classroom. Teachers' background and formation largely influence the implemented curriculum. Finally, the attained curriculum refers to "the mathematics that the student has learned and the attitudes that the student has acquired as a result of being taught the curriculum in school" (Mullis & Martin, 2007, p.11). The attained curriculum may be considered as the final product of the educational process. The study of these three steps was a central focus of the SIMS.
Such an investigation clearly has common features with Chevallard's study of didactic transposition. The intended curriculum contains the knowledge to be taught, the implemented curriculum involves the taught knowledge and the attained curriculum refers to the learnt knowledge.

2.2 Analysis tools

This section introduces the theoretical lenses that guide our analysis of the didactic transposition of the derivative concept. We justify the choice of them and explain how they are interrelated in our research.

2.2.1 Networking of theories as a research practice

In the research in Mathematics Education, many different theories and theoretical frameworks have been developed. To deal with the diversity of theories, at CERME4 (Congress of European Research in Mathematics Education) in 2005, the Networking Theories Group was initiated. It gathers members from France, Germany, Israel, Italy, UK, and Spain and it is coordinated by the German Bikner-Ahsbahs. They started from the shared assumption that the existence of different theories is a resource for the research in Mathematics Education. Thus, rejecting the idea of merging all into one big theory, they considered the possibilities of connecting theories. The group has grown and networking theories has become a real research practice. We make reference to the volume published by Bikner-Ahsbahs and Prediger (2014) to give the following definition.

> "By networking, we mean research practices that aim at creating a dialogue and establishing relationships between parts of theoretical approaches while

respecting the identity of the different approaches" (Bikner-Ahsbahs & Prediger, 2014, p.118).

They present a scale of networking strategies, according to the degree of integration (see Fig. 2.1.4). The extreme positions are *ignoring other theories* and *unifying globally*, which are not considered as strategies. Indeed, the former would mean that theories exist as isolated entities, without the possibility of learning from each other; whereas the latter would lead to have a whole huge theory, in contrast with the aim of maintaining the peculiarities of each component theory.

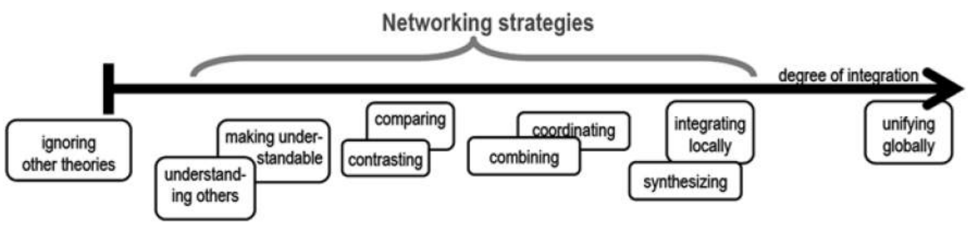

Figure 2.1.4 - A landscape of strategies for connecting theoretical approaches (Bikner-Ahsbahs & Prediger, 2014, p.119).

Thus, networking strategies are defined as

"[...] the connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline" (Bikner-Ahsbahs & Prediger, 2014, p.119).

Among all possible networking strategies, we want to stress those that *combine* and *coordinate* theoretical approaches for a networked understanding of an empirical phenomenon or a piece of data. Combining and coordinating means looking at the same phenomenon from different theoretical perspectives as a way for going in-depth the phenomenon under analysis. In particular, the strategy of coordinating is used "when a conceptual framework is built by fitting together elements from different theories for making sense of an empirical phenomenon" (Bikner-Ahsbahs & Prediger, 2014, p.120). Many different methods are proposed by the Networking Theories Group for supporting processes of networking. Among them, there is the method of *parallel analysis* of the same piece of data through different theoretical lenses.

Drawing on the research practice of networking of theories, we adopt the coordinating strategy to network three elements coming from three different theories.
First of all, since we are interested in studying the practices that involve the derivative concept, we refer to a fundamental notion of ATD: that of **praxeology** (Chevallard, 1999). This theoretical tool allows us to detect the involved types of task, the techniques to solve them and the related justifications, together with the theory elements that

support the justifying arguments. Nonetheless, it provides us with a quite static and general picture of the employed practices. Perhaps, also if we consider different textbooks, examination tasks or curricula regulations, the picture we get of the adopted praxeologies can be very similar. In other words, it does not help us accounting for more deep differences in the dynamics underlying the praxeologies. How are the **perspectives** (Vandebrouck, 2011) activated? How are the **semiotic resources** (Arzarello, 2006) employed? We integrate these latter lenses in our data analysis, using parallel analysis as a networking method. In the next paragraph, we present each of these theoretical tools and justify their coordination. Starting from some fundamental assumptions in the study of our research problem, the integration of the perspectives and the semiotic bundle and their coordination with the praxeology construct give us the possibility to identify and describe the dynamics of practices we intend to study in the Algebra/Calculus transition.

2.2.2 Chevallard's notion of praxeology and model of didactic moments

Chevallard (1999) introduces and develops the notion of praxeological organization, or more briefly **praxeology**, in his theory. At the basis of the ATD there is the assumption that

> "*every* human activity which is regularly accomplished can be subsumed under a *unique* model, which the word *praxeology* here summarizes"[2] (Chevallard, 1999, p.223).

A praxeology is constructed and depends strongly on the notions of *task* and *type of task*. Given the type of task $\mathcal{T}$, a $\mathcal{T}$-related praxeology firstly specifies a *technique* τ, that is a way to accomplish the tasks t belonging to $\mathcal{T}$. Thus, a $\mathcal{T}$-related praxeology firstly contains a practical-technical block, denoted by $[\mathcal{T}/\tau]$. Within a given institution, for any type of task, the related technique is always accompanied by at least an embryonic form of a rational speech about itself. This rational speech, named *technology* and denoted by θ, is aimed to "rationally" justify the technique, that is to ensure that the technique really gives us what we want to find. Nevertheless, a technology θ for the technique τ has also another function that is to explain, to make intelligible, to clarify τ. A third possible function of a technology θ consists of producing techniques for the type of task $\mathcal{T}$. Finally, the technological speech, in turn, contains some statements we can ask reason of. Thus, we have a higher level of justification-explication-production related to technology that is called *theory* and denoted by Θ. Therefore, within a praxeological organization, besides the practical-technical block $[\mathcal{T}/\tau]$, a technological-theoretical one $[\theta/\Theta]$ takes shape. A $\mathcal{T}$-related praxeology is then composed of four elements $[\mathcal{T}/\tau/\theta/\Theta]$. Moreover, Chevallard (1999) classifies mathematical praxeologies into a sequence of increasing complexity. Built up around a single kind of problem, we have pointwise praxeologies, which can be successively gather according to their theoretical background to give rise to local, regional or global praxeologies that cover respectively a whole mathematical theme, a sector or a domain. In this thesis, we will refer mainly to pointwise

[2]Our English translation of the passage: "*toute* activité humaine régulièrement accomplie peut être subsumée sous un modèle *unique*, que résume ici le mot de *praxéologie*".

and at most local praxeologies. Furthermore, following Chevallard (1999), we distinguish between two different levels of praxeology: the **mathematical praxeology** and the **didactic praxeology**. He writes:

> "Given a mathematical theme of study θ, we are going to consider in sequence a) *the mathematical reality* that can be built in a mathematical classroom where the theme θ is studied, b) *the way* in which this mathematical reality can be built, that is the way in which the study of the theme θ is accomplished. The first object - 'the mathematical reality that...' - is nothing else but a *mathematical praxeology* or *mathematical organization*, which we will denote OM_θ. The second object - 'the way in which...' - is what we will call a *didactic organization*, which we will denote in a similar way OD_θ."[3] (Chevallard, 1999, p.232)

Mathematical praxeologies

A type of task T is generally expressed through a verb and a particular content, for example "*expanding* the given literal expression" or "*calculating* the value of a function in a given point"[4]. A task t is a type of task which has been specified in all its parts, for example "expanding the literal expression $(2x+1)(2x-1)+(x-1)^2$" or "calculating the value of the function $f(x)=x^2-x+1$ in the point $x_0=4$". To make an example, let us consider the praxeology for this latter type of task.

Task	Calculating the value of the function $f(x)=x^2-x+1$ in the point $x_0=4$.
Technique	Substitute x with 4 in the analytic expression of f, obtaining $f(4)=4^2-4+1=13$.
Technology	The point (x_0, y_0) lies on the curve of equation $f(x)$ iff $f(x_0)=y_0$.
Theory	Properties of the algebraic curves.

The mathematical praxeologies are in continuous evolution. As Chevallard claims: "The institutions are walked through by a praxeological *dynamics* [...] The praxeologies indeed get old: their theoretical and technological components lose their reliability and become opaque, whereas new technologies emerge."[5] (Chevallard, 1999, p.230). A

[3]Our English translation of the passage: "Étant donné un thème d'étude mathématique θ, on considérera successivement a) *la réalité mathématique* qui peut se construire dans une classe de mathématiques où l'on étudie le thème θ, b) *la manière* dont peut se construire cette réalité mathématique, c'est-à-dire la manière dont peut s'y réaliser l'étude du thème θ. Le premier objet - 'la réalité mathématique qui...' - n'est rien d'autre qu'une *praxéologie mathématique* ou *organisation mathématique*, qu'on notera OM_θ. Le second objet - 'la manière dont...' - est ce qu'on nommera une *organisation didactique*, qu'on notera, de manière analogue, OD_θ".

[4]Chevallard also makes examples that fall not only within the mathematical context, but here we exclusively deal with tasks of mathematical nature.

[5]Our English translation of the passage: "Les institutions sont parcourues par toute une *dynamique* praxéologique [...] Les praxéologies, en fait, vieillissent: leurs composants théoriques et technologiques perdent de leur crédit et deviennent opaques, tandis que des technologies nouvelles émergent."

praxeological dynamics can be internal or external to a praxeology. On the one hand, it can consist of an internal dialectics between the practical-technical block and the technological-theoretical one. It occurs, for instance, when a technique is constructed step by step thanks to frequent abductions to theory. On the other hand, a praxeology can enter another developing one, contributing to its construction or improvement: this is an external praxeological dynamics. To make a simple example, in all textbooks of grade 9-10 within the parabola study we find the formula $V\left(-\frac{b}{2a},-\frac{\Delta}{4a}\right)$ for the vertex. The students become used to automatically apply it. However, with an increasing work on functions, in grade 11, a high-level common mathematical praxeology is fostered by teachers in classroom. It exploits the previously shown praxeology related to the type of task "calculating the value of a function in a given point".

Task	Determining the coordinates of the vertex of the parabola $y = x^2 - x + 1$.
Technique	Calculate the abscissa of the vertex, that is $x_V = -\frac{b}{2a}$, obtaining $x_V = 1/2$. Then, substitute x with x_V in the analytic expression of the parabola, obtaining $y_V = (1/2)^2 - 1/2 + 1 = 3/4$.
Technology	The vertex is a point of the parabola curve.
Theory	The point (x_0, y_0) lies on the curve of equation $f(x)$ iff $f(x_0) = y_0$. Properties of the algebraic curves.

The praxeology to find the vertex of a parabola has then evolved thanks to the embedding of a praxeology coming from the theory of algebraic curves. The new technique leads the students to conceptualize the parabola as an algebraic curve. So it makes the level of the praxeology higher with respect to a blind application of the formula $V\left(-\frac{b}{2a},-\frac{\Delta}{4a}\right)$ and it makes work better the concept of function.

Within our research we take into account two types of task involving the derivative notion.

Type of task $T_{tangent}$	"determining the equation of the tangent line to a generic function in a point"
Type of task $T_{f'}$	"representing the derivative function"

Indeed, these are the most worked types of task in the Italian secondary school for introducing the derivative concept, as the gradient of the tangent line, and the derivative function. Moreover, the construction of a praxeology for these types of task seem to us a suitable context to trigger the activation of a local perspective.

Didactic praxeologies

One of the main didactic tasks of a maths teacher is to transpose to students the mathematical praxeology for each type of task. In a maths course, when a teacher thinks about the way to teach a specific notion, a key question arises: "How to approach the issue τ_T?", where τ_T is the technique or set of techniques to accomplish the type of task T related to the given notion. The didactic praxeologies or didactic organizations OD are the teacher's possible answers to this kind of questions.
It is to study the didactic praxeologies that Chevallard has introduced the **model of didactic moments** (Chevallard, 1999). At a macro level, this model permits to observe how the teacher structures the lessons around a particular notion and a particular type of task. Chevallard identifies six didactic moments in the teacher's intervention in the classroom. For practical reasons we will list them here, but it must be stressed that they do not occur in a particular order and, generally, they need more than an episode to be accomplished.

- The moment of the first meeting with the type of task T must be read not necessarily in a chronological sense but better as a significant meeting that really poses T as a problematical type of task.

- The moment of exploration of the type of task T entails the construction of a technique τ (at least in embryonic form) which allows to accomplish T.

- The technological-theoretical moment aims to the realization of the technological-theoretical environment related to the praxis $[T/\tau]$.

- The moment of practising the mathematical praxeology is a moment for adapting and perfecting it.

- The institutionalization moment consists in a work of synthesis and formulation of the mathematical praxeology, and also in a work of amalgamation of it with those previously studied.

- The assessment moment occurs on two levels: (i) the evaluation of the mastery acquired with the mathematical praxeology, and (ii) the evaluation of the validity of the praxeology itself.

During the process of construction of a new technique, with a proper technology, for a specific given task, the old techniques, with their own technologies, emerge and take part in the formulation of the new practice. In other words, it starts up an interesting dynamics of old and new praxeologies which can belong to different mathematical domains. In our specific case, new praxeologies have to be constructed within the Calculus domain and old praxeologies, previously acquired in Geometry or Algebra, resurface and play a fundamental role in the construction process, both at technical and technological level. The common feature of old involved praxeologies lies in the fact that all of them present algebraic techniques to accomplish the related task. On the contrary, the techniques referring to the fresh developing praxeology are proper to Calculus (such as limits). So, our focus is on this dynamics between old and new praxeologies, in order to describe how the old Algebra-related praxeologies intervene in the construction of the new Calculus-related ones. In this sense, Chevallard's frame provides us with a complete, though static, picture of the situation and with names to properly call the different actors on the stage. Nevertheless, our research question itself has a dynamical nature. We intend to describe *how* a certain dynamics develops. To do that, we need other theoretical lenses which can help us to find out what prepares the ground for the setting of a new technique and, more generally, what determines the evolution in time of mathematical praxeologies.

2.2.3 Activated perspectives on the involved functions

At a finer level of analysis, we focus on the different perspectives - pointwise, global and local - which are activated on functions by the proposed mathematical tasks and the related developed praxeologies. In particular, with the aim of investigating the praxeological dynamics, here is our first hypothesis:

> HP1: *A change in perspectives marks the evolution of a praxeology.*

For example, the type of task $\mathcal{T}_{tangent}$ has usually already been worked from a pointwise perspective or a global one with the determination of the tangent to a conic. In that occasion, algebraic techniques have been exploited, such as the $\Delta = 0$ technique or the so-called "doubling rule". With a generic function f, these old techniques, together with the involved perspectives, are not successful anymore. The praxeological evolution occurs thanks to a change towards a local perspective on f. As far as the type of task $\mathcal{T}_{f'}$ is concerned, the pointwise technique to find $f'(x_0)$, which involves the limit of the incremental ratio, must be reworked in a global technique to determine $f'(x)$. Also in this case, the praxeological dynamics is caused by a shift in perspectives on both the functions f and f'.
Using the perspectives as an analysis tool, we are interested in detecting their activation in the work done on functions.
More precisely, we identify a **pointwise perspective** when the work on a function is centred on a specific point or a finite set of points. We recognise a **global perspective** when the interest is on a specific interval or on the whole domain of the function (even when the domain is not explicit). In particular, we also speak of global perspective when a pointwise property is considered for a generic value of x belonging to an interval or to the domain (**universal pointwise perspective**). Finally, we recognise a **local perspective** when the focus is on "what the function does" in a neighbourhood of the point, without any specification of the extremes of the considered interval.
Identifying a perspective entails first of all to clarify on which object the perspective is activated. Indeed, it has to be pointed out that a certain perspective is always activated with respect to a particular function. To make an example, the claim "$f'(2) = 3$" may be read as a pointwise information about f' in $x = 2$, but also as a local property about

the slope of f in $x = 2$.
In addition, this example shows us that the way in which a property is stressed is extremely important in order to grasp the activated perspective on a specific function. Indeed, it is certainly necessary to identify that a certain property of f or f' is claimed, but also it is essential to analyse the way in which such a claim is made. Sometimes indeed we grasp the real perspective from which a certain proposition is made only if we observe carefully how it is formulated. For instance, some local properties such as "f is discontinuous in x_0" often reveals the pointwise corresponding image that the function has a hole in x_0. On the contrary, sometimes a perspective is not formulated in an uttered claim, but it is implicitly conveyed by a drawing or a gesture. For example, if one claims "$f'(2) = 0$" we are led to suppose that he is saying something pointwise on f'. However, if at the same time he moves his hand horizontally on the graph of f, we can interpret his claim as a local consideration on f.
Therefore, for our analysis, it is extremely important not only what is said but also *how* it is said. For this reason, we introduce our third lens: the semiotic bundle that can give us information about the activated perspectives. This is our second hypothesis:

> HP2: *A change in the used semiotic resource triggers a praxeological evolution.*

It happens, for instance, when a graphical technique is converted into symbols and the resulting praxeology evolves in a more abstract one.

2.2.4 Employed semiotic resources

Another lens to investigate the dynamics within and between mathematical praxeologies is the semiotic analysis. We base it on the concept of **semiotic bundle**, introduced by Arzarello (2006) to study the relationships among the semiotic resources that are involved in a mathematical activity. Arzarello starts from the definition of semiotic system given by Ernest (2006). It establishes that a semiotic system is the composition of:

- a set of signs (uttered, spoken, written, drawn or electronically encoded);
- a set of rules for producing and transforming signs;
- a set of relationships between signs and their meanings.

A semiotic system has important semiotic functions, e.g., transformational and symbolic (see Arzarello *et al.* 1994). The transformational function consists of "the possibility of transforming signs within a fixed system or from a system to another, according to precise rules" (Arzarello, 2006, p.272). Treatment and conversion within and between semiotic registers of representation, introduced by Duval (1993), are transformations of this kind. The symbolic function instead refers to "the possibility of interpreting a sign within a register, possibly in different ways, but without any material treatment or conversion on it" (Arzarello, 2006, p.272).

Arzarello (2006) widens Ernest's definition for two reasons:

(i) to encompass the variety of semiotic resources used by students and teachers, such as gestures, glances, drawings and extra-linguistic modes of expressions;

(ii) to study the relationship within and between the registers simultaneously active and their dynamics, since their activation is multimodal.

Therefore, he defines a semiotic set as composed of three elements:

- a set of signs (in the sense of Peirce[6]) produced with different intentional actions;
- a set of modes for producing and transforming signs;
- a set of relationships among signs and respective meanings.

A semiotic bundle or bundle of semiotic sets, in turn, is made up of

- a collection of semiotic sets;
- a set of relationships between them.

According to Arzarello, "speech, gestures and written representations (from sketches and diagrams to mathematical symbols)" are examples of "three different types of semiotic sets" and all together, with the relationships among them, "constitute a semiotic bundle, which dynamically evolves in time" (Arzarello, 2006, p.284). With regards to mathematics learning, he proposes two different kinds of analysis:

> "The first one is *synchronic analysis*, which studies relationships among different semiotic sets activated simultaneously by the subject. The second is *diachronic analysis*, which studies the relationships among semiotic sets activated by the subject in successive moments" (Arzarello, 2006, p.287).

The semiotic bundle and the perspectives

Different semiotic resources can be activated while working on functions: speech, gestures and written speech, symbols, sketches or drawings. They can exploit different registers of representation (algebraic, symbolic, graphical, etc.) on functions and reveal or hide a particular perspective on them. It is important to stress that a semiotic bundle is not a juxtaposition of the composing semiotic sets. On the contrary, we distinguish the components only for sake of analysis, but actually the semiotic sets are deeply intertwined to form a unitary system. Arzarello makes the example of the unity speech-gesture as semiotic bundle and recalls McNeill: "we should regard the gesture and the spoken utterance as different sides of a single underlying mental process" (McNeill, 1992, p.1).
When speech and gestures concur to underline the same perspective on a function, this

[6]In a letter to Lady Welby, Peirce defined a sign as follows. "I define a Sign as anything which is so determined by something else, called its Object, and so determines an effect upon a person, which effect I call its Interpretant, that the latter is thereby mediately determined by the former" (Peirce, 1977, pp.80-81; *Letter to Lady Welby*, 1908).

unity may enhance such a perspective and foster its adoption. However, it is also possible that two or more different semiotic resources simultaneously active highlight different perspectives on a function. For instance, while the local claim "f is discontinuous in x_0" is uttered, a pointwise gesture is made on the graph of f in the point x_0. In this case, the semiotic bundle composed of speech, graph and gesture presents some conflicting features with respect to the adopted perspective. It may interfere with the intention of conveying a local perspective on the function.
In regards to this relation between semiotic resources and perspectives, we want to remark that sometimes some semiotic bundles are suitable, more than others, to disclose a certain perspective on functions. For example, accompanying global remarks on a function with its graph may enhance the global perspective on it. Furthermore, recalling the categorization of gestures in McNeill (1992), a pointing gesture might convey a pointwise perspective, whereas a continuous iconic gesture, for example along the graph, may prompt to a global perspective on it.
Therefore, in our semiotic analysis it seems extremely important to us to investigate how the perspectives network with the semiotic bundle. To this purpose, two points appear very useful to detect and to discuss within the analysis:

- which semiotic resources are combined together to convey a specific perspective (in particular, the local one);

- the concordance/discordance of the perspectives conveyed by the different resources composing the semiotic bundle.

2.3 Focus on the teacher

With the interest in studying the didactic transposition of the derivative concept in the secondary school, at the core of our analysis are the teachers' praxeologies in classroom. This is the second step of the didactic transposition process: from the mathematics to be taught to the taught mathematics. To better understand the didactic choices, techniques and justifications that form the didactic praxeology of a teacher, we have to take into account the influence of the first work of transposition made by the noosphere: from the scholarly mathematics to the mathematics to be taught. Indeed, all the materials provided by the minister of education, educational programmes, and other noosphere agents, represent institutional constraints for the teacher who designs her lessons and implements them in her classroom. She has to conform her goals to those of the national curriculum, she can avail of the textbook as a supporting resource, and, especially at the last year of upper secondary school (grade 13), one of her main concerns is to prepare students for the final national assessment. All these constraints, along with the experience of teaching she has, her knowledge of the classroom and the specific school context, generate in the teacher certain beliefs. Such orientations might strongly influence her praxeologies.

2.3.1 The influence of the beliefs on teachers' praxeologies

With the purpose of describing teachers' praxeologies, we have to take into account the influence of teachers' practical knowledge, that is teachers' knowledge about their work. A description of teachers' practical knowledge is given by Elbaz (1983). It includes experiential knowledge on the field about the students' learning approaches, interests, needs, strengths and difficulties, about teaching techniques and classroom management, about the social context of the school. It also encompasses teachers' theoretical knowledge of the subject and of pedagogical issues related to an individual's development and learning. A teacher elaborates her practical knowledge in terms of personal value and beliefs. Many researches in Mathematics Education show that the teaching of mathematics is affected in a significant way by the belief system and knowledge (e.g., Ernest, 1989; Schoenfeld, 2011; Furinghetti & Morselli, 2011). According to Ernest (1989), beliefs are one of the most influential elements on mathematics teachers' practice, along with the social context in which the teaching takes place and the reflection on the teaching-learning process. Schoenfeld (2011) bases his theory of decision making on three components: resources (including knowledge), orientations (or beliefs) and goals. He sees them as the elements whose interaction and combination determine the decision-making of a teacher, especially when she has to cope with an unexpected episode in the lesson development. Furinghetti and Morselli's study on the teaching of proof (Furinghetti & Morselli, 2011) has focused in particular on teachers' beliefs, pointing out that they can be internally oriented (themselves as persons, as learners, as teachers) or externally oriented (the nature of mathematics, the nature of mathematics teaching and learning). Furthermore, they stress the important question of *inconsistencies* between beliefs and instructional practices, which has been studied in literature. Following Furinghetti and Morselli (2011), to unravel the problem of inconsistencies, we will consider *leading beliefs*, that are beliefs that seem to drive the way the teacher deals with a particular mathematical object/process.

Concerned with the teachers' practical knowledge, several teacher-centred studies, namely researches on teachers' education and professional development, have focused on the identification of the knowledge that is necessary to handle for teaching of mathematics. One of the first researches in this direction was conducted by Shulman (1986). He detected three content-specific dimensions that characterized "the knowledge that grows in the minds of teachers" (Shulman, 1986, p.9).

(a) The *subject matter content knowledge* includes knowledge of the subject and its organizing structures.

(b) The *curricular knowledge* is the knowledge of all the curriculum material designed for the teaching of particular subjects and topics at a given level, along with the related indications and contraindications.

(c) The *pedagogical content knowledge* consists of the knowledge of "the most useful ways of representing and formulating the subject that make it comprehensible to others. [...] Pedagogical content knowledge also includes an understanding of what

> makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Shulman, 1986, p.9)

Starting from Shulman's studies, Ball and Bass (2003) have elaborated a practice-based theory of *Mathematical Knowledge for Teaching* (MKT), which has been defined as the "mathematical knowledge needed to carry out the work of teaching mathematics" (Ball, Thames & Phelps, 2008, p.395). A refinement to Shulman's categories has been proposed (see Fig. 2.3.1).

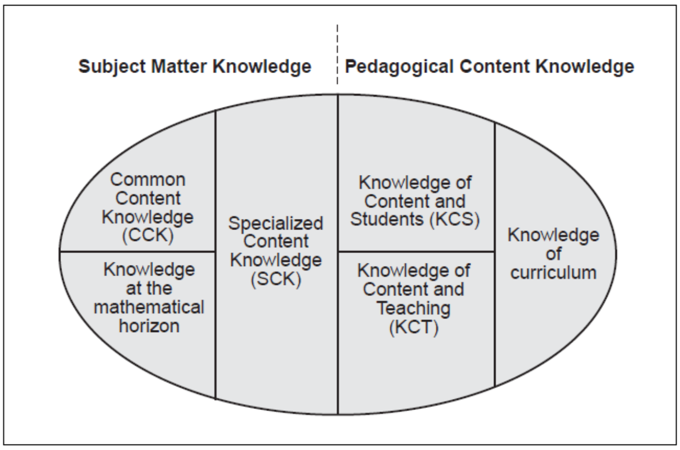

FIGURE 2.3.1 - SCHEME OF THE MKT MODEL (IN HILL, BALL & SCHILLING, 2008, P.377).

The subject matter knowledge includes the *common content knowledge* (CCK), the *specialized content knowledge* (SCK) and the *knowledge at the mathematical horizon*. CCK is the knowledge teachers need in order "to be able to do the work that they assign their students" (Ball, Thames & Phelps, 2008, p.399), as any other mathematicians who know that particular content. SCK represents "the mathematical knowledge that allows teachers to engage in particular teaching tasks", such as knowing how to "accurately provide mathematical explanations for common rules and procedures" (Hill, Ball & Schilling, 2008, p.377). Finally, what we can call *horizon content knowledge* concerns the knowledge of the relationships between the different topics of the curriculum.
The pedagogical content knowledge instead is subdivided into the *knowledge of content and students* (KCS), the *knowledge of content and teaching* (KCT) and the knowledge of curriculum, where KCS is "focused on teachers' understandings of how students learn particular content" (Hill, Ball & Schilling, 2008, p.378), and KCT "combines knowing about teaching and knowing about mathematics" (Ball, Thames & Phelps, 2008, p.401). The lines between the sub-domains of MKT can be subtle. For instance,

> "[...] recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns,

and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS)." (Ball, Thames & Phelps, 2008, p.401).

This model has been used by many researchers in the last decade. The Spanish interpretation operated by the SIDM group (from the Spanish "Research Seminar into Mathematics Educations") is particularly interesting. They center their study on significant knowledge only for mathematics teachers. Thus, starting from Ball's and colleagues' research they have worked out the *Mathematics Teacher's Specialized Knowledge* (MTSK) model (Carreño *et al.*, 2013; Flores *et al.*, 2014). One of its peculiarities seems to us the re-organization of the structure of the MKT around the mathematics teacher's belief system (see Fig. 2.3.2).

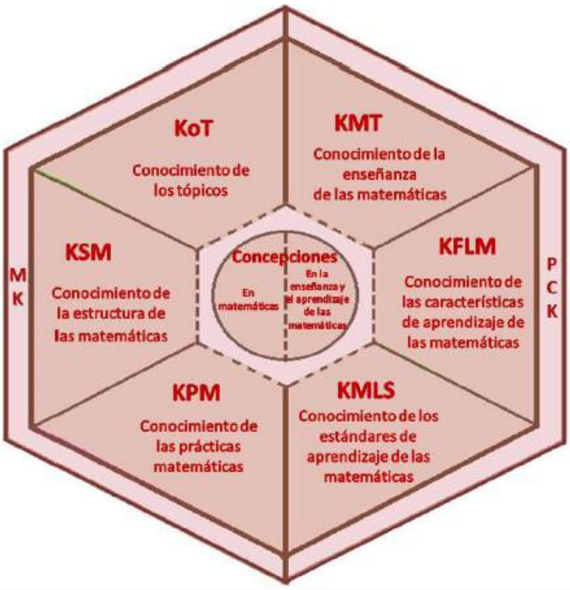

Figure 2.3.2 - Scheme of the MTST model (in Flores *et al.*, 2014).

Ball's team model and its adaptations in Mathematics Education research have the quality of being a finer and effective characterization of the mathematical knowledge needed to teach. However, their main limit lies in the fact that they cannot give reason of the dynamics that necessarily arise when the community of the teachers and the community of researchers meet for example in a teachers' education program. Arzarello and colleagues (Aldon *et al.*, 2013; Arzarello *et al.*, 2014) highlight that the teachers' professional development is described in literature

"[...] in terms of communities of practice, communities of inquiry, adaptive systems, collective participation, sustained conversation and egalitarian dialogue. The cornerstone of these studies is the notion of critical reflection,

> conceived not only as a fundamental attitude to be instilled in teachers but also as a professional responsibility (Arzarello *et al.*, 2014, p.350).

When the focus of attention is centred on the teachers, complexity and dynamism emerge. To deal with them, French and Italian researchers (Aldon *et al.*, 2013; Arzarello *et al.* 2014) distance themselves from the quite static MKT model and propose a theoretical model grounded on Chevallard's notion of didactic transposition occurring at a meta level.

2.3.2 The meta-didactic transposition process

Arzarello and colleagues construct "a descriptive and interpretative model, which considers some of the main variables in teacher education (the community of teachers, the researchers, the role of the institutions), and accounts for their mutual relationships and evolution over time" (Arzarello *et al.*, 2014, p.348). They call the overall resulting process *meta-didactic transposition*, and so also the model takes the same name: Meta-Didactic Transposition model (in short, MDT).
MDT model takes into account the practices of mathematics researchers and those of mathematics teachers, when the respective communities come into contact. It is based on Chevallard's Anthropological Theory of the Didactic (Chevallard 1985, 1992, 1999; Bosch & Gascón 2006), but with the intention to adapt and extend ATD to the context of teacher education.
The model accounts for the complex dynamics which characterize the activities involving the community of researchers and that of teachers. Moreover, it considers the institutional constraints imposed on both the communities (e.g., ministerial goals for teachers' education programmes, intended/implemented/attained curricula and textbooks).
Thus, five variables are intertwined in determining the meta-didactic transposition process: the institutional aspects, the meta-didactic praxeologies, the double dialectics, the brokering processes and the dynamics between internal and external components.

The teachers' and researchers' communities involved in the MDT process, are subjects within a certain institution. Teachers belong to the actual schools where they teach, and researchers refer to the School as a higher institution: the noosphere that decides curricula, has particular teaching traditions, produces textbooks, and so on. Indeed, when the researchers in Mathematics Education come into contact with the teachers' community hold simultaneously two different positions. They belong to the university or the department where they work, but in that particular occasion they act as teachers' educators. MDT model considers meta-didactic praxeologies, which consist of the tasks, techniques, and justifying discourses that develop during the process of teacher education. For instance, the task can be stimulating the teachers' reflection, and the relative technique can be the collective discussion. In so doing, it is possible that the two communities of educators and teachers come to share a common theoretical framework that justifies the mathematical tasks, techniques and argumentation under scrutiny. The discussion is about the didactic praxeologies that different teachers can adopt dealing with a specific mathematical problem. For this reason, the described praxeology occurring at a meta level is called a meta-didactic praxeology. As Arzarello and colleagues observe,

> "[A meta-didactic praxeology is] the result of the interaction between the reflections of the community of researchers about the didactic praxeologies previously designed and developed, and the concrete practices used by the teachers in their professional activities" (Arzarello *et al.*, 2014, p.354).

At the beginning of a meta-didactic transposition process two distinguished communities are identifiable: that of researchers (in the role of designers and educators), and that of the teachers (in the role of teachers-students). The researchers and the teachers have their own praxeologies, that a priori can be very different. The MDT process aims to develop teachers' existing praxeologies towards new ones, which consist of a blending of the two initial praxeologies: a shared praxeology. The latter can have effect also on the initial praxeologies of the two communities: the teachers can return back to their classroom with new teachers' praxeologies and the researchers can come back to think about the training phases and redesign them, developing new researchers' praxeologies. Figure 2.3.1 schematically illustrates the actors and the dynamics in the MDT process.

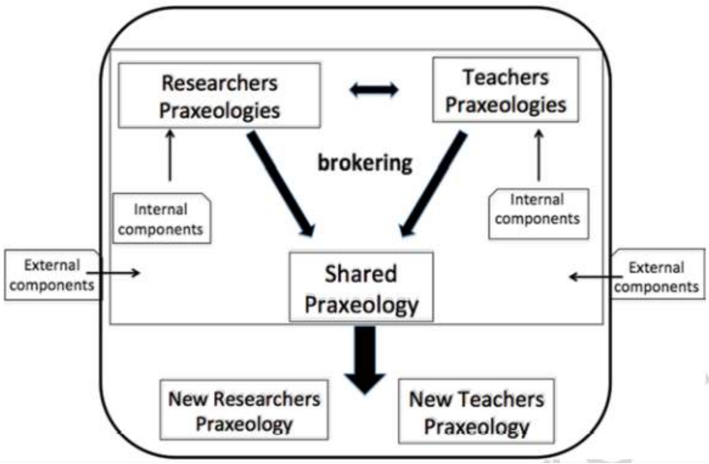

Figure 2.4.1 - The Meta-didactical Transposition model (in Arzarello *et al.*, 2014., p.355)

An important result of the MDT process is that some of the components characterizing the two communities' praxeologies change their status. Typically, some components which are internal to the researchers' praxeologies enter the teachers' praxeologies, and from external for the latter they become gradually internal. It occurs that the researchers' praxeologies change as well, as a result of the interaction with the community of teachers. It is also possible that the educational process makes some components, which are initially external to both the communities, become internal (for instance, under new regulations for the curriculum or for the final assessment).
The MDT process is often facilitated by the mediation of the *brokers*: they are subjects that belong to both the communities, e.g., the teacher-researchers.

Moreover, the success of the process relies in the so-called *double dialectic*. The first dialectic happens at the didactic level in the classroom, while the second dialectic is at the meta-didactic level. The former is about the mathematical meaning that the students have constructed in the classroom. The latter is given by the interpretation that the teacher gives to the first dialectic, as an effect of her praxeology in the classroom, and the interpretation of the first dialectic according to the researchers' community, who help the teachers in reflecting upon it.

In this thesis we focus on the teacher's role within the process of didactic transposition of the derivative concept in the classroom. A certain interaction between the researchers and the teachers has occurred. It has been essentially made of a posteriori meetings in which the teachers reflected upon what happened in classroom, asking the help of the researchers who observed the lessons and analysed them. Even though it was not one of the aims of the thesis project, during the reflection on the praxeologies observed in the classroom, the theoretical framework, initially internal only to the community of the researchers, has been partially shared with some of the involved teachers. To describe such an interaction, we will make reference to the MDT model, naturally grounded on the ATD theory.

2.4 Research questions and overall methodology

Let us retrace our research problem. Our starting point was to study the articulation of Algebra and Calculus in the secondary school. We began with an epistemological analysis to understand in what the two domains differ in terms of mathematical work. We finally came to express the Algebra/Calculus transition as a change in the perspectives activated on functions. In particular, we identified the introduction of a local perspective on functions as one of the main features that distinguish Calculus practices from the algebraic ones. Contextualizing our research in the secondary school, where such a transition began to be studied, we formulated our research problem (see Section 1.4). We recall it here:

What role is given to the local perspective when the Algebra/Calculus transition occurs in the secondary school?

In particular, we have chosen to investigate the transition through a specific mathematical theme which lives between the two domains: the derivative. We have presented the different "scholar" approaches to the derivative concept and discussed the conceptions and the obstacles that the research in Mathematics Education has pointed out so far in relation to this concept. The notion of didactic transposition, introduced in this chapter, allows us to specify the generic phrase "when the Algebra/Calculus transition occurs in the secondary school". More precisely, indeed, we are interested in the **didactic transposition of the derivative concept in the secondary school**, as a particular transposition phase of the Algebra/Calculus transition. We intend as didactic transposition both the process that transposes the knowledge produced at a certain level

onto another one and the product of this process. Usually, we try to deduce the process from the product it has worked out.
Moreover, the tools constituting our theoretical framework, presented in Section 2.2, enable us to reformulate the research problem into the following overall research question:

(RQ) How does the local perspective intervene in the development of derivative-related praxeologies in the secondary school?

In particular, the derivative-related praxeologies we are interested in are referred to two specific types of task:

$T_{tangent}$:	determining the equation of the tangent line to a generic function in a point;
$T_{f'}$:	representing the derivative function.

These two types of task involve the development of Calculus praxeologies. Thus, we expect that they provide us with suitable contexts in which observing the activation of a local perspective. Therefore, to answer our main research question (***RQ***) we pose these first sub-question:

(RQ.1) What role is given to the local perspective on functions in the secondary teaching of the derivative?

This first research sub-question will guide our analysis of the didactic transposition of the derivative concept in

- the regulations of the national curriculum;
- the textbooks theory and expected resolution of exercises;
- the expected resolution of the final examination problems.

The analysis of such institutional materials will provide us with an overlook of the intended curriculum teachers have to refer to for treating the derivative notion.
Then, as we have stressed in Section 2.3, our main concern is the second step of the didactic transposition: from the knowledge to be taught to the taught knowledge by the teachers in their classrooms. Through the analysis of three case studies of three teachers we will try to answer the following sub-questions:

(RQ.2) How do teachers construct the derivative-related praxeologies with and for their students?

(RQ.1+2) What role do teachers give to the local perspective on functions in the construction of such derivative-related praxeologies?

To answer these latter sub-questions, the semiotic analysis appears to us extremely important, because teachers have, in addition to the textbooks, the possibility to use simultaneously and dynamically different semiotic resources to work on functions.

It is our intention also to have insight on the attained curriculum, even though we are aware that this will be just a narrow window on the effects of the teachers' praxeologies have had on students. In the analysis of the activities[7] specifically designed and proposed to the students, we ask:

(RQ.3) In which ways different praxeologies developed in classroom can affect the students' praxeologies, in terms of local perspective?

[7]In this dissertation, as we will further specify in Chapter 5, the term "activity" denotes a problem or a set of problems that the students have to solve, by working alone or in team, for constructing or consolidating the meaning of the involved mathematical objects.

Part II

Analysis

Chapter 3

Analysis of the intended curriculum

The intended curriculum is constituted by all the institutional material produced by the noosphere: official programmes, textbooks, recommendations to teachers, didactic materials, etc. In particular, we focus firstly on the Italian National guidelines of 2010 for scientific high schools, secondly on the two most adopted textbooks in Piedmontese scientific high schools, and thirdly on the Italian final examination given to experimental courses of scientific high schools in June 2013.
We address to the scientific upper secondary instruction because our research questions, in particular the intention to examine if the local perspective is present and at what extent, can be framed in a broader meta-question: are the students well-prepared to face Analysis concepts as they are treated within the scholarly mathematics at university? Supposing that the most part of students who decide to attend university courses of mathematics comes from a scientific upper secondary instruction, we chose to concentrate on national guidelines, textbooks and examination designed and devoted to scientific high schools only.

3.1 The derivative in the National guidelines

In order to analyse the institutional regulations, we refer to the *Indicazioni nazionali degli obiettivi specifici di apprendimento per i licei* (National guidelines of the specific learning objectives for high schools, issued by the Italian Ministerial Decree n.211 of October 7, 2010) which represents part of the Italian secondary curricula. In particular, we focus our attention on the guidelines specifically devoted to scientific high schools (Attachment F to the M.D.211), in the section "Mathematics" (pp.337-341). Notice that the contents are not presented in a schematic way, grade by grade. Instead, all the topics to cover are merged in a discursive text, divided in three subgroups: grade 9-10, grade 11-12 and grade 13. A brief general introduction opens the section making explicit the overall competences and objectives. Some concepts and methods are gathered together and it seems relevant to us that one of these groups is constituted by

> "the elements of algebraic calculus, the elements of the Cartesian analytic

> geometry, a good knowledge of the elementary functions of Calculus, the elementary notions of the differential and integral calculus" (p.337).

Algebraic calculus and analytic geometry belong to the same group of concepts and methods as the study of functions and the differential and integral calculus.
The introduction is closed by the main general guideline: "Few concepts and fundamental methods, acquired in depth" (p.338).
The foreword is followed by the specific learning goals. Within them, let us focus on the indications concerning the derivative concept. Thus, we refer to grade 11-12 and grade 13.

In grade 11-12, within the theme "Relations and functions" we read

> "An important theme of study will be the *concept of speed of variation* for a process represented by a function" (our underlining, p.340).

The suggestion is then to work on the speed of variation as preliminary to the concept of derivative. In particular, this is a physical approach to the cognitive root of the "slope" of a function in a point. This is because, in a graph position-time representing a physical phenomenon, the speed with which the process varies in an interval of time Δt corresponds to the slope of the segment matching the points $(t_0, s(t_0))$ and $(t_0 + \Delta t, s(t_0 + \Delta t))$. Anyway, in the curriculum there is no further indication, so the teachers are free to approach the concept of speed of variation as best they think.

In grade 13, within the same theme "Relations and functions" the derivative holds a central position along with the main concepts of Calculus.

> "The student will acquire the main concepts of infinitesimal calculus – in particular continuity, *differentiability* and integrability - also in relation to the problematics which generated them (instantaneous speed in mechanics, tangent to a curve, determination of areas and volumes). It will be not required a particular training in the computational techniques, which will be limited to the capability of differentiating functions that are already known, simple products, quotients and compositions of functions, rational functions and to the capability of integrating entire functions as polynomial and other elementary functions" (our underlining, p.341).

Differentiability, as well as continuity and integrability, is presented as a fundamental property of the infinitesimal calculus. We recognize in this passage the previously stated rule "Few concepts and fundamental methods, acquired in depth" (p.338). Indeed, it is recommended not to insist on difficult calculations.
Afterwards, the topic of differential equations is covered. This concept has been introduced in the regular curriculum since 2010, while before it was only present as possible in-depth study.

> "Another important theme of study will be the *concept of differential equation*, what we mean with its solutions and its main properties, as well as some important and significant examples of differential equation, with particular regard to Newton's equation of dynamics" (our underlining, p.341).

This concept makes derivatives and integrals work as tools, whereas above they have been introduced as objects. We refer to the dialectic tool-object described by Douady (1986) in her research. She states that "A student possesses mathematical knowledge if he is able to provoke its functioning as explicit tools in the problems he has to solve [...] if he is able to adapt it when the usual conditions for its use are not precisely verified for interpreting problems or for posing questions with regard to it"[1] (Douady, 1986, p.11). The importance of employing the Calculus concepts as tools is also stressed ahead in the curriculum regulations.

> "It will be a matter of understanding the role of infinitesimal calculus *as fundamental conceptual tool* in describing and modelling phenomena that come from physics or of different nature. Moreover, the student will become familiar with the general idea of optimization and with its applications in several contexts." (our underlining, p.17).

3.1.1 Remarks

Differentiability appears in the grade 13 curriculum, along with the other important local properties of continuity and integrability. It can be even anticipated in grade 11-12, working with the concept of speed of variation, especially in a physical context. However, the **local feature of these properties is not mentioned nor hinted**. In fact, it is more than simply implicit, because its evidence comes up only after a deep reading and interpretation of the hidden links made by the curriculum. The approaches to adopt are left to the teachers' choices. It is all up to the teachers, who get a great freedom but also a great responsibility, to introduce the local perspective on these properties, and in particular on differentiability.
Therefore, it becomes extremely interesting, in order to study the didactic transposition of the derivative notion, to analyse how the national guidelines are interpreted firstly by the textbooks' authors and secondly by those who have to prepare the national final examination in mathematics. Worked out in this chapter, such an analysis will set the stage for Chapter 4, where we will focus on the teachers who, availing of all the institutional materials, have to teach the notion in classroom.

3.2 The derivative in the textbooks

We are going to analyse the introduction to the derivative concept and to the derivative function in two Italian textbooks.

- Bergamini, M., Trifone, A. & Barozzi, G. (2013). *Matematica.blu 2.0*, vol. 5, Libro digitale multimediale. Zanichelli: Bologna.

[1] Our English translation of the original passage: "*Un élève a des connaissances en mathématiques s'il est capable d'en provoquer le fonctionnement comme outils explicites dans des problèmes qu'il doit résoudre [...] s'il est capable de les adapter lorsque les conditions habituelles d'emploi ne sont pas exactement satisfaites pour interpréter des problèmes ou poser des questions à leurs propos*".

- Sasso, L. (2012). *Nuova matematica a colori*, Edizione BLU per la Riforma. Quinto anno. Vol. 5. Petrini De Agostini Scuola: Novara.

Here are the criteria of our choice. In April 2012, the *Fondazione Giovanni Agnelli* has published the data of an inquiry conducted on the high schools in Piedmont (Fondazione Giovanni Agnelli, 2012). It is a classification of the schools based on their ex students' results in the first year of university. It takes into account four criteria: the school effect (specific contribution given to the students' preparation by the institute), the students' effect (some individual characteristic of students which can influence their results), the territorial effects (territorial context of the school), the addresses effect (socio-cultural context of the institute).
We selected in this list the first 25 scientific high schools in Piedmont and consulted the list of the adopted textbooks for the school year 2014-2015, which is available on each school website. By comparing the adoptions of 129 classrooms (for the detailed table of adoptions, see Appendix A), it turns out that 83 of them have preferred the textbook written by Bergamini, Trifone and Barozzi (2013) - also in different editions - and 27 instead use the one written by Sasso (2012). They both follow the national guidelines of 2010. Let us analyse how each of these textbooks introduces the derivative concept and the derivative function as well.

3.2.1 *Matematica.blu 2.0* by Bergamini, Trifone, Barozzi

In this textbook, after some chapters devoted to the functions and their properties, the limit of functions and sequences, we find the chapter titled "The derivative of a function". Let us focus particularly on the first section which is similarly named "The derivative of a function" (pp.1618-1623). See Figure 3.2.1 for a conceptual map of this section.

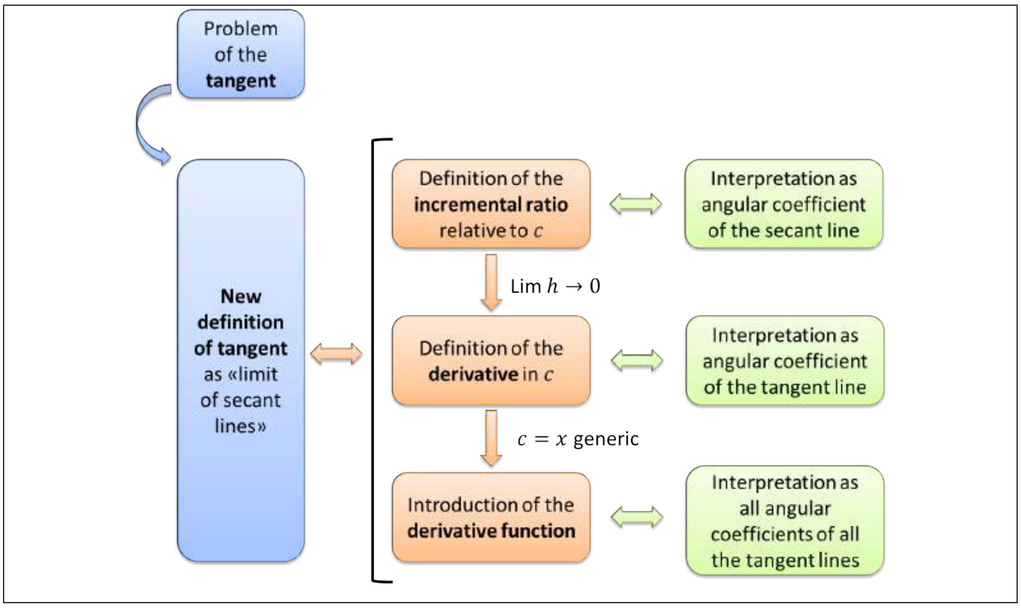

FIGURE 3.2.1 - CONCEPTUAL MAP OF THE SECTION "THE DERIVATIVE OF A FUNCTION".

Approach to the derivative concept and its definition

The derivative concept is approached through the problem of determining the equation of the tangent line to a curve in a point. The property for the circle, and in general for the conics, is recalled: "the tangent in a point P is that straight line which intersects the conic itself only in P" (p. 1618). Then, the authors observe that this property is not valid for a generic curve and make the graphical counterexample in Fig. 3.2.2 and the verbal counterexample of a parabola cut in a single point by its axis.

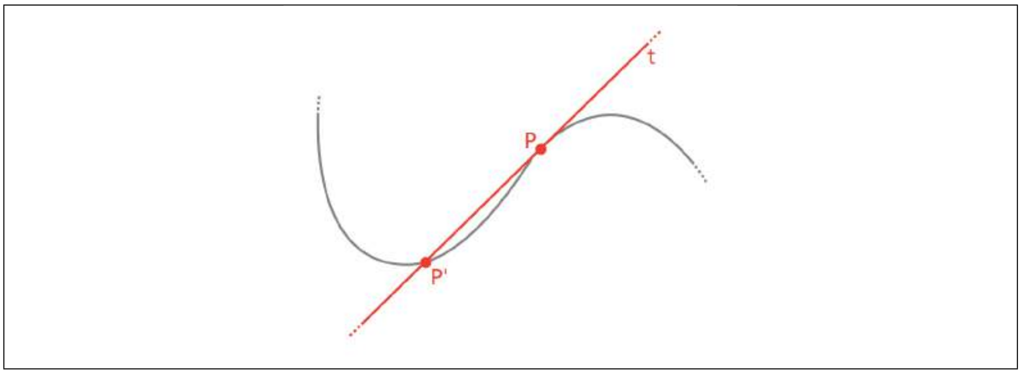

FIGURE 3.2.2 - THE TANGENT t IS TANGENT TO THE CURVE IN THE POINT P, BUT IT INTERSECTS IT ALSO IN THE POINT P' (IN BERGAMINI ET AL., 2013, P.1618).

It is then recalled "the notion of limit, thinking of the process according to which it is possible to approximate the tangent line through secant lines that approach it more and more" (p. 1618). The following definition is given, accompanied by the graph in Fig. 3.2.3.

DEFINITION. TANGENT LINE TO A CURVE - The tangent line t to a curve in a point P is the limit position, if it exists, of the secant PQ as Q tends (both from the left and from the right) to P.

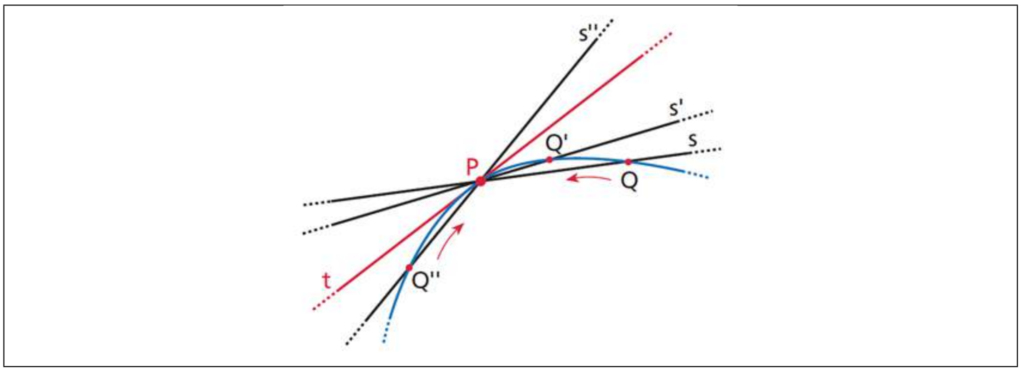

FIGURE 3.2.3 - FIGURE SUPPORTING THE DEFINITION OF TANGENT LINE TO A CURVE (IN BERGAMINI ET AL., 2013, P.1618).

Therefore the generic definition of a tangent line is explicitly provided by the textbook as the "limit position" of the secant lines intersecting the curve in the tangency point and in another one which "tends to" it. These secant lines are said to "approach" the tangent "more and more". This idea is graphically conveyed in Figure 3.2.3 through two small arrows towards the tangency point (from the left and from the right). Thus, the recall of the notion of limit, the expressions "tends to", "approach more and more" are verbal hints to a local perspective on the generic curve. The only description of Fig. 3.2.3 is given directly in the definition. The link between the two semiotic resources (words and graph) is largely left to the reader's interpretation of the figure. It is not that simple, especially because all the graphical data provided as a support look essentially pointwise: the points P, Q, Q', Q''. It is implicit, for example, that Q, Q', Q'' can be seen as a sequence that approaches P.

In order to find the equation of the so-defined tangent line to the function $y = f(x)$, the first step proposed by the textbook consists of defining the incremental ratio, which is immediately given through symbols. Given the point $A(c, f(c))$ belonging to the graph of f, the abscissa c is increased of the quantity h, obtaining the point $B(c+h, f(c+h))$. The increments $\Delta x = x_B - x_A = h$ and $\Delta y = y_B - y_A = f(c+h) - f(c)$ are considered. The incremental ratio is then defined as $\dfrac{\Delta y}{\Delta x} = \dfrac{f(c+h) - f(c)}{h}$, with reference to the graphical representation in Fig. 3.2.4.

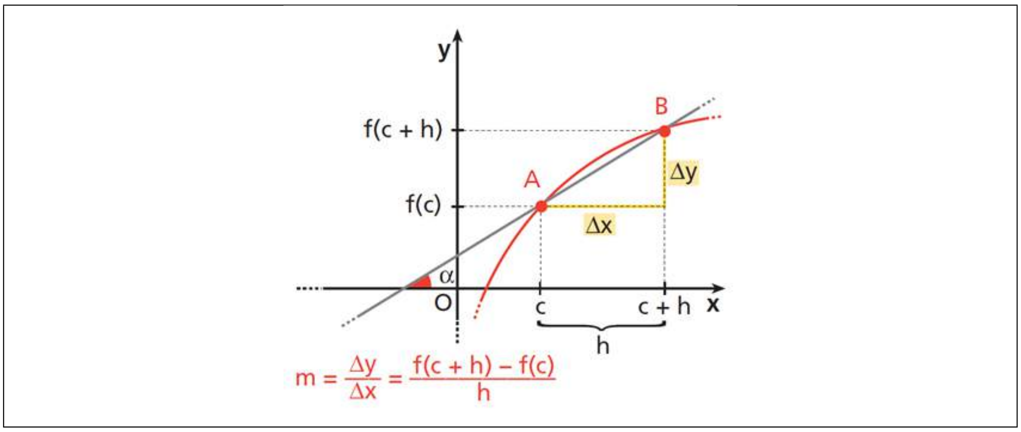

FIGURE 3.2.4 - FIGURE SUPPORTING THE DEFINITION OF INCREMENTAL RATIO (IN BERGAMINI ET AL., 2013, P.1619).

The incremental ratio is interpreted as the angular coefficient[2] of the straight line passing through A and B. A detailed example with the specific parabola $y = 2x^2 - 3x$ in $c = 1$ is provided. In the provided definition and example, the **pointwise and global perspectives prevail** within the semiotic bundle *words+symbols+graph*.
Afterwards, in the paragraph "The derivative of a function" the idea of a point approaching another one on the curve, introduced with the tangent definition, is developed further. Specifically, the approaching is algebraically interpreted within a Cartesian reference system. Indeed, on the graph of equation $y = f(x)$ the points $A(c, f(c))$ and $B(c+h, f(c+h))$ are taken into account. A note on the border underlines that "the point A is fixed while the point B varies as h varies" (p.1620). To convey the idea of the variation of h, different secant lines are drawn for three decreasing values of h. Each secant line is drawn along with the tangent in A on a Cartesian plane reproducing the graph of the function $y = f(x)$ and the points A and B, the latter depending on h. The three resulting Cartesian planes are then put into sequence, as if they were successive frames of a film, in order to convey the idea of movement (see Fig. 3.2.5).

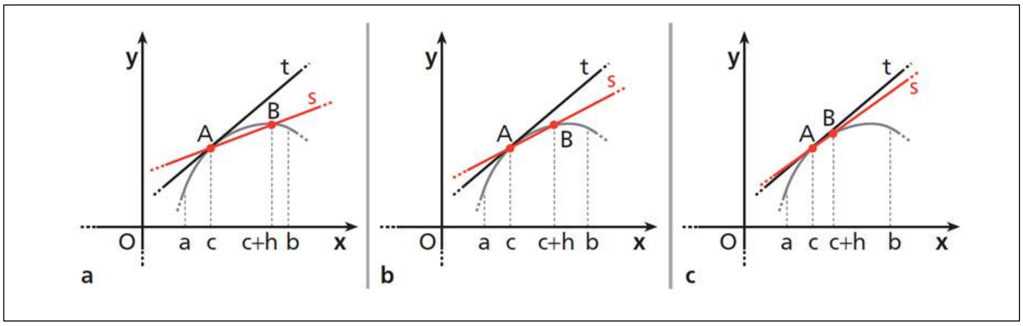

Figure 3.2.5 - a, b and c represent three different possible secant position of the straight line AB, compared with the tangent t in A (in Bergamini et al., 2013, p.1620).

The following technological speech (p.1620), that we report word by word, explains the Fig. 3.2.5.

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[2] In Italian, the coefficient of x in the equation $y = mx + q$ is called "coefficiente angolare", with reference to the property $m = \tan\alpha$, where α is the angle that the line forms with the positive direction of x-axis. Normally, Italian textbooks, teachers and students refer to m as "coefficiente angolare" since the first time the straight line is studied in the Cartesian plane (grade 8). Although the slope of a line is strictly related to the angle which it forms with x-axis, the used name partially hides this relation. Students usually do not link automatically, or at least not directly, the value of m with the slope of the line. We think that the Italian name can evoke a certain mental image, different from the image linked to the English word "gradient" or the French word "coefficient directeur" for example. We believe that this fact can differently influence the mathematical discussion and activity in classroom. Therefore, in the transcriptions or speech elaborations, we generally prefer keeping the literal translation "angular coefficient", instead of the correct English translation "gradient" or "slope".

> By giving to h values that get smaller and smaller, the point B approaches more and more the point A. When $h \to 0$, the point B tends to overlap the point A and the straight line AB tends to become the tangent line to the curve in A. The angular coefficient of the secant line AB, that is the incremental ratio, tends to the angular coefficient of the tangent line, which is called *derivative* of the function in the point c.

Notice that the **local perspective is introduced with an idea of approaching**. This is realized in a graphical register as an interval (in this case, a right interval) of the point, whose width becomes smaller and smaller. Graphically such an idea of movement is conveyed by the juxtaposition of three successive frames, in which the width of the interval h gets decreasing values. In words, the approaching is expressed as

- "B approaches more and more the point A";
- "B tends to overlap A";
- "the straight line AB tends to become the tangent";

and it is associated with the value of h, the width of the interval, which becomes

- "smaller and smaller";
- in symbols "$h \to 0$".

Thus, words and symbols are used together to explain the graphical representation in Fig. 3.2.5, expressing:

- the idea of movement as an approaching;
- the idea of something which becomes smaller and smaller;
- the idea of something which becomes something else.

The semiotic bundle composed of *words+symbols+graph* aims to introduce a local perspective on f. However, the **activation of the local perspective is left to the reader's understanding of the relationships** existing among the three semiotic resources, briefly illustrated in the technological speech. First of all, the reader has to see in the different graphs given in sequence the movement of the secant line towards the tangent line; then, he has to relate this movement to $c+h$ which approaches c on x-axis; finally, he has to express this approaching symbolically as $h \to 0$. And especially the final step is a difficult jump, because from the global interval $[c; c+h]$ drawn in Fig. 3.2.5, the reader has to pass to the local neighbourhood where $h \to 0$. Thus, after some work of pointwise nature (detection of A, B and the secant line AB) and of global nature (the successive intervals $[c; c+h]$), it is this final step that allows to really adopt a local perspective.

The definition of derivative is then given as follows, accompanied by the graphical representation in Fig. 3.2.6.

Definition. Derivative of a function - Let $y = f(x)$ be a function defined on the interval $[a; b]$, we call derivative of the function in the point c internal to the interval, and it is denoted with $f'(c)$, the limit, if it exists and is finite, as h goes to 0, of the incremental ratio of f relative to c:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

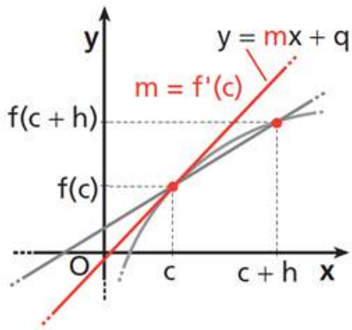


Figure 3.2.6 - Figure supporting the definition of the derivative of a function. (in Bergamini et al., 2013, p.1620).

The successive paragraph, "The derivative calculation", provides an example of calculation of the derivative of a function in a point. More precisely, the task consists of calculating the derivative of the function $y = x^2 - x$ in $c = 3$.

Hence, we can remark that the definition transposed by Bergamini et al. (2013) is the Def. 1 discussed in Paragraph 1.2.2. The tangent line is the limit position of a sequence of secant lines to the curve and the derivative is the limit of the incremental ratio of the function. In conclusion, the **local perspective remains implicit in the relationships internal to the semiotic bundle** composed of graph, words and symbols.

Introduction of the derivative function

Immediately after the derivative calculation in the given point $c = 3$, within the same paragraph, the derivative function is introduced as follows.

We can calculate the derivative of a function also in a generic point. In this case, the obtained value $f'(x)$ is a function of x and, for this reason, we speak also of derivative function. [...] The derivative function, as x varies, provides the angular coefficient of all the tangent lines to the given function.

The paragraph "The derivative calculation" closes with an example: the calculation of the derivative of the function $f(x) = 4x^2$ in the generic point x. The first step of the resolution gives the algebraic technique without further explanations:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The only hint to the local perspective on f is the presence of the symbol of limit, which derives from a direct application of the technique given for $f'(c)$. Thus, the **local perspective is implicitly present** in the definition of the derivative function **thanks to the use of the symbol** $\lim$.

Jumping to page 1851, within the chapter devoted to the study of function, we find the graphical representation of the derivative function. The section "The graphs of a function and of its derivative" deals with the graphical relationships between the graph of a function $f(x)$ and that of its derivative $f'(x)$. Such relationships are summarized in Fig. 3.2.7 (a,b,c,d).

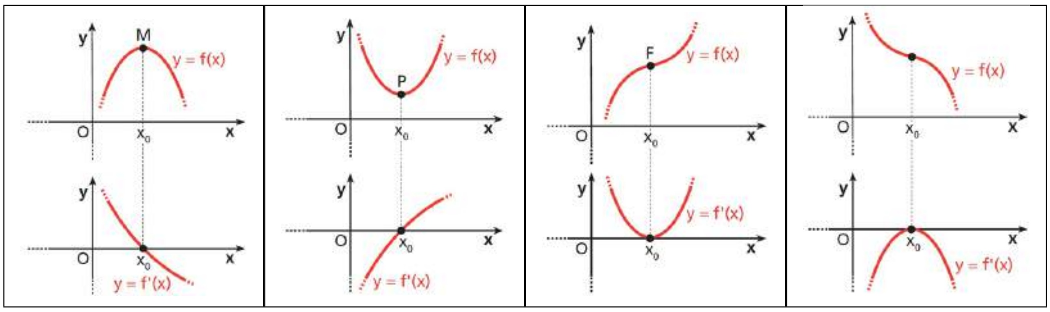


Figure 3.2.7(a) - In x_0 the function $f(x)$ has a relative maximum. In the same point, the derivative $f'(x)$ is zero and so its graph intersects x-axis in x_0. On the left of x_0, the derivative is positive, on the right it is negative (p.1851).

Figure 3.2.7(b) - In x_0 the function $f(x)$ has a relative minimum: so $f'(x_0) = 0$. The graph of the derivative intersects x-axis in x_0: on the left of x_0, the derivative is negative, on the right it is positive (p.1851).

Figure 3.2.7(c) - In x_0 $f(x)$ has an ascending horizontal inflection point, so $f'(x_0) = 0$. The graph of the derivative intersects x-axis in x_0, and $f'(x)$ is positive both on the right and on the left of x_0. Therefore in x_0 there is a minimum for $f'(x)$ (p.1851).

Figure 3.2.7(d) - In x_0 $f(x)$ has an descending horizontal inflection point: with the same considerations as in the previous case, for $x \neq x_0$, $f'(x)$ is negative. Therefore in x_0 there is a maximum for $f'(x)$ (p.1851).

Notice that, with the purpose of giving to the derivative function a graphical representation, some **local considerations about the involved functions are made**. They are directly formulated on f', whose sign is distinguished on the left and on the right of the point x_0. Thus, they indirectly entail a local graphical reading of the maximum, minimum and horizontal inflections of f.
In this case, all the semiotic sets involved in the semiotic bundle, namely graph, words and symbols, concur to underline a local perspective on the functions f' and indirectly on the function f.

Exercises

Let us give a quick overlook on the proposed exercises. We refer to the types of task related with the theoretical paragraphs we have just analysed above. In particular, we focus on the exercises that specifically refer to the introduction of the derivative concept, of the derivative function and of its graph. More precisely, the tasks analysed are 101: 86 of them make direct reference to the section "The derivative of a function" (pp.1655-1659) and the remaining 15 are taken from the section "The graphs of a function and of its derivative" (pp.1905-1909).
As a first step in our analysis, we detect the different types of task proposed. Some preliminary exercises are aimed to the handling of the elements that compose the formal definition of incremental ratio. Then, we find more calculative exercises which, given the analytic expression of a function $f(x)$, require to determine its incremental ratio for fixed or generic values of c and h. After, some tasks require to calculate the derivative of a given function in a given or generic point c. Sometimes a guided exercise is provided: we consider it very useful to identify the expected techniques, modes of application and justifying arguments. Finally, within the exercises related to the study of functions, some tasks ask to deduce the graphical representation of the derivative function, starting from the graph of a function and some others ask the converse: to deduce the graphical representation of a function, starting from the graph of its derivative.
Thus, seven types of task have been detected. Let us present at least one representative task for each of them.

<u>Type of task T_1</u> (18 tasks): Computing and handling the elements of the formal definition of incremental ratio.

Ex.8 p.1655 - Given the function $y = \dfrac{x^2-1}{x}$, consider $c = 2$ and $h = 0,1$ and determine Δx and Δy.

<u>Type of task T_2</u> (9 tasks): Determining the incremental ratio of a given function f in a given point c, for a generic increment h.

Ex.20 p.1656 - $f(x) = \dfrac{x-5}{x} \quad c = 4$.

<u>Type of task T_3</u> (8 tasks): Determining the incremental ratio of a given function f in a generic point c, for a generic increment h.

Ex.28(a) p.1656 - $f(x) = \sqrt{x}$.

<u>Type of task T_4</u> (21 tasks): Calculating the derivative of a given function f in a given point c, by applying the definition.

Ex.33(a) p.1657 - $f(x) = x^3 + 4x + 1 \quad c = 1$.

Type of task $\mathcal{T}_5$ (4 tasks): Proving that it is not possible to calculate the derivative of the given function f in the given point c.

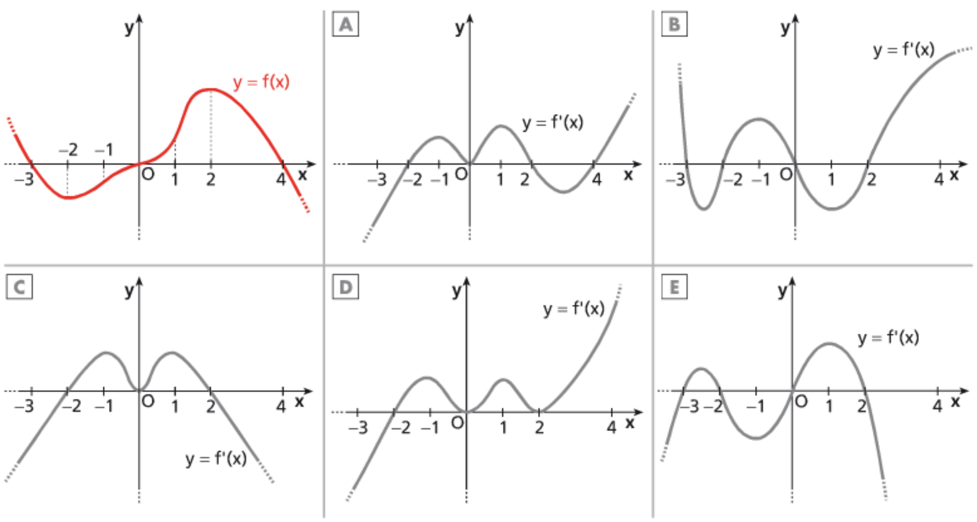
Ex.43 p.1658 - $f(x) = \sqrt{x-1} \quad c = 1$.

Type of task $\mathcal{T}_6$ (26 tasks): Calculating the derivative of a given function f in the generic point c.

Ex.59 p.1658 - $f(x) = -e^{1+x}$.

Type of task $\mathcal{T}_7$ (5 tasks): Given the graph of $y = f(x)$, determining the graph of its derivative $y = f'(x)$.

Ex.272 p.1906



Ex.274 p.1906 - Draw in the same Cartesian plane the graph of the function $y = 2xe^{2x}$ and that of its derivative and then find the coordinates of their intersection point.

Type of task $\mathcal{T}_8$ (10 tasks): Given the graph of $y = f'(x)$, determining the graph of the function $y = f(x)$.

EX.279(B) P.1908

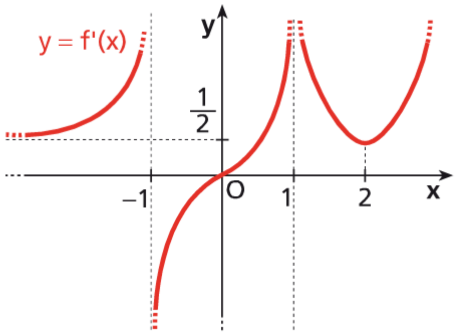


Table 3.1 schematically summarizes the semiotic resources the student is expected to use and the perspectives he is expected to adopt in solving the types of task $\mathcal{T}_i$, for $i = 1, \dots, 8$.

Type of task (% of tasks)	**expected semiotic resource**	**expected perspectives**
$\mathcal{T}_1$ (18%)	symbols	pointwise on f
$\mathcal{T}_2$ (9%)	symbols	pointwise, global on f
$\mathcal{T}_3$ (8%)	symbols	pointwise, global on f
$\mathcal{T}_4$ (21%)	symbols	pointwise on f and f'
$\mathcal{T}_5$ (4%)	symbols	pointwise on f and f'
$\mathcal{T}_6$ (25%)	symbols	global(=univ. pointwise) on f and f'
$\mathcal{T}_7$ (5%)	graphs (& symbols)	pointwise, global and local on f and f'
$\mathcal{T}_8$ (10%)	graphs	pointwise, global and local on f and f'

TABLE 3.1 - EXPECTED USE OF THE SEMIOTIC RESOURCES AND PERSPECTIVES TO SOLVE THE TYPES OF TASK INVOLVING THE DERIVATIVE CONCEPT AND THE DERIVATIVE FUNCTION.

A brief analysis of the tasks, that involve the derivative and the derivative function at the initial stage of introduction and understanding of the concept, leads us to some conclusive remarks. Bergamini et al. (2013) propose several exercises requiring the use of symbols. However, a relevant 15% of them requires almost exclusively to activate the graphical resource. The way in which the latter is expected to be worked encompasses a good handle of the perspectives on functions, also the local one. Indeed, some tasks (belonging to $\mathcal{T}_7$ and $\mathcal{T}_8$) propose graphs of discontinuous or non-differentiable functions. In the part devoted to the theory, we have found **some local considerations in the graphical work on the derivative function**, thus we suppose that the students are expected to use them for solving the similar given tasks.

3.2.2 *Nuova Matematica a colori* by Sasso

In this textbook, the theme "Differential calculus" comes after the theme "Limits and continuity". The derivative is introduced after the limits of real functions of a real variable, the limits of sequences and the continuity. We are going to analyse the first section devoted to the derivative, whose title is "The derivative concept" (pp.258-264). See Fig. 3.2.8 for a conceptual map of this section.

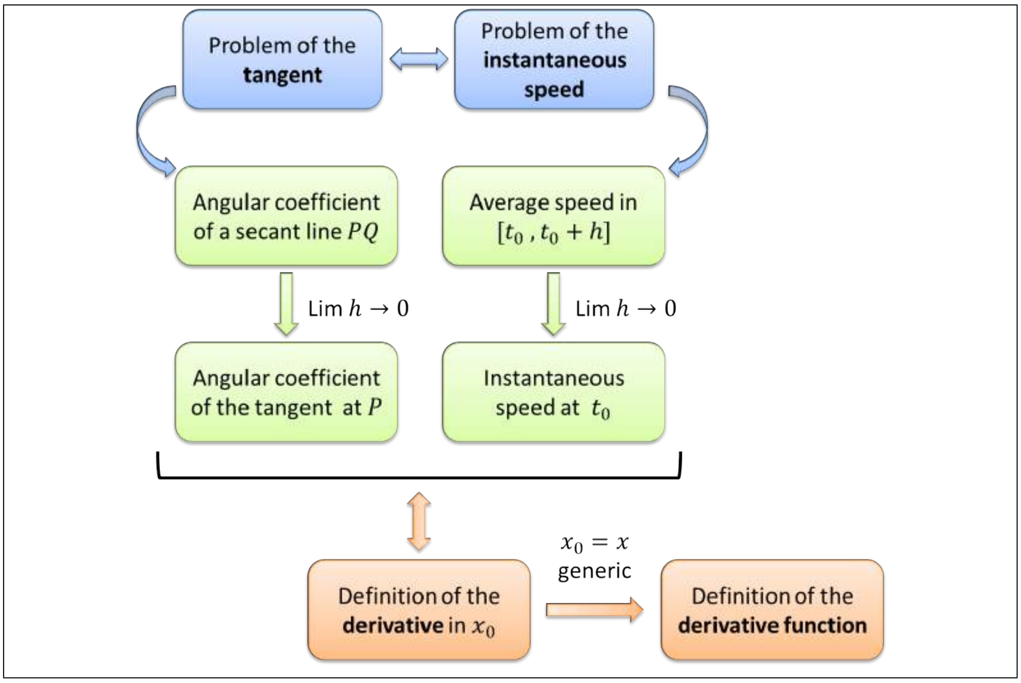


FIGURE 3.2.8 - CONCEPTUAL MAP OF THE SECTION "THE DERIVATIVE CONCEPT".

Approach to the derivative concept and its definition

The concept is approached through the presentation of two problems "which, also historically, led to its birth" (p. 258): the problem of the tangent line and the problem of the instantaneous speed. A text box on the border gives some historical hints about Leibniz and Newton and the period in which they worked out the differential calculus.

As for the problem of the tangent, the text develops in a very discursive way. It is recalled that the problem of the tangent equation has been already solved within the analytic geometry in the case of the conics. But here the question is: "What is the tangent line to a curve in one of its points P?" (p. 258). Then, the author discusses some possible attempts to define it. The first definition might be: "it is the only straight line passing through P that does not intersect the curve at other points" (p. 258). Nonetheless, he proposes two graphical counterexamples (see Fig. 3.2.9): the former concerns a tangent line that intersects the curve not only in P, but also in another point; the latter shows one of the infinite straight lines passing through the origin in the graph of the absolute value of x.

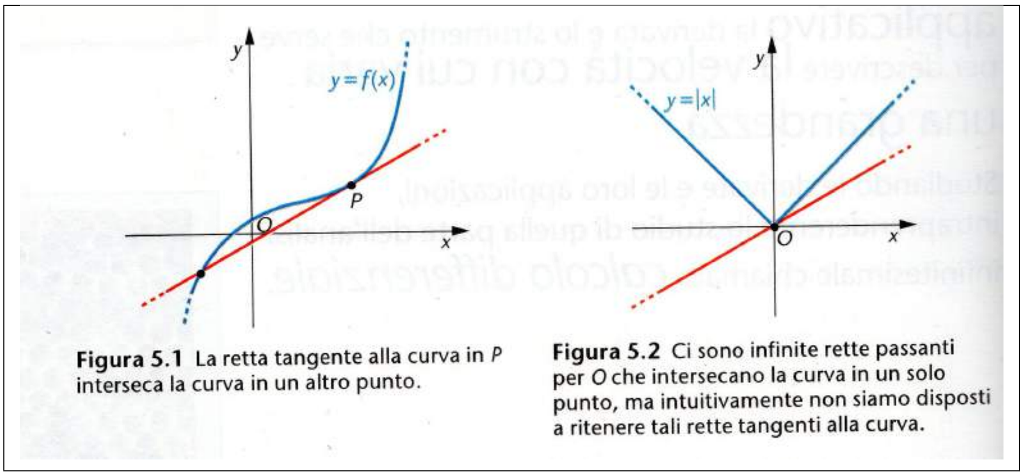


Figure 3.2.9 - On the left: the tangent to the curve in P intersects the curve at another point; on the right: there exists an infinite number of straight lines passing through O which intersect the curve in a single point, but intuitively we cannot consider these straight lines as tangent to the curve (in Sasso, 2012, p.258).

The second attempt is the same definition that is valid for the conics: "[the tangent is] the straight line that has a double intersection with the curve in P" (p. 258). Then, the author points out a difficulty by making the symbolic example of the system between $y = \sin x$ and $y = mx$. The solving equation would be $\sin x = mx$ which is not a polynomial equation. So, "there is no way to count the multiplicity of the solutions of this equation, and consequently to impose that the solution $x = 0$ is double" (p. 258). Hence, the author highlights the need to consider a new idea, from a new dynamical point of view. From now on, let us transcribe word by word what is written in the textbook.

Given the function $y = f(x)$ and a point $P(x_0, f(x_0))$ belonging to its graph, in order to define the tangent line of f in P let us firstly take into account a straight line which passes through P and which is secant to the curve in another point Q, "close" to P, with abscissa $x_0 + h$ (see Fig. 3.2.10). [A text box on the border recalls that on-line this figure is available in its dynamical version.] We know that the angular coefficient of the straight line PQ is expressed by the formula:

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x_0 + h) - f(x_0)}{h}$$

and that the straight line PQ has equation:

$$y - f(x_0) = m_{PQ}(x - x_0).$$

Let us imagine that h tends to 0. The point Q moves on the graph of f and approaches P, till overlapping it when $h = 0$. Contextually the secant line turns around P, till it reaches a "limit" position which intuitively we can identify with that of the tangent line. Let us consider the limit towards which the angular coefficient of the straight line PQ tends as h goes to 0:

$$\lim_{h \to 0} m_{PQ} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If this limit tends to a finite value, we can define the tangent line as the straight line which passes through P and which has this angular coefficient.

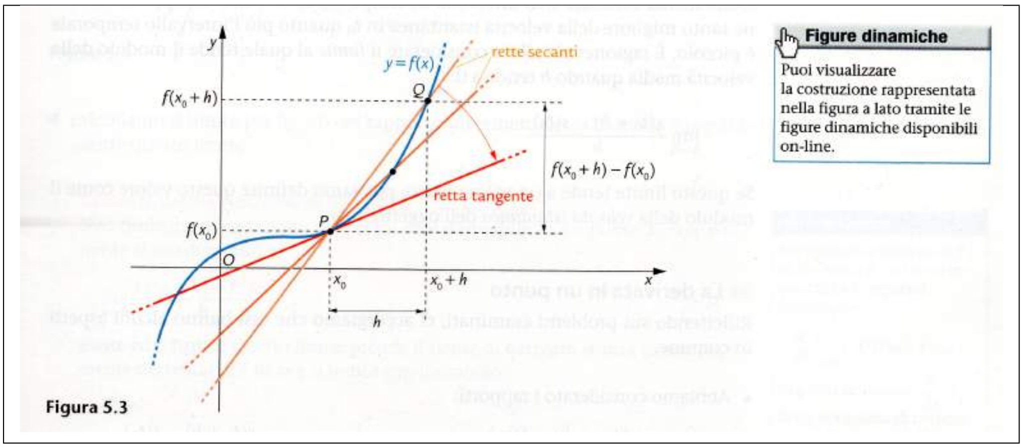


FIGURE 3.2.10 - FIGURE SUPPORTING THE PROCEDURE DESCRIBED ABOVE TO FIND THE TANGENT TO A GENERIC FUNCTION $y = f(x)$ AT P (IN SASSO, 2012, P.259).

The author serves of verbal and graphical resources, along with symbols. Observe some of the chosen terms or expressions concerning the point Q on the curve:

- "close" to P;
- Q moves on the graph of f;
- it approaches P;
- till overlapping it when $h = 0$;

and concerning the straight line PQ:

- turns around P;
- till it reaches a "limit" position.

We only find unusual the expression "when $h = 0$", since h is never actually equal to zero.
Nevertheless, all these terms convey an idea of movement that on the graph can be given only by means of a rounded arrow towards the tangent position (see Fig. 3.2.10). The **local perspective is thus approached by using terms which convey an idea of movement**. Such terms however refer to a static figure that is dynamical in the on-line version. Let us suppose that an ordinary reader looks only at the figure on the textbook. Then, linking the "dynamical" text to what happens on the graph can be very difficult. Indeed, the given graphical context is necessarily global, because in a drawing on a piece of paper the interval h detects inevitably a global portion of the curve of f. In such a global context, the reader has to identify a local movement in a neighbourhood of the point x_0. And he is expected to do so only with the help of the accompanying text. Thus, the local perspective is potentially contained in the text and mostly in the relation that the reader manages to establish between the text and the figure. So, **the adoption of a local perspective is left to the capability of the reader to imagine the local movement** in the right way and in all its details. We realize that this process is not simple at all, without the intervention of a mediation figure like the teacher is in the classroom.

As far as the instantaneous speed is concerned, the author makes the example of a body that falls down. The question he poses is: "What is the speed of the object in a given instant?" (pp.259-260). Knowing the position function $s(t)$ of an object, the average speed in the interval $[t_0, t_0 + h]$ is written as the ratio:

$$\frac{s(t_0 + h) - s(t_0)}{h}.$$

Then, the author introduces the limit as $h \to 0$. He justifies this operation by claiming that the more the interval of time $[t_0, t_0 + h]$, on which the average speed is calculated, is small, the more the average speed will approximate the instantaneous speed in t_0. At a first glance, it seems that the author is basing on an idea of approximation. Nonetheless, this statement is very close to the case of the tangent line. Indeed, the mentioned approximation is not that of the curve described by $s = s(t)$, but that of the average speed which becomes the instantaneous speed. Therefore, as before with the case of the tangent, the basic idea is again that the secant vector, representing the average speed, becomes the tangent vector, which is the instantaneous speed in t_0. Thus, within the physics example of the instantaneous speed, **the local dimension is realized with the introduction of the limit operation** and justified by using not terms of movement but an equivalent idea of change.

Afterwards, in the paragraph "Derivative in a point", the textbook compares the two problems by highlighting in them the same pattern (increment h, ratio, limit of the ratio as h goes to 0), which is retraced step by step for any function f in any point x_0 of its domain. The definition of incremental ratio is stated and the final limit is called derivative. The graphical situation in Fig. 3.2.11 is illustrated and the following formal definition is given (p. 261).

DERIVATIVE OF A FUNCTION IN A POINT A function of equation $y = f(x)$ is said differentiable in a point x_0, belonging to its domain, if $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad (3.1)$$ exists and is finite. This limit takes the name of first derivative (or simply derivative) of f in x_0 and it is denoted with the symbol: $f'(x_0)$

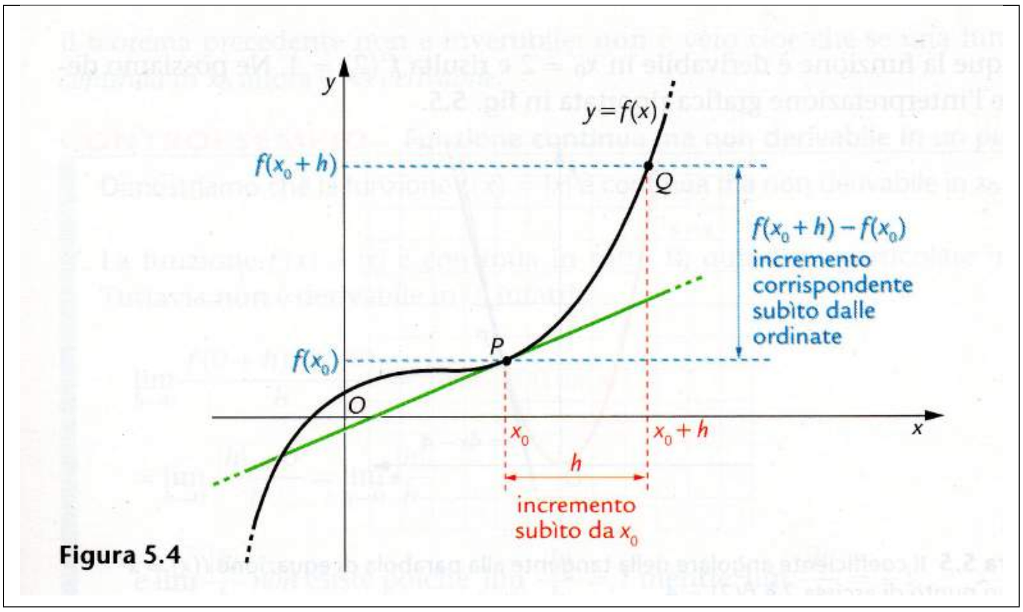


FIGURE 3.2.11 - FIGURE ACCOMPANYING THE DEFINITION OF DERIVATIVE IN A POINT (IN SASSO, 2012, P.261).

The paragraph "Derivative in a point" is closed by an example, where the type of task is explicitly given: "Calculation of the derivative of a function in a point according to the definition" (p. 261). The specific task chosen to illustrate the technique (3.1) involves $f(x) = x^2$ in $x_0 = 2$. In the proposed resolution, the technique (3.1) is directly applied with $f(2+h) = (2+h)^2$ and $f(2) = 2^2$. A graphical representation shows that the result 4 corresponds to the gradient of the tangent to the parabola in $P(2, 4)$.
At the end of the paragraph, an alternative definition of derivative is given as an in-depth

note, by setting $x = x_0 + h$. $f'(x_0)$ is then alternatively defined as $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

Hence, we can notice that Sasso (2012) approaches the derivative as the limit of the incremental ratio. So, he proposes a didactic transposition of the definition we denoted as Def. 1 in Paragraph 1.2.2. The tangent line to a generic curve in P is presented as the limit position of a secant line PQ when Q approaches P. **The local perspective on f is implicit** in the use of terms involving

- the idea of movement, such as "close", "approaches", "overlapping";
- the idea of "becoming instantaneous" as the considered increment of time becomes smaller and smaller.

Introduction of the derivative function

Moving on to the end of the section "The derivative concept", the closing paragraph deals with the "Derivative function and higher order derivatives". Let us transcribe word by word the shift from the pointwise definition of $f'(x_0)$ to the global definition of $f'(x)$ (p. 263).

> Given a function f, we can define a new function f', called (first) derivative function of f, that with each point, where f is differentiable, associates its derivative. Formally, if D is the domain of the function f and D' is the subset of D in which f is differentiable, the function f' is defined as follows:
>
> $$f' : D' \to \mathbb{R}, \quad \text{for which} \quad x \mapsto f'(x).$$

An example follows this definition, where the type of task is specified: "[Determining the] derivative of a function according to the definition" (p. 264). Then, chosen $f(x) = x^2$, the author calculates the derivative of f "in the generic point of abscissa $x \in \mathbb{R}$". An algebraic technique to find the derivative function is suggested. The author explicitly writes:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{Limit (3.1) with} \quad x_0 = x. \tag{3.2}$$

In the given definition and example, there is no local consideration on the function f. **The local perspective is not expressly highlighted by the algebraic technique** (3.2). The symbol of limit is present, but it derives from the formal definition just given for the derivative in a point (3.1) and then it is mainly treated through syntactical manipulations. Also the technological hint in the example, namely "Limit (3.1) with $x_0 = x$" does not actually entail local remarks on the function f involved in the limit calculation. **The local perspective remains implicit in the symbols** used, specifically $\lim$ and $h \to 0$.

To find a graphical representation of the derivative function we have to jump to page 440, to the section "Deducible graphs" in the chapter "Study of function". Here the paragraph "From the graph of a function to that of its derivative" is entirely devoted to how graphically representing the derivative function $y = f'(x)$.
A general technological speech illustrates step by step the properties that the graph of $y = f'(x)$ must have. We list them below, as the textbook does (pp.440-441).

To sketch a possible graph of the derivative of a function $y = f(x)$ it is enough to draw upon the following considerations.

a. The function $y = f'(x)$ has for domain the set of values of x for which the function $y = f(x)$ is differentiable.
b. The function $y = f'(x)$ is even if $y = f(x)$ is odd, it is odd if $y = f(x)$ is even (try to prove it as an exercise).
c. The zeros of the function $y = f'(x)$ (that are the abscissas of the points in which $[f']$ intersects x-axis) are the stationary points of $y = f(x)$ (that are points of relative maximum or minimum or inflection points with horizontal tangent).
d. The sign of the function $y = f'(x)$ is: positive in the intervals in which $y = f(x)$ is increasing and negative in the intervals in which $y = f(x)$ is decreasing.
e. The points of relative maximum or minimum of the function $y = f'(x)$ correspond to the inflection points of the function $y = f(x)$.
f. The intervals of variation of the function $y = f'(x)$ correspond to the intervals where the function $y = f(x)$ is concave or convex: if $y = f(x)$ is convex in an interval, then $y = f'(x)$ is increasing in that interval, if instead $y = f(x)$ is concave, then $y = f'(x)$ is decreasing.

This list of properties precedes the following example, whose task is "Let's deduce from the graph of the function $y = f(x)$, drawn in fig. 7.18 [see Fig. 3.2.12], the graph of the function $y = f'(x)$" (p. 441).

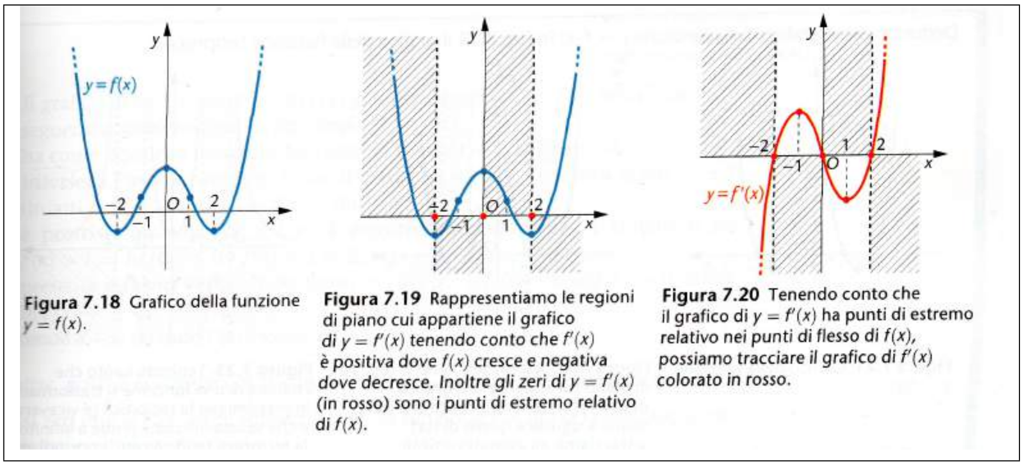


Figure 3.2.12 - From left to right: graph of the function $y = f(x)$; let's represent the regions of plane in which the graph of $y = f'(x)$ lies taking into account that $f'(x)$ is positive where $f(x)$ increases and negative where it decreases. Moreover the zeros of $f'(x)$ (in red) are the points of relative maximum/minimum of $f(x)$; considering that the graph of $y = f'(x)$ has points of relative maximum/minimum at the inflection points of $f(x)$, we can draw in red the graph of $f'(x)$ (in Sasso, 2012, p.441).

The technique is illustrated through the graphical steps in Fig. 3.2.12, and the related captions explain how the supporting properties are used. In particular, the technique consists of detecting the points of maximum, minimum and inflection for f and highlighting them on x-axis. Basing on the previous properties c. and d., the sign of the function $y = f'(x)$ is marked by blackening the zones where f' cannot pass and the intersections of f' with x-axis are marked by three full dots on it. In the last step, the graph of f' is drawn by taking into particular account the property e. to correctly place the maximum and minimum points.

In the resolution of this task, graphical and verbal resources are integrated to form the semiotic bundle. This entails some global and pointwise considerations on the graph of f and on the graph of f'. The main remarks indeed are made on

- the significant points of f (maximum and minimum points, inflection points) and consequently of f' (zeros and maximum/minimum points);
- the variation intervals of f in relation to the sign intervals of f'.

These are pointwise and global adaptations of the properties stated above. **No real local work is done on the involved functions**. For instance, the tangent line is not mentioned as a possible help in understanding the trend of the two functions.

Exercises

Let us move on to a quick analysis of the exercises. With regard to the theoretical referential just analysed above, we focus on the exercises that specifically refer to the

introduction of the derivative concept, of the derivative function and of its graph. More precisely, we have analysed 56 tasks: 51 of them make direct reference to the section "The derivative concept" (pp.294-296) and the remaining 5 are taken from the section "Deducible graphs" (pp.477-478).
The first step of the analysis consists of subdividing all the tasks into types of task. Specifically, some preliminary exercises are aimed to the understanding of the formal definition and its geometrical meaning. After, we find more calculative exercises which, given the analytic expression of a function $f(x)$, require to determine its derivative in a given value of x_0 or the derivative function as well. It is expressly demanded to apply the definition (through the limit of the incremental ratio as h goes to 0). The first of each group of exercises presents a guided resolution. It is useful to have insight into the expected techniques, modes of application and justifying arguments. Finally, in regard to the section devoted to the deducible graphs, some exercises consists of deducing the graphical representation of the derivative function, starting from the graph of a function. So, five types of task have been detected. Let us present at least one representative task for each of them.

Type of task T_1 (5 tasks): Recognizing the formal writing of the definition of the derivative in a point.

EX.2 P.294 - To calculate according to the definition the derivative of the function $y = f(x)$ in the point $x_0 = 2$, which of the following limits has to be calculated?

A. $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(h)}{h}$

B. $\lim\limits_{h\to 0} \dfrac{f(2-h)-f(2)}{h}$

C. $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

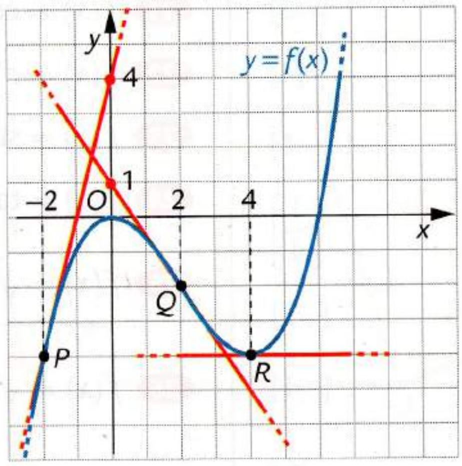
D. $\lim\limits_{h\to 0} \dfrac{f(2-h)+f(2)}{h}$

EX.6 P.294 - Each of the limits you find in the first column represents the derivative of a function f in the specified point x_0. Make the right matches.

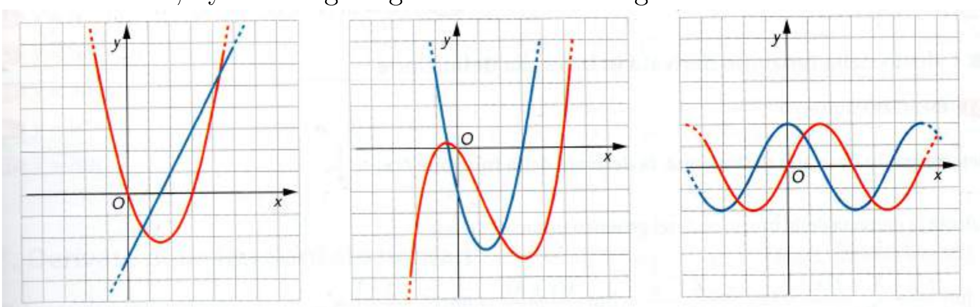
a. $\lim\limits_{h\to 0} \dfrac{(2+h)^3-8}{h}$	A. $f(x) = x^2 \quad x_0 = 4$
b. $\lim\limits_{h\to 0} \dfrac{(-2+h)^3+8}{h}$	B. $f(x) = x^2 \quad x_0 = -4$
c. $\lim\limits_{h\to 0} \dfrac{(4+h)^2-16}{h}$	C. $f(x) = x^3 \quad x_0 = 2$
d. $\lim\limits_{h\to 0} \dfrac{(-4+h)^2-16}{h}$	D. $f(x) = x^3 \quad x_0 = -2$

Type of task T_2 (4 tasks): Interpreting graphs by using the derivative concept.

Ex.7 p.295 - Observe the figure below, where the tangent lines have been drawn to the graph of the function f at the three points P, Q and R.
Basing on the geometrical meaning of the derivative, complete the following equivalences:
$f'(-2) = \ldots \qquad f'(2) = \ldots \qquad f'(4) = \ldots$



Ex.8 p.295 - Each of the following figures shows the graph of a function and the graph of the derivative function. Identify which of the two is the graph of the derivative, by recalling its geometrical meaning.



<u>Type of task $\mathcal{T}_3$</u> (22 tasks): Calculating the derivative of a function in a given point, according to the definition.

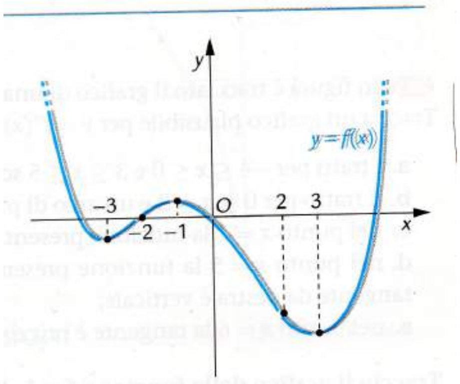
Ex.11 p.296 - $f(x) = x^2 - 2x \qquad x_0 = 3.$

<u>Type of task $\mathcal{T}_4$</u> (20 tasks): Calculating the derivative of a function, according to the definition.

Ex.35 p.296 - $f(x) = \dfrac{2}{x}$.

Type of task $\mathcal{T}_5$ (5 tasks): Given the graph of a function f, sketching the graph of its derivative.

Ex.241 p.477 - The accompanying figure shows the graph of a function $y = f(x)$; moreover the abscissas of the points of relative maximum/minimum and of inflection are indicated. Sketch the graph of $y = f'(x)$.



In Table 3.2 we detect which semiotic resources the student is expected to use and which perspectives he is expected to adopt in solving the types of task $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ and $\mathcal{T}_5$.

Type of task (% of tasks)	**expected semiotic resource**	**expected perspectives**
$\mathcal{T}_1$ (9%)	symbols	pointwise on f and f'
$\mathcal{T}_2$ (7%)	graphs	pointwise, global or local on f and f'
$\mathcal{T}_3$ (40%)	symbols	pointwise on f and f'
$\mathcal{T}_4$ (35%)	symbols	global(=univ. pointwise) on f and f'
$\mathcal{T}_5$ (9%)	graphs	pointwise, global and local on f and f'

Table 3.2 - Expected use of the semiotic resources and perspectives to solve the types of task involving the derivative concept and the derivative function.

After a brief analysis of the tasks involving the derivative and the derivative function, at a first level of introduction and understanding of the concept, we can make some conclusive comments. Sasso (2012) proposes several symbolical exercises, and some of them foster a reasoned manipulation of the symbolic writing. Nonetheless, a significant 16% of these introductive exercises almost exclusively employs the graphical resource. The latter, contrary to the symbolic one, a priori may foster the adoption of all the perspectives on the involved functions, even the local one. In particular, a local reasoning is needed in the last group of tasks ($\mathcal{T}_5$) where the given function can also be discontinuous or non-differentiable at some points. In the guided exercise, **some local remarks are**

present and they are fundamental to solve the task. Thus, we suppose that they are expected from the student in all the similar proposed tasks.
Therefore, **only the graphical work seems to suitably foster the possible adoption of a local perspective**.

3.2.3 Remarks

We have analyzed two of the most widespread textbooks in scientific high schools in Piedmont. In particular, we have focused on the approach to the derivative concept and its definition, the introduction to the derivative function and the exercises relative to these theoretical issues. Though different schemes of exposition, both the textbooks define the derivative as the limit of the incremental ratio of a generic function, when the increment of the abscissa goes to zero. It corresponds to a first transposition of the DEF. 1 given and discussed in paragraph 1.2.2. The written speech justifying this definition involves the presentation of the tangent line as the limit position of a sequence of secant lines, when the interval between the two intersections with the curve wears thin. To be operational this definition entails firstly a pointwise and global work: the choice of another point on the curve, different from that of tangency, and so of an interval between the two abscissas. For both the textbooks, this work is done in the semiotic bundle composed of graph, words and symbols. The local perspective enters this semiotic bundle through the symbolic resource, namely with the introduction of the symbols $\lim$ as $h \to 0$. The words try to convey the local perspective through terms of movement, and in particular of approaching. On the static graph, instead, the idea of a point approaching another one is really difficult to express. A possibility (that of Sasso, 2012) is to add arrows on the graph, to indicate that a point moves towards another one. Another possibility (that of Bergamini et al., 2013) consists in juxtaposing a sequence of different Cartesian planes, where the points on the curve are chosen each time more close. In both cases, the adoption of the local perspective is left to the reader's capability of correctly interpreting such graphical devices in relation to the words that accompany them. In other terms, it is up to the reader to understand and establish the correct relationships among the different resources of the semiotic bundle *graph+words+symbols*. The result is that the local perspective is potentially, but implicitly, contained in the semiotic resources provided by the textbook.
What can directly foster its adoption is the representation of the derivative function, not in the algebraic register, but in the graphical one. Especially within the examples and the exercises, a local work is needed in order to interpret the behaviour of a function in the neighbourhood of a stationary point or a point of discontinuity or non-differentiability.
Therefore, the local perspective becomes explicit when the graphical resource is accompanied by a speech that enhances the reading of the graph in a neighbourhood of a point (on the left / on the right of it) rather than the idea of movement or the sudden introduction of the symbol $\lim$.

3.3 The derivative in the final examination

The data we take into account in this paragraph are contained and discussed in greater detail in a comparative study between France and Italy concerning the teaching of functions at secondary level (Derouet & Panero, 2014). In this study, the Italian final examination given to experimental courses in scientific high schools in June 2013 has been compared with the corresponding *baccalauréat S Metropole* of June 2013. In the context of this thesis, we find interesting to focus on some of the exercises and problems of the Italian examination in question which involve derivatives.
The mathematical test is composed of two problems and ten questions. Notice that the student has to solve one problem and five questions, then the commission will correct only the exercises that have been indicated by the student. Let us consider the two problems proposed in June 2013. The original text (MIUR, 2013) is in Appendix B.

PROBLEM 1
A function $f(x)$ is defined and differentiable, along with its derivatives of first and second order, in $[0, +\infty[$ and the figure [see Fig. 3.2.13] shows the graphs Γ and Λ respectively of $f(x)$ and its second derivative $f''(x)$. The tangent to Γ in its inflection point, whose coordinates are $(2; 4)$, passes through $(0; 0)$, while the straight lines $y = 8$ and $y = 0$ are horizontal asymptotes respectively for Γ and Λ.
1. Prove that the function $f'(x)$, that is the first derivative of $f(x)$, has a maximum and determine its coordinates. Knowing that for each x in the domain it is $f''(x) \leq f'(x) \leq f(x)$, what is the possible graph of $f'(x)$?
2. Suppose that $f(x)$ represents, obviously with suitable measure units, the growth model of a certain type of population. What information about its evolution one can deduce from the graphs in figure and in particular from the fact that Γ presents an horizontal asymptote and an inflection point?
3. If Γ is the graph of the function $f(x) = \dfrac{a}{1 + e^{b-x}}$, prove that $a = 8$ and $b = 2$.
4. Under the hypothesis of the point 3., calculate the area of the region of plane delimited by Λ and by the x-axis on the interval $[0, 2]$.

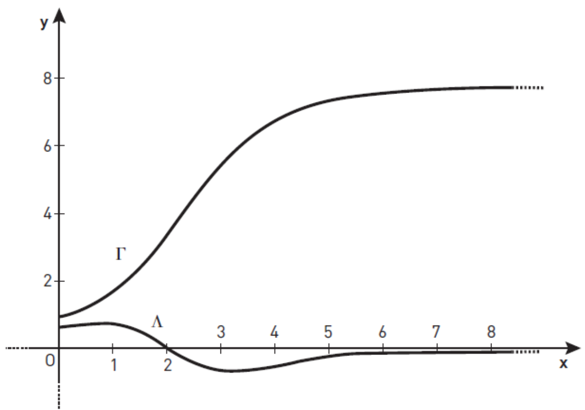


FIGURE 3.2.13 - THE GRAPH GIVEN AS A REFERENCE FOR PROBLEM 1 TEXT.

Our first remark is that almost the whole problem turns around the derivative function. Indeed, f' is the object of the study, while f and f'' are tools to accomplish it. Let us examine which types of task related to the derivative notion are proposed. We provide an expected answer for each of them, in order to understand how the derivative intervenes in the resolution.

Question 1. Given the graph of one of its primitives f and that of its derivative f'' and some pointwise information about the tangent line in the inflection point of f,

- proving the existence of a maximum for f';
- determining the coordinates of the maximum of f';
- drawing a possible graph for f', knowing that $f''(x) \leq f'(x) \leq f(x)$ for all x in the domain.

Expected answer to question 1:

In order to find the maximum of f', we search for x for which the derivative, that is f'', is zero, so it means $f''(x) = 0$. We observe that the graph of f'', represented by , cuts the x-axis in $x = 2$ and changes of sign, thus $x = 2$ is the abscissa of a stationary point of f'. By studying the sign of $f''(x)$, we determine if it is a maximum or a minimum.

x	0		2		$+\infty$
$f''(x)$		$+$	0	$-$	
$f'(x)$		$\nearrow$		$\searrow$	

Hence, f' admits a maximum in $x = 2$. The ordinate of this maximum is $f'(2)$. Now, $f'(2)$ is the gradient of the tangent to the curve of f, that is , at the inflection point of abscissa 2.

$$f'(2) = \frac{f(2) - 0}{2 - 0} = \frac{4}{2} = 2$$

since the tangent to at the inflection point, whose coordinates are $(2; 4)$, passes through $(0; 0)$. Therefore, the coordinates of the maximum of f' are $(2; 2)$.
Afterwards, from the variation of f,

x	0		$+\infty$
$f(x)$		$\nearrow$	
$f'(x)$		$+$	

we deduce that f' is positive on $[0;+\infty[$. From the sign of f'',

x	0		2		$+\infty$
$f''(x)$		$+$	0	$-$	
$f'(x)$		↗		↘	

we deduce that f' is increasing on $[0;2]$, then decreasing on $[2;+\infty[$ and it admits a maximum in $x=2$ (namely the point $(2;2)$ we found before). Finally, from the variation of f'',

x	0		≈ 1		≈ 3		$+\infty$
$f''(x)$		↗		↘		↗	
$f^{(3)}(x)$		$+$	0	$-$	0	$+$	

we deduce that f' is convex, then concave, then convex again. Moreover, since $f''(x) \leq f'(x) \leq f(x)$, the curve of f' lies in the region of plane delimited by the two curves Γ and Λ (Fig. 3.2.14).

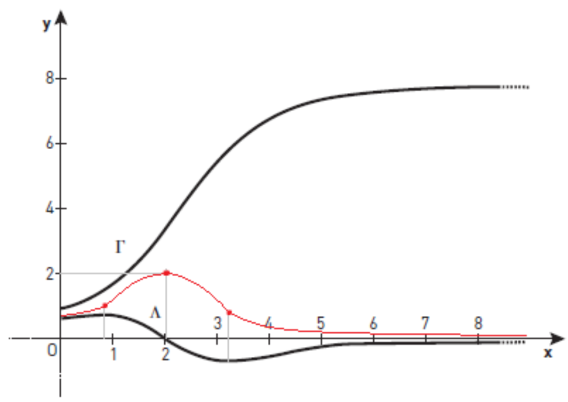


Figure 3.2.14 - Possible graphical representation of f'.

<u>*Question 3.*</u> Determining two parameters involved in the analytic expression of a function, having two conditions.

Expected answer to question 3:

Thanks to the data given in the statement and the answer to question 1, we know that $f(2)=4$ and $f'(2)=2$.

Further, if $f(x) = \dfrac{a}{1+e^{b-x}}$, then $f'(x) = a\dfrac{e^{b-x}}{(1+e^{b-x})^2}$.

We can observe that $f'(x) = f(x)\dfrac{e^{b-x}}{1+e^{b-x}}$.
So, we have to solve the following system:

$$\left\{\begin{array}{rcl} f(2) & = & 4 \\ f'(2) & = & 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \dfrac{a}{1+e^{b-2}} & = & 4 \\ \\ f(2)\dfrac{e^{b-2}}{1+e^{b-2}} & = & 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \dfrac{a}{1+e^{b-2}} & = & 4 \\ \\ 4\dfrac{e^{b-2}}{1+e^{b-2}} & = & 2 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcl} \dfrac{a}{1+e^{b-2}} & = & 4 \\ \\ 2e^{b-2} & = & 1+e^{b-2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \dfrac{a}{1+e^{b-2}} & = & 4 \\ \\ e^{b-2} & = & 1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcl} \dfrac{a}{1+e^{b-2}} & = & 4 \\ b & = & 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \frac{a}{2} & = & 4 \\ \\ b & = & 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} a & = & 8 \\ b & = & 2 \end{array}\right.$$

Question 4. Calculating the area under the curve of the derivative of second order, by using an integral.

Expected answer to question 4:

The required area $\mathcal{A}$ is equal to the following definite integral:

$$\mathcal{A} = \int_0^2 f''(x)\,\mathrm{d}x.$$

Now, according to the fundamental theorem of the integral calculus, if G is a primitive of g, then $\int_a^b g(x)\,\mathrm{d}x = G(b) - G(a)$.
Here, one primitive of f'' is f', so

$$\mathcal{A} = f'(2) - f'(0) = 2 - \frac{8e^2}{(1+e^2)^2}.$$

In solving these types of task, the notion of derivative is expected to be recalled both as an object and as a tool (Douady, 1986). Indeed, on the one hand, the student has to apply the definition of derivative itself as the gradient of the tangent line. It occurs, for instance, when it is expressly required to determine the ordinate of the maximum point of f' in question 1.. On the other hand, the student is also expected to introduce and to use the derivative as a tool in order to solve tasks that do not directly allude to the derivative. It happens in question 3., when the student has to put into play the derivation formula $D\left[\dfrac{u}{v}\right] = \dfrac{u'v - uv'}{v^2}$, in order to establish the system of two equations in two unknowns. And it is also the case of question 4., when f' has to be seen as the primitive of f'' and so it intervenes as a tool in the solving formula, namely $\mathcal{A} = \int_0^2 f''(x)\mathrm{d}x = f'(2) - f'(0)$. Thus, at a technical level the derivative is employed in different forms:

- as a direct application of the definition;

- as a tool in other formulas/methods specifically employed to solve the task.

On the technological-theoretical side, the function f' changes frequently its role within the properties and the definitions to recall. In particular, it must be considered as both the derivative of f and the primitive of f''. This is a complex involvement of the derivative concept and of the derivative function.
As for the perspectives, notice that, actually, within this problem, **it is not necessary to activate a local perspective on the involved functions**. The game remains pointwise/global, and it is the same for its derivatives of first and second order, f' and f''.
What is largely fostered here is the coordination of different semiotic resources: about f and f'' graphical, discursive and symbolic information is given, while about f' we have not access to any kind of representation. Its graph is directly required in question 1., whereas a symbolic expression has to be found in order to establish the system in question 3.

PROBLEM 2
Let f be the function defined for all x positive by $f(x) = x^3 \ln x$.
1. Study f and sketch its graph γ in a Cartesian orthonormal reference system Oxy; after having verified that γ presents both an inflection point and a minimum point, calculate, using a calculator, their abscissas rounded to the third decimal place.
2. Let P be the point in which γ intersects x-axis. Find the equation of the parabola which has axis parallel to y-axis, passes through the origin and is tangent to γ in P.
3. Let R be the region delimited by γ and by x-axis on the right-open interval $]0, 1]$. Calculate the area of R, showing the followed reasoning and express it in mm^2, having supposed that the linear measure unit corresponds to $1\ dm$.
4. Draw the symmetrical curve of γ with respect to y-axis and also write its equation. Do the same for the symmetrical curve of γ with respect to the straight line $y = -1$.

Notice that this task is less centred on the derivative. The object of the study is mainly f and, even though f' and f'' are not mentioned, they become useful as tools to study f. The proposed types of task which involve the derivative, along with the expected answers, are the following.

<u>*Question 1.*</u> Studying a given function in order to draw its possible graph (in particular, studying its variation and its concavity).

Expected answer to question 1.

In order to study the variation of the function f, we have to study the sign of the derivative f' of f. For all $x \in D_f$,

$$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2 = x^2(3 \ln x + 1).$$

Hence,

$$f'(x) \geq 0 \Leftrightarrow x^2(3 \ln x + 1) \geq 0 \Leftrightarrow 3 \ln x + 1 \geq 0 \Leftrightarrow \ln x \geq -\frac{1}{3} \Leftrightarrow x \geq e^{-1/3}$$

x	0		$e^{-1/3}$		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$	

Therefore, f admits a minimum in $x = e^{-1/3}$. The minimum M has coordinates $(e^{-1/3}; f(e^{-1/3}))$, so $M(e^{-1/3}; -\frac{1}{3}e)$.

In order to study the concavity of the function f, we have to study the sign of the second derivative f'' of f. For all $x \in D_f$,

$$f''(x) = 2x(3 \ln x + 1) + x^2 \frac{3}{x} = 2x(3 \ln x + 1) + 3x = x(6 \ln x + 5).$$

Hence,

$$f''(x) \geq 0 \Leftrightarrow x(6 \ln x + 5) \geq 0$$

and, since $x \in D_f =]0; +\infty[$, x is always positive, then

$$f''(x) \geq 0 \Leftrightarrow 6 \ln x + 5 \geq 0 \Leftrightarrow \ln x \geq -\frac{5}{6} \Leftrightarrow x \geq e^{-5/6}$$

.

x	0		$e^{-5/6}$		$+\infty$
$f''(x)$		$-$	0	$+$	

Thus, f is concave on $\left]0; e^{-5/6}\right]$ and convex on $\left[e^{-5/6}; +\infty\right[$.
Therefore, γ admits an inflection point at the abscissa $x = e^{-5/6}$. The inflection point F has coordinates $(e^{-5/6}; f(e^{-5/6}))$, so $F(e^{-5/6}; -\frac{5}{6}e^{-5/6})$.

Question 2. Determining the equation of a parabola, knowing that it passes through two points and that it is tangent to a given curve in one of these points.

Expected answer to question 2.

The parabola, whose symmetry axis is parallel to y-axis, has equation $y = ax^2 + bx + c$, where the real numbers a, b, c are to determine. If the parabola has to be tangent to the curve γ in P, both the curves must have the same tangent in P. It means that, in the point of abscissa $x = 1$, the derivative of g, where $g(x) = ax^2 + bx + c$, must be equal to the derivative of f, where $f(x) = x^3 \ln x$:

$$g'(1) = f'(1) \Leftrightarrow 2a \cdot 1 + b = 1 \cdot (3 \ln 1 + 1) \Leftrightarrow 2a + b = 1. \tag{3.3}$$

By imposing the following conditions: the parabola passes through the point P and through the origin, and the equation (3.3) is satisfied, we obtain the system:

$$\begin{cases} 0 = a \cdot 1 + b \cdot 1 + c \\ 0 = a \cdot 0 + b \cdot 0 + c \\ 2a + b = 1 \end{cases} \Leftrightarrow \begin{cases} a + b + c = 0 \\ c = 0 \\ b = 1 - 2a \end{cases} \Leftrightarrow \begin{cases} a + 1 - 2a = 0 \\ c = 0 \\ b = 1 - 2a \end{cases}$$
$$\Leftrightarrow \begin{cases} a = 1 \\ c = 0 \\ b = 1 - 2a \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ c = 0 \\ b = -1 \end{cases}$$

Thus, the parabola representing the function g has equation $y = x^2 - x$.

Question 3. Calculating an area, through an improper integral.

Expected answer to question 3.

The area of R corresponds to the following integral (since f is negative on the given interval):

$$Area_R = -\int_0^1 x^3 \ln x \,\mathrm{d}x = -\lim_{t \to 0^+} \int_t^1 x^3 \ln x \,\mathrm{d}x.$$

We apply the formula of integration by parts:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

where

$$f(x) = \ln x \Rightarrow \quad f'(x) = \frac{1}{x}$$
$$g'(x) = x^3 \Rightarrow \quad g(x) = \frac{x^4}{4}.$$

We obtain:

$$\begin{aligned} Area_R &= -\lim_{t \to 0^+} \left(\left[\frac{x^4}{4} \ln x \right]_t^1 - \int_t^1 \frac{1}{x} \frac{x^4}{4} \,\mathrm{d}x \right) = -\lim_{t \to 0^+} \left(-\frac{t^4}{4} \ln t - \frac{1}{4} \int_t^1 x^3 \,\mathrm{d}x \right) \\ &= -\lim_{t \to 0^+} \left(-\frac{t^4}{4} \ln t - \frac{1}{4} \left[\frac{x^4}{4} \right]_t^1 \right) = -\lim_{t \to 0^+} \left(-\frac{t^4}{4} \ln t - \frac{1}{16}(1 - t^4) \right) \\ &= \frac{1}{4} \lim_{t \to 0^+} t^4 \ln t + \frac{1}{16} \lim_{t \to 0^+} (1 - t^4) = \frac{1}{4} \lim_{t \to 0^+} t^4 \ln t + \frac{1}{16}. \end{aligned}$$

The limit $\lim_{t\to 0^+} t^4 \ln t$ gives an indeterminate form $[\infty \cdot 0]$. Therefore,

$$\begin{aligned}\lim_{t\to 0^+} t^4 \ln t &= \lim_{t\to 0^+} \frac{\ln t}{t^{-4}} = \left[\frac{\infty}{\infty}\right] \quad \text{U.F.}\\ &= \lim_{t\to 0^+} \frac{t^{-1}}{-4t^{-5}} = \lim_{t\to 0^+} \frac{t^4}{-4} = \frac{0^+}{-4} = 0^- \quad \text{(thanks to the rule of de l'Hôpital)}\end{aligned}$$

Then, we obtain that the $Area_R = \frac{1}{16}\, dm^2 = 0,0625\, dm^2 = 625\, mm^2$.

In the methods to accomplish all these types of task, the derivative intervenes as a tool (Douady, 1986), in particular at the symbolic level. In the statement, there is no allusion to the use of the derivative in the solving techniques. The need to recall it emerges through the argumentation of the resolution processes, namely on the technological-theoretical plane. In question 1., the student has to mobilize a fundamental property related to the derivative concept, that is the relation between the variation of the function and the sign of its derivative. In question 2., it is necessary to make operative the definition of derivative as the gradient of the tangent line, since the latter must be the same for both the involved functions. Anyway, it should be usual for the students to employ the derivative as a tool within these situations. In question 3., instead, the use of the derivative is completely up to the student and not suggested by the context, which consists of an integral calculation. Here, resorting the derivative reveals essential in the following techniques:

- applying the formula of integration by parts for calculating the integral $\displaystyle\int_t^1 x^3 \ln x \, \mathrm{d}x$;
- applying the rule of de l'Hôpital for solving the indeterminate form, given by the $\lim_{t\to 0^+} t^4 \ln t$.

Contrary to Problem 1, information about f is given in a verbal-symbolic form and also the introduction of f' by the student may occur at the syntactical level.
Even though the student has to resort the derivative as a tool, **it does not lead to the adoption of a local perspective** on the studied function, namely f. Applying properties and formulas in a pointwise or global way reveals sufficient.
Anyway, in this case, it is the integral notion that should activate a local perspective on the function f. Indeed, the integral to calculate is improper, since the function $f(x) = x^3 \ln x$ is not defined in $x = 0$, which is the left extreme of the given interval. Thus, without any indication, the student has to introduce an intermediary t, varying between 0 and 1, to calculate the area as the integral $\int_t^1 x^3 \ln x \, \mathrm{d}x$, and finally to make t tend to 0^+. This is the only moment, in the resolution of the two problems, in which a local perspective is effectively required on the given function f. Nevertheless, we can conclude that the local aspects are really put into play only in one over the eight questions proposed throughout the two problems, and it happens within the determination of an integral and not through the use of the derivative.

3.3.1 Remarks

Many of the proposed tasks of the analysed final examination involve the derivative concept. The expected resolution entails the differentiation both as an object and as a tool (Douady, 1986). This is not just a matter of calculations. Indeed, some exercises make the derivative work in the graphical frame, with frequent changes from the algebraic-symbolic register to the graphical one.
Nevertheless, recalling and employing the derivative and its properties to solve the tasks do not require to adopt a local perspective on the involved functions. However, it is the work with the integral that entails this adoption. Thus, we can say that the local perspective is expected to be activated, but only in one over the eight proposed tasks (not all compulsory, because it is the student that chooses which problem, composed of four questions, to solve). Moreover, it occurs within an integration task, not directly involving the derivative. In conclusion, the local feature of the differentiability property is not evaluated in the final examination we have analysed.

Chapter 4

Analysis of teachers' practices: three case studies

This chapter is the core of the thesis. We focus on teachers' practices in classroom. First of all, we are going to introduce the three phases of the research: interviews with the teachers, observation of their practices around the derivative concept in classroom and activities with the students. These three phases have provided us with audiotapes, videos, written productions. We analyse the data through the theoretical framework introduced in Chapter 2 (see Section 2.2). The aim is examining how the local perspective enters the semiotic bundle and intervenes in the mathematical praxeology developed by the teacher in the classroom.

4.1 Research methodology

Three teachers joined up our research project. They have different working experiences in different high schools. We revealed them the basic intention of our research project: to study the teaching practices related to the derivative concept. Initially, we did not tell them anything about our analysis lenses, so that the lessons follow their natural course. The lessons necessarily occurred in an unusual context, due to the presence of an external observer. Our concern was to influence the regular lesson development no more than that. The data collection consists of three phases: we conducted a preliminary interview; we observed each teacher's lessons; we proposed two activities to the students. The data collected in the last phase are analysed in depth in the next chapter, whereas in this chapter our focus is on the teachers.

4.1.1 Interviews

Each teacher has been interviewed. The conversation has been audio-recorded upon the interviewee's consent.
We made explicit that the context was that of a PhD thesis in Mathematics Education. In particular, we presented the project as a research on the teaching of the derivative

concept. The teachers seemed interested in the topic. Indeed, they all expressed their personal concern about the difficulty in teaching and learning this notion. Then, we explained our methodological approach as external observers in the classroom. Their lessons would have been videotaped upon their consent. We showed our intention of proposing two activities to the students. We did not reveal the content not to influence their natural practices. Nonetheless, we promised to discuss together about the activities, before proposing them in the classroom.
Finally, we made explicit to the teachers the purpose of the preliminary interview. In the light of the following observation phase, we needed to get insights into their usual practices, in favour of a better interpretation of data. We made extremely clear from the start that our goal was not to compare or judge them. Rather, we were interested in having a range of practices as varied as possible.

Therefore, we organized a semi-structured qualitative research interview, following Kvale's guidelines (1996). We prepared a sequence of themes we wanted to cover around the teaching of the derivative topic.

- Use of the textbook;
- Usual or ideal introduction of the derivative concept;
- Usual or ideal introduction of the derivative function;
- Possible influences from previous practices.

Around these themes, we prepared a list of essential questions, to pose or cover during the interview. Nonetheless, we were opened to changes of sequence and formulation of the questions, "in order to follow up the answers given and the stories told by the subjects" (Kvale, 1996, p.124).

- What textbook has been adopted in your school?
- Do you like it? Do you follow it?
- How did you introduce limits with your students? What kind of work did you do on them?
- Do you think it may influence the students' approach to the derivative concept?
- How do you usually introduce the derivative notion?
- How do you usually introduce the derivative function?
- What are the main difficulties among the students?

Usually the ice-breaking question concerns the introduction of the derivative concept or the adopted textbook.

Beside this preliminary interview, we had other opportunities to talk more informally with the teachers during and after the observation phase. In those occasions, our questions have been more probing and interpreting, rather than introducing and direct (Kvale, 1996). It means that our preliminary questions formulated like "Can you tell me something about...?" or "How do you do that?" transform into the form "Could you say something more about what happened?" or "Is it correct that you feel/mean that...?". These last questions depend on what we observed during the lessons (e.g., a reaction to some student's intervention, a particular choice of task/resource).

From the research point of view, the interviews provided us with some insights into the so-called declared or expected practices. More deeply, they revealed us the teachers' beliefs about the topic and the related practices (Furinghetti & Morselli, 2011; see Paragraph 2.3.1). They permitted us to interpret some teacher's behaviours or to justify some of her choices.

We are going to present some significant extracts from the interviews in the opening paragraph of each teacher's case. We do not transcript them, because our goal is not to analyse their speech, but to collect their beliefs about the proposed themes. Therefore, you will find a personal elaboration of the answers content.

4.1.2 Observation in classroom

To collect data of teachers' real practices in classroom, we adopted the method of participant observation (DeWalt, DeWalt & Wayland, 1998). It is commonly used in anthropological fieldwork, but it has been more and more employed also in Mathematics Education as "the most common way of collecting and interpreting data from the classroom" (Arzarello et al., 1998, p.249). The participant observer "develops a split between observing and observed subjects in a dialogical relation" (Arzarello et al., 1998, p.250). Using the distinction made by DeWalt et al. (1998), our observation method is characterized by a "moderate participation" that occurs when the observer "is present at the scene of the action but does not actively participate or interact, or only occasionally interacts, with people in it" (DeWalt et al., 1998, p.262). It is what has happened in each classroom for about 10 hours of lessons. Upon the teachers' and students' consent, we videotaped the lessons for later analysis. The most significant parts of the videos have been selected (through the criteria described in the next paragraph), cut and transcribed. The transcription includes all the employed semiotic resources: not only the speech, but also screen shots of gestures, drawn graphs, sketches, written symbols and so on. You will find the transcription of these excerpts integrated within the paragraph discourse in the case of each teacher. The letter "T" denotes the teacher, the letter "S" followed by a number (e.g., S1, S2, ...) denotes a particular student, while "Ss" denote an undetectable group of students or almost the totality of the class. Outside the transcription lines, instead, we use the name initials to refer to the teachers, namely M., M.G. and V..

4.2 Analysis methodology

In this paragraph we are going to explain how we have chosen the extracts to analyse and how we have combined the theoretical lenses introduced in Chapter 2 for the analysis.

4.2.1 Choice of the practices to analyse

The observation of each teacher in classroom left us with more than 10 hours of video-taped lessons. Obviously, not every practice is significant to be analysed. Our focus is on the two types of task $\mathcal{T}_{tangent}$ and $\mathcal{T}_{f'}$ we recall below.

Type of task $\mathcal{T}_{tangent}$	"determining the equation of the tangent line to a generic function in a point"
Type of task $\mathcal{T}_{f'}$	"representing the derivative function"

These types of task turn out to be extremely important and delicate for the three teachers since the first interview. They usually work on them to accomplish the respective didactic tasks: "introducing the derivative concept" and "fostering the conceptualization of the derivative function". In particular, we are interested in analysing the phases through which the teacher guides the students' mathematical work, by introducing or constructing the techniques for $\mathcal{T}_{tangent}$ and $\mathcal{T}_{f'}$.
It is in those circumstances, indeed, that the teacher (consciously or not) introduces a new perspective on the involved functions (typically f and f') and, at the same time, she must deal with the previously worked perspectives. The perspectives on f' are all new for the students, whereas much work has already been done on f from a pointwise and global point of view. Beside them, dealing with the derivative concept leads to adopt a local perspective on f, which is fresh for the students at grade 13. It entails an exchange between old and new knowledge, seen as both *savoir* and *savoir faire*.
Moreover, there are some semiotic resources (such as graphs or symbols) that can be chosen by the teacher in order to approach and solve the task. Availing of these resources, in turn, can foster the activation of certain perspectives on the involved functions or even restrain some others.

4.2.2 Lenses of analysis and their combination

Following Chevallard's model of didactic moments, we focus in particular on three phases of the teacher's design and development of the praxeology. Referring to the types of task $\mathcal{T}_{tangent}$ and $\mathcal{T}_{f'}$, we detect the moment of the first meeting with $\mathcal{T}$, the moment of exploration and construction of a technique and the technological-theoretical moment. Even if the teacher has a clear and organized scheme in her mind, the three moments are often intertwined in a classroom practice. We give a general overlook on the teacher's didactic organization, by stressing the types of task, tasks and problems she proposes

and solves with the students. We try to interpret the role of each part of the organization within the developing praxeology.
During the process of construction of a new technique, with a proper technology, for a specific given task, the old techniques, with their own technologies, emerge and take part in the formulation of the new practice. Our analysis allows us to determine which previously worked praxeologies intervene, in what moment and at what level.
Then, we conduct a more fine analysis of some selected video extracts. We primarily base on activated perspectives and semiotic resources. Essentially, we characterize teacher's and students' speech, gestures, symbols, sketches or drawings in terms of pointwise, global or local perspectives. The analysis is accompanied by our personal remarks about the dialectics between perspectives and semiotic resources, within the rising praxeology. Thanks to this kind of analysis we try to explain how previously worked praxeologies intervene in the formulation of the new ones. In particular, before starting the analysis of each case, we want to schematically recall in Table 4.1 the conics-related praxeology. In all our case studies, it is part of the students background and the teacher has to deal with it, be her choice to recall, to review or to reject it.

$O\mathcal{M}_{\text{conics}}$	
Type of task	Determining the equation of the tangent line to a conic in one of its points P
Techniques	1) $\Delta = 0$ 2) Doubling rule 3) $m_{tg} = -(m_{radius})^{-1}$ (for the circle)
Technology	1) The tangent line is defined as the straight line that has two coincident intersections with the conic 2) / 3) The tangent to a circle centred in O in one of its points P is the perpendicular to the radius OP, passing through P; they know by heart the relation of anti-reciprocity for the angular coefficients of two perpendicular lines.
Theory	1) Analytic equation of a generic straight line passing through a given point P, system algebraic solving rules, second order equations, first and second order inequalities 2) / 3) Analytic equation of a generic straight line passing through a given point P, first order equations

Table 4.1 - Mathematical praxeology for the tangent to a conic: constructed two years before, it is common to our three case studies.

4.3 The case of M.

M. teaches maths and physics in a high school in Savigliano (Cuneo). She has been working in this school for several years, and in particular with *Quinta B*'s students since they attended their third year. The school has adopted the textbook *Nuova Matematica a colori. Edizione BLU per la riforma* written by Sasso (2012). M. has contributed to its choice, finding it "very close to her way of teaching". She follows it for both preparing lessons and giving exercises to students. Nevertheless, she also chooses interesting activities from other resources, such as the modules *Multi ForMat* by Maraschini & Palma (2000). Moreover, in her practices, she tells to be influenced by *L'analisi infinitesimale negli studi preuniversitari: proposte didattiche* (Dupont, 1983), that is a booklet written by Prof. Pascal Dupont for the university courses of Mathematics Education attended by M..

4.3.1 From the interviews: M.'s beliefs

In the preliminary interview with M., in November 2012, we spoke about the work done with her students on limits, as preparatory to the introduction of the derivative concept.

How did you introduce limits with your students? What kind of work did you do on them? And how do you think this may influence the students' approach to the derivative concept?
M. approached the limit definition with the neighbourhood notion. She does not use the definition in $\epsilon - \delta$, because she thinks it seriously complicates things for students. In her experience, she noticed that students always get concerned about this definition and make great efforts just to remember it by heart, without trying to understand it. By referring to neighbourhoods, without any explicitness in terms of ϵ and δ, M. and her students graphically tried to conceptualize the limit notion. They started from a neighbourhood on y-axis and went towards a neighbourhood on x-axis. They also approached the few theorems they studied, such as the limit uniqueness and the squeeze theorem, with the limit definition given through neighbourhoods.
Afterwards they dealt with the so-called remarkable limits. She always says that there are only two limits of special interest:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

They proved these two and deduced all the other limits from them. In order to justify the second remarkable limit, M. had the opportunity to speak about Nepero's number definition as the limit of the sequence $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n$, which they had intuitively studied with tables of values in excel.

How do you usually introduce the derivative notion?
As the textbook does (Sasso, 2012), M. usually introduces the derivative concept by

proposing two problems that, also from an historical point of view, have led to the introduction of this notion. The first is the geometrical problem of the tangent line and the second is the physical problem of the instantaneous speed. They solved both of them by introducing a limit: as $\Delta x \to 0$, in the case of $\frac{\Delta y}{\Delta x}$, and as $\Delta t \to 0$, in the case of $\frac{\Delta s}{\Delta t}$. Then, M. compares the two problems, showing that they actually are the same problem. In the geometrical case, M. denotes $f'(x_0)$ the obtained limit, when it exists and it is finite. She calls it derivative of the function f in the point of abscissa x_0. She stresses its geometrical meaning: it corresponds to the angular coefficient[1] of the tangent line to the function f in x_0.

In classroom, since the first lesson about the derivative concept, M. immediately spoke about "derivative function". After we discussed about the way she introduced the derivative function to students.

Let's talk about the choices you made about the introduction of the derivative function: moment of the lesson, methods, ...?
She answered: "I always think about it: starting from the derivative function in a point, which is a number, one must obtain the function. This is really a delicate step... It is a sort of conquest: every time, I make a calculation that I could actually make only once. I find that if you [teacher] give to students a justification of what you are doing, they get it better..."
As far as the methods are concerned, M. thinks that the key consists of working on both the algebraic and the graphical side. She prefers explaining by hand, at the blackboard, the whole process to obtain the graph of f', starting from that of f. That is because she believes this can help the students in conceptualizing the derivative of a function as a function itself, with its own graphical representation. So, she wants them to do the whole process by hand on their notebooks, before doing it with GeoGebra. She adds: "I love GeoGebra and I really believe that working dynamically with this kind of software can be extremely useful both to strengthen and to discover. Indeed, within some contexts, such as the Euclidean geometry and the geometrical transformations, I rely on GeoGebra to help visualization. On the contrary, with the derivatives, I see it as a mean to confirm all the steps done by hand".

4.3.2 Type of task $T_{tangent}$: determining the equation of the tangent line to a generic function in a point

From the interviews, it transpires that M. includes the textbook among the didactic resources to design her lessons. Indeed, she tells us that she has chosen Sasso's textbook, proposed it to her colleagues and, then, adopted it along with the whole school. In classroom, the structure she actually gives to the first lesson is exactly the textbook one (see Paragraph 3.2.2, in particular Fig. 3.2.8). She also gives the reference to the students, so that they can follow: "The things we are going to see now are on your

[1] For the use of the expression "angular coefficient", see note n.4 in Chapter 3.

textbook, approximately... the page is 258". Then, she suggests a "title" for the lesson, by saying:

> "*If you want to note something down, now we are going to do the introduction to the derivative concept: the problem of variations. Derivatives are one of the fundamental Calculus concepts and they will help us to solve several problems. [...] I'm going to introduce two problems that from an historical point of view led, in the 17th century, through their resolution, to the derivative concept. They are the problem of the tangent and the problem of the instantaneous speed.*"

During the construction of the praxeology for the problem of the tangent, the teacher holds the control of the lesson. The students intervene only for posing questions, if they do not understand something on the blackboard, or if they are questioned by the teacher. By presenting the problem of the tangent and the problem of the instantaneous speed, with some examples, the teacher works on the technological-theoretical block of the praxeology. Afterwards, she formalizes the derivative concept:

> "*I start by defining the derivative of a function in a point.*"

She defines it as the limit of the incremental ratio of the function, when the abscissa x_0 is incremented of h, as h goes to 0. Thus, she gives a technique to find the angular coefficient of the tangent line to a generic function in a point. Returning back to the problem of the tangent, she makes an example with the function $y = x^2$ in the point of abscissa $x_0 = 2$, in order to show the technique just introduced and to practice it.
M.'s didactic organization is summarized in Table 4.2.

Tasks t, type of tasks T and problems given in classroom	**Construction of the OM for $\mathcal{T}_{tangent}$**
Problem 1 (of the tangent), $\mathcal{T}_{tangent}$: we have a generic function, we want to determine the tangent line in a specific point.	The teacher wants to stress the similarities between this two kinds of problems, working at a technological-theoretical level.
Problem 2 (of the instantaneous speed): we launch a body upward, we want to determine its speed in the instant 3 seconds.	The teacher prepares the tools (in particular, the limit) for constructing a new technique $\tau_{tangent}$, justifying their necessity.
$\mathcal{T}_{m_{tg}}$: formalizing the derivative concept.	By introducing a subtype of task $\mathcal{T}_{m_{tg}} \subset \mathcal{T}_{tangent}$, the teacher gives a technique $\tau_{m_{tg}}$ to find the angular coefficient of the tangent line.
$t_{m_{tg}}$: calculating the first derivative of the function $y = x^2$ in the point of abscissa $x_0 = 2$.	Within an example, the teacher practices the technique $\tau_{m_{tg}}$.
$t_{tangent}$: writing the equation of the tangent line to the function $y = x^2$ in $x_0 = 2$.	Within the same example, the teacher frames the technique $\tau_{m_{tg}}$ inside the general technique $\tau_{tangent}$.

Table 4.2 - M.'s didactic organization around the type of task $T_{tangent}$.

Notice that other previous praxeologies intervene: $O\mathcal{M}_{\text{straight line}}$ (Table 4.3), in the mathematical domain, and $O\mathcal{M}_{\text{average speed}}$ (Table 4.4), in the physical domain. In both of them, we find an algebraic formula involved at the technical level: that of the angular coefficient of a straight line $m = \dfrac{\Delta y}{\Delta x}$ and that of the average speed $v = \dfrac{\Delta s}{\Delta t}$.

$O\mathcal{M}_{\text{straight line}}$	
Type of task	Determining the equation of a straight line passing through two points $P(x_P, y_P)$ and $Q(x_Q, y_Q)$
Technique	$r : y - y_P = m_{PQ}(x - x_P)$ or $y - y_Q = m_{PQ}(x - x_Q)$ where $m_{PQ} = \dfrac{y_P - y_Q}{x_P - x_Q}$
Technology	Graphical and intuitive justification.
Theory	- Through two points in the plane it passes one and only one straight line. - Coordinates in the Cartesian plane.

Table 4.3 - Previous mathematical praxeology related to the straight line.

$\mathcal{OM}_{\text{average speed}}$	
Type of task	Determining the average speed of a body moving from A to B.
Technique	$v = \dfrac{s_B - s_A}{t_B - t_A}$
Technology & Theory	Average speed is the rate of change of distance with time.

Table 4.4 - Previous physical praxeology related to the average speed.

The teacher develops the praxeology $\mathcal{OM}[\mathcal{T}_{mtg}/\tau_{mtg}/\theta_{mtg}/\Theta_{tangent}]$. Three didactic moments are detectable:

1. meeting with the type of task, trough two significant problems;
2. construction of the technological-theoretical block for $\mathcal{T}_{mtg}$;
3. elaboration of a technique for $\mathcal{T}_{mtg}$.

The approach to the two problems presented and the technique τ_{mtg} introduced have a local character, essentially because of the use of limit symbol. We are interested in how this local perspective on f is introduced by the teacher.

Construction of the technological-theoretical block for $\mathcal{T}_{mtg}$

M. poses the first problem: she draws a generic function f, she chooses a point $P(x_0, y_0)$ on this graph, traces the tangent line in P (Fig. 4.3.1) and gives the type of task $\mathcal{T}_{tangent}$:

"*We want to determine the tangent line to the curve in the point P.*"

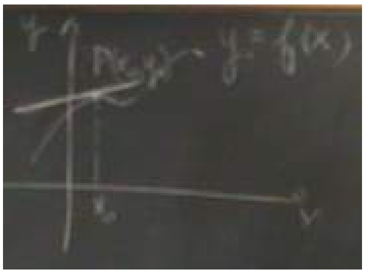


Figure 4.3.1 - M.'s formulation of the type of task $T_{tangent}$ in the graphical register.

Then, she stops to reflect with the class upon the same problem when the function is a conic. We are going to analyse, from both a perspective-based and a semiotic point of view, the teacher's and the students' utterances.

	What happens (teacher-students dialogue)	Perspectives on f	Semiotic resources
1	T: "*Let's think about it for a while... The problem of the tangent is not a problem we have never faced before. Think about the analytic geometry. What have we done with the conics?*"		
2	S1: "*Doubling rule.*"		
3	S2: "*Delta equal to zero.*"		
4	T: "*Ok. If we have a conic, so a second order curve, we have seen the possibility to... Someone said "doubling rule". When did it work? When the point belonged to...*"	pointwise	speech indicators
5	S1: "*To the curve.*"	pointwise	speech indicators
6	T: "*The point belonged to the curve. I've never proved it, I gave it to you as a rule, but it has a link, we could see it, with the derivative. The other possibility was for example to impose the tangency condition... If we have to define what it means that a straight line is tangent to a curve, how did we do it? By imposing the tangency condition, that is by saying that the points, the solutions, between the straight line and the conic were?*"	pointwise	speech indicators
7	S3: "*Coincident.*"	pointwise	speech indicators
8	T: "*Eh! Coincident! So we imposed delta equal to zero. But when can we do it? Obviously when, from the system between the straight line and the conic, I obtain an equation...?*"		
9	S4: "*Of second order.*"		
10	T: "*Of second order. Right?*"		
11	Ss: "*Yes.*"		
12	T: "*But if I have a generic curve, so a function such as $y = \sin x$, am I able to solve the system [which gives] $\sin x = mx$?*"		
13	Ss: "*No...*"		

14	T: "*Absolutely not. So, you see, we need for a more general method, that gives us the possibility to find the tangent in a point to any curve.*"		

With a brief speech, M. recalls to the students the algebraic methods used in the conics-related praxeology (see Table 4.1), in particular the doubling rule and the $\Delta = 0$ method [*1-3*]. As for the former, she lets transpire that a justification lies within the derivative theory, whereas, for the latter, she stresses that its application is strictly related to conics [*4-11*]. From a technological-theoretical point of view, M. underlines that these methods have a pointwise character [*6-8*], and she explicitly highlights the need for a more general method [*12-14*]. Thus, she poses the new praxeology in a general position with respect to the previous conics-related one.
Afterwards, she deletes the picture in Fig. 4.3.1 and draws it again. Here are the words and written signs she uses to prepare the ground for what she calls a "dynamical idea" [line *15*].

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
15	T: "*We can see it [the problem] with an idea, I'd dare to say, of dynamical kind. Let's start not from the tangent, but from the secant line to the curve. That is, let's start from the curve $y = f(x)$. We take a point x, then we take a point x plus... I'd call it Δx. Or better, let's call the point x_0...*" (she completes the drawing in Fig. 4.3.2(a)) "*So, we consider the point P and the point Q. The point P has coordinates $(x_0, f(x_0))$ [...] then, we consider another point, I took it on the right [of P], but I could take it on the left, it doesn't matter. With the increment, I mean not necessarily a positive increment. So, $x_0 + \Delta x$, and its image is clearly $f(x_0 + \Delta x)$.*"	global pointwise	graph + symbols + speech indicators speech indicators
16	T: "*Let's write the equation of the secant line.*" (Tracing the straight line through P and Q) "*Do we know how to write it?*"	pointwise	graph
17	S5: "*Yes [...] We find m...*"		
18	T: "*We find the angular coefficient... as what?*"		
19	S5: "*As the delta between the y of the two points and the x of the two points.*"	global	oral symbols

20	T: "*Sure.*" (Writing $m = \dfrac{\Delta y}{\Delta x}$) "*We know it well: delta y over delta x, so the ratio between the vertical increment and the horizontal one.*" (She adds Δy and Δx on the graph, as in Fig. 4.3.2(b))	global	symbols + speech indicators + graph

Figures

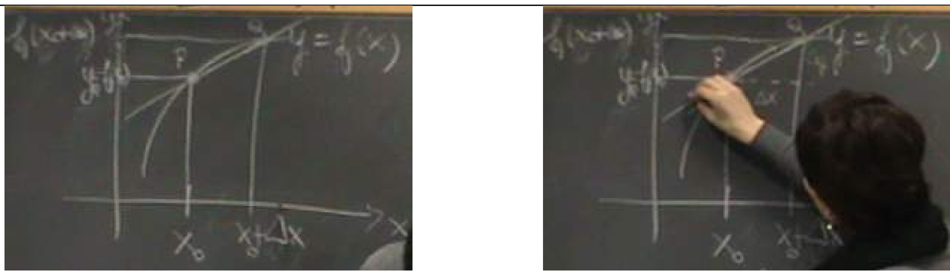


Figure 4.3.2(a) and 4.3.2(b) - M.'s reformulation of the type of task $T_{tangent}$ always in the graphical register.

Student S5 recalls the praxeology $O\mathcal{M}_{\text{straight line}}$ (see Table 4.3) to find, within the analytic geometry, the equation of a straight line [*17* and *19*]. M. writes it and makes it explicit for the case in which they are (Fig. 4.3.3). She adds:

> *"Our $\dfrac{\Delta y}{\Delta x}$, written in terms of the function, is called 'incremental ratio', that from a geometrical point of view represents the angular coefficient of the secant line."*

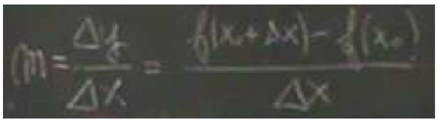


Figure 4.3.3 - M. implements the recalled m technique as incremental ratio.

By starting from the secant, giving two points P and Q on the function, M. fosters the students to recall the straight line-related praxeology, and in particular, the formula for the angular coefficient m. Notice that she starts from a pointwise perspective on the given function, by choosing two points on it [*15-16*]. Then, she moves to a global perspective, by calculating the incremental ratio of f between the two points [*19-20* and Fig. 4.3.3]. Another remark is about the symbols placed on the graph: these two semiotic sets are strictly tied. M. refers to the increment as Δx, both in the speech and in the written symbols, and this may induce the students to recall the definition of m as $\dfrac{\Delta y}{\Delta x}$ [*17* and *19*]. Afterwards, she introduces the dynamical idea and goes towards a local perspective on f. Let us see how.

	What happens (teacher-students dialogue)	Perspectives on f	Semiotic resources
21	T: "*If I take, instead of* Q*, a closer point* Q'*, and I trace the secant line.*" (She traces the line PQ', Fig. 4.3.4) "*Then, I take a point* Q'' *and then again the secant line.*" (She traces the line PQ'', Fig. 4.3.4)	pointwise	graph + speech indicators
22	T: "*What happens as* Q *gets closer and closer to* P*?*"	local	speech indicators
23	S4: "*It [the line] becomes tangent instead of secant.*"	local	speech indicators
24	T: "*Yes! As* Q *gets closer and closer to* P*, the angular coefficient of the secant, let's call it* m_{PQ}*, becomes that of...?*"	local	speech indicators
25	Ss: "*The tangent.*"		
26	T: "*That of the tangent. How can we speak about this approaching...?*"	local	speech indicators
27	Ss: "*Delta* x *goes to zero.*"	local	speech indicators
28	T: "*Delta* x *goes to zero or, if you want, with the... limit!*" (She writes the limit in Fig. 4.3.5)	local	speech indicators + symbols

Figures

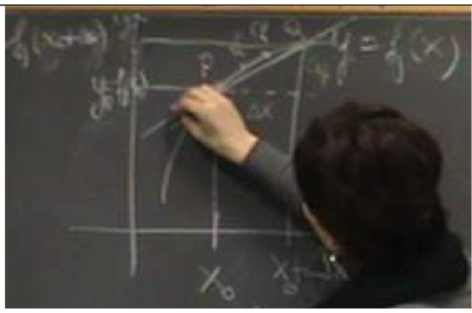


Figure 4.3.4 - M.'s dynamical idea to make the secant become the tangent.

Figure 4.3.5 - M. introduces the limit for $\Delta x \to 0$.

With the idea of introducing a sequence of points on the function $Q, Q', Q'', \ldots$ that get closer and closer to P, M. fosters the students to recall the limit theory. This dynamical idea, supported by the drawing, justifies the need for a technique that involves the limit as the increment goes to zero. So the work is pointwise on f when the teacher chooses the points Q' and Q'' [*21*]. It becomes local when M. supports her drawing with the speech, by saying "As Q gets closer and closer to P" [*22*]. And it moves on to a work on written symbols, with the "localization" of the global sign Δx, previously introduced

[*15*]. The shift is made through the speech, proposed by students [*27*], after M.'s request of formalizing "this approaching" [*26*]. The sign Δx, which so far has represented any global interval between two any points on x-axis, now is seen in a local perspective, as representing the interval between two points that get closer and closer, that is a (right) neighbourhood of the point x_0. The limit sign, that intervenes in line [*28*], Fig. 4.3.5, comes to be justified by M. with the process of "localization" of the sign Δx. We observe that this process remains implicit in the graph, with which M. is supporting her speech. This is something we can say at a very fine level of analysis, but we think it is very important. The local perspective on f, which is present in the speech through indicators of "localized movement", remains implicit in the graph. Indeed, during the speech, Δx is never modified and it remains a rather big interval on the drawing. Moreover, no gesture is made on the graph in order to show that Q is approaching P. The speech consists in reading of the previously drawn graph, without further actions on it. The local perspective on f, that is really exalted in the speech with implicit link to the represented graph, is then almost immediately delegated to the symbols ($\lim$, $\Delta x \to 0$). The symbols chosen on the graph (e.g., Δx, that evokes analogies in physics with Δt) and the speech for reading the drawing (e.g., expressions like "closer and closer" or "approaching") serve up to students the elements to recall the limit technique and the $\Delta x \to 0$ writing.

The same process is activated by M. on the second problem, that of the instantaneous speed. It is shown on an example:

> *"We launch a body upwards [...] we want to determine its speed in the instant 3 seconds".*

M.'s formulation and resolution process are shown in Fig. 4.3.6. We will skip the transcript details, because of the analogies with the previous problem, in terms of drawings and justifying speech.

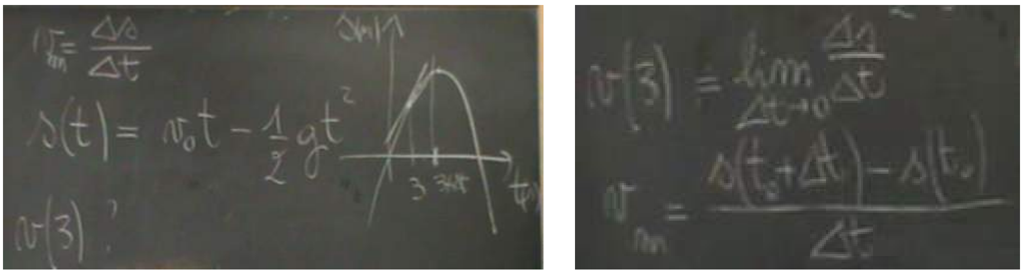


FIGURE 4.3.6 - M. INTRODUCES AND SOLVE THE PROBLEM 2.

M. closes the presentation of the two problems by saying what follows.

	What happens (teacher-students dialogue)	Perspectives on f	Semiotic resources
27	T: "*It is very important the problem of variations, that is the relationship between the variation of two magnitudes. Sometimes it is important to know what the function exactly does in a point, so I give you an abscissa and you calculate the correspondent ordinate. But it is also important the trend, in general, of a function. Think about the angular coefficient. Since this quantity*" (she points at the limit) "*represents the angular coefficient of the tangent line, we know that if the angular coefficient is positive, then the straight line is increasing, so the variation [of the function] will be an increasing variation.*"	pointwise global	speech indicators

Notice that explicitly speaking the teacher is marking the difference between the value of the function in a point, that is a pointwise information, and the variation, as a global information on the function. She is not speaking about local implications, yet, even if the employed technique involves the limit. Thus, we can say that the local dimension on f remains implicit at the technical level, but not made explicit on the technological-theoretical side.

Elaboration of a technique for $\mathcal{T}_{m_{tg}}$

Let us consider the formalization that M. gives to the derivative concept. She introduces it as a technique $\tau_{m_{tg}}$ for the sub-type of task $\mathcal{T}_{m_{tg}}$, that is determining the angular coefficient of the tangent line to a generic function in a point. In line [*28*] we find her speech. The teacher makes explicit that so far they have always "worked nearby". As a marginal observation, she makes explicit the local character of their work on f.

	What happens (teacher-students dialogue)	Perspectives	Semiotic resources	on f, f'
28	T: "*I start by defining the derivative of a function in a point. You see, so far, chosen a point, we've always worked nearby.*"	pointwise local	speech indicators	f' f
29	T: "*Given a function $y = f(x)$, we define 'derivative' the limit of the incremental ratio of the function, calculated in that point.*" (Then, she draws again the situation but calling h the increment Δx, Fig. 4.3.7 on the right)	local	speech indicators + graph + symbols	f

30	T: "*Then, we define as derivative of the function <u>in the point x_0</u>, that is the derivative of the function <u>calculated in the point x_0</u>, the <u>limit as h goes to 0 of the incremental ratio</u> Δy over Δx, that is...*" (She writes the limit in Fig. 4.3.7) "*We denote it with $f'(x_0)$, derivative of the function f <u>calculated in the point x_0</u>.*"	pointwise local	speech <u>indicators</u> + symbols	f' f
31	T: [...] "*So, we can say that the geometrical meaning of the derivative is what? It represents the angular coefficient of the tangent line to the function in the point x_0. Obviously, I mean the point of abscissa x_0.*"			

Figures

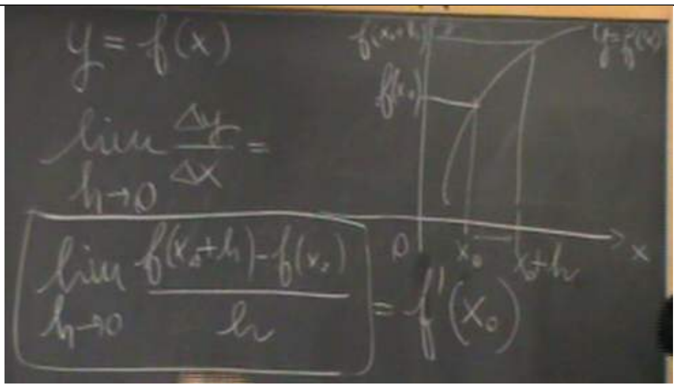


Figure 4.3.7 - M.'s formalization of the derivative concept.

With the written formalization of the derivative (Fig. 4.3.7), M. wants to give a technique to determine the angular coefficient of the tangent line to a function, as the remark [*31*] stresses. So, she devolves to the derivative a technical role in the praxeology for determining the tangent line to a generic function. The praxeology $O\mathcal{M}[\mathcal{T}_{m_{tg}}, \tau_{m_{tg}}, \theta_{m_{tg}}, \Theta_{m_{tg}}]$ is now built. M. proposes to work on an example, in order to practice the technique she has just introduced. She gives and solves the task $t_{m_{tg}} \in \mathcal{T}_{m_{tg}}$, which consists of determining the derivative of the function $y = x^2$ in the point of abscissa $x_0 = 2$. She finds $f'(2) = \underline{4}$ (Fig. 4.3.8) and she makes a final remark about the tangent line [*32-37*].

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
32	T: "*Actually, look at the drawing... Since <u>it is 4</u>, what does it represent?*"	pointwise	speech <u>indicators</u>	f'
33	Ss: "*The tangent.*"			
34	T: "*Not the tangent, don't say me the tangent! It represents?*"			

35	Ss: "The angular coefficient..."			
36	T: "*The slope, the angular coefficient of the tangent line in the point of abscissa* 2." (She writes m_{tg} under the result 4 and traces the tangent line in the point (2, 4) on the graph, Fig. 4.3.8)	pointwise	speech indicators	f
37	T: "*So, if it [the angular coefficient] is positive, you can see that the function will be...?*" (Her hand follows the function going upwards) "*Increasing, right?*"	global	speech indicators + iconic gesture	f

Figures

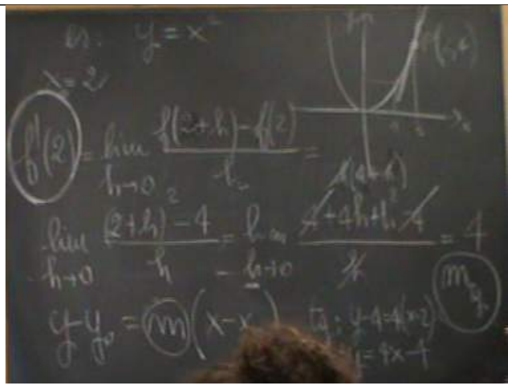


FIGURE 4.3.8 - M. GIVES AND SOLVES THE TASKS $t_{m_{tg}} \subset t_{tangent}$.

Notice that, as in line [*27*], the speech that accompanies the graph and the symbols underlines the link between a pointwise perspective on f' and a global perspective on f. The established relation is again between the value of $f'(x_0)$ and the global variation of f in the interval that contains the point of abscissa x_0 [*37*].

M. finally includes the subtask $t_{m_{tg}}$ within the broader task $t_{tangent}$ [*38*]. This last step completes the technique for the type of task $T_{tangent}$ [*45*].

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
38	T: "*If we would like to write the equation of the tangent line?*"			
39	Ss: "*We make it pass through...*", "*We impose the passage through the point* (2, 4)*...*", "*...with the angular coefficient...*"			
40	T: "*What expression of the straight line do we use?*"			
41	Ss: "$y - y_0$*...*"			

42	T: "*Exactly.*" (She writes it on the blackboard, Fig. 4.3.8) "*Where we replace x_0 and $f(x_0)$ with what?*"	pointwise	symbols	f
43	Ss: "*The coordinates of the point.*"	pointwise	speech indicators	f
44	T: "*The coordinates of the point, so in our case* $(2, 4)$*, and then we replace* m *with?*" (She circles m_{tg} under the limit result and $f'(2)$ at the beginning of the limit calculation) "*You see, here we can also write* $f'(2)$*, instead of* m*.*" (She finds the equation $y = 4x - 4$)	pointwise	symbols	f'
45	T: "*In general, the equation of the tangent line, as you find it on textbooks, is* $y - y_0 = f'(x_0)(x - x_0)$*.*" (Writing it on the blackboard)	pointwise	symbols	f'

The conclusion of the exercise, in which M. finds the equation of the tangent line, recalls the whole straight line-related praxeology. The tangent line is then seen as the straight line passing through the given point of abscissa x_0 on f and having $f'(x_0)$ as angular coefficient.

The new praxeology $O\mathcal{M}_{tangent}$ is summed up in Table 4.5. The technological speech is made of sentences explicitly uttered by the teacher in classroom. As for the theoretical knowledge, the first three are new pieces of knowledge, whereas all the others are old pieces of knowledge to recall.

$\mathcal{OM}_{tangent}$	
Type of task $\mathcal{T}_{tangent}$	Determining the equation of the tangent line to a generic function in a point.
Technique $\tau_{tangent}$	$tg : y - f(x_0) = f'(x_0)(x - x_0)$ where $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$
Technology $\theta_{tangent}$	Given a point $P(x_0, f(x_0))$, consider a small increment h of the abscissa x_0. You obtain another point on the curve $Q(x_0+h, f(x_0+h))$. Trace the secant line PQ, its angular coefficient is given by $m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x_0 + h) - f(x_0)}{h}$. Imagine a sequence of points $Q, Q', Q'', \ldots$ that get closer and closer to P. It means that h goes to 0. The limit position of the secant $PQ, PQ', PQ'', \ldots$ is the tangent in P. So $\lim_{P \to Q} m_{PQ} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m_{tg}$, which we denote with $f'(x_0)$. Finally, the tangent line is the straight line $y - y_0 = m(x - x_0)$ passing through P with angular coefficient $m_{tg} = f'(x_0)$.
Theory $\Theta_{tangent}$	- The definition of the tangent line to a generic function in a point as the limit position of a line cutting the function in that point and into another one, which gets closer and closer to it. - The problem of the tangent. - The problem of the instantaneous speed. - The analytic equation of a straight line, $m = \frac{y_1 - y_2}{x_1 - x_2}$ - The limits theory.

TABLE 4.5 - M.'S MATHEMATICAL PRAXEOLOGY FOR THE TYPE OF TASK $\mathcal{T}_{tangent}$.

Remarks

The initial teacher's speech aims at showing to the students that the old algebraic praxeologies, namely the conics-related ones, do not fit a more general problem. The new limit technique to solve the problem of the tangent is presented as a more general method with respect to the previous algebraic ones. Two praxeologies are recalled, in particular that of the straight line equation, which still belongs to the Algebra domain. The local perspective has to be induced by the teacher. Centered on the limit technique, the lecture prevails as dicactic technique along with the IRE (Initiate-Response-Evaluate) model of questioning the students. The limit is proposed by the students as a technique but induced by the target questions and the symbolical situation prepared by the teacher. She uses the graphical resource supplied with symbols and accompanied by terms of "localized movement". Anyway, the local dimension, which is implicitly contained in

the speech, is almost immediately delegated to the limit symbol. The graph remains a pointwise and global support. The formalization phase is local in so far as the limit is used. From the introduction of this symbol on, all the local expressions we can detect in the speech are directly associated with the limit technique ("as h goes to 0", "limit of the incremental ratio", etc.). All the terms of "localized movement" disappear. Moreover, it is stressed the fact that the tangent gives information about the variation of the given function, enhancing a global perspective on it.
As a consequence, $O\mathcal{M}_{tangent}$ has a strong practical-technical block, with local features embedded in the use of the limit symbol. Instead, the technological-theoretical block turns out to be based on a tangent definition provided by the teacher to foster the introduction of the limit.

4.3.3 Type of task $T_{f'}$: representing the derivative function

During the first lesson, after having introduced the derivative concept, M. speaks about "derivative function". As she tells us after, the approach has to be both algebraic and graphical.
Let us now consider what happens in the classroom. We recall that the derivative notion has just been defined as:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Immediately after M. addresses the type of task $\mathcal{T}_{f'}$, by saying:

> *"Let's try to calculate the derivative of some functions, now".*

M.'s didactic methodology consists of deducing the generic technique $\tau_{f'}$ by working on particular cases. The first one is a numerical example, namely the task $t_{m_{tg}}$ we have hinted to before (see Fig. 4.3.8), through which M. wants the students to work on the just elaborated technique $\tau_{m_{tg}}$. Once solved $t_{m_{tg}}$, M. generalizes it into another task $t_{f'} \in \mathcal{T}_{f'}$:

> "*Let's calculate the first derivative of the function $y = x^2$ in a generic point of abscissa x.*"

Thus, she begins to explore the type of task $\mathcal{T}_{f'}$ through a particular case. This is the moment of construction of the block $[\mathcal{T}_{f'}/\tau_{f'}]$, where $\mathcal{T}_{f'}$ is approached from the algebraic side. Indeed, in this phase of the work in classroom, the type of task $\mathcal{T}_{f'}$ is specified in *algebraically representing the derivative function.* That is why we denote the type of task and the related praxeology with the superscript "alg": $\mathcal{T}_{f'}^{alg}$ and $O\mathcal{M}_{f'}^{alg}$. The two algebraic phases of the work are summarized in Table 4.6.

Tasks t, type of tasks T and problems given in classroom	**Construction of the $O\mathcal{M}$ for $\mathcal{T}_{f'}^{alg}$**
$t_{m_{tg}}$: calculating the first derivative of the function $y = x^2$ in the point of abscissa $x_0 = 2$.	This can be seen as a transitional task: the teacher gives and solves it not only to work on the just introduced technique $\tau_{m_{tg}}$, but also to create the basis for the next task.
$t_{f'}$: calculating the first derivative of the function $y = x^2$ in the generic point of abscissa x.	Generalizing the previous task from x_0 to a generic x, the teacher works on the technological plane, from $\theta_{m_{tg}}$ to $\theta_{f'}$. She gives an algebraic technique for $\mathcal{T}_{f'}$.

Table 4.6 - M.'s didactic organization for working on the type of task $T_{f'}$ in the algebraic register.

In a successive moment, M. will work on the derivative function also in the graphical register of representation, so that we will have also a $\mathcal{T}_{f'}^{gra}$ (see ahead subparagraph "Elaboration of a technology, passing through the graphical technique").

We are going to analyse essentially two didactic moments:

1. the elaboration of the algebraic technique, starting from the technological speech;

2. the elaboration of a technology, passing through the graphical technique.

The introduction of a new global perspective on the derivative function is realized by M. with particular examples which have generic relevance. M. chooses a pointwise task on f' and she has to transform it into a universal pointwise one, in order to move towards a global perspective on the derivative function.
In this first approach to the problem, the perspective on the function f is still implicitly local, because of the limit sign used in the technique.

Elaboration of the algebraic technique, starting from the technological speech

Let us focus on the particular task $t_{m_{tg}}$, that we have previously met within the praxeology $O\mathcal{M}_{tangent}$ (see Fig. 4.3.8). Here we transcribe the resolution phase, skipped in the previous paragraph, in order to show the functionality of this task as *generic example* in the teacher's intention.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
1	T: "*Let's make an example: $y = x^2$.*" (She draws the graph, Fig. 4.3.9) "*Everybody knows it. We would like to calculate the first derivative of this function <u>in the point of abscissa $x_0 = 2$</u>.*" (She detects the point on the graph, Fig. 4.3.9)	global pointwise	graph speech <u>indicators</u>	f f'

2	T: "*So, what shall I do? It means that I calculate f' in 2.*" (She writes $f'(2)$) "*What shall I do now? The limit as h goes to 0 of the incremental ratio. Notice that, instead of x_0, I've already detailed it: $f(2+h)-f(2)$ over h.*" (She writes the limit, Fig. 4.3.10) "*Does it work? Are you all with me?*"	pointwise local and pointwise	speech indicators + symbols	f' f
3	Ss: "*Yes.*"			
4	T: "*Equal... And now I do the calculation.*" (She replaces $f(2+h)$ with $(2+h)^2$ and $f(2)$ with 2^2, then she develops the square of the binomial, and eliminates $\cancel{4}$ and $\cancel{4}$ in the numerator, Fig. 4.3.10)	pointwise	symbols	f
5	T: "*Notice that the only care we must have, maybe it seems strange, is that the variable of this limit isn't x anymore, but it is?*"	local	speech indicators + symbols	f
6	Ss: "*h.*"			
7	T: "*h. After all, you can see it also here*" (She underlines h in the limit sign) "*We haven't any problem here, but later when we'll generalize, there will be also x. So, be careful to the variable of the limit, ok? So, h goes to 0.*"	local	speech indicators + symbols	f
8	T: "*What is the value of this limit?*"			
9	Ss: "*4*", "*0*"			
10	T: "*Too much answers: 4, 1, 0, infinity. Everything and more.*"			
11	Ss: "*0*", "*0 over 0*"	local	oral symbols	f
12	T: "*So calculate it!*" (She puts h in evidence, finding $h(4+h)$ at numerator; she eliminates h at numerator with h at denominator, Fig. 4.3.11) "*It becomes $4+0$, and then?*"	local	symbols + speech indicators	f
13	Ss: "*4*"			
14	T: "*The result is 4.*"	pointwise	symbols	f'

Figures

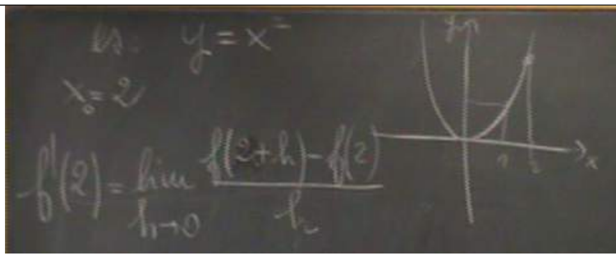


Figure 4.3.9 - M. poses the task $t_{m_{tg}}$.

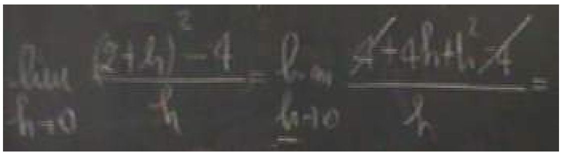


Figure 4.3.10 - M. applies the technique $\tau_{m_{tg}}$ in order to solve the task $t_{m_{tg}}$.

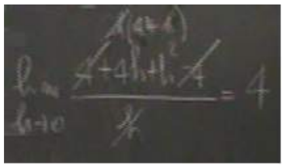


Figure 4.3.11 - M. solves the task $t_{m_{tg}}$.

The teacher proposes a pointwise task on f' [*1*], so the perspective on f' remains on the point x_0 for the whole solving process [*1-14*]. During this process, instead, the perspectives on f frequently change. Notice that the implemented technique $\tau_{m_{tg}}$ [*2*] (Fig. 4.3.10) potentially has a local dimension on f, due to the presence of the limit sign. This local dimension is here neutralized by the pointwise one, since the function f is continuous and differentiable in the point x_0. Indeed, a pointwise perspective is sufficient to replace x_0 in the expression of f [*4*]. M. makes an explicit reference to the limit variable h [*5-7*], that is potentially a local remark on f, because it gives a local dimension to the global incremental ratio. Nevertheless, we stress that this local remark actually remains at the syntactic level: M.'s speech refers to the involved variables. Her main concern indeed is the possible confusion between x, that is the variable of the function, and h, that is the variable of the limit. This remark is made to prepare to the next step, where h and x will coexist in the same calculation [*7*].
In solving the limit [*8-14*], algebraic praxeologies intervene, especially the algebraic technique to solve the indeterminate form [0/0] (see Table 4.7).

$O\mathcal{M}_{[0/0]}$	
Type of task	Solve the indeterminate form [0/0] (in limits involving polynomial functions).
Technique	Algebraically decompose the polynomials at numerator and at denominator. Eliminate the common factor. Solve the simplified limit.
Technology	The indeterminate form [0/0] is due to the presence of an infinitesimal common factor above and below the fraction sign.
Theory	Decomposition rules of polynomials.

Table 4.7 - Previous mathematical praxeology related to the resolution of the indeterminate form [0/0].

Now, let us see how this task takes the role of generic example, leading to the algebraic representation of the derivative function.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
15	T: "*Now, to summarize, what have we done? The concept of derivative, but calculated in a point.*" (She points to an imaginary point in front of her, with her left hand, Fig. 4.3.12 on the left) [...] "*The derivative of a function in a point, what does it give?*"	pointwise	speech indicators + pointing gesture	f'
16	S6: "*A coefficient.*"			
17	T: "*A number, exactly. But now let's make a step forwards. We have $y = x^2$ and we have calculated the angular coefficient of the tangent line in the point $x_0 = 2$.*" (She repeats the same gesture of before with the left hand, as in Fig. 4.3.12 on the left) "*If now I ask you: "What is the value of the angular coefficient in the point with abscissa $x_0 = 5$?" One should again work hard and do all the calculation. Right? In $x_0 = 1$... and so on.*" (She moves her hands as if something is rolling in front of her) "*You see, it's not so convenient, also from a practical point of view.*"	pointwise	speech indicators + symbols + pointing gesture	f'
18	T: "*So, what shall I do? The calculation in a generic point x*" (She joins upwards the fingers of her right hand and then turns it down to her left hand that is open upwards, Fig. 4.3.12 on the right) "*Ok? That is, instead of calling it x_0, I call it x.*" (She turns again her right hand on the open left hand, as before, Fig. 4.3.12 on the right)	global(=univ. pointwise)	speech indicators + symbols + gestures	f and f'
19	T: "*And now we must be really careful! I call it x. Which outcome I expect?*"			
20	S1: "*A function.*"	global	speech indicators	f'
21	T: "*Can it be a number?*"			

22	S4: "*With* x."	global	oral symbols	f'
23	T: "*Yes, it will be a function of* x. *So, you understand that we can speak about "derivative function", which will be again a function of* x."	global	speech indicators + symbols	f'
24	S2: "*And then we can replace inside it...*"	pointwise	speech indicators	f'
25	T: "*Perfect! S2 is saying "Of course, then, if I want the coefficient of the tangent in the point* $x_0 = 5$, *it will be sufficient to put* $x = 5$ *in the derivative function". Let's do it!*"	pointwise	speech indicators + symbols	f'

Figures

FIGURE 4.3.12 - M.'S DIFFERENT GESTURES FOR "THE POINT x_0" (ON THE LEFT) AND "THE GENERIC POINT x" (ON THE RIGHT).

After this comment, the teacher solves the same task following the same steps with $x_0 = x$ instead of $x_0 = 2$. She obtains $f'(x) = 2x$ (see Fig. 4.3.13).

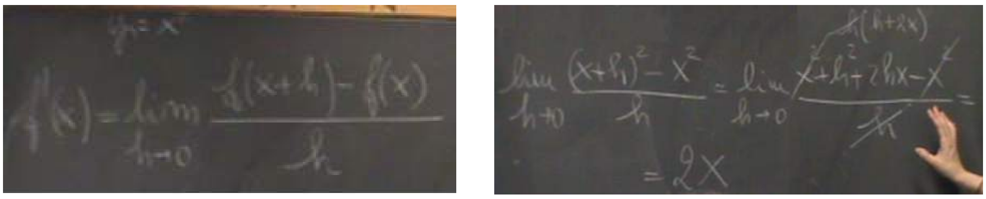


FIGURE 4.3.13 - M. POSES AND SOLVES THE TASK $t_{f'}$.

The first utterance and the first gesture used by the teacher [*15*] (Fig. 4.3.12 on the left) stress that the starting perspective is pointwise on f'. Then, she underlines that making other numerical examples is actually useless, since every one of these examples ($x_0 = 5$, $x_0 = 1$,...) would always entail the same calculations done for $x_0 = 2$ [*17*]. The case of $x_0 = 2$ is becoming a generic example. M. introduces a syntactical technique: the replacement of x_0 with x [*18*]. The previous argument [*17*] can be seen as a technology for this technique. More precisely, the non-convenience of working on the pointwise sign x_0 is the teacher's justification for shifting to the generic sign x, which is universal pointwise. The generic sign x represents "every value of x_0". Even M.'s gesture for accompanying

the universal pointwise expression "the generic point x" [*18*] is different from the one used before for referring to "the point x_0" [*15*] (see Fig. 4.3.12 on the right). Finally, the teacher makes the students reflect upon the global expectations on the result [*19-23*]: $f'(x)$ is expected to be globally a function of x, in which we can replace x with any number we wish [*24-25*]. In this passage, another technique can be detected for finding out the derivative of f in a particular abscissa: the replacement of x with this abscissa in the expression of $f'(x)$. On the technological side, this technique is supported by the whole previous argument about the derivative function.
Notice that M. devolves the shift from pointwise to global perspective on f' to the syntactical process, which is typical of the algebraic writing: from a particular x_0 to a generic x. This is the classical algebraic technique of the replacement of a variable with its value and vice versa.

The new praxeology for the type of task $T_{f'}$ in its algebraic formulation is built (see Table 4.8). The technological speech is made of sentences explicitly uttered by the teacher in the classroom.

$OM_{f'}^{alg}$	
Type of task $T_{f'}$	Algebraically representing the derivative function.
Technique $\tau_{f'}$	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Technology $\theta_{f'}$	If we want to know $f'(x_0)$ for many values of x_0 the calculations become long and not so convenient. So, we can take the generic point x, calculate $f'(x_0)$ in $x_0 = x$ and after replace the desired x_0-value in the obtained expression. $f'(x_0)$ is a number, whereas $f'(x)$ is a function.
Theory $\Theta_{f'}$	- Shift from a pointwise value to the global function. - Algebraic writing of an expression in the generic x.

Table 4.8 - M.'s mathematical praxeology for the type of task $T_{f'}$ in the algebraic register.

Elaboration of a technology, passing through the graphical technique

M. proposes and solves with the students two graphical activities concerning the properties of f' and its relation with f. The technical work prevails, since teacher and students are engaged in globally constructing the graph of f', starting from that of f. In parallel, a local perspective must be adopted on f, looking at the tangent at every point. This graphical technique finally leads to compare f and f', so to reflect on the technological-theoretical side. In one of the a posteriori interviews, M. comments:

> "*According to me, this kind of graphical work should clarify the concept of derivative function, showing that it is a function with its own graph. Moreover, it is important to me the fact that they [students] can already see a link between f and f'... I believe that they must face it [the problem] with their own forces, also working hard on the blackboard to get their results.*"

Let us now see briefly what the two graphical activities consist of.

Tasks t, type of tasks T and problems given in classroom	**Construction of the OM for $T_{f'}^{gra}$**
Task t_1: reconstructing the graph of a function f, given the table of values $x \mid f(x) \mid m(x)$ (Fig. 4.3.14)	The teacher guides a student (named S1) in solving the tasks at the blackboard. They develop an embryonic graphical technique and they refine the technological-theoretical speech by specifying what makes this technique work.
Task t_2: sketching the graph of the function $y = m(x)$	
Task t_3: drawing the graph of the function $f(x) = 1/3x^3 - 1/3x^2 - 2/3x$	The teacher guides a student (named S2) in solving the task. They refine the technology for the graphical technique by means of reference to the theory.
Task t_4: calculating f' and drawing its graph	

Table 4.9 - M.'s didactic organization for working on the graphical type of task $T_{f'}$.

x	$f(x)$	$m(x)$
1	1	1
2	3/2	1/2
3	3	2
4	9/2	0
5	4	-1
6	3/2	-4
7	1/2	?
8	3/2	1/2

Figure 4.3.14 - Table of values given in task t_1.

The first activity (tasks $t_1 + t_2$) is proposed by the teacher to a student (named S1) and involves both blackboards they have in the classroom, initially to deal with different registers of representation on functions: namely the numerical and the graphical ones. Unfortunately, we have not the video of the first part of the activity, because we could not record, but we have notes of some relevant utterances and reconstructions of the graphical steps at the blackboard. Since the activities last almost two hours, we prefer presenting them by points. Thus, from line [*26*] to line [*42*], each line [n] refers to a summarized action or sentence, and not to a single transcription.

	What happens (teacher-students actions)	**Perspectives**	**Semiotic resources**	**on f, f'**
26	T: "*Let's draw point by point this situation on the Cartesian plane.*" S1 detects the given points $(x, f(x))$ (see Fig. 4.3.15)	pointwise	speech indicators + num. symbols + graph	f

27	T: "*Using another color, represent $m(x)$. In each point let's represent the tangent, since we know its gradient. Without lengthen it, remain in the neighborhood of the point.*" In each $(x, f(x))$, S1 draws a segment with the given slop $m(x)$ (as in Fig. 4.3.16).	local	speech indicators + graph	f
28	The teacher suggests to connect the points, maintaining the tangency to the segments. S1 starts drawing the graph in Fig. 4.3.17.	global	graph	f
29	S1: "*If [m] is* 0*, it means that [the tangent line] is constant, it's like $y = k$.*" She detects a maximum point (see Fig. 4.3.17)	local	graph + oral symbols	f
30	S1: "*[m] is undefined, the function [f] in that point isn't differentiable. Maybe it is not continuous in that point...*" The teacher recalls the example of the absolute value function $y = \|x\|$ in $x = 0$. S1 detects a corner in $(7, 1/2)$ (see Fig. 4.3.17)	local	speech indicators + graph	f

Figures

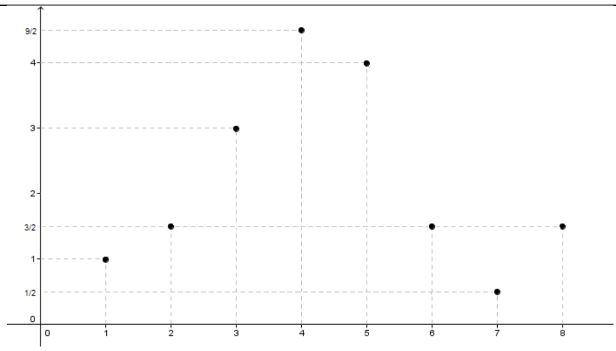


FIGURE 4.3.15 - (OUR RECONSTRUCTION) S1 DETECTS THE GIVEN POINTS $(x, f(x))$ IN THE CARTESIAN PLANE.

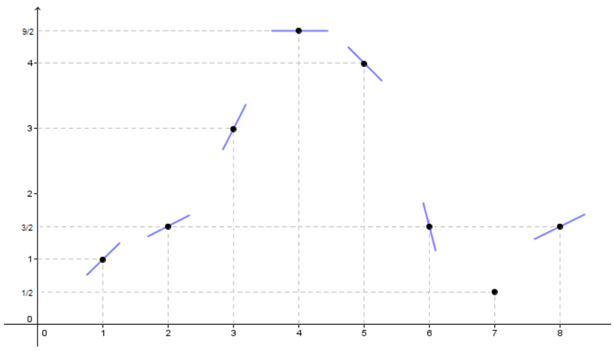


FIGURE 4.3.16 - (OUR RECONSTRUCTION) S1 TRACES THE SEGMENTS OF TANGENT WITH THE GIVEN SLOPE m.

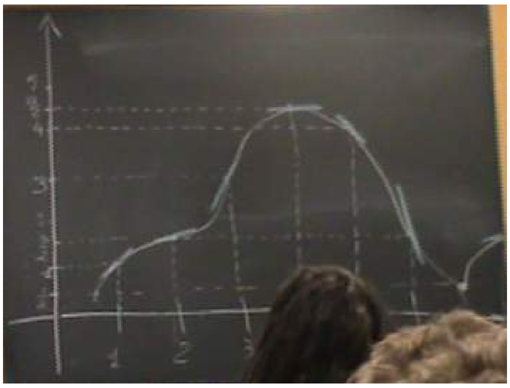


Figure 4.3.17 - S1 interpolates the detected points in order to have a possible graph of f.

Starting from pointwise numerical data (Fig. 4.3.14), the work on f is firstly locally and then globally graphical. The only unusual information is the slop on f at some points and the teacher guides the student in using it [*27-28*]. This is mainly a technical graphical work, as said before, but we have two moments [*29*] and [*30*] in which the student S1 stops and tries to justify what she's going to draw. It happens at the maximum abscissa, $x = 4$, and at the abscissa $x = 7$, where the slope is not defined. The given task on f is global (reconstructing the graph [*28*]), but for accomplishing it the student has to look locally at f, using the tangent line [*28-30*].

	What happens (teacher-students actions)	**Perspectives**	**Semiotic resources**	**on f, f'**
31	T: "*What does $m(x)$ represent? It is the gradient of the curve tangent in the point with abscissa x. Practically, it is a function of the variable x.*" T asks S1 to make an attempt of drawing $y = m(x)$, referring to it as the "derivative function".	global	symbols + speech indicators	$m = f'$
32	Guided by the teacher, S1 interprets the data in the table (Fig. 4.3.14) graphically: "*I put $m(x)$ on y-axis.*"	pointwise	graph	$m = f'$
33	T: "*S1, try to connect the points $(x, m(x))$.*" S1's first attempt is in Fig. 4.3.18.	global	speech indicators + graph	$m = f'$
34	S1 wonders how the graph of m could be in $x = 7$: "*It goes towards... infinity?*"	local	graph + speech indicators	$m = f'$

35	T prompts her attention to the corner she has drawn in f and, in particular, to the right and the left values of m: "*Listen: in a point, right and left limits are finite but different.*"	local	speech indicators + graph	$m = f'$
36	S1: "*There is a jump!*"	local	speech indicators + graph	$m = f'$

Figures

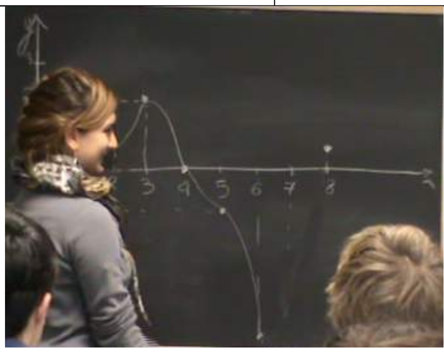


FIGURE 4.3.18 - S1 TRIES TO CONNECT THE POINTS OF m IN THE CARTESIAN PLANE.

The teacher changes the perspective on $m(x)$ [*31*]. So far, they use its values as local information about the slop of f [*27-30*], but now they are going to use them as pointwise data on the function m itself [*31-32*]. In other words, they start to work on m as the derivative function f'. Task t_2, as task t_1, is global (drawing the graph of f' [*33*]), but it entails local doubts and reflections [*34-36*].

Then the teacher helps S1 in sketching the graph of the function m, and intervenes on the technological plane. Indeed, she firstly recalls the relation between the slope of f and the sign of m in the tangent equation that she writes generically as $y = mx + q$. In particular, she discusses the sign of the gradient m. Within a global perspective, she makes the sign of m explicit in each portion of the graph of f, namely on the intervals $[1, 4)$ and $(4, 7)$ (see Fig. 4.3.19). In parallel, M. completes the graph of m on the other blackboard (see Fig. 4.3.20).

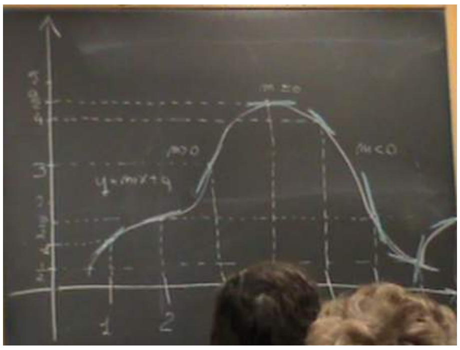


Figure 4.3.19 - M.'s discusses the sign of the tangent gradient.

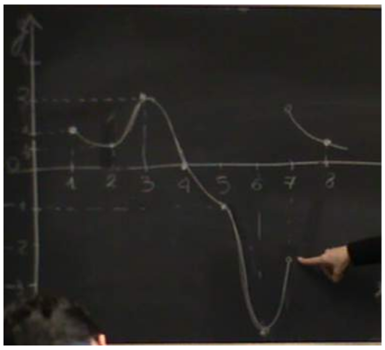


Figure 4.3.20 - M.'s completes the possible graph of $y = m(x)$.

The second activity (tasks $t_3 + t_4$) is developed by the teacher who involves another student (named S2) at the blackboard. As before, both blackboards are used for the treatment of the involved functions in different registers of representation. The right blackboard is devoted to the numerical-algebraic work, while the left one is reserved to the graphical conversion. M. explicitly gives the tasks.

> "*Let f be defined by $f(x) = \frac{1}{3}x^3 - \frac{1}{3}x^2 - \frac{2}{3}x$. On f, let's do this work: let's determine the values of the function at some points, then determine the rates of instantaneous variation [gradient m] in each of these points and try to understand where the function is increasing, decreasing and where it is stationary. [...] The idea is to draw the third degree function and then to draw its derivative function, in order to see the existing links [between the two graphs].*"

Notice that the tasks are initially pragmatic ("let's determine", "the idea is to draw"...) to become more analytic and arguing-oriented ("try to understand", "in order to see"...). Moreover, the tasks which the students are already able to do (determining the values of a function, drawing its graph and recognizing properties on intervals) are pointwise and global. Instead, the last question, that is the goal of the activity, requires a local perspective, which the students are entering.

	What happens (teacher-students actions)	**Perspectives**	**Semiotic resources**	**on f, f'**
37	In order to determine some values of f, T suggests the numerical technique of the table of values $\lvert x \lvert f(x) \rvert$ (see Fig. 4.3.21) and calculates each value by substitution.	pointwise	numerical symbols	f
38	T draws the graph of f, by interpolating the determined points and availing students' knowledge about "*<u>the end behavior</u> and <u>the variation</u> of a cubic*" (see Fig. 4.3.22).	pointwise local global	graph + speech <u>indicators</u>	f

39	T: "*Let's do the limit of the incremental ratio, we will find the derivative function, that is $m(x)$, and then we'll be able to fill the table in, because we'll substitute our values of x.*" T asks to a student, S2, to algebraically calculate $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ in the generic point x.	local global(=univ. pointwise)	speech indicators + symbols	f f'
40	S2 obtains $x^2-\dfrac{2}{3}x-\dfrac{2}{3}$ and denotes it with $f'(x)$, recognizing a parabola.	global	symbols	f'
41	S2 fills the table in by substitution (see Fig. 4.3.23). Then, he finds the vertex $V\Big(-\dfrac{b}{2a}, f'\Big(-\dfrac{b}{2a}\Big)\Big)$ and the intersections A and B with x-axis, by solving $f'(x)=0$.	pointwise	symbols	f'
42	S2 draws the parabola on the left blackboard near the graph of f (see Fig. 4.3.24).	global	graph	f'

Figures

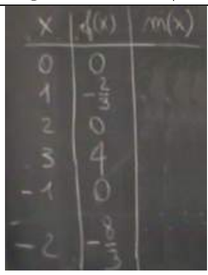


Figure 4.3.21 - Table of values of f.

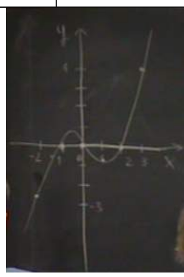


Figure 4.3.22 - Graph of the cubic f.

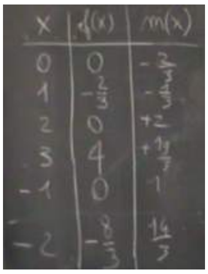


Figure 4.3.23 - Table of values of m.

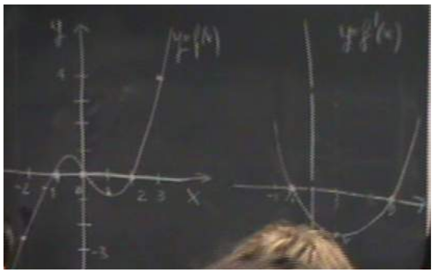


Figure 4.3.24 - Graph of f and f' side by side.

Beside the algebraic-graphical dialectics (the work on the two blackboards), it takes place another interesting dialectics. Different techniques, which are pointwise [*37* and *41*], local [*39*], and global [*38*, *40* and *42*], successfully intertwine to give to $m = f'$ the same status of function that f has. Notice that the teacher supports with explicit justification [*39*] only the local technique of the limit of the incremental ratio. With her words, M. reminds the students that calculating this limit in a generic point x leads to the derivative function.

Nevertheless, the students have internalized the limit technique as a pointwise algebraic tool that allows them to find the derivative. It lack a local interpretation of the limit formula. The teacher activate the local perspective on f, acting on the technological-theoretical level. She promotes the work on the graphs they now have side-by-side, starting from the results of calculation, namely the intersections of f' with x-axis, x_A and x_B. She translates these pointwise data about f' into local information about f. She locally adjusts the graph of f, deleting and drawing it again in a neighbourhood of x_A and x_B (see Fig. 4.3.25). Finally, the teacher uses gestures to stress the relation between positive/negative intervals of f' and increasing/decreasing intervals of f (see Fig. 4.3.26). Thus, she stresses the correspondence of global properties of the two graphs, especially the sign of f' and the variation of f.

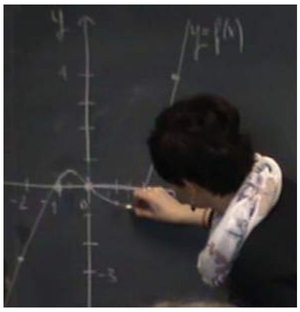


Figure 4.3.25 - M. locally redraws the graph of f.

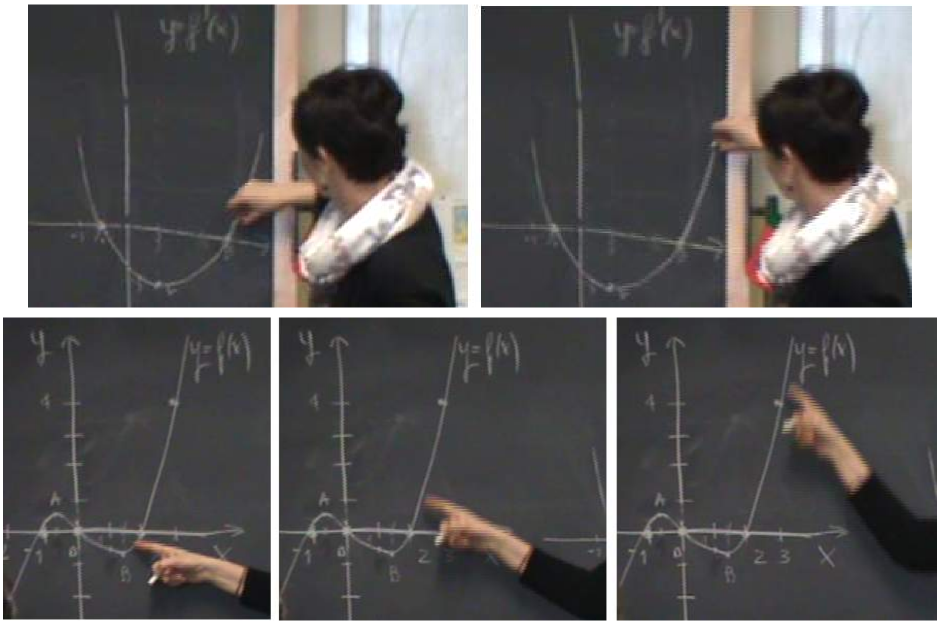


Figure 4.3.26(a) and 4.3.26(b) - M.'s global gestures in order to show the relation between sign of f' (above) and variation of f (below).

Therefore, we have an empirical praxeology for the type of task $T_{f'}^{gra}$. It is summed up in Table 4.10.

$OM_{f'}^{gra}$	
Type of task $T_{f'}$	Graphically representing the derivative function.
Technique $\tau_{f'}$	- In those intervals where $f(x)$ is increasing, draw $f'(x)$ above x-axis; - In those intervals where $f(x)$ is decreasing, draw $f'(x)$ below x-axis; - If you find a corner in f, put on y-axis the m-values of the tangent line on the right and on the left of the corner.
Technology $\theta_{f'}$	Graphically representing $y = f'(x)$ means graphically representing $y = m(x)$ where $m(x)$ corresponds to the values of the angular coefficient m of the tangent to f, at each point x. On the graph of f, trace the tangent line at some points. Generally, it will have equation $y = mx+q$. When $m > 0$ the tangent line is increasing and so the function; when $m < 0$ the line is decreasing and so the function; when the tangent line is horizontal, neither increasing nor decreasing, $m = 0$. When you have a corner in the graph of f, it means that you have different value of m on the right and on the left of the corner.
Theory $\Theta_{f'}$	Relationship between the sign of $m = f'$ and the variation of f.

Table 4.10 - M.'s mathematical praxeology for the type of task $T_{f'}$ in the graphical register.

Remarks

The proposed technique to algebraically determine the derivative function bases on the syntactical shift from the pointwise sign x_0 to the universal pointwise sign x. The teacher explicitly says: "Instead of calling it x_0, I call it x". The functions f and f', which depend on x, acquire then a global feature. It occurs through the universal pointwise employ of symbols and speech. In particular, $f'(x)$ is defined as a function of x: sign which is introduced as generic, and implicitly used in the global sense of variable.
Working on the graphical task is really important for the teacher, in particular to get a global view on the derivative function. Given or reconstructed the graph of a function f, the task is drawing the graph of $y = m(x)$, where $m(x)$ is the gradient of the tangent to f at the point x. The empirical method consists of interpolating some points $(x, m(x))$. The graph of the derivative is presented as the graph having on y-axis the values of m and on x-axis the corresponding values of the abscissas where m is calculated or given. The final fostered perspective on f' is global, since the final goal is completing the possible graph of the derivative function, respecting two fundamental conditions. The former is a pointwise condition: it has to pass through the determined points $(x, m(x))$. The

latter is global: the positive/negative intervals of $m = f'$ must correspond to the increasing/decreasing intervals of the function f.
The work on the graph of f' has also some local implications on f: it permits to locally redraw the graph of f, for example in a neighbourhood of the maximum/minimum points.

4.4 The case of M.G.

M.G. is a supply teacher of maths and physics in the same M.'s high school, in Savigliano (Cuneo). It is her first year in this school, and in particular with *Quinta A*'s students. The students have had a different teacher during the previous years. As we said before, the school has adopted the textbook *Nuova Matematica a colori. Edizione BLU per la riforma* written by Sasso (2012). M.G. has not contributed to its choice, but she says: "It positively surprised me... Finding a book, whose approach is the one you would follow. The graphical approach is present, also concerning the exercises part. And it has interesting exercises". She consults it before preparing lessons and uses it for giving exercises to students. Nevertheless, she also chooses original exercises from other textbooks, such as *Nuovo Lezioni di matematica* by Lamberti, Mereu and Nanni (2012), even more difficult to stimulate students.

4.4.1 From the interviews: M.G.'s beliefs

In the preliminary interview with M.G., in January 2013, we spoke about the general situation of the class and about the work done on limits as preliminary to the introduction of the derivative concept.

How did you introduce limits with your students? What kind of work did you do on them?
To a direct question about the $\epsilon - \delta$ definition, M.G. answers that her students "know by heart all the four definitions". They started by an intuitive graphical definition and by the meaning of "going nearer". She expresses her concern: "But then, the definitions had to be learnt because, unfortunately, at the final examination they are required". So, starting from a definition involving a generic idea of neighbourhoods, she got to the formal definition. She insisted on the direction: from a neighbourhood on y-axis to a neighbourhood on x-axis. She adds: "I insisted a little bit on the memorization [...] because I think that if you continue studying something by heart, soon or later something enters your mind. And, making a transverse speech, I needed to know if the students were studying. So making them repeat word by word this definition gave me an idea about it".

How do you usually introduce the derivative notion?

She is planning the lesson: she intends to start with a revision of the problem of the tangent in the analytic geometry, and then to move on to the general problem through the derivative. She justifies this choice by a general consideration about the students background: she does not really know what they have done during the previous years. She needs to ascertain that they know how to solve the problem of the tangent. She reveals her lessons planning in more details: "I make an introduction, with the excuse of revising [the conics]. [...] Focusing on the problem of the tangent, we recall all the ways to find the tangent [...] why I come to an equation of second order. And then if the curve is different? *Patatrac*. The idea will be: first of all, I draw it by hand with a ruler. The problem is solved? No. Then, I introduce a Cartesian reference system and I ask to determine the equation. And I haven't second order instruments anymore, which allow me to... I would like to get it [the tangent] through the secant line... as the limit of the secant line".

In one of our following meetings, we spoke about the derivative function.

How do you usually introduce the derivative function?

She works algebraically on it, but for introducing the graph she says: "I always try to go on the graph, because I see the graph, even if the definition may elude me. Nevertheless, I prefer waiting a while before introducing the graph of the derivative function". Such a wait is justified by her students' fragility in using the graph as a support. It still sounds strange for the students, who do not seem used to do it.

4.4.2 Type of task $T_{tangent}$: determining the equation of the tangent line to a generic function in a point

M.G. introduces her first lesson about the derivative concept with an exercise taken from the textbook (Sasso, 2012, ex.428 p.131; see the table below)[2].

Exercise n. 428 page 131 of Sasso's textbook.
Let γ be a circle centred in the origin and passing through the point $P(4,3)$. **a.** Write the equation of γ. **b.** Write the equation of the straight line r, which is tangent to the circle in P. **c.** Let Q be a point on the circle γ, belonging to the first quarter of plane and having abscissa x. Express the angular coefficient m_{PQ} of the straight line PQ in function of x.

[2]Considera la circonferenza γ, avente centro nell'origine e passante per il punto $P(4,3)$.
a. Scrivi l'equazione di γ.
b. Scrivi l'equazione della retta r, tangente alla circonferenza in P.
c. Considera un punto Q sulla circonferenza γ, appartenente al primo quadrante, di ascissa x ed esprimi in funzione di x il coefficiente angolare m_{PQ} della retta PQ.
d. Calcola $\lim_{x\to 4} m_{PQ}$. Come si può interpretare il risultato ottenuto in relazione a quanto ricavato al punto **b**?

d.	Calculate $\lim_{x \to 4} m_{PQ}$. How can you interpret the obtained result in relation to what you found at point **b**?	

This exercise is proposed by Sasso at the end of the unit about limits of real functions of a real variable. It belongs to the "Problems that lead to the calculation of the limit of an algebraic function", in the section "Geometrical problems".
Indeed, it involves a circle and the tangent line in one of its points. M.G. chooses it as a link between the conics revision and the problem of the tangent. The conics-related praxelogy (see Table 4.1) intervenes in this first phase of the lesson and it is somehow compared with a new praxeology which involves the limit technique. The teacher quickly reviews how to solve questions **a** and **b**, and instead she insists on questions **c** and **d**. For the particular task proposed in question **b**, that is writing the equation of the tangent line to the circle in P, the teacher compares two different techniques to find the angular coefficient m_{tg}:

1. $m_{tg} = -\dfrac{1}{m_{OP}}$, where O is the center of the circle;
2. $m_{tg} = \lim_{x \to x_P} m_{PQ}$, where Q is another point of the quarter of circle in the first quarter of plane, whose abscissa is x.

Then, the teacher notices that the second technique can be a valid solving strategy when one has a generic function, since in that case the methods seen with the conics (such as technique 1.) do not work anymore. Therefore, M.G. introduces this technique testing it on an example which the students can manage, so that they are able to compare the obtained results. Then, she formalizes the new technique for a generic curve $y = f(x)$, and practices it in the case of particular elementary functions. More precisely she solves the task: determining the equation of the tangent to $y = e^x$ in $x_0 = 1$. In this case, she distinguishes between tangent and normal line and she also calculates the equation of the normal line in $x_0 = 1$. The phases of the work are summarized in Table 4.11.

Tasks, type of tasks and problems given in classroom	**Construction of the OM for $T_{tangent}$**
First task: ex. 428 page 131 of the textbook, questions **c** and **d** in particular.	The teacher explores the type of task $T_{tangent}$ in the case of a conic. She elaborates a new technique for finding m_{tg}, and justifies it by interpreting it in the geometrical context. The tangent is then seen as the limit position of the secant line.
Problem: the case of a generic curve $y = f(x)$.	The new technique is formalized by the teacher.
Second task: determining the tangent and the normal to the function $y = e^x$ in the point $x_0 = 1$.	The new technique is practised by the teacher.

TABLE 4.11 - M.G.'S DIDACTIC ORGANIZATION FOR WORKING ON THE TYPE OF TASK $T_{tangent}$.

Within this first lesson, the teacher builds the praxeology $O\mathcal{M}_{tangent}$ in the case of a generic curve. We can distinguish the following didactic moments:

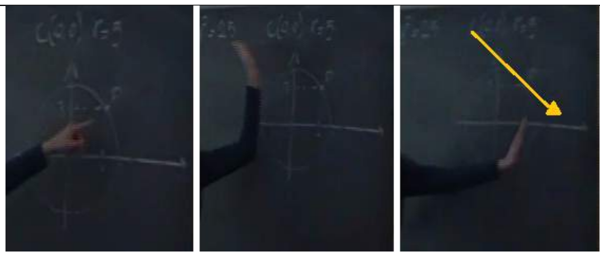
1. meeting with the type of task $\mathcal{T}_{tangent}$, through the exercise;
2. formalization on the new technique $\tau_{tangent}$ accompanied by the construction of a new technological-theoretical support for the tangency condition;
3. practice of the new praxeology on an example.

This episode is also significant for the introduction of a local perspective on the involved function, that M.G. intends to realize on the particular example of the circle, by geometrically interpreting the limit. She has to deal with the pointwise character of the previously learnt techniques, specifically the one quickly reviewed at the point **b** of the first proposed task.

Meeting with the type of task $\mathcal{T}_{tangent}$

M.G. continues the exercise they started the previous lesson. So, she begins from quickly reviewing the questions **a** and **b**. It follows the transcript.

	What happens (teacher-students dialogue)	**Perspectives on γ**	**Semiotic resources**
1	T: "*I take the results, let's work on the idea. [...] Given the circle γ : $x^2+y^2=25$, centred in the origin and passing through the given point $P(4,3)$*" (she represents the situation graphically) [...] "*We were asked to write the equation of the tangent line in P*" (she points to P) "*to the curve.*" (She moves her hand on an imaginary tangent in P, Fig. 4.4.1)	pointwise	speech indicators + graph + pointing gesture

2	T: "*What have we done? Classic method. We exploited the geometrical fact*" (she traces the radius OP) "*we calculated* m*, the angular coefficient of* OP*, which was* $3/4$*.*" (She writes $m_{OP} = \frac{3}{4}$) "*And then we said that the direction of the tangent line to the curve in* P" (she traces the tangent line in P) "*will have been perpendicular to the radius, so to* m *of* OP*. And so* m *of the tangent line will be* $-4/3$*.*" (She writes $m_{tg} = -\frac{4}{3}$) "*Afterwards, in order to determine the equation of the tangent line, I have the angular coefficient, I make the bundle pass through the point with this angular coefficient, and I find the equation. And we found* $y - 3 = -\frac{4}{3}(x - 4)$*.*" (She writes the equation)	pointwise	graph + symbols
3	T: "*In the following point, instead, there was a different request. We were asked to consider another point* Q *that belongs to the circle*" (she chooses a point Q, Fig. 4.4.2) "*and to the first quarter of plane.*"	pointwise	speech indicators + graph
4	T: "*And then we were asked to write the angular coefficient of the straight line passing through* P *and through* Q*.*" (She writes m_{PQ}) "*For* m_{PQ}*, no problem: angular coefficient between two points. I use the formula* $y_Q - y_P$ *over* $x_Q - x_P$*.*" (She writes it after the expression "$m_{PQ} =$") "*We noticed that we had to use only the upper part of the circle, with the sign plus*" (she writes "$+\sqrt{25 - x^2}$" near the upper semicircle) "*So,* m_{PQ} *is equal to* $\sqrt{25 - x^2} - 3$ *over* $x - 4$"	pointwise	speech indicators + symbols
Figures	 **FIGURE 4.4.1** - M.G.'s gestures to explicit the task $t_{tangent}$.	 **FIGURE 4.4.2** - M.G.'s choice of Q.	

The work done by M.G. on the function is pointwise since she is referring to the point P and another point Q on the circle γ. She firstly considers the radius OP and its direction [*1*]. The coordinates of P on γ are used to find m_{OP}: thus, pointwise information on γ are exploited. Then, the teacher employs the pointwise algebraic technique $m_{tg} = -\dfrac{1}{m_{OP}}$ to obtain the angular coefficient of the tangent line in P [*2*]. Finally, she comes back to the equation of the bundle and makes "it pass through the point" P. The whole reasoning is P-centred, so the perspective activated on γ is pointwise. When the teacher moves on to consider another point Q on γ [*3*] the perspective is still pointwise. Also the information the teacher uses to find m_{PQ} [*4*], namely the coordinates of P and Q, are pointwise. The semiotic resources that support this pointwise reasoning are pointing gestures on the graph, while referring to P or Q. To recall the algebraic technique $m_{tg} = -\dfrac{1}{m_{radius}}$, M.G. leans on the graph. In particular, she traces the radius OP and then the tangent line in P. The drawing helps recalling the relation of perpendicularity between the two.
The teacher makes a global remark on the curve γ by defining the equation of the upper semicircle. In this case, she is supported by symbols: "with the sign plus", $+\sqrt{25 - x^2}$.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
5	T: "*Then, which was the next request? [...] It was about calculating the <u>limit as x goes to 4</u> of m_{PQ}.*" (She writes it on the blackboard, and then she adds very quickly what follows) "*We calculated the limit, it was an <u>indeterminate form 0 over 0</u>, rationalization, computations, at the end we got a numerical value, that was?*"	local	speech <u>indicators</u> + symbols
6	S1: "$-\dfrac{4}{3}$."		
7	T: "$-\dfrac{4}{3}$" (She writes it on the blackboard) "*And then we left with the question: "does this $-\dfrac{4}{3}$ has something to do with this $-\dfrac{4}{3}$ that I found with the geometrical fact?" I mean m_{tg} which was $-\dfrac{4}{3}$. So, once we have done all the algebraic steps, let's try to understand what we do when we do this limit, when we write this m.*" (She points to the limit and to m_{PQ})		

8	T: "*m of PQ represents the direction of PQ as straight line.*" (She points to P and to Q) "*What is the position of the straight line through P and Q*" (she traces the straight line PQ) "*with respect to the circle? It isn't tangent.*"	pointwise	speech indicators + pointing gestures + graph
9	S2: "*It's secant.*"	pointwise	speech indicators
10	T: "*It's secant. So* m_{PQ} *is m of the secant line. What do I do geometrically, as algebraically x goes to 4? Where is x? x is the abscissa of the point Q.*" (She writes $Q(x, \sqrt{25-x^2})$ and detects x on x-axis)	pointwise	speech indicators + symbols + graph
11	T: "*And when x goes to 4, it means that x is approaching 4.*" (She draws an arrow from x to 4 on x-axis, as in Fig. 4.4.3) "*If x goes to 4*" (She follows the arrow on x-axis, as in Fig. 4.4.3) "*but x is linked to Q*" (She points to Q) "*What does Q do?*"	local	speech indicators + symbol + graph + iconic gesture
12	S2: "*It goes down.*"		
13	T: "*It goes down, walking on the circle.*" (She moves her left finger on the arrow on x-axis and, at the same time, the chalk in her right hand on the circle go from Q towards P, Fig. 4.4.4) "*So, it walks on the circle, approaching more and more what?*"	local	speech indicators + iconic gestures + graph
14	Ss: "*P.*"		
15	T: "*P.*"		

Figures

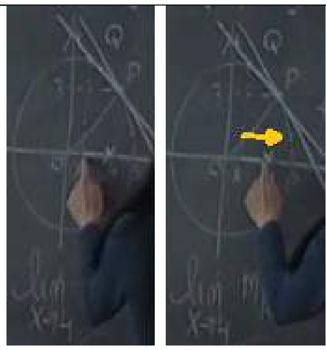


Figure 4.4.3 - M.G. follows the arrow from x to 4 with her left finger.

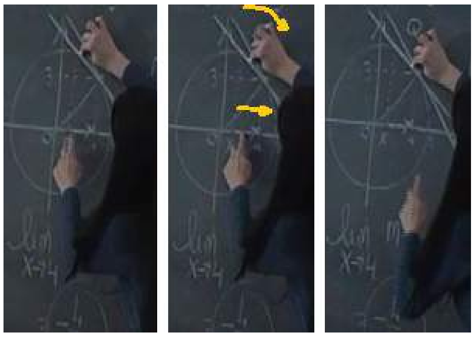


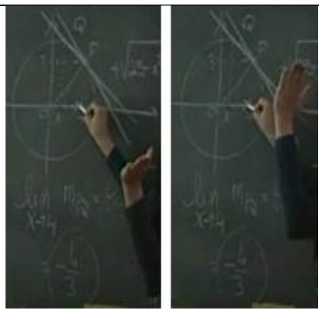
FIGURE 4.4.4 - M.G.'s simultaneous gestures to simulate the movement of x towards 4 and of Q towards P.

The teacher starts introducing a local perspective, which is initially based on the use of algebraic symbols ($\lim_{x\to 4}$, [0/0]), accompanied by local verbal expressions such as "limit as x goes to 4" [*5*]. Then, she proposes a geometrical interpretation of the employed technique involving the limit [*8-15*]. Starting again from pointwise considerations [*8-10*], she stresses the pointwise nature of m_{PQ} and of the algebraic technique to find it. She exploits the graph and the symbols support, with pointing gestures to P and to Q. Afterwards, she graphically interprets the verbal expression "as x goes to 4" as the local movement of the point Q towards the point P on the curve [*11-15*]. To simulate this movement, M.G. follows with her finger an arrow drawn on x-axis going from x to 4 (Fig. 4.4.3) [*11*] and simultaneously she walks on the curve with the chalk moving from Q to P (Fig. 4.4.4) [*13*]. Such local gestures towards P are accompanied by the local verbal expressions "x is approaching 4" or "Q is approaching more and more P". The students here intervene only to answer the teacher's inputs.

	What happens (teacher-students dialogue)	**Perspectives on γ**	**Semiotic resources**
16	T: "*As x goes to 4*" (she moves the chalk in her right hand, Fig. 4.4.5) "*what does the angular coefficient of the secant line do?*"	local	speech indicators + gesture
17	S2: "*It's approaching* 0."	local	speech indicators
18	T: "*It goes... to what?*" (She walks through the tangent in P with the chalk)	local	speech indicators + gesture
19	S3: "*To that of the new straight line.*"	local	speech

20	T: "*To that of the tangent line. So the secant line*" (she tilts her left arm as in Fig. 4.4.5 on the left) "*as x goes to 4*" (she follows the arrow from x to 4 on x-axis with the chalk in her right hand, as she did in Fig. 4.4.3) "*goes to the tangent line.*" (She moves her left arm as in Fig. 4.4.5 on the right) [...] "*In this limit position, the secant line is becoming the tangent line*" (She moves her arm on the blackboard representing the two straight lines, see Fig. 4.4.6)	local	speech indicators + iconic gestures
21	T: [...] "*But I'm working with m. So the angular coefficient of the secant line, as x goes to the abscissa of the point of tangency, goes to the angular coefficient of the tangent line. Is it right?*"	local	speech indicators
22	Ss: "*Yes.*"		
23	T: "*Now, we have seen that if our problem is determining the equation of the tangent line to a curve, and if it is a conic it is simple but when it is a different curve it isn't simple anymore, this can be the idea. The idea is that, since I cannot calculate immediately the angular coefficient of the tangent line, I walk away from the point, I position myself in a secant place, in order to approach it again, through a limit.*" (She points to P, then to Q and finally again to P)	local pointwise	speech indicators pointing gestures

Figures



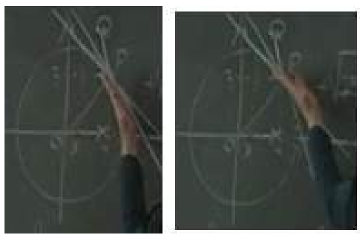


Figure 4.4.5 and 4.4.6 - M.G.'s gestures to represents the secant line that becomes tangent.

M.G. continues her interpretation of the limit as x goes to 4. Before she has considered what happens to the point Q, now her focus is on what happens to m_{PQ} [*16-19* and *21*], and more generally to the secant line [*20*]. The verbs used by the teacher and the students are verbs of local movement such as "goes to", "is approaching" and "is becoming". M.G. accompanies them with a movement of the arm to simulate the shift from the secant to the tangent line.

From a praxeological point of view, the teacher identifies the technique involving the limit as the technique for a new and more general praxeology to determine the tangent to a function in a point [*23*]. She justifies it through the comparison with an old algebraic technique on the example of the circle, and through a graphical interpretation.
According to our theoretical lenses, we can notice in comment [*23*] a certain discordance between the pointwise pointing gestures and the simultaneously local verbal expressions. The adoption of the local perspective may be somehow inhibited by the concomitance of pointwise gestures.

Formalization of the new technique $\tau_{tangent}$ and remarks on the technological-theoretical plane

M.G. formalizes the concept on the blackboard (Fig. 4.4.7), exploiting the idea on a generic function $y = f(x)$.

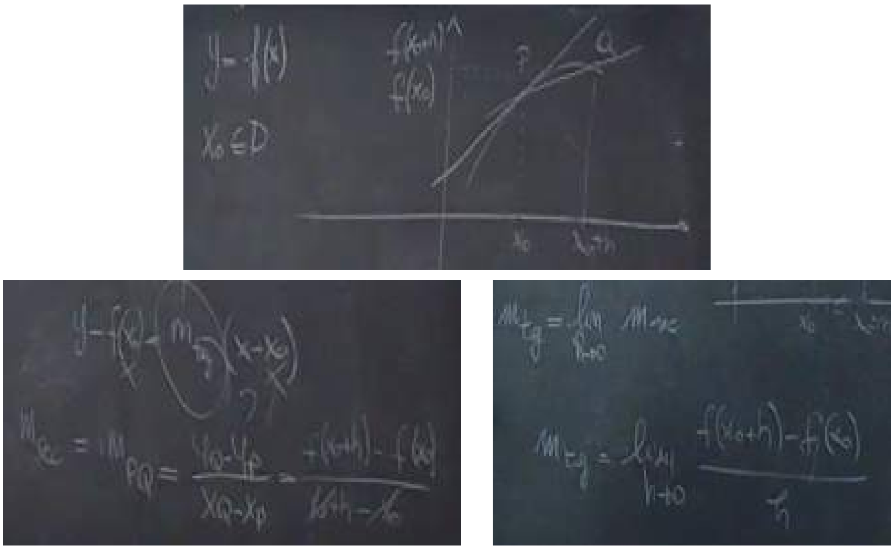


Figure 4.4.7 - M.G. formalizes the new technique in the case of a generic function f.

She finally makes some theoretical remarks on the formal expression of m_{tg}.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
24	T: "*First doubt: This calculation, m_{tg}, depends on an operation, which is the limit operation. "Limit of something". Let's suppose that I'm able to calculate all the limits. Am I sure to find a result?*" (She points to m_{tg}) "*Do the limit computation always give an outcome? Not necessarily. Actually, this limit could exist or not. We will investigate about this fact.*" (She specifies the two cases, with the symbols "$\exists$" and "$\nexists$", as in Fig. 4.4.8) "*It's sure that we have to add something about this calculation. If the limit exists [...] all outcomes are acceptable? [...] There's a value that we can't accept, that is?*"	local	symbols
25	Ss: "*Infinity*", "*Zero*"	local	oral symbols
26	T: "*Infinity.*" (She writes the symbol "∞" and circles it, in the case of existence, Fig. 4.4.8) "*Why? Because let's recall what we wanted to find: the angular coefficient of the tangent line, because we wanted to come back to the bundle. The bundle was $y - f(x_0) = m_{tg}(x - x_0)$. I can't think to replace an infinite value as angular coefficient here. What kind of straight line would I get? [...] So, it must be a finite value. Then if the limit of the angular coefficient of the secant exists and is finite, I obtain the angular coefficient of the tangent.*"	local	symbols

Figures

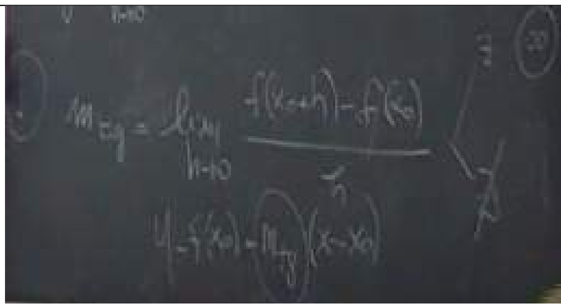


Figure 4.4.8 - M.G.'s symbols to summarize the possible features of the limit value.

The teacher rests on the features of the limit value: it can exist or not and, if existing, it can be finite or infinite. Her interpreting speech [*24*-*26*] contributes to the new praxeology at a technological level. The perspective is implicitly local in the involved symbols of limit and infinity. Notice that some students think that zero is a problematic value [*25*] (as it has already happened in line [*17*]). The teacher does not comment such interventions now. Perhaps she prefers to close the discussion and then to return back to the case in which the limit is zero.

Practice of the new praxeology on an example

M.G. concludes the first theoretical part. She has shown a new technique for the type of task $T_{tangent}$ and made some technological and theoretical remarks about it. Afterwards, she wants to practice the new technique on an example. Here is the situation she chooses and how she manages it.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
27	T: "*Let's see a practical example. Because, now it seems that we have a lot of unknowns, but actually it's not so. I consider a function: e^x. I take a point that belongs to the domain: the domain here is all $\mathbb{R}$, I can take whatever I want, I take 1, $x_0 = 1$. And I want to determine the equation of the tangent to the curve in the point of abscissa 1.*" (She writes the data at the blackboard: "$y = e^x$" and "$x_0 = 1$")	global and pointwise	speech indicators + symbols
28	T: "*Drawing.*" (She traces the graph of e^x, she detects the point 1 on x-axis and also $f(1)$ on y-axis, Fig. 4.4.9) "*What is the problem? Determining the equation of the tangent line, that is this line.*" (She traces the tangent line, Fig. 4.4.9)	global and pointwise	graph
29	T: "*And I already have the solution: $y - f(1) = m_{tg}(x - 1)$. And again the only doubt is here*" (She circles m_{tg}) "*It's to find m_{tg}, because I know $f(1)$: it's $e^1 = e$.*"	pointwise	speech indicators + symbols
30	T: "*But I know the strategy: I distance myself from the point and then I come back.*" (She moves her hands like in Fig. 4.4.10) "*How much I go far? Boh... We say, of h? We position in $1 + h$.*"	global	speech indicators + gesture + symbols
31	T: "*I don't have to calculate h. I use h only to distance myself from the point a little bit, and then ZAC to come back with h that goes to* 0." (She repeats the gesture in Fig. 4.4.10 on the drawing)	local global	speech indicators + sounds gestures

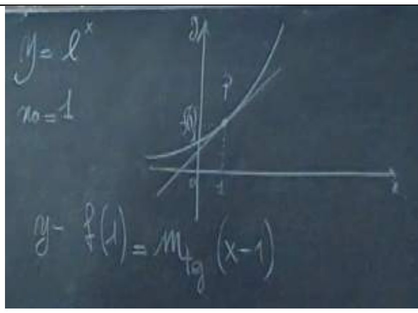


Figure 4.4.9 - M.G. poses the particular task $t_{tangent}$ with $f(x) = e^x$ and $x_0 = 1$.

Figure 4.4.10 - M.G.'s gesture to indicate the strategy for finding m_{tg}: "I distance myself from the point and then I come back" [*30*].

M.G. considers the example of $y = e^x$ and the task consists of determining the equation of the tangent line to the curve at the point of abscissa $x_0 = 1$. While the teacher chooses this specific point, she makes a global remark on the function domain [*27*]. Then, she draws the graphical situation. Therefore, the starting perspectives are global and pointwise on $f(x) = e^x$. Also the teacher's wide gesture (see Fig. 4.4.10) and the expression "I distance myself from the point and then I come back" in line [*30*] have a global character. After, she introduces some local perspective in her speech: she refers to a "little" distance and to h going to zero [*31*]. However, her gesture is the same as in Fig. 4.4.10: wide and global. So, we can notice a mismatch between the gesture and the accompanying speech with respect to the perspective that they convey. Perhaps, the activation of the local perspective could be somehow inhibited by the concomitance of global gestures.

M.G. does all the calculation to find m_{tg} in $x_0 = 1$, using the new technique (see Fig. 4.4.11). At the end of the lesson, she comes back again to formalization and gives some nomenclature, speaking of derivative for the first time [*32-35*].

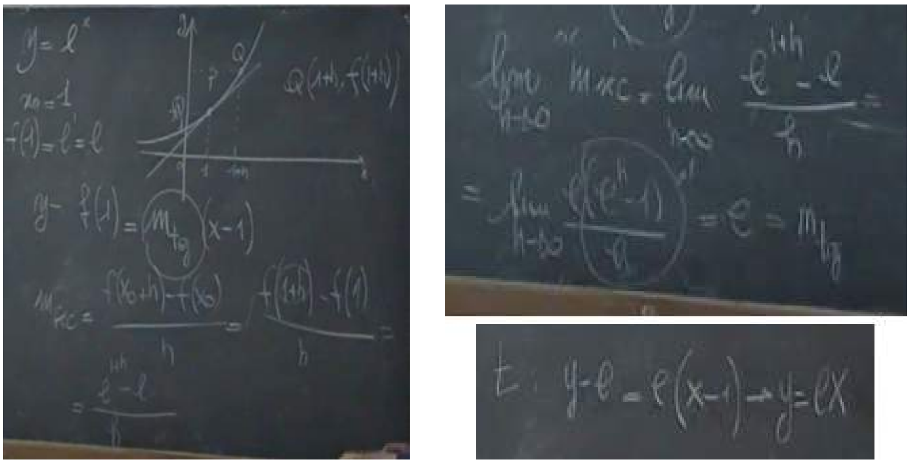


FIGURE 4.4.11 - M.G. SOLVES THE GIVEN TASK $t_{tangent}$.

	What happens (teacher-students dialogue)	Perspectives	Semiotic resources	on f, f'
32	T: "*Now, we cannot hope that in mathematics no one has found a way for avoiding to write every time "m_{sec}" and "m_{tg}". Because this way exists.*"			
33	T: "*Let's write one more time what we've done.*" (She writes from right to left "$m_{tg} = \lim_{h\to 0} m_{sec}$", see first row in Fig. 4.4.12) [...] "*What is m_{sec}? Let's interpret algebraically the meaning. It is $f(x_0+h)-f(x_0)$ over h. Then, we do the limit as h goes to 0. And if this limit exists and is finite, I obtain the angular coefficient of the tangent line, which however... is denoted with $f'(x_0)$.*" (She writes again from right to left "$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$", see second row in Fig. 4.4.12) "*And it is called 'derivative of f <u>in x_0</u>'. Derivative of the function <u>in the point x_0</u>.*" (See Fig. 4.4.12)	local pointwise	symbols speech <u>indicators</u>	f f'
34	S2: "*Why f'?*"	global	symbols	f'

35	T: "*Ah! They had to find a name. They had to find a symbol that represents this situation [...] What is better stressed here? That the angular coefficient of the tangent line depends on the function f*" (She points to f in the expression "$f'(x_0)$", see second row in Fig. 4.4.12) "*Because if I change function, the angular coefficient changes. But it depends also on x_0*" (She circles with her finger "x_0" in the expression "$f'(x_0)$", see second row in Fig. 4.4.12) "*If I change point, the tangent line changes.*" [...] "*Then let's give it a name that puts into evidence what it uses: f and x_0. And let's say that it has a geometrical interpretation as angular coefficient.*"	global and pointwise	symbols	f'

Figures

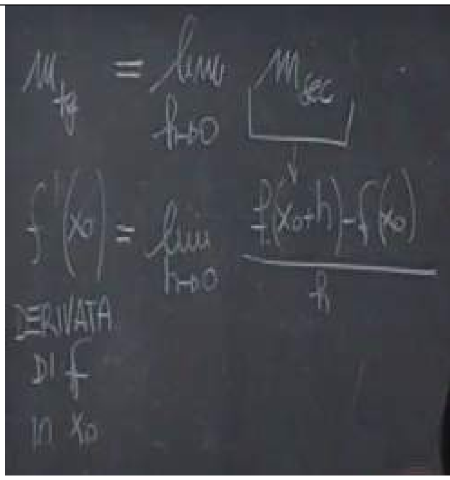


FIGURE 4.4.12 - M.G.'s introduction of the name "derivative".

Beside a local perspective on f which is implicit in the used symbols, the teacher introduces the function f' and a pointwise perspective on it. Thanks to S2's question [*34*], she better stresses the pointwise dependence of $f'(x_0)$ on the point x_0 [*35*].

The new praxeology $O\mathcal{M}_{tangent}$ is summed up in Table 4.12, where we try to use the teacher's expressions. In the theory part, only the first definition is new for the students.

$OM_{tangent}$	
Type of task $T_{tangent}$	Determining the equation of the tangent line to a generic function in a point.
Technique $\tau_{tangent}$	$tg : y - f(x_0) = m_{tg}(x - x_0)$ where $m_{tg} = \lim_{h\to 0} m_{sec} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ that is denoted with $f'(x_0)$ when it exists and is finite.
Technology $\theta_{tangent}$	Given a point $P(x_0, f(x_0))$, consider an increment h of the abscissa x_0. You obtain another point on the curve $Q(x_0 + h, f(x_0+h))$. Trace the secant line PQ, its angular coefficient is given by $m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x_0 + h) - f(x_0)}{h}$. Imagine that the abscissa of Q goes to the abscissa of P. It means that h goes to 0. The limit position of the secant PQ is the tangent in P. So $m_{tg} = \lim_{P\to Q} m_{sec} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$, that if existing and finite is denoted with $f'(x_0)$. Finally, the tangent line is the straight line belonging to the bundle $y-y_0 = m(x-x_0)$ that passes through P with angular coefficient $m_{tg} = f'(x_0)$.
Theory $\Theta_{tangent}$	- The definition of the tangent line to a generic function in a point as the limit position of the secant line cutting the function in that point and into another one, which walks on the curve getting closer and closer to it. - The problem of the tangent. - The analytic equation of a straight line, especially $m = \frac{y_1 - y_2}{x_1 - x_2}$ - The limits theory.

Table 4.12 - M.G.'s mathematical praxeology for the type of task $T_{tangent}$.

Remarks

Almost the entire lesson is centred on the technique to find m_{tg}. The teacher has revised the problem of the tangent to conics; the exercise she chooses as a connection with the derivative topic is about the tangent to a circle. The new technique involving the limit is presented in parallel with the previous algebraic ones. In particular, the comparison occurs at the level of the result. The limit as a technique has not to be introduced, because it is already given by the exercise statement. The teacher's concern becomes that of justifying why and how it works. The lesson is indeed characterized by the teacher's lecture. The students take notes and only intervene to answer specific questions, according to the IRE model. The teacher has to work on the technological-theoretical plane to interpret the symbols meaning. She uses the graphical resource, the symbolic manipulation, terms of "localized movement" and gestures. The local dimension is present is

everyone of these resources, even if somehow not in perfect simultaneity. It occurs that a local speech accompanies pointwise gestures on the graph or global gestures in the air. For the students, whom local perspective is still rather weak, this slight discordance may be disorientating.
The resulting praxeology $O\mathcal{M}_{tangent}$ is local insofar as the limit symbol is used in the formula for m_{tg}. The technology is strongly based on the graphical interpretation of the technique. The local perspective on the curve is implicitly present in the idea of approaching. As for the theory, instead, the tangent definition is marginally touched by the teacher, while working on m_{tg} technique.

4.4.3 Type of task $T_{f'}$: representing the derivative function

M.G. has closed the first lesson on the derivative concept by giving to the students some exercises about the determination of the equation of the tangent line to a function $y = f(x)$ in a point x_0 of the domain. The students have used the technique $\tau_{tangent}$ developed in classroom.
At the beginning of the second lesson, M.G. declares that the goal is learning to calculate derivatives more quickly, finding together some important rules. She says:

> "*Now we're going to see the derivatives of elementary functions. [...] We're going to build the table of our almost ten commandments, that has to be learnt by heart. [...] We're going to prove them, almost all of them. [...] They all come from the exercises you have done for homework but, if every time that I have to calculate a derivative I have to take the incremental ratio and do the limit, it becomes something long. By proving these little theorems, one discovers that it's possible to do calculation more quickly.*"

The first case presented by the teacher is the constant function. But before finding the expression of its derivative, M.G. opens a parenthesis about the different status of $f'(x_0)$ and $f'(x)$. Then, she obtains the derivative of the function $f(x) = k$ as $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, where "$x$", she says, "is generic". See Table 4.13 for the details of M.G.'s didactic organization.

Tasks t, type of tasks T and problems given in classroom	**Construction of the $O\mathcal{M}$ for $T_{f'}^{alg}$**
$t_{f'}$: calculating the derivative of the function $y = k$.	This task allows to directly work the technique $\tau_{f'}$ with the generic x on a particular function f.
Parenthesis: the different nature of $f'(x_0)$ and $f'(x)$.	This parenthesis represents the corresponding technology of the technique they are going to use.

Table 4.13 - M.G.'s didactic organization for working the type of task $T_{f'}$ in the algebraic register.

Within this brief episode, the teacher provides the praxeology $OM_{f'}^{alg}[T_{f'}/\tau_{f'}/\theta_{f'}/\Theta_{f'}]$ for dealing with the type of task $T_{f'}$ in the algebraic register. Let us distinguish two didactic moments:

1. construction of the technological-theoretical support of an algebraic technique;

2. elaboration and practice of the algebraic technique on an example (significant meeting with the type of task $T_{f'}$).

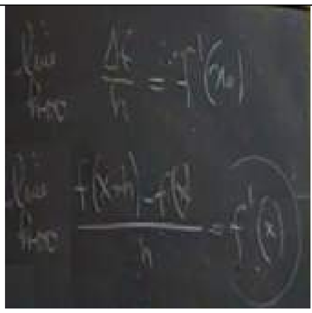
Several lessons later, M.G. will devote a whole hour to solve the graphical task of finding the graph of the derivative function, given the graph of a function (see ahead subparagraph "The graphical technique is shown on an example").

M.G. wants to work on a global technique to find the derivative function. She makes reference to "a generic x", but she has to deal with the pointwise character constructed on f' with the previous tasks. In the algebraic approach to the problem, the perspective on the involved function f is still implicitly local, thanks to the limit symbol involved in the technique.

Construction of the technological-theoretical support of an algebraic technique for $T_{f'}$

The teacher takes into account the constant function: $y = k$. When she asks "*What is its derivative?*", she realizes that she has not introduced the derivative as a function yet. Thus, she opens the following parenthesis.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
1	T: "*I must open a parenthesis... We have said that if the limit as h goes to* 0 *of the incremental ratio exists and it is finite,*" (she writes "$\lim_{h\to 0} \frac{\Delta f}{h}$", Fig. 4.4.13) [...] "*this limit is the angular coefficient of the tangent line, but it is also the derivative of the function calculated in x_0.*" (She writes "$= f'(x_0)$")	local pointwise	speech indicators + symbols	f f'

2	T: "*You have noticed that in the second group of exercises x_0 was not given. And, actually, if I calculate the limit as h goes to* 0 *of... Not in x_0 anymore, but in any x belonging to the domain, so a generic x...*" (She writes "$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$") "*I obtain the derivative of f in x,*" (She adds "$= f'(x)$") "*where x is generic.*"	global(=univ. pointwise)	speech indicators + symbols	f and f'
3	T: "*And so, here*" (pointing to $f'(x_0)$) "*the result is numerical, finite,* $5, 7, 21, ...$ *a number.*"	pointwise	symbols	f'
4	T: "*Instead, here*" (pointing to $f'(x)$) "*as a result I have an expression that depends on x. And you have see it in the second group of exercises where you haven't* 7 *anymore, but you have an algebraic expression... $y = 2x - 1$, for example. And actually this result is a function.*"	global	speech indicators + symbols	f'
Figures	 **Figure 4.4.13** - M.G. explains the difference between $f'(x_0)$ and $f'(x)$.			

The teacher provides the technique to algebraically determine the derivative function of a given f in line [*2*]. Her exact words are: "*I calculate the limit as h goes to* 0 *of... Not in x_0 anymore, but in any x belonging to the domain, so a generic x...*". The technique consists of replacing x_0 with x. The teacher opposes the pointwise character of $f'(x_0)$ [*3*] and the global character of $f'(x)$ [*4*]. She bases this distinction on a symbolic and syntactical difference. She observes that the limit calculated for a given x_0 gives you a number, whereas the limit calculated for "any x belonging to the domain", "a generic x" [*2*], gives you "an expression that depends on x" [*4*]. So, she uses the generic abscissa x to have a universal pointwise perspective on f and f'.
Our theoretical tool allows us to make a further remark about the engaged semiotic resources. M.G.'s technological speech bases on the syntactical difference between the

incremental ratio written in x_0 and the same incremental ratio written in a generic x. Nevertheless, the symbols she uses at the blackboard (see Fig. 4.4.13) do not properly support this technological intention. Indeed, the symbolic expression $\dfrac{\Delta f}{h}$ chosen by M.G. does not contain any reference to x_0. There is no evident difference between it and the symbolic expression $\dfrac{f(x+h)-f(x)}{h}$ written below, where instead the dependence on x is stressed. The use of the symbols, in this case, instead of supporting the speech, may hide the difference the teacher is speaking of.

The algebraic technique is shown on an example

After the brief parenthesis, the teacher continues to solve the exercise of finding the derivative of the function $y = k$.

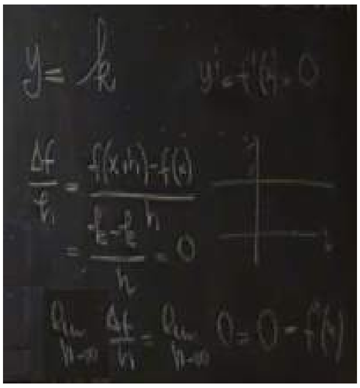


Figure 4.4.14 - M.G.'s determines the derivative of $y = k$.

Some minutes later, to conclude the lesson, M.G. draws a generic function in the Cartesian reference system (Fig. 4.4.14). In order to stress one more time the difference between $f'(x_0)$ and $f'(x)$, she chooses a graphical support.

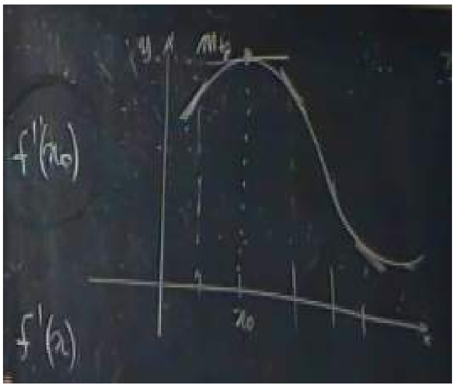


Figure 4.4.15 - M.G.'s graphical support to explain the difference between $f'(x_0)$ and $f'(x)$.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
5	T: "*When I calculate the derivative in a point, when I calculate $f'(x_0)$*" (she writes "$f'(x_0)$") "*we have the angular coefficient of the tangent line to the curve in that point. x_0 is here*" (she detects an abscissa x_0 on x-axis, as in Fig. 4.4.15(a)) "*In this point*" (she marks the point on the curve) "*I have the angular coefficient, the m of the tangent in x_0.*" (She traces the tangent and writes "m_{tg}" on it, see Fig. 4.4.15(b))	pointwise	speech indicators + symbols + graph	f' and f
6	T: [...] "*All our speech has been centered not on a specific x_0, but on a generic x.*" (She makes two different gestures, see Fig. 4.4.16(a) and 16(b))	global(=univ. pointwise)	speech indicators + oral symbols + gestures	f and f'
7	T: "*What is the difference between $f'(x_0)$ and $f'(x)$?*" (she writes "$f'(x)$" under $f'(x_0)$) [...] "*$f'(x)$ is the angular coefficient of the tangent line to the curve not in a fixed x_0, but in any x of the domain.*" (She completes the drawing by adding some tangent lines at some abscissas x, see Fig. 4.4.15) "*So I have all the possible directions as the point x varies in the domain. [...] I have a function as a result, namely f' depends on x, f' is a function.*"	global	symbols + speech indicators + graph	f'
8	S1: "*So as x varies we have different results.*"	global	speech indicators	f'
9	T: "*Yes, different results. And so if I want to calculate it in x_0*" (she points to $f'(x)$) "*I calculate the derivative for all x and then I measure it on x_0, by substituting x_0 every time.*"	global → pointwise	speech indicators	f'

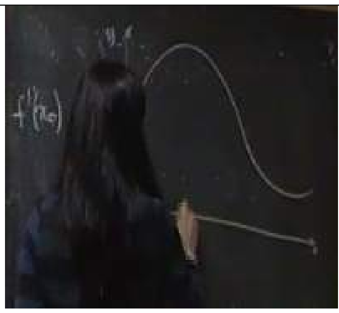


Figure 4.4.15(a) - M.G. detects a specific point of abscissa x_0 on the curve.

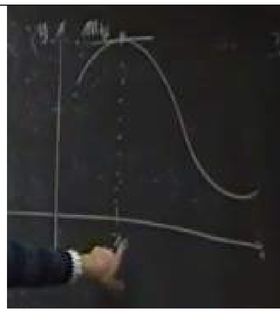


Figure 4.4.15(b) - M.G. detects the tangent at the abscissa x_0.

Figure 4.4.16(a) - M.G.'s gesture to indicate "a specific x_0" [6].

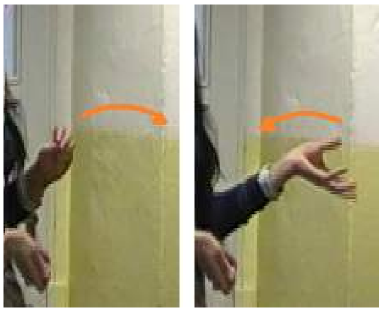


Figure 4.4.16(b) - M.G.'s gesture to indicate "a generic x." [6].

The teacher returns on the distinction between $f'(x_0)$ and $f'(x)$. This time she exploits graphical and gesture resources. With a fixed x_0 on x-axis [5] she associates the gesture in Fig. 4.4.16(a). It iconically recalls the vertical alignment between x_0 on x-axis and the point $(x_0, f(x_0))$ on the curve representing the function f. The perspective is pointwise on f and on f'. With "any x in the domain" [7] she associates the gesture in Fig. 4.4.16(b). Her left hand moves backwards and forwards horizontally in the air, like an x that varies on x-axis. This global gesture accompanies the universal pointwise teacher's attempt to detect some x and correspondingly some tangents on the graph of f [7] (see Fig. 4.4.15). Such a combined use of graph and gestures gives a global perspective on f. As for the derivative function f', the drawn graph cannot help. The global perspective on f' is made explicit through the speech: "all the possible directions as the point x varies in the domain" [7]. The universal pointwise sign x here is explicitly used in the global sense of variable.

In Table 4.14 we summarize the praxeology given by M.G. for determining the derivative function in the algebraic register.

$OM_{f'}^{alg}$	
Type of task $T_{f'}$	Algebraically representing the derivative function.
Technique $\tau_{f'}$	$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
Technology $\theta_{f'}$	$f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ gives a number, while the same limit in a generic x of the domain gives a function. So, we can calculate the limit in the generic point x and after replace the wanted x_0-value in the obtained expression.
Theory $\Theta_{f'}$	- Shift from pointwise to global perspective. - Algebraic writing of an expression in the generic x.

Table 4.14 - M.G's mathematical praxeology for the type of task $T_{f'}$ in the algebraic register.

The graphical technique is shown on an example

Let us jump to the thirteenth lesson. Throughout this period, M.G. has worked on the study of function, introducing the study of variation through the sign of the first derivative and the study of concavity through the sign of the derivative of second order. Then, she proposes to deduce the graph of the derivative function $y' = f'(x)$ starting from the only graph of the function $y = f(x)$. She uses both the blackboards in the classroom: on the right blackboard she draws the graph of $y = f(x)$, while on the left blackboard she deduces the graph of $y' = f'(x)$. Here is the task, as it is uttered by the teacher:

> *"Given the graph of a function, how can I deduce the graph of the derivative? And this is a well-posed question because the derivative of a function is a function itself. Therefore, being a function, it has its own graph."*

She chooses a graph on the textbook (namely the example on page 441 of Sasso's textbook) and she draws it on the right blackboard by saying: *"I have highlighted the important things"* (see Fig. 4.4.17).

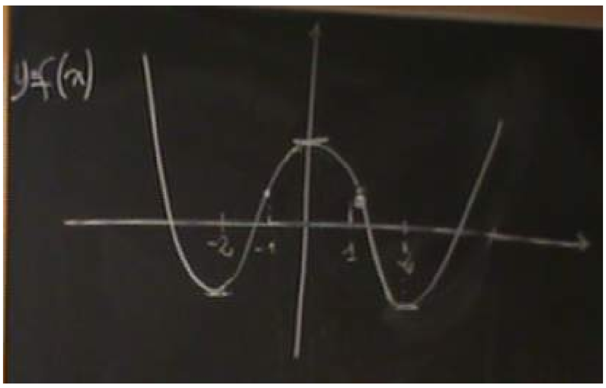


Figure 4.4.17 - The starting graph of f.

In the following lines we analyze step by step the technique developed by M.G. in order to solve the task. Basically, her intention is to lead a classical study of function (see Table 4.15) whose object is the derivative function $y' = f'(x)$, by reading the necessary information directly from the given graph of $y = f(x)$ [*10*].

$OM_{\text{study of function}}$	
Type of task	Studying the function $y = f(x)$ in order to draw its possible graph.
Technique	- Domain: impose the existence conditions for the expression $f(x)$. - Intersections with axis: solve the system of $y = f(x)$ with $x = 0$ and of $y = f(x)$ with $y = 0$. - Sign: study $f(x) > 0$. - Vertical asymptotes: verify if $\lim_{x \to x_0} f(x) = \infty$, for the specific x_0 where f is not defined. - Horizontal asymptotes: verify if $\lim_{x \to \infty} f(x) \in \mathbb{R}$; - Oblique asymptotes: it is $y = mx + q$ if $\lim_{x \to \infty} f(x)/x = m \in \mathbb{R}^*$ and if $\lim_{x \to \infty} (f(x) - mx) \in \mathbb{R}$. - Stationary points: solve $f'(x) = 0$. - Variation: study $f'(x) > 0 \quad \Rightarrow f$ is increasing; f is decreasing elsewhere. - Inflection points: solve $f''(x) = 0$. - Concavity: study $f''(x) > 0 \quad \Rightarrow f$ is convex; f is concave elsewhere. - Sketch a graph with the found information.
Technology	- $Dom(f) = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$. - Axis equations are $x = 0$ and $y = 0$. - f is positive/negative for all x for which $f(x)$ is positive/negative. - Asymptotes definition. - The stationary points are the derivative zeros. - f is increasing/decreasing where the derivative is positive/negative. - f changes concavity where the derivative has a stationary point. - f is convex/concave where the derivative is increasing/decreasing.
Theory	Functions (algebraic expression) and limits of functions. Definition of derivative as the gradient of the tangent line to the function in a point. Definition of derivative of second order as the derivative of the derivative.

Table 4.15 - Previous mathematical praxeology consisting in studying a function to draw its graph.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
10	T: "*Well, to represent the graph of a function, I have to study it. So, domain, intersections, symmetries, sign, asymptotes, searching for maxima and minima, and so on. All deduced from it.*" (She points to the graph of f in Fig. 3.5.17)	global, point-wise and local	speech indicators	f'
11	T: "*Domain of f': I run on the function with the tangent and I check if, by chance, I run into some corner, cusp, non-differentiable point. I run with the tangent, do you see it? TI-TI-TI-TI-TI-TI...* (She quickly traces imaginary tangents to f, as in Fig. 4.4.18) "*I'd say that I can always represent it. So, $D' = D = \mathbb{R}$.*"	local global	graph + gesture + sound speech indicators + symbols	f f'

Figures

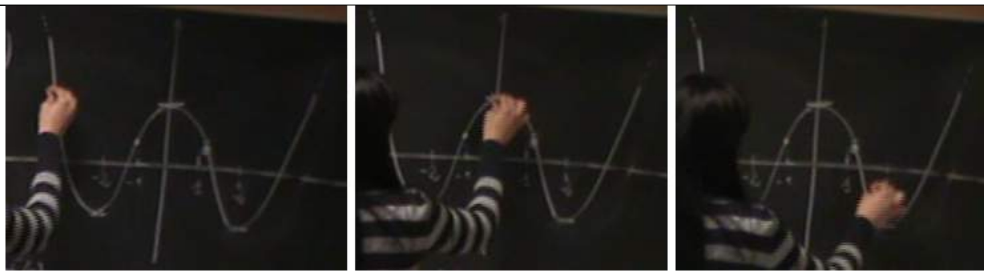


FIGURE 4.4.18 - M.G.'S GESTURE ON THE GRAPH OF f TO FOLLOW THE TANGENT, AS SHE SAYS TI-TI-TI.

The teacher starts from a global property of f': the domain [*11*]. To find information about it, she looks at the graph of f under a local perspective. She "runs" on the graph, reproducing the tangent. Every tangent corresponds to a sound "TI" and to an imaginary short segment she traces with the chalk (Fig. 4.4.18). The local perspective on f is now stressed with a gesture on the graph.

	What happens (teacher-students dialogue)	Perspectives	Semiotic resources	on f, f'
12	T: "*Intersections of f': I am wondering when this new function $f'(x)$ is equal to* 0." (She writes "$f'(x) = 0$") "*When $f'(x)$ is equal to* 0*? Can I know the solutions of this thing? Is it a known information? When I studied that graph,*" (she points to the graph of f) "*I solved this equation finding out the stationary points.*" (Her left hand moves horizontally as in Fig. 4.4.19)	pointwise local	speech indicators + symbols speech indicators + gesture	f' f
13	T: "*The stationary points of the function are the intersections of the derivative with x-axis.*"	local pointwise	speech indicators	f f'
14	T: "$x = -2, x = 0, x = 2$*: stationary points for f, points of intersection with x-axis for f'.*" (She writes it on the blackboard and marks the zeros of f' on the graph, see Fig. 4.4.20) "*So the same values play a different role in the function and in the derivative.*"	local pointwise	speech indicators + symbols speech indicators + symbols + graph	f f'

Figures

Figure 4.4.19 - M.G.'s gesture to represent a stationary point.

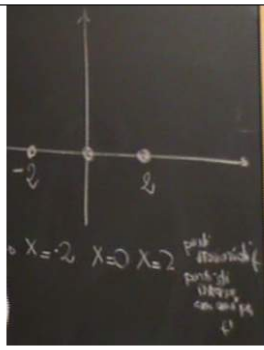


Figure 4.4.20 - Graphical translation on f' of the information about its zeros.

The teacher focuses on another pointwise property of f': the intersections with x-axis. She works firstly on symbols, by imposing the equation "$f'(x) = 0$". This equation allows her to recall the praxeology for studying a function (see Table 4.15) [*12*]. Indeed, solving $f'(x) = 0$ is the algebraic practice to detect the stationary points of f. M.G. uses another gesture to stress a local perspective on f. Her left hand moves horizontally to reproduce the horizontal tangent at a stationary point (Fig. 4.4.19). The relation between the zeros of f' and the stationary points of f is expressed in words [*13*]. Finally, it is graphically converted in full dots on x-axis corresponding to $x = -2, x = 0, x = 2$

(Fig. 4.4.20).

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
15	T: "*Sign of f'...*" (She writes "$y' > 0$ $f'(x) > 0$", see Fig. 4.4.21 on the left) "*Can I find the answers to this inequality, by reading them on the graph of the function? Think about the normal study of function, when you impose $f'(x) > 0$, what do you find out?*"	global	symbols	f'
16	Ss: "*Decrease and increase.*" "*Maxima and minima.*"	global and local	speech indicators	f
17	T: "*I find out if the function is decreasing or increasing. Then I can say if the stationary points are maxima or minima, but firstly I can say if the function increases or decreases. Ok? What does the function do? It decreases till the minimum, increases till the maximum, decreases and increases.*" (She retraces the graph with the chalk)	global	speech indicators + gesture	f
18	T: "*Where the function is increasing the derivative is positive, when the function is decreasing the derivative is negative.*" (She draws the scheme in Fig. 4.4.21 on the right)	global	speech indicators + scheme	f and f'
19	T: "*This new graph has positive sign from -2 to 0, it is drawn above x-axis, I delete below.*" (She keeps her left hand above x-axis and blackens the region of plane below, as in Fig. 4.4.22)	global	speech indicators + graph + gesture	f'

Figures

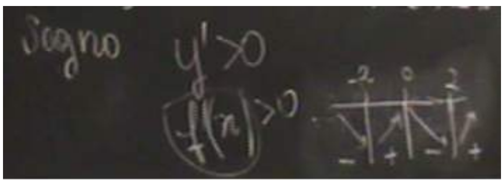


Figure 4.4.21 - Symbols and scheme for deducing f' sign from f variation.

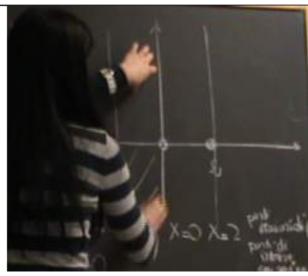


Figure 4.4.22 - M.G.'s blackens the region of plane where f' graph does not pass.

The third considered aspect is global: the sign of f'. The teacher immediately expresses it in symbols: "$f'(x) > 0$" (Fig. 4.4.21 on the left). With this inequality she reminds to students the praxeology for studying a function [*15-16*]. The teacher relates the global sign of f' with the global variation of f [*18*]. To read the variation of f, she continuously retrace with the chalk the profile of f, checking where it decreases or increases [*17*]. Finally, M.G. graphically converts the information about the sign of f', by blackening the regions of plane above or below x-axis (see Fig. 4.4.22).

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f', f''**
20	T: "*Increase or decrease of f': what object do I study in order to understand if a function increases or decreases?*"	global	speech indicators	f'
21	S2: "*The derivative.*"			
22	T: "*The sign of the first derivative! The first derivative of the first derivative is the derivative of second order*" (she writes it in symbols, see Fig. 4.4.23 on the left) "*which I am still able to read on the graph of the function, in terms of concavity. The derivative of second order of f is positive when it [f] is convex.*" (She realizes the scheme in Fig. 4.4.23)	global global	symbols speech indicators + graph + scheme	f', f'' f, f''
23	T: [...] "*f' increases from $-\infty$ to -1 and also from 1 to $+\infty$*" (see Fig. 4.4.23 on the right) "*and decreases elsewhere.*"	global	speech indicators + symbols	f'

Figures

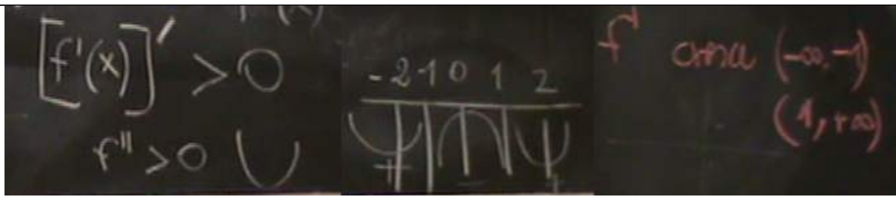


Figure 4.4.23 - Symbols and scheme to study the variation of f'.

A further aspect to take into account is the variation of f', that is a global property. M.G. makes again reference to the praxeology for studying a function, by asking "*What object do I study in order to understand if a function increases or decreases?*" [*20*]. When a student answers "*The derivative*" [*21*], she specifies "*The sign of the first derivative*" [*22*]. Then, with a syntactical game of symbols, she writes "$[f'(x)]' > 0$" and expresses it in terms of f: "$f''(x) > 0$" (Fig. 4.4.23). It allows her to recall the praxeology for studying a function and to look at the convexity/concavity of the graph of f. As she has done for the sign, she makes use of a scheme (Fig. 4.4.23) to relate the convexity/concavity of f with the sign of the f''. Returning back to the symbolic initial expression "$[f'(x)]' > 0$", she reads it in terms of f', deducing the needed information about its variation [*23*] (Fig. 4.4.23). She keeps this global data and shifts the attention to local inflection points of f [*24*].

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f', f''**
24	T: "*I'm going to mark the inflection points: -1 and 1. These inflection points give me some information about the derivative?*"	local	graph	f
25	T: "*Where $f''(x) = 0$, that is in the inflection points of f, what does f' do?*" (She writes "$f''(x) = 0$" and then "$[f'(x)]' = 0$", Fig. 4.4.24)	pointwise local	symbols graph	f',f'' f
26	T: "*While here I read an inflection point for f, here I read a stationary point for f'. They are the possible maximum and minimum. So in -1 I have a maximum and in 1 a minimum.*" (She marks them with a short horizontal line, see Fig. 4.4.25)	local	speech indicators + graph	f and f'

Figures

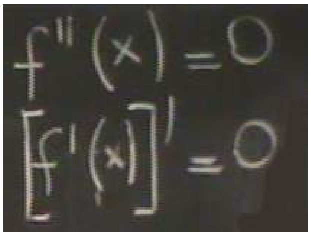


Figure 4.4.24 - Symbols for studying the stationary points of f'.

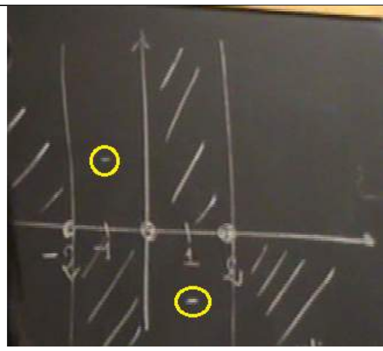


Figure 4.4.25 - M.G.'s horizontal signs on the graph of f' to indicate the stationary points.

The teacher uses the same syntactical transformation on the symbolic expression "$f''(x) = 0$" to get "$[f'(x)]' = 0$" [*25*]. It allows her to relate the inflection points of f to the stationary points of f' [*26*]. The teacher uses a local horizontal sign to mark the stationary points on the graph of f' (Fig. 4.4.25).

By recollecting all the information deduced on f', the teacher sketches its possible graph (see Fig. 4.4.26).

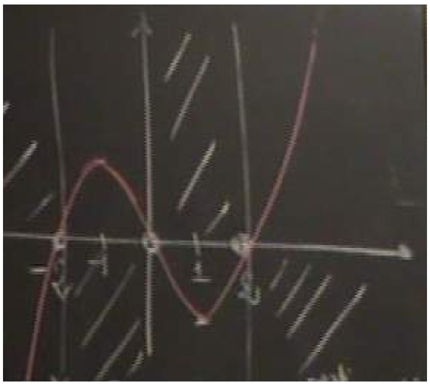


Figure 4.4.26 - The final graph of f'.

We can summarize the praxeology for $T_{f'}$ in the graphical register in Table 4.16.

$O\mathcal{M}_{f'}^{gra}$	
Type of task $T_{f'}$	Graphically representing the derivative function, starting from the graph of a function f.
Technique $\tau_{f'}$	Following the steps of the study of function (from the previously acquired $O\mathcal{M}_{\text{study of function}}$, see Table 4.15).
Technology $\theta_{f'}$	The algebraic request about f' can be interpreted in terms of f.
Theory $\Theta_{f'}$	The theory behind the praxeology for studying a function. The reversibility of the implications f' property $\Rightarrow$ f property.

Table 4.16 - M.G.'s mathematical praxeology for the type of task $T_{f'}$ in the graphical register.

Remarks

In order to algebraically represent the derivative function, the teacher proposes the syntactical technique of replacing the pointwise sign x_0 with the universal pointwise sign x. The explicit technique is expressed in words as follows: "calculate the limit as h goes to 0 not in x_0 anymore, but in any x belonging to the domain, so a generic x". It entails a global perspective on the functions f and f', which depend on x. Such a global perspective is reached on f, thanks to the graphical resource, and it is obtained on f' as universal pointwise perspective, through the use of symbols and speech. We noticed a slight disharmony between this two semiotic resources when used together. It occurs that the speech underlines the different pointwise and universal pointwise nature of $f'(x_0)$ and $f'(x)$, but the used symbols actually hide this difference.
The function $f'(x)$ is defined as "all the possible directions as the point x varies in the domain". The universal pointwise sign x is explicitly used in the global sense of variable and a global perspective is obtained on the function f'.
The graphical work is a very delicate moment, according to the teacher (see the interviews in Paragraph 4.4.1). However, it is very important in order to make students aware

that the derivative function is a function itself, with its own graph. Working with this semiotic resource can enhance the global perspective on f'. Given the graph of a function f, the task is deducing the graph of $y' = f'(x)$. The teacher uses the praxeology related to the study of function, which they have recently practised in details in the classroom. Thus, $O\mathcal{M}_{\text{study of function}}$ turns out to be embedded in the new praxeology, both technically and technologically. By means of gestures and graphical resources, the study of $y' = f'(x)$ allows to activate a local perspective on f, whereas the perspectives on f' are alternatively pointwise and global. Notice that this praxeology allows to get a final global perspective on f' without passing through universal pointwise considerations. This approach implicitly gives for granted that the graph of the derivative function associates the abscissa x with the gradient of the tangent line to f at the abscissa x.

4.5 The case of V.

V. teaches maths and physics in a high school in Turin. She has been working in this school for several years, and in particular with *Quinta E*'s students since they attended their third year. The school has adopted the textbook *Manuale blu di Matematica 2.0* written by Bergamini, Trifone and Barozzi[3], but V. prefers to follow her notes for preparing lessons and to use the textbook only for the exercises. In particular, to address the theory part, she refers to the volume *Lezioni di Analisi Matematica 1*, written by Geymonant (1981) for Analysis university courses. This can be due to the fact that she has a PhD in Analysis and that she gave lectures as teaching assistant in engineering courses.

4.5.1 From the interviews: V.'s beliefs

In the preliminary interview with V., in October 2012, we spoke about the work done with her students on limits, as fundamental to analytically treat the derivative concept.

How did you introduce limits with your students? What kind of work did you do on them? And how do you think this may influence the students' approach to the derivative concept?
Starting from sequences and limits of sequences, thanks to IT supports such as GeoGebra or Excel, V. made her students perceive by intuition the limit concept for $n \to +\infty$. She introduced the horizontal strip as a tool and they used expressions like "from a certain point onward, it [the function graph] enters the strip", "I can make this strip as small as I want", "it [the function graph] flattens more and more", "I can put it as close as I want" and so on. It led to the formal definition with ϵ, which entails big difficulties for the students. When they moved on to the limits of functions, setting $x \to +\infty$ worked well, by analogy with the discrete case. A difficulty came out while dealing with x going to a point, because one has to take into account the continuity property. It is the continuity definition itself that allows you to calculate the limit of a continuous function f, by

[3] This is the previous edition of *Matematica.blu 2.0* (Bergamini et al., 2013).

simply substituting the given x-value x_0 into the expression $f(x)$. Remarking that the result is the same even if the function has a hole in the point $x = x_0$, they established the rule of "never being" in the single point $x = x_0$ while calculating the limit value. They finally took into account those limits that lead to the indeterminate form [0/0]. They began with polynomial examples and worked on the algebraic treatment, and then V. introduced limits such as $\lim_{x\to 0}\frac{\sin x}{x}$. They discussed about the possibility to approximate them with expressions containing only polynomials, to transform the situation into a known case. To decide which polynomial function to use, they employed a graphical support and used the zoom function. Thus, V. formalized a "f[function]-g[polynomial] equivalence" by giving the definition of asymptotic equivalence as follows.

> DEF. Let f and g be two functions that goes to zero as $x \to 0$. We say that f and g are asymptotically equivalent when $x \to 0$, if $\lim_{x\to 0}\frac{f(x)}{g(x)} = 1$ and we write $f(x)\underbrace{\sim}_{x\to 0}g(x)$.

Now, her expectation is that the imprint given to the work on limits has influenced the students' conceptualization of the tangent line as the straight line that best approximates the function in a point.

How do you usually introduce the derivative notion?
To introduce the derivative notion, V. often makes the students reflect upon the tangency concept, asking them "What is a tangent line?". Indeed, she thinks this is the problem: "although we have spoken of tangent line in lots of occasions, the students don't know how to define it". V. usually raises the issue of the tangent line definition in the classroom and tries to exploit the reactions: someone recalls the $\Delta = 0$ method, some others the geometrical properties of the conics. Finally, V. recalls some previously learnt concepts such as the instantaneous speed, the angular coefficient of a straight line, the trigonometric tangent of an angle. Normally, V. defines the tangent line to a curve in a point as the limit position of a straight line that cuts the curve in that point and in another one which gets closer and closer to it. She would like to use the interactive whiteboard (IWB) in classroom to show the process with GeoGebra.

In one of the following meeting with V., in November 2012, we spoke about the derivative function.

How do you usually introduce the derivative function?
From a graphical point of view, often by using GeoGebra, V. usually shows how the derivative graph can be automatically generated, given the graph of a function. She shows the procedure once with GeoGebra, but then she expects that the students reproduce it with paper and pencil. V. wants them to develop a graphical reasoning ability. And one of the reasons why her colleagues and her like the adopted textbook is exactly that it proposes a lot of graphical exercises.

4.5.2 Type of task $T_{tangent}$: determining the equation of the tangent line to a generic function in a point

To the question "How do you introduce the derivative notion?", V.'s answer is that she usually starts with the tangent line definition. In classroom, indeed, she immediately poses the type of task $\mathcal{T}_{tangent}$.

> "*Given a function, given a point on this function, we want to determine the equation of the tangent line to the function in that point.*"

She does not give any graphical representation of the "given function", probably because she wants it to be as generic as possible. Afterwards, she goes straight to the main problem, by asking:

> "*First of all, which properties must a tangent line have?*"

An open discussion about the tangent line definition arises, in which the students, as V. expected, recall all the operational definitions concerning the conics. V.'s main goal, as she later confesses us, consists of uprooting them, showing their inefficacy with a generic function. The previous praxeology to which the teacher and the students refer is the conics-related praxeology, we summed up in Table 4.1.
Finally, in order to accomplish the general given type of task, V. proposes to find the angular coefficient of the tangent, following the more generic definition they are searching for. Table 4.17 sums up the phases developed by V. to accomplish the type of task $\mathcal{T}_{tangent}$.

Tasks *t*, type of tasks *T* and problems given in classroom	**Construction of the $O\mathcal{M}$ for $\mathcal{T}_{tangent}$**
$\mathcal{T}_{tangent}$: determining the equation of the tangent line to a function in a point	
Problem: defining the tangent line to a function in a point	The teacher and the students discuss on the general theory $\Theta_{tangent}$
$\mathcal{T}_{m_{tg}}$: finding the angular coefficient of the tangent line	The teacher leads the students in constructing the technology for the type of task $\mathcal{T}_{m_{tg}}$, from which they develop the technique $\tau_{m_{tg}}$

Table 4.17 - V.'s didactic organization for working on the type of task $T_{tangent}$.

The teacher and the students detect the subtype of task $\mathcal{T}_{m_{tg}}$ inside the main type of task $\mathcal{T}_{tangent}$. They firstly develop the praxeology $O\mathcal{M}[\mathcal{T}_{m_{tg}}/\tau_{m_{tg}}/\theta_{m_{tg}}/\Theta_{tangent}]$. We can detect the three didactic moments:

1. first meeting with the type of task;

2. construction of the technological-theoretical block;

3. elaboration of the technique.

Moreover, this episode is relevant for the introduction of a new local perspective on a generic function, that V. realizes by using the object tangent line. So far, this object has been tied to the conics study, where its definitions are pointwise and global. In the construction process of a new more generic praxeology, V. has to deal with the pointwise and global character of these previously learnt definitions and the related techniques and technologies.

Construction of the technological-theoretical block for $T_{m_{tg}}$

Let us focus now on the discussion arisen in the classroom after V.'s question: "*Which properties must a tangent line have?*". We will not transcript all the statements, but only the most significant ones.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
1	S1: "*[the tangent line must] intersect [the function] in a single point.*"	pointwise	speech indicators
2	S2: "*[the tangent line] must be perpendicular.*"	pointwise	speech indicators
3	S3: "*For the circle...*" (his hands form a T, as in Fig. 4.5.1)	pointwise	pointing gesture
4	S4: "*But, if it is so, not all the points has a tangent line ... I'm imagining a sloped function*" (tilting his hand) "*then maybe the tangent line in that case could intersect the function in another point, right?*"	global	speech indicators
5	T: [...] "*So, are you thinking of something like this?*" (She sketches the curve in Fig. 4.5.1)	global	sketch
6	S4: "*Yes, there is the tangent line but it touches other points of the function.*"	global	speech indicators
7	T: "*For example, if I search for the tangent line here?*" (She points at the maximum point on the curve, see Fig. 4.5.2) "*How do I imagine it?*"	pointwise	pointing gesture
Figures			

Figure 4.5.1 - S3's gesture which reminds of radius and tangent perpendicularity, for the circle.

Figure 4.5.2 - V.'s first non-example: the tangent line could intersect the curve in another point.

Basing on their previous experience with the tangent line and the conics [*3*], some students propose pointwise properties [*1-2*], that are the theoretical basis on which the old conics-related praxeology is grounded (see Table 4.16). The students approach this type of task as those previously treated when the curve was a conic: they firstly try to adapt their old knowledge. Then, thanks to S4's global remark [*4*], the teacher uses a global graphical non-example [*5-7*] (Figure 4.5.2) to show that the tangent line to a function in a point could intersect the function again in another point.
We can notice that the teacher has not intentionally given any representation of the generic function they are exploring. So, the students have the possibility to use every kind of register or semiotic resource to express their utterances. On the one hand, the students' words are accompanied by gestures that refer to conics-related practices, such as that in Fig. 4.5.1. On the other hand, the teacher tries gradually to fix students' utterances and gestures into written sketches on the board, so that the whole class can discuss about them. The teacher chooses the graphical register of representation for this purpose, and produces a conversion from the students' speech to the sketch in Fig. 4.5.2. Proposed by the students but drawn by the teacher, written signs of this kind become powerful instruments in the teacher's hands: she stresses their potential features, in order to foster students' discussion.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
8	S5: "*To avoid what S4 said, we can take a suitable interval*" (moving his two indexes up and down together, as in Fig. 4.5.3) "*where the tangent line satisfies our conditions...*"	local	speech indicators + iconic gesture
9	T: "*So, we limit the zone.*"	local	speech indicators
10	S5: "*At that point, if I want a tangent line to a point in that interval, I can do it without any other intersection of the line in that interval.*"	local	speech indicators
11	T: [...] "*So, we take a point, wherever we want, this one* (x_0, y_0), *we limit to a suitable neighbourhood*" (she sketches the situation at the white board, see Fig. 4.5.4) "*and what do we require there?*"	local	speech indicators + sketched signs
12	S6: "*There, that the line intersects [the function] only in that point.*"	local pointwise	speech indicators
13	S5: "*It is not enough.*"		

14	T: "*It is not enough. Why?*"		
15	S7: "*It could be like this*" (he draws in the air a line intersecting the function, see Fig. 4.5.5)	pointwise	iconic gesture
16	T: "*It could do so*" (She draws the situation in Fig. 4.5.6)	pointwise	graph

Figures

Figure 4.5.3 - S5's gesture expressing the intention of limiting the zone of study.

Figure 4.5.4 - V.'s graphical interpretation of S4's intention [*8*] and [*10*].

Figure 4.5.5 - S7's gesture to figure out that the straight line could be secant, instead of tangent.

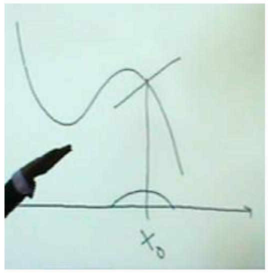


Figure 4.5.6 - V.'s second counterexample: according to the first definition given by the students, the straight line could also intersect the curve.

Students modify their proposal in a local direction [*8-10*], but the pointwise perspective is still too strong [*12*]. After S5's realization [*13*] and S7's pointwise remark [*15*], the teacher makes a pointwise graphical counterexample [*16*] (Fig. 4.5.6) to show that the students' definition is not effective.
Once again, the teacher converts the students' speech and gestures into a written sketch. We can notice that S5's gesture (Fig. 4.5.3) represents an efficient semiotic resource for the students to speak about locality. Indeed, we will find a similar gesture ahead in the discussion, produced by another student. The teacher exploits it, adding two vertical lines on the sketch at the whiteboard (Fig. 4.5.4) and accompanying the gesture and the written sign "| |" with a rigorous mathematical speech [*11*]. This is an example of *semiotic game*: "The teacher mimics one of the signs produced in that moment by the students (the basic sign) but simultaneously he uses different words. [...] Namely, the teacher uses one of the shared resources (gestures) to enter in a consonant communicative attitude with his students and another one (speech) to push them towards the scientific

meaning of what they are considering" (Arzarello & Paola, 2007, p.23). In V.'s case, the relation of signs is more complex. Indeed, the teacher exploits one of the shared gestures, but without repeating it. In recalling it, she changes the semiotic resource, converting the gesture into the written sign "| |" and accompanying it with a meaningful mathematical speech.

	What happens (teacher-students dialogue)	**Perspectives on** f	**Semiotic resources**
17	S6: "*That it is also, let me say, perpendicular. I don't know exactly how...*"	pointwise	speech indicators
18	T: "*Why perpendicular?*"		
19	S8: "*To what?*" [...]		
20	S9: "*It could be perpendicular to the radius of the circle which best approximates the curve, couldn't it?*" [...]	pointwise local	speech indicators
21	T: "*So, the tangent to the circle which best approximates the function in that point. It is possible to do so, but it arises the problem to find the circle which best approximates ... It is more difficult from the beginning.*"	local	speech indicators

The pointwise property [*17*] is due to students' reference to the tangent line in the case of a circle [*20*]. This is one of the tangent images that V. wants to undermine. Nevertheless, it is interesting that S9 tries to relate what she knows with the new given situation, introducing a local view [*20-21*]. V. concludes that this definition is not of immediate and simple application.

	What happens (teacher-students dialogue)	**Perspectives on** f	**Semiotic resources**
22	S10: "*There are two methods...*"		
23	T: "*Two methods for what?*"		
24	S10: "*To calculate the tangent:* $\Delta = 0$ *that was valid for hyperbolas, parabolas and circles; and then the method of doubling. I thought about them, but I don't understand how to...*"		
25	T: [...] "$\Delta = 0$. *Is it possible to apply it here? [...] Why does* $\Delta = 0$ *method work for the curves we studied two years ago?*"		
26	S5: "*Because they are of second degree.*"		
27	T: "*And then?*"		
28	S1: "*They have* Δ*.*"		

29	T: "*Since the equation is of second degree, at the end I have a second degree solving equation and then, by imposing* $\Delta = 0$*, what am I requiring?*"		
30	S1: "*That it has only one solution.*"	pointwise	speech indicators
31	S3: "*Two coincident solutions.*"	pointwise	speech indicators
32	T: "*Two coincident solutions. [...] But if I take something like this*" (she refers to the sketch in Fig. 4.5.3) "*which is its degree?*"		
33	S4: "*At least third.*"		
34	T: "*Clearly the solving equation will be of third degree. [...] And I would like to say that it has two coincident solutions, but I can't because there isn't* Δ."	pointwise	speech indicators

Since it was successful with conics, someone reminds the algebraic $\Delta = 0$ method, which they studied in the previous years [*22-24*]. V. points out the ineffectiveness of that process for a generic function [*25-34*] and stresses its pointwise aspect [*30-31* and *34*]. The discussion moves on to the possibility of applying the conics-related techniques, in particular $\Delta = 0$ method. From the utterance [*17*] to [*34*], we point out that the only semiotic resource they use is the speech. The teacher helps the students to argue why the old $\Delta = 0$ technique worked with conics, but cannot be adapted to this more generic type of task.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
35	S11: "*It must all lie in the same half-plane, except for the point.*"	global	speech indicators
36	T: "*What do you mean?*"		
37	S11: "*A function detects two half-planes.*"	global	speech indicators
38	T: "*Yes. They aren't half-planes, but regions of plane.*"	global	speech indicators
39	S11: "*Ok. And the straight line must always lie in the same region of plane.*"	global	speech indicators
40	T: "*Yes. Always?*"	towards local	speech indicators + intonation

41	S11: "*In the interval*" (measuring a short distance with his hands, as in Fig. 4.5.7)	local	speech indicators + iconic gesture
42	T: "*Locally. All we are saying is only local.*" (She sketches two vertical lines on the white board, see Fig. 4.5.8) [...] "*Ok, S11. And if I draw a function like this*" (She sketches the curve in Fig. 4.5.9) "*and I ask you to find the tangent in this point*" (indicating the inflection point) "*Is there the tangent in that point or not?*"	local	speech indicators + sketched signs + pointing gesture
43	S6: "*It tends to coincide with the function.*"	local	speech indicators
44	S5: "*It is like when we studied* $\sin x$ *that was asymptotically equivalent to* $y = x$*, isn't it?*"	local	speech indicators
45	T: "*Did we have the tangent in that case?*"		
46	S7: "*There exists the tangent but the reasoning based on the regions of plane falls.*"		

Figures

Figure 4.5.7 - S11's gesture to locally refer to an interval.

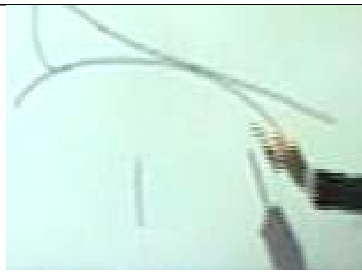


Figure 4.5.8 - Teacher's conversion of S11's gesture into written sign "| |"

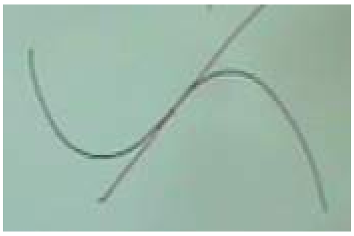


Figure 4.5.9 - V.'s third counterexample: the tangent line in an inflection point "passes through" the curve.

Another student makes a global proposal [*35-39*], then corrected into a local one [*40-41*]. V. makes explicit that they are thinking locally and shows a local graphical counterexample [*42*] (Fig. 4.5.9) of the tangent line in an inflection point. S11's gesture (Fig. 4.5.7) recalls the previous S5's gesture (Fig. 4.5.3). As in line [*11*], this shared gesture to express a local perspective is followed by the teacher's conversion into the written sign "| |" (Fig. 4.5.8). Students are now completely disarmed on the pointwise and global fronts. Every reference to conics-related praxeology has fallen. Nevertheless, this last counterexample [*42*] (Fig. 4.5.9) reminds them of the asymptotic equivalence property [*43-46*] and gives them the possibility to refer to another more recent praxeology, summed up in Table 4.18.

$OM_{\text{asymptotic equivalence}}$	
Type of task	Solving limits such as $\lim_{x\to 0} \frac{\sin x}{x}$.
Technique	Firstly, observe that it gives an indeterminate form [0/0]. Thanks to a graphical software, draw the graphs of $y = \sin x$ and $y = x$. Zoom them in a neighbourhood of the origin and notice that they seem to coincide. Then, we can say that $\sin x \sim x$ for x belonging to a neighbourhood of $x = 0$. So, $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{x} = 1$.
Technology	If you have to compare two infinitesimal quantities in a neighbourhood of $x = 0$, so that their ratio gives [0/0], you need to compare the speed with which they go to zero. If they turn to coincide in the given neighbourhood, you can say that their are asymptotically equivalent.
Theory	- Analytic equation of a straight line, angular coefficient as incremental ratio. - Given definition of asymptotic equivalence: DEF. Let f and g be two functions that goes to zero as $x \to 0$. We say that f and g are asymptotically equivalent when $x \to 0$, if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ and we write $f(x) \underbrace{\sim}_{x\to 0} g(x)$. - Limits theory.

TABLE 4.18 - PREVIOUS MATHEMATICAL PRAXEOLOGY CONSTRUCTED BY V. RELATED TO THE ASYMPTOTIC EQUIVALENCE BETWEEN FUNCTIONS.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
47	S1: "*The tangent can be the line which best approximates the given curve in a neighbourhood of the point, can't it?*"	local	speech indicators
48	S5: "*But it is what we did to have the asymptotic equivalence!*"		

49	S11: "*Limit as x goes to x_0 of $f(x)$ over the straight line is equal to 1.*" (He goes and write it on the whiteboard, see Fig. 4.5.10)	local	speech indicators + symbols
50	T: [...] "*If you remember, this is the condition we used when we spoke about asymptotic equivalence.*"		
51	S1: "*It is right or not? Let's get to the point.*"		
52	T: "*The fact is that it must work, but it is an approach that I have never tried before. Let's try together now.*"		

Figures

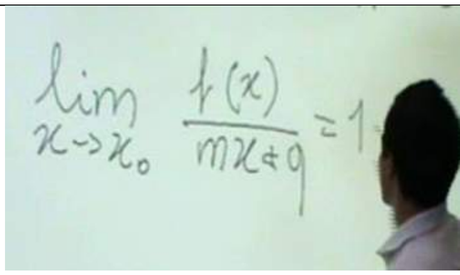
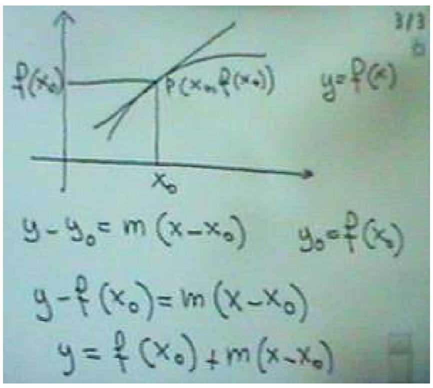
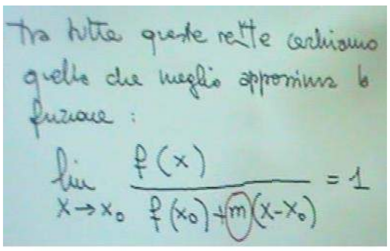


Figure 4.5.10 - S11 writes the relation he has just expressed in words.

Eventually, one of the students proposes a local property: "*The tangent line can be the line which best approximates the given curve in a neighbourhood of the point, can't it?*" [*47*]. This is one of the possible ways to correctly define the tangent line to a function. Along with S11's attempt of formalizing [*49*] (Fig. 4.5.10), it represents a turning point for the work in the classroom. Indeed, only in this moment the teacher formulates in written symbols the type of task $T_{tangent}$ (Fig. 4.5.11). V. poses the type of task $T_{m_{tg}} \subset T_{tangent}$ by circling the unknown m in the equation of a generic straight line which passes through the point $(x_0, f(x_0))$: $y - f(x_0) = m(x - x_0)$.





Among all these straight lines let's search for the one that best approximates the function:
$\lim_{x \to x_0} \frac{f(x)}{f(x_0)+m(x-x_0)} = 1$

Figure 4.5.11 - The teacher analytically formulates the type of task $T_{tangent}$

Since, so far, the problem was only formulated in words, the main semiotic resource has been the speech. The first reformulation of the problem in written symbols marks a significant change in the classroom mathematical activity. The shift towards the semiotic resource of written symbols prepares the ground to the first technology produced for the type of task $T_{tangent}$. Such a change in the semiotic activity corresponds to the abandon of the conics-related praxeology (which has a pointwise-global character) in favour of the asymptotic equivalence-related praxeology (which has a local character).
The teacher succeeds in abandoning the conics-related praxeology, which entailed algebraic techniques: namely $\Delta = 0$ method and $m_{tg} = -(m_{radius})^{-1}$. It is a student (S11) who proposes a symbolic formulation of the best approximation of a function through a straight line. He employs the definition which comes from the theory of the asymptotic equivalence. This is a local condition, which we denote with $\theta'_{m_{tg}}$, since it represents a first attempt to get an analytic expression for m_{tg}:

$$\theta'_{m_{tg}} : \lim_{x \to x_0} \frac{f(x)}{f(x_0) + m(x - x_0)} = 1.$$

Whereas no component of the conics-related praxeology can be applied for treating the type of task $T_{tangent}$, the theoretical component of the asymptotic equivalence-related praxeology can be potentially employed in the construction of the technological level for the rising praxeology.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
53	S7 : "*But... It is a sort of identity, isn't it? I mean, if I replace x with x_0, since it goes to it, [the part with] m becomes zero and all becomes $f(x)$ [$f(x_0)$] divided by $f(x)$ [$f(x_0)$] which is equal to 1. It's a sort of identity, of something obvious, isn't it?*"	pointwise	speech indicators
54	T: "*It is not so obvious... I'm saying that the function is approximated by a straight line in the best way.*" [...]	local	speech indicators
55	T (referring to the equivalence in Fig. 4.5.11 on the right): "*There exists and it is simple the way to obtain m from it. But now I cannot see the way to do it rigorously. Because [...] I'd need to algebraically transform this equivalence, which is the condition I want, so that I get m equal to something.*" [...]		
56	S11: "*So m is equal to the limit as x goes to x_0 of $f(x) - f(x_0)$ over $x - x_0$, isn't it?*"	local	speech indicators + oral symbols

Figures

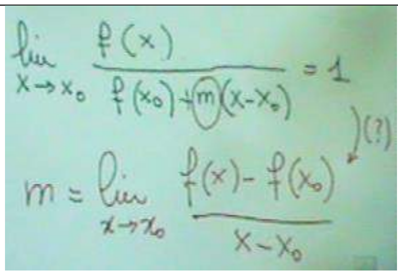


Figure 4.5.12 - V. writes S11's intuition in symbols.

Within this symbolic formulation, again S11 intuitively gets the correct technique $\tau_{m_{tg}}$: "*m is equal to the limit as x goes to x_0 of $f(x) - f(x_0)$ over $x - x_0$*" [*56*]. The students go through a phase of "analytic babbling"[4]. V. writes S11's proposal in symbols at the whiteboard (Fig. 4.5.12). She adds an interrogative point between the technology $\theta'_{m_{tg}}$ and the technique $\tau_{m_{tg}}$. That is because they do not manage to get the guessed technique through a rigorous algebraic manipulation. It depends on the fact that $\theta'_{m_{tg}}$ is not the correct technology, as they will realize the lesson after. Some doubts start to emerge on $\theta'_{m_{tg}}$ validity as well as on its pointwise character [*53*].

Technology refinement to justify the technique for $\mathcal{T}_{m_{tg}}$

At the beginning of the second lesson on the derivative concept, V. comes back to the $\mathcal{T}_{tangent}$ symbolic formulation (Fig. 4.5.11). She recalls also one of the last students' interventions [*53*] about the validity of $\theta'_{m_{tg}}$.

	What happens (teacher-students dialogue)	**Perspectives on f**	**Semiotic resources**
57	T (drawing the situation in Fig. 4.5.13): "*This is a straight line that obviously doesn't approximate the function well. But, whatever value I give to m, for all the straight lines which pass through this point, the condition [$\theta'_{m_{tg}}$] is true.*"	pointwise	speech indicators + graph

[4]The expression is borrowed by the ArAl project, born to promote an early approach to algebraic thinking within the group GREM (Department of Mathematics, University of Modena-Reggio Emilia, Italy). The ArAl project elaborated the metaphor of "algebraic babbling", which brings the learning methods for the algebraic language closer to those used for natural language. Here, we use it in reference to the "analytic" language of Calculus.

58	T: "*Why it [θ'_{mtg}] doesn't give me the idea of asymptotic estimate? Because the asymptotic estimate is valid for infinitesimal quantities, which go to* 0*. Thus here first of all I need an indeterminate form* 0/0*, the two quantities must go to zero, and then I compare the speed with which they go to zero.*"	local	speech indicators

Figures

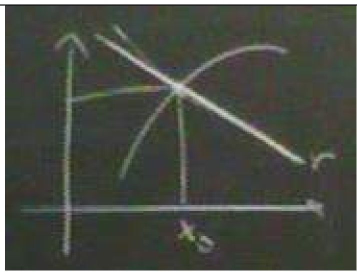


Figure 4.5.13 - V.'s graphical counterexample for θ'_{mtg}.

The moment of construction of the right technology θ''_{mtg} for the new technique τ_{mtg} is complex and thorny. Thus, the teacher comes back to rein the lesson. She firstly shows the pointwise character of the previously given condition θ'_{mtg}, which does not fit with their request of local estimate [*57*]. Then, by recalling the principle of the asymptotic equivalence, she proposes another graphical-symbolical formulation of the type of task $\mathcal{T}_{tangent}$ (Figure 4.5.14). More specifically, she applies to the x-axis a vertical translation of vector $(0; f(x_0))$. While miming this geometrical transformation with an upward movement of her hand, she says:

> "*In order to make the two quantities infinitesimal, what do I do? Simply, it corresponds to apply a translation. It is like we start from this point.*"

And she draws the red straight line on the graph as the new x-axis (see the graph in Fig. 4.5.14). Then, she focuses the students' attention on the quantities AB and CB, giving names to the points on the graph. These two quantities are now infinitesimals as x goes to x_0 and the asymptotic equivalence can be established between them. By expressing in symbols the quantities AB and BC (see Fig. 4.5.14 on the right), she can eventually write the right technology (Figure 4.5.15), which effectively justifies the technique τ_{mtg} (Figure 4.5.16). We denote it θ''_{mtg}:

$$\theta''_{mtg} : \lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x - x_0)} = 1.$$

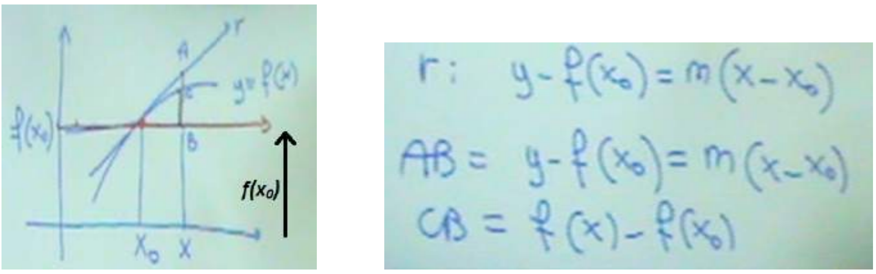


Figure 4.5.14 - V.'s second analytic formulation of the type of task $T_{tangent}$: the x-axis is vertically translated by vector $(0; f(x_0))$.

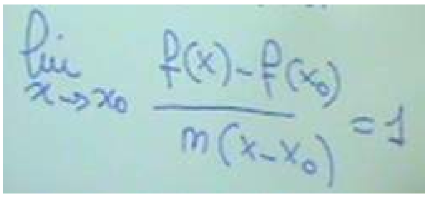


Figure 4.5.15 - The correct technology $\theta''_{m_{tg}}$.

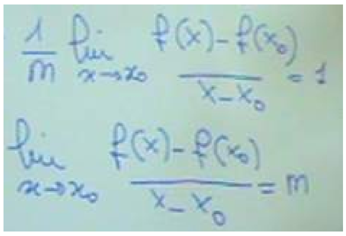


Figure 4.5.16 - From the correct technology $\theta''_{m_{tg}}$ to the technique $\tau_{m_{tg}}$.

The construction of the $O\mathcal{M}[T_{m_{tg}}/\tau_{m_{tg}}/\theta''_{m_{tg}}/\Theta_{tangent}]$ is now complete. A process of synthesis allows the teacher and the students to integrate this result into the formulation of the type of task $T_{tangent}$ and completely solve it.

Table 4.19 summarizes the new praxeology for $T_{tangent}$. Notice that the technological speech is made of sentences explicitly uttered by the teacher in the classroom. As far as the theoretical knowledge is concerned, only the first one is a new piece of knowledge, whereas all the others are old pieces of knowledge to recall. We can observe that the asymptotic equivalence property is embedded in the theoretical component of this new praxeology and it is fundamental for finding the new technology $\theta_{tangent}$.

$O\mathcal{M}_{tangent}$	
Type of task $T_{tangent}$	Determining the equation of the tangent line to a generic function in a point.
Technique $\tau_{tangent}$	$tg : y - f(x_0) = m(x - x_0)$ where $m = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.
Technology $\theta_{tangent}$	Among all the straight lines which pass through the point $(x_0, f(x_0))$, the tangent is the one that best approximates the function. [...] The infinitesimal quantity $f(x) - f(x_0)$ is asymptotically equivalent to the infinitesimal quantity $m(x - x_0)$, so the following condition is satisfied: $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{m(x - x_0)} = 1$. Then, m is a constant so I can bring it out of the limit sign, finding $\dfrac{1}{m} \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = 1 \iff m = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.
Theory $\Theta_{tangent}$	- The definition of the tangent line to a generic function in a given point as "the line which best approximates the given curve in a neighbourhood of the [given] point" [47]; - The analytic equation of a straight line; - The asymptotic equivalence property; - The limits theory.

Table 4.19 - V. and her students' mathematical praxeology for the type of task $T_{tangent}$.

Remarks

The teacher makes the students recall old algebraic praxeologies, namely the conics-related one. Her didactic technique is the mathematical discussion, which is centred on the tangent definition. The old algebraic techniques are rejected one by one in the case of a generic function. Another more recent praxeology, which is linked to the asymptotic equivalence property, intervenes and gets a determinant role. It is related to limits, so it belongs already to the Calculus domain. Indeed, it has already a local imprinting. Therefore, we can observe that the local dimension is conferred to the problem of the tangent and to the whole praxeology thanks to the embedding of the asymptotic equivalence-related praxeology. More precisely, the theory $\Theta_{\text{as.eq.}}$ enters the theory $\Theta_{tangent}$ and determines the technology $\theta_{m_{tg}}$. As a consequence, $O\mathcal{M}_{tangent}$ has a strong technological-theoretical block, with a pronounced local character. The definition of tangent, reached through the discussion, allows to approximate the given function in a neighbourhood of a point. Thus, the resulting perspective on the function is local. The teacher and the students explicitly gain it, firstly through the speech, the written signs, the gestures and, only after a little struggle, they move to symbols.

4.5.3 Type of task $T_{f'}$: representing the derivative function

In the previous paragraph we have analysed how V. has introduced the derivative concept. Its final formalization (see Fig. 4.5.17) has been the following:

> "*f is differentiable in* x_0 *(*$x_0 \in dom f$*) if it exists and is finite* $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$*, which we denote with* $f'(x_0)$".

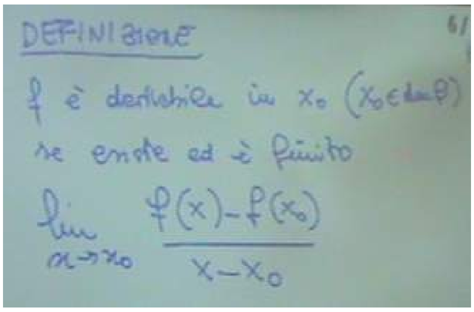


Figure 4.5.17 - V.'s formalization of the derivative concept.

She has given some exercises to the students, as homework, but they have found some difficulties, due to the different textbook notation. So, V. starts her third lesson about derivatives by saying:

> "*We have seen a theoretical and introducing part. But now we are really interested in learning how to do calculations. What's the problem? You have tried to do some calculation for today, and you have seen that calculating each time the limit of the incremental ratio is something tiring. [...] So, we'd really like to have an automatic method for finding the derivative. What we are going to learn today is how calculating the derivative without having to do every time the limit of the incremental ratio. [...] We will use another notation, since it is more convenient*".

Thus, she changes the limit variable by posing $h = x - x_0$ (Fig. 4.5.18) and, doing so, she gives an alternative technique $\tau'_{m_{tg}}$, which is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

to find the derivative of a function f in its point x_0.

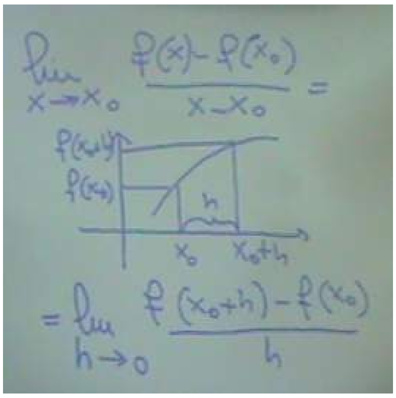


Figure 4.5.18 - V.'s second formalization of the derivative concept.

V. proposes a first task $t_{f'}$:

> "*Calculate the limit of the incremental ratio for the function* $f(x) = x^2$ *[...] Calculate the derivative for* x^2 *in any point* x_0 *of it*".

Then, she interprets the result in terms of function (without making it explicit) and starts filling in a table $f|f'$. The didactic task which V. wishes to accomplish is giving rules for quick calculation, and not specifically introducing the derivative function. Nevertheless, it is after a student's intervention that V. comes to speak about derivative as a function. The derivative function is represented in the algebraic register. Therefore, we can say that the type of task $T_{f'}$ has been here specified as *algebraically representing the derivative function*. We add the superscript "alg" to stress that a particular register of representation has been chosen to accomplish the type of task. The phases of the work are summarized in Table 4.20.

Tasks t**, type of tasks** T **and problems given in classroom**	**Construction of the** OM **for** $T_{f'}^{alg}$
Problem: showing the equivalence between the technique $\tau_{m_{tg}}$ given in classroom and the textbook one.	The teacher works at technological level to show that the two techniques are actually the same one. As a final result, she gives an alternative technique $\tau'_{m_{tg}}$.
$t_{f'}$: calculating the limit of the incremental ratio for the function $f(x) = x^2$, that is the derivative in any point x_0 of it.	The teacher gives an algebraic technique $\tau_{f'}$.
Problem (a student's doubt): do the independent and dependent variables change, shifting from f to f'?	The teacher answers the student's question by speaking of the derivative as a function, with its algebraic expression. It is an attempt on a technological level for $T_{f'}$.
$t_{tangent}$: writing the equation of the tangent line to the parabola x^2 in the point of abscissa $x = 2$.	This task actually belongs to $T_{tangent}$, since it refers to the tangent equation, but the teacher gives it in this moment in order to practice the pointwise formula on f', $m = f'(x_0)$.

Table 4.20 - V.'s didactic organization for working on the type of task $T_{f'}$ in the algebraic register.

Within this episode, the teacher gives the praxeology $OM_{f'}^{alg}$ in the algebraic register. We can distinguish V.'s actions in the three didactic moments:

1. first meeting with the type of task;
2. elaboration of an algebraic technique;
3. construction of a technological-theoretical block.

As for the perspectives, V. has to deal with the pointwise character on f', which is implicit in the given definition (see $\tau_{m_{tg}}$ and $\tau'_{m_{tg}}$). Nevertheless, the introduced technique $\tau_{f'}$ goes towards a global perspective on f', essentially for the use of the generic x_0 sign. We are interested in how this global perspective on f' is managed by the teacher.
Anyway, notice that the local perspective constructed on a generic function f, while working on $T_{tangent}$, remains implicitly present in the use of the limit symbol at the technical level.

Elaboration of an algebraic technique

Let us start focusing on the particular task $t_{f'}$ in which V. practices the alternative technique $\tau'_{m_{tg}}$ (Fig. 4.5.18). We concentrate especially on the final speech, when the teacher gives generality to the whole process.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
1	T: "*Let's try to make some examples. We are going to calculate the limit of the incremental ratio for the function $f(x) = x^2$. Try to write it on your own. So, try to calculate the derivative for x^2 in any point x_0 of it.*"	local global(=univ. pointwise)	speech indicators oral symbols	f f, f'
2	S1: "*Any point?*"			
3	T: "*Any point... As usual, let's call it x_0.*"			
4	[The students work alone for a while, the teacher walks through the classroom] S2: "*It gives $2x_0$!*"	global(=univ. pointwise)	symbols	f'
5	S1: "$2x_0 + 1$."	global(=univ. pointwise)	symbols	f'
6	S2: "*+1, it's true!*"			
7	T: "*Why +1? It should be $2x_0$... Let's try.*" (She makes all the steps at the whiteboard, see Fig. 4.5.19) [...]	global(=univ. pointwise)	symbols	f'
8	T: "*Obviously, if you calculate this limit, you get zero over zero.*" (She writes [0/0], see Fig. 4.5.19) "*Necessarily, always. If not so, I have done something wrong, if I don't get an indeterminate form.*"	local	speech indicators + symbols	f
9	T: "*What have I discovered? I've discovered that when I have the function x^2, its derivative is... point by point... is $2x_0$.*"	global(=univ. pointwise)	speech indicators + symbols	f, f'
10	T: "*So, if I write a function here, and its derivative here*" (she starts composing a table at the blackboard, Fig. 4.5.20) "*I've discovered that the derivative of the function x^2 is $2x$.*" (She fills the table in) "*This is an automatic process, because if I have x^2, from this moment on, I won't calculate the limit of the incremental ratio anymore. I know that its derivative is $2x$.*"	global	table + symbols	f, f'

11	T: "*I've calculated it once and for all, in the general case of any point x_0, so I have it.*"	global(=univ. pointwise)	speech indicators + oral symbols	f, f'

Figures

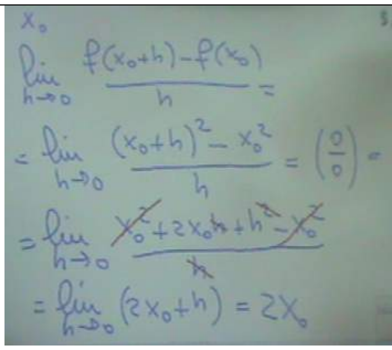


FIGURE 4.5.19 - V. SOLVES THE TASK $t_{f'}$.

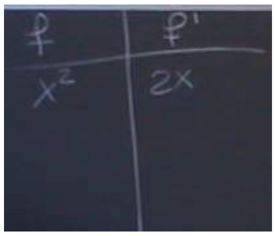


FIGURE 4.5.20 - TABLE $f|f'$ TO COLLECT THE MAIN RULES.

Notice that the given task [*1*] is universal pointwise on the involved functions ("any point x_0") and the techniques the students dispose of are pointwise on f'. V.'s utterance [*3*] seems to reveal that the class is somehow familiar with the work on generic signs. While solving the exercise [*4-8*], V. keeps a local perspective on f: not only she uses the limit sign, but also she stresses [*8*] its local implications from a technological point of view. When she gets the result $2x_0$, she globally interprets it as "the derivative point by point" [*9*], in a universal pointwise sense. V. suddenly replaces x_0 with the global variable x [*10*]. This semiotic technique is implicit in the change of signs from line [*9*] to line [*10*]. V. uses the table $f|f'$ as a resource to systematize. As for the perspectives, x_0 is used as a universal pointwise sign, representing every abscissa x_0 of the domain, while x has the global meaning of variable. From a technological point of view, at this stage V. does not make explicit the shift from x_0 to x. It follows an opaque praxeology, whose technique and related technology are only hinted.

Construction of the technological-theoretical block

The doubt of one of the students makes V. move on to a technological level.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
12	S3: "*The independent variable changes from f to f'... Is it x_0 or is it always the same?*"	global	speech indicators + oral symbols + intonation	f, f'
13	T: "*It is a point x.*"	global	oral symbols	f, f'

14	S3: "*The variable from f to f' changes... I mean, as the graph changes, it changes also the x. If not so, I make confusion... Is it right or wrong?*"	global	speech indicators	f, f'
15	T: "*It is wrong, but it's right that you wonder about it. Let's take $f(x) = x^2$, which I'm able to draw, that is the parabola.*" (She draws the curve, see Fig. 4.5.21)	global	graph + symbols	f
16	T: "*What have we discovered and proved? That if I take any point x_0*" (she chooses a point x_0, see Fig. 4.5.21) "*then the angular coefficient of the tangent line in the point of abscissa x_0 [...] is $2x_0$.*"	global(=univ. pointwise) pointwise	speech indicators symbols + graph	f
17	T: "*So, if I draw the tangent line here*" (she traces the tangent in the corresponding point on the parabola, see Fig. 4.5.21) "*this straight line has $2x_0$ as angular coefficient.*" (She writes $m = 2x_0$)	pointwise	graph + symbols	f
18	T: "*What does it mean? That, at this point, I can make x_0 vary as I want.*" (She moves her hand forwards and backwards, as in Fig. 4.5.22) "*... At this point, I can write x instead of x_0, for convenience.*"	global	speech indicators + symbols + iconic gestures	f, f'
19	T: "*And point by point I have a formula, that is the following*" (she writes $f'(x) = 2x$) "*which point by point*" (she moves the stick as in Fig. 4.5.23) "*tells me the value of the angular coefficient of the tangent line.*"	global(=univ. pointwise) global	speech indicators + symbols iconic gestures + graph	f' f

Figures

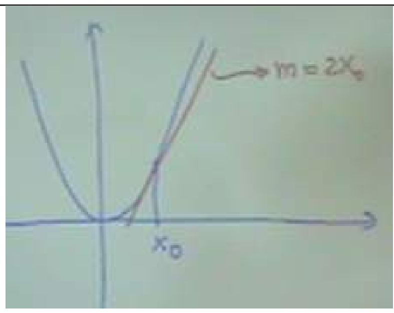


Figure 4.5.21 - Graphical example chosen by V.

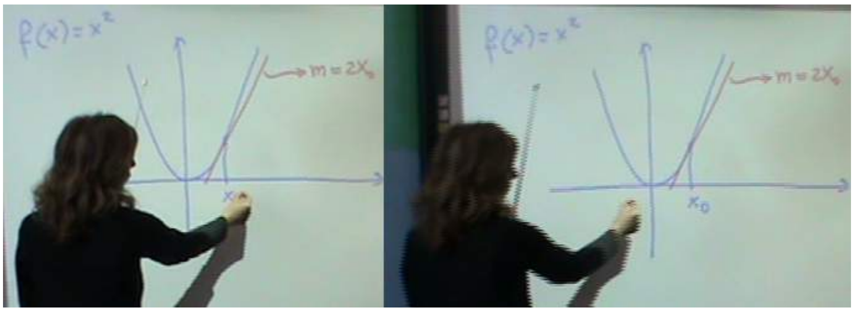


FIGURE 4.5.22 - V.'S GESTURE TO ACCOMPANY THE WORDS "I CAN MAKE x_0 VARY AS I WANT" [*18*].

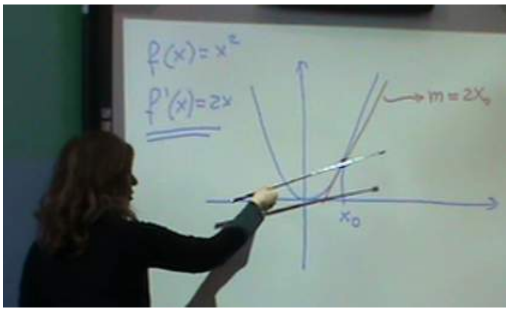


FIGURE 4.5.23 - V.'S GESTURE TO SHOW THE TANGENT "POINT BY POINT" [*19*].

As we can infer from S3's interventions [*12* and *14*], the opaque praxeology introduced by V. induces doubts and confusion in the students. The teacher clarifies the generic role of x_0 and justifies, at a technological level, the shift from x_0 to x which was not so explicit in lines [*9-10*]. V. starts from stressing the pointwise basic character of the sign x_0, by choosing a particular point on x-axis and the corresponding one on the parabola $y = x^2$ [*15*]. In line [*16*], we can notice an incongruity which occurs typically when one uses the graphical resource to speak of something generic. One declares to consider a generic point on the curve, any value of the abscissa x_0. However, when detected on the drawing, the point or the abscissa necessarily becomes a precise point on the curve or a specific value of the abscissa. In order to regain generality and variability, V. uses the speech "I can make x_0 vary as I want [...] I can write x instead of x_0" [*18*] and continuous gestures on the graph (see Fig. 4.5.22). The previous hint to technology and technique [*9-10*] is a little bit developed here. She justifies the change of x_0 into x (the used technique) as a convenience. Actually she is giving to the generic universal pointwise sign x_0 the global status of variable. It happens through V.'s utterance [*18*] and the continuous gesture (Fig. 4.5.22), combined together. As a consequence, the perspective on the functions $f(x)$ and $f'(x)$ would be global in the sense of variable. Nevertheless, when the teacher makes explicit the global perspective on $f'(x)$, she enhances the universal pointwise character of the formula $f'(x) = 2x$, which "point by point tells me the value of the angular coefficient of the tangent line" [*19*]. Instead, her gestures on function f (see Fig. 4.5.23) are continuous and global.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
20	T (aiming to $S3$): "*So, you are right to pose the question, but one needs to understand well what x is.*"	global	speech indicators + symbols	f, f'
21	S3: "*$f'(x)$ gives me the angular coefficient...*"		oral symbols	
22	T: "*Yes, as x varies. So, the variable is the same. Point by point, here I have a function that point by point automatically, as a machine, tells me the angular coefficient of the tangent line.*"	global global(=univ. pointwise)	speech indicators	f'
23	S3: "*Only, I don't understand the passage... If we know that m is $2x$, $f(x)$ corresponds to y, while m corresponds to the tangent. How can they be equivalent? I don't understand.*"	global	oral symbols	f, f'

In the teacher's words [*22*] we find the definition of the derivative f' as "*a function that point by point automatically, as a machine, tells me the angular coefficient of the tangent line*". The explicit perspective on f' is global in the sense of universal pointwise. However, the new doubt of the student S3 is about the status of f' as a function [*23*]. He has a clear global image of the function f, thanks to the graphical register used in the teacher's drawing, but he cannot understand how also the angular coefficient m (and so f') could behave like the function f does.

Finally, V. gives an exercise in order to "clarify the ideas":

> "*Write the equation of the tangent line to the parabola x^2 in the point of abscissa $x = 2$.*"

The students work alone for a while, then V. solves it at the whiteboard (Fig. 4.5.24). Then, she aims again to S3.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
24	T: "*Is it clearer now?*"			
25	S3: "*But if x is equal to 2, why does the second x remain x?*"	pointwise	symbols	f'
26	T: "*No, it's not $x = 2$... Ah, in the point! This would be x_0, yes, ok...*" (she replaces x with x_0 at the beginning, Fig. 4.5.25)	pointwise	speech indicators + symbols	f'
27	Ss (surprised): "*Aaaaaah!*"		exclamation	

28	T: "*Yes, but... Ok... All right?* <u>*In the point of abscissa* 2.</u>"	pointwise	speech <u>indicators</u>	f'

Figures

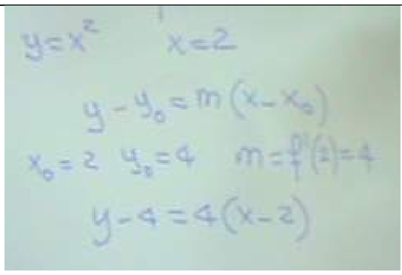


Figure 4.5.24 - V. solves the given task $t_{tangent}$.

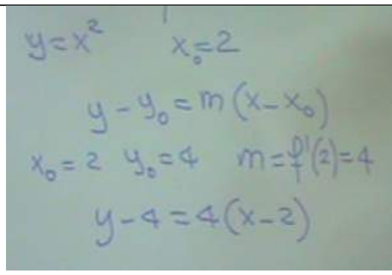


Figure 4.5.25 - V. changes $x = 2$ in $x_0 = 2$.

The doubt of the student is once again due to a misunderstanding on the written symbols, but it is relevant that the symbols in question are x and x_0 [*25*]. The other students seem to feel the same kind of uneasiness [*27*]. This example, which returns to a pointwise perspective on f', seems not helping the student to grasp a global point of view on the derivative function. On the contrary he would like to replace x with 2 [*25*] in a completely pointwise perspective even on f.
The praxeology that V. activates for writing the derivative function, in the algebraic register, is summed up in Table 4.21. Notice that the technological speech is made of sentences explicitly uttered by the teacher in the classroom.

$OM_{f'}^{alg}$	
Type of task $T_{f'}$	Writing the derivative function in the algebraic register of representation.
Technique $\tau_{f'}$	Calculate $f'(x_0) = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$, then replace x_0 with x in the result.
Technology $\theta_{f'}$	$\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ is equivalent to $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ thanks to the change of variable $h = x - x_0$. "I can make x_0 vary as I want (gesture in Fig. 4.5.21)... At this point, I can write x instead of x_0, for convenience" [*18*]
Theory $\Theta_{f'}$	- The definition of derivative function as "a machine" that given an x-value gives me the corresponding $f(x)$-value [*22*]. - The limits theory. - The definition of the derivative in a point.

Table 4.21 - V. and her students' praxeology for the algebraic type of task $T_{f'}$.

Some hints about a graphical technique

Six lessons after, V. returns on the difference between the derivative in a given point and the derivative in x. Within this context, she proposes a graphical task we call $t_{f'}^{gra}$: given the graph a cubic function in GeoGebra, constructing the graph of its derivative. The name $t_{f'}^{gra}$ denotes that the task belongs to $\mathcal{T}_{f'}$ in a graphical register. It can be formulated as *graphically representing the derivative function*. The teacher quickly shows the solution of the task to the students. In the following transcription, we will refer to Fig. 4.5.26 which V. is constructing in the Dynamic Geometry Environment (DGE) of GeoGebra.

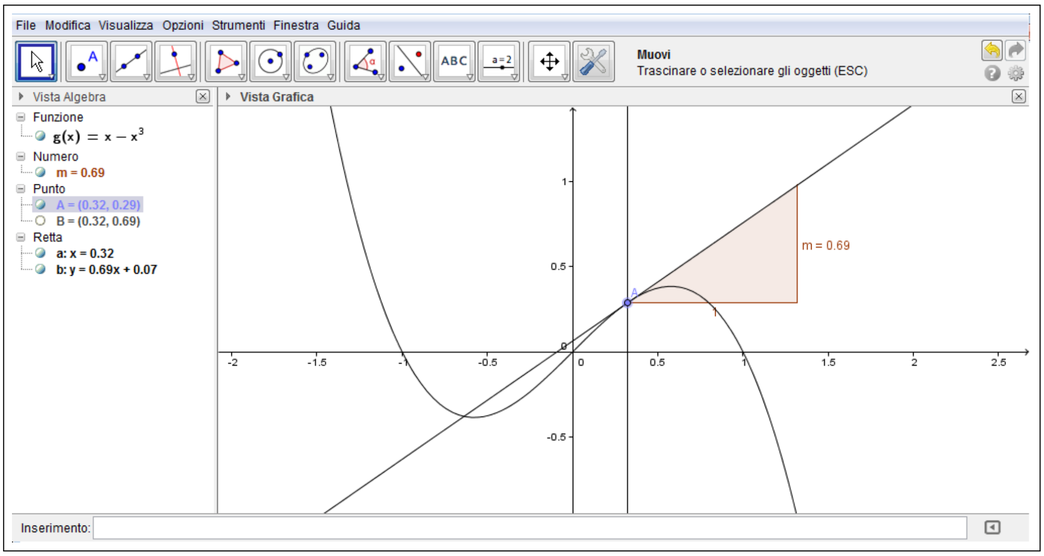


FIGURE 4.5.26 - GEOGEBRA SCREENSHOT, REFERENCE FOR THE LINES *29-37*.

	What happens (teacher-students dialogue)	**Perspectives**	**Semiotic resources**	**on f, f'**
29	T: "*Let's suppose to have the function $x - x^3$, for example.*" (She keys "$y = x - x^3$" in GeoGebra and presses Enter) "*This is its graph.*"	global	symbols + graph in DGE	f
30	T: "*I consider a point of the function*" (she uses the command "point on" to detect the point A on the graph) "*and I consider the tangent in this point to this function.*" (She uses the command "tangent line to" selecting A and the graph)	pointwise	speech indicators + graph in DGE	f
31	T: "*If I drag this point*" (she drags A, Fig. 4.5.27) "*this is the tangent in that point.*"	global	continuous gesture in DGE	f

32	T: "*Now, if I want the angular coefficient... Well, where is the slope?... Slope of this straight line.*" (She uses the command "slope" selecting the tangent) "*This number represents the derivative, the angular coefficient, in that point.*" (She points downwards with her index, Fig. 4.5.28)	pointwise	speech indicators + graph in DGE + gesture	f'
33	T: "*Now, if I define another point B equal to... Wait! Perpendicular line...*" (she uses the command "perpendicular" selecting x-axis and A) "*In correspondence to the abscissa of the point*" (she positions the cursor on the intersection between x-axis and the perpendicular line, Fig. 4.5.29) "*I give m-value as ordinate.*"	pointwise	speech indicators + graph in DGE + pointing gesture in DGE	f'
34	T: "*This point A has a generic abscissa x*" (she positions the cursor again as in Fig. 4.5.29) "*I assign the ordinate m in correspondence to this point... So I trace the point [...] $B = (x_A, m)$...*"	pointwise	speech indicators + pointing gesture in DGE + oral symbols	f'
35	S1: "*It is not enough to write 0.32?*" (He refers to the numerical value of A abscissa)	pointwise	numerical symbols	f'
36	T: "*No, because, by doing so, I would fix it*" (she points to A on the whiteboard, Fig. 4.5.30) "*instead I want it to vary.*" (She moves her hand backwards and forwards on x-axis, as in Fig. 4.5.31)	pointwise $\rightarrow$ global	speech indicators + gestures	f'
37	S1: "*Ah! Yes.*"			

Figures

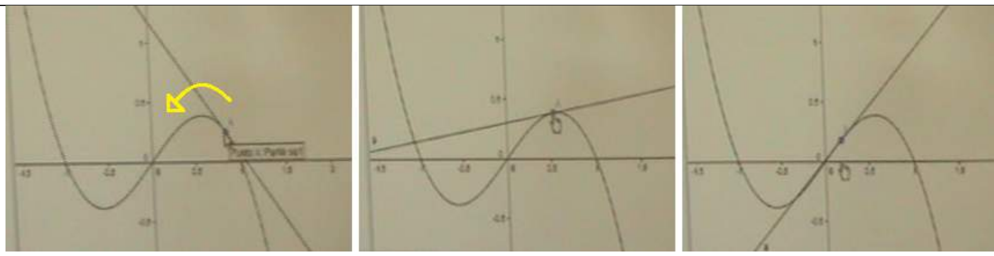


Figure 4.5.27 - V. drags the point to make the tangent "surf" on the graph.

Figure 4.5.28 - V.'s pointing gesture as she says "in that point" [*32*].

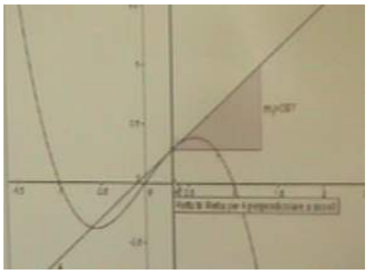


Figure 4.5.29 - V. positions the cursor on A abscissa.

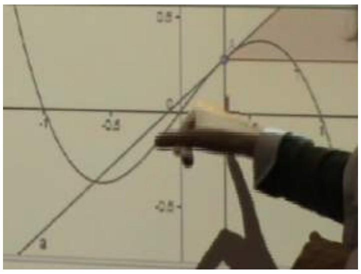


Figure 4.5.30 - V. points to the abscissa of A with her finger.

Figure 4.5.31 - V. moves her hand backwards and forwards as she says "I want it to vary" [*29*].

V. introduces a new resource: the Dynamic Geometry Environment of GeoGebra. If suitably used, it has a great potential and the teacher wants to exploit it. It occurs that many of the usual semiotic resources are now active in the Dynamic Geometry Environment. Sometimes their effect is the same they would have on the whiteboard, except that the result is obtained more quickly (e.g., drawing a graph or a straight line). Nonetheless, the effect of the semiotic resources used in DGE is also incredibly enhanced thanks to the dynamical feature of the software (e.g., dragging as continuous gesture). First of all, the teacher makes GeoGebra draw the global graph of a cubic function [*29*] (see Fig. 4.5.26). Then, she exploits the command of GeoGebra to detect a point A on the graph [*30*]. This action would have be exclusively pointwise in a paper and pencil environment or at the whiteboard. Made in DGE, the same action has is potentially universal pointwise since the point can only move on the given graph and, if one drags it, the point A becomes the generic point on the curve. When V. drags it [*31*] as in Fig. 4.5.27, she easily activates as global perspective on f. Notice that V. has already made the same continuous gesture on a parabola with the shadow of the stick [*18*] (see Fig. 4.5.23). In that case, the speech was essential to explain the gesture ("I can make it vary as I want") and the universal pointwise perspective had to be forced with the words "point by point". Here, the shift from pointwise to universal pointwise is already provided by the software. Afterwards, V. focuses on the derivative function. She detects the slope of the tangent [*32*] and she would like to define the point B which describes the derivative curve as A moves on the given graph. She defines it in words through its

coordinates [*33-34*]. Unfortunately she does not remember the command to indicate the abscissa of a point in GeoGebra (i.e., $x(A)$). One of the students makes the pointwise proposal of giving to the abscissa of B the numerical value that the abscissa of A has in that particular position on the graph [*35*]. V. reacts contrasting the pointwise abscissa of a fixed A and its global variation [*36*]. The teacher accompanies her rebuttal with two different gestures on the screen (Fig. 4.5.30 and 4.5.31), which are the same she made before with the parabola case [*18*] (see Fig. 4.5.22).
V. suspends the activity and returns back to it with an already prepared file the lesson after. She sent us the file, because we could not attend that lesson. In that occasion, she drags A on the given curve and shows that the point B consequently moves, describing the derivative trace (see Fig. 4.5.32). Notice that the trace is a pointwise tool, with a high potential to activate a universal pointwise, and so a global, perspective.

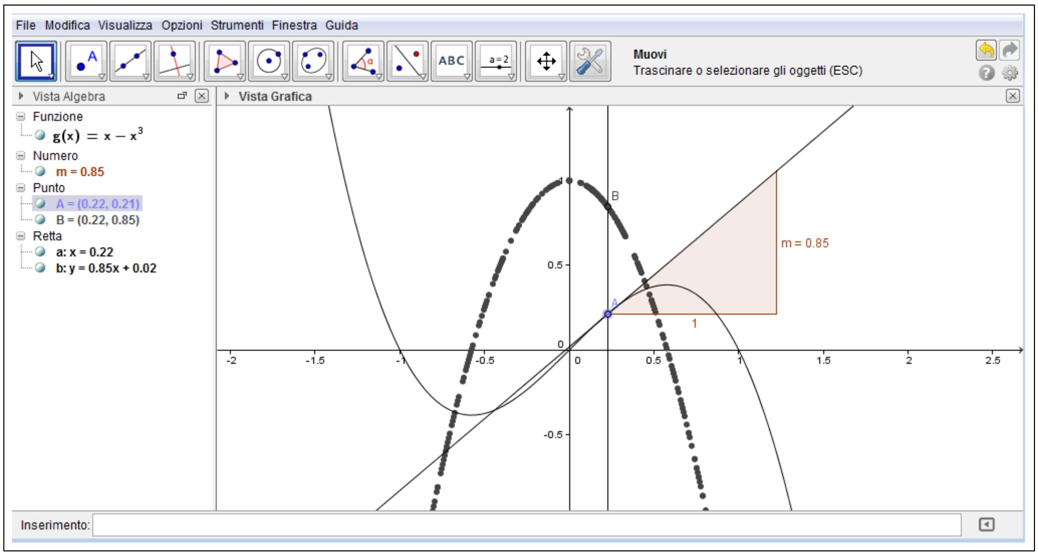


FIGURE 4.5.32 - GEOGEBRA SCREENSHOT (COMPLETE).

Remarks

The algebraic technique given by the teacher consists in the syntactical change of the universal pointwise sign x_0 in the global sign x. It is expressed through the words "I can make x_0 vary as I want [...] I can write x". The perspective on the involved functions f and f' which depend on the variable x turns out to be global. On f, the global perspective is highlighted through the graph and the continuous gestures on it. On the derivative function instead the employed semiotic resources are mainly symbols and speech, which are used in a universal pointwise perspective. The definition of $f'(x)$ as a function as to be forced verbally: "a function that point by point automatically, as a machine, tells me the angular coefficient of the tangent line". Notice that it is a global definition with a universal pointwise character.
The graphical task is proposed and solved in GeoGebra. Given the graph of a function, it consists of constructing the graph of the derivative function. The technical steps, which

correspond to GeoGebra commands, are based on the pointwise property:

$$B \in f' \quad \text{if} \quad B(x(A), m)$$

where A is a generic point of the given f. The graph of the derivative is presented as the locus of the points B, as A (and so, as x) varies. The final enhanced perspective on f' turns out to be global, thanks to the use of the command "trace", which draws the function point by point. The activation of the universal pointwise perspective is facilitated in the dynamical geometry environment of GeoGebra. By choosing a point A on the curve to differentiate, A is automatically a generic point of the curve of f. Since B depends on A, in the same way, it becomes the generic point of the curve of f'. The shift to the universal pointwise perspective on f' is already provided by the software.

Chapter 5

Analysis of students' work

In this chapter we move on to analyse the activities proposed in the three classrooms at the end of the observation phase.
In this dissertation, the term "activity" denotes a problem or a set of problems that the students have to solve, by working alone or in team, for constructing or consolidating the meaning of the involved mathematical objects. In this sense, we refer to the meaning that activity has in the UMI's (Unione Matematica Italiana) reflections about the mathematical laboratory (UMI, 2004).
The aim of the proposed activities is to get insights into the effects of the teacher's praxeologies in the students' autonomous work. In particular, we focus on the techniques and the technologies they use, the perspectives they activate and the semiotic resources they employ for solving the problems.
More precisely, Activity 1 (Section 5.3) is specifically focused on the graphical differentiation, it involves a graphical work on functions and requires a written justification of the solution. Instead, Activity 2 (Section 5.4) concerns the algebraic writing of the tangency condition and promotes an algebraic work on functions.
We present the a priori and a posteriori analysis of both the activities. In the a priori analysis, we explain how the activities have been designed, by exploiting different semiotic resources. Moreover, we illustrate at what extent they could help us in investigating the students' acquisition of a local perspective on functions. The a posteriori analysis refers to the results of some students.

5.1 The classrooms and their background

The students who worked on our activities were attending the last year of scientific high school. They were 18-19 years old (grade 13). Activities 1 and 2 were proposed to each classroom after about 10 hours of lessons on the derivative topic. In the previous chapter only some particular moments of the observed lessons have been selected for the analysis. Therefore, as first step, we summarize in Table 5.1 the mathematical contents taught by each teacher in her classroom, before the proposal of the activities.

M. in ***Quinta B***	M.G. in ***Quinta A***	V. in ***Quinta E***
- Problem of the tangent line and problem of the instantaneous speed - Angular coefficient of the tangent line as the limit of the angular coefficient of a secant line to a function - $m_{tg} = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$ - Equation of the tangent line to a function f at the point of abscissa x_0: $y - f(x_0) = f'(x_0)(x - x_0)$ - Determination of the derivative of a given $f(x)$ at a given abscissa x_0 - Derivative function as the derivative of a given $f(x)$ at a generic abscissa x - Second derivative as derivative of the derivative: relation between the sign of f'' and the concavity of f - Non-differentiable points - Calculation of the derivative of some elementary functions - Graph of the derivative function - Theorems involving the derivatives (sum, product, quotient, inverse) without proof - Application of the theorems to the study of rational and irrational functions	- Revision of the equation of the tangent line to conics - Tangent line as the limit of the secant line to a function f - Determination of the angular coefficient m_{tg} as limit of m_{sec}: $m_{tg} = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ - Definition of the derivative of a function f at a point x_0 as $f'(x_0) = m_{tg}$ - Equation of the tangent line and of the normal line - Distinction between the number $f'(x_0)$ and the function $f'(x)$ when x is generic - Calculation of the derivative of some elementary functions - Theorems involving the derivatives (sum, product, quotient, inverse) without proof - Derivative function $f'(x)$ as the angular coefficient of the tangent line in any x of the domain, all the possible directions as the point varies on the curve - Non-differentiable points - Optimization problems - Relation between sign of f' and variation of f: verification on two simple examples of rational functions - Second derivative and relation between the sign of f'' and the concavity of f - Complete study of functions - From the graph of the function, deducing the graph of the derivative	- Definition of the tangent as the straight line that best approximates the function in a point - Alternative definition of the tangent as the limit of the secant line to a function - Angular coefficient of the tangent line as $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ - Equation of the tangent line to a function f at the point x_0: $y - f(x_0) = f'(x_0)(x - x_0)$ - Non-differentiable points - $f'(x_0) = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ - Calculation of the derivative of some elementary functions - Derivative function as a "machine" that automatically associates with x the value of the angular coefficient of the tangent line to the function in x - Theorems involving the derivatives (sum, product, quotient, inverse) with proof - Relation between zeros of f' and maximum/minimum points of f, sign of f' and variation of f - Application of the derivative to the study of a function - Brief demonstration of the graph of the derivative function with GeoGebra - Optimization problems

Table 5.1 - Table of contents developed in classroom by each teacher before the activity proposal.

5.2 Methodology and data collection

The students solved both the proposed activities working in small groups of 3-4 people. We wished to encourage the discussion in the group. For this purpose, we asked the teacher's suggestion to form homogeneous groups, so that the components were at a similar level of competences. Two or three groups have been videotaped while solving the activities. Thus, the collected data consist of each group's written production and some groups' video containing the exchanges and the reflections that led to the final production.
As the researcher who followed the activities in classroom, I had the only function to introduce the work and to videotape the students. I made a general intervention only if needed to understand the text of the problems. Generally, if possible, always the same groups have been videotaped during both the activities, in order to give continuity to our data.
In this phase, the teacher was asked to be a simple observer. She could walk through the classroom observing what happened in the working groups, and she was required to intervene the less she can. However, she had the right to help a group in difficulty, by reminding them the definitions and the concepts introduced during the previous lessons.

5.3 Activity 1

Activity 1 is specifically designed around the process of graphical differentiation.

5.3.1 Description

The activity lasts one hour and is divided in three parts. The students are asked to write a letter to two imaginary students having the same age, who are studying to pass the final high school examination. This context and the proposed problems were inspired by the research of Yoon, Thomas and Dreyfus (2011). The request of writing a letter has mainly the function of making the students think about a systematization and a justification of their solution and method.
The first and the second part (see "Scheda 1" and "Scheda 2" in Appendix C) are centred on the following graphical type of task: given the graphs of three functions (on the same Cartesian plane), determining which of them represents respectively a function f, its first derivative f' and its second derivative f''.
In the third part (see "Scheda 3" in Appendix C), the students are supposed to create a situation similar to that of Problems 1 and 2. The task is to draw the graphs of three functions on the same Cartesian plane, so that they represent respectively a function f, its first derivative f' and its second derivative f''.
A general prerequisite is represented by some pointwise, global and local properties of functions, such as domain, sign, zeros, end behaviour, and at least a graphical idea of variation and of maximum and minimum points. Specific prerequisites are the concept of derivative in a point x_0, the geometrical construction and the analytic equation of the

tangent line to a function in a point, and some local properties, such as continuity and discontinuity, differentiability and non-differentiable points.

5.3.2 *A priori* analysis

Firstly, let us describe the methodology we use to analyse the collected data. We use the same lenses employed in analysing teachers' practices. Therefore, our focus is on three components:

1. the praxeologies adopted in order to solve the proposed problems;
2. the perspectives assumed on the involved functions;
3. the semiotic resources activated in order to solve the activity.

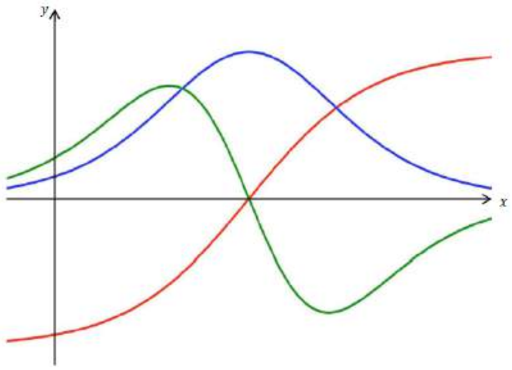
Every problem is designed taking into account these three great components. In particular, the pieces of knowledge to recall for solving the activity are relationships between the function f and its derivative f'. They are summarized in Table 5.2, with the related perspectives involved on f and f'.

f	f'	**Perspective(s)**	**Type of relation**
maximum/minimum point	zero	local-pointwise	*l-p*
variation in $[a, b]$	sign in $[a, b]$	global-global	*g-g*
concavity in $[a, b]$	variation in $[a, b]$	global-global	*g-g*
inflection point	maximum/minimum point	local-local	*l-l*
discontinuity point	discontinuity point	local-local	*l-l*
non-differentiable point	discontinuity point	local-local	*l-l*
horizontal/oblique asymptote	horizontal asymptote ($y = 0/y = k$)	local-local	*l-l*

Table 5.2 - Relations $f - f'$ and related perspectives required in Activity 1.

The first relation links a local property of f to a pointwise property of f'. We denote the relations of this kind with the acronym *l-p* (*local-pointwise*). The second relation and the third one are between global properties of f and f': we denote them with *g-g*. The last four relations are between local properties of f and f': we denote them with *l-l*. It is also possible that students use these relations without activating the expected perspectives. For example, let us imagine that a student notices that a function f is not differentiable in x_0 and that he links this information to the fact that its derivative is not defined in x_0. We know that if f' is not defined in x_0 it is obviously discontinuous there, but we can not assume this implication for granted in the student's reasoning. In this case, we can only say that the student has adopted a local perspective on f and a pointwise perspective on f'. Thus, a relation of type *l-l* is used in a *l-p* perspective.

Problem 1



TASK 1 - The drawing shows the graphs of three functions. Determine the right combination of them: a function f; its derivative: f'; the derivative of f': f''.

This problem is taken from a study carried out by Yoon, Thomas and Dreyfus (2011). Three graphs are given in the same Cartesian reference system. They are coloured in red, blue and green, so that students can refer to them more simply. Any algebraic expression is given, only the graphical representation. As for the semiotic resources, the drawing in the graphical register prevails in the question and a written justification is required in the answer.

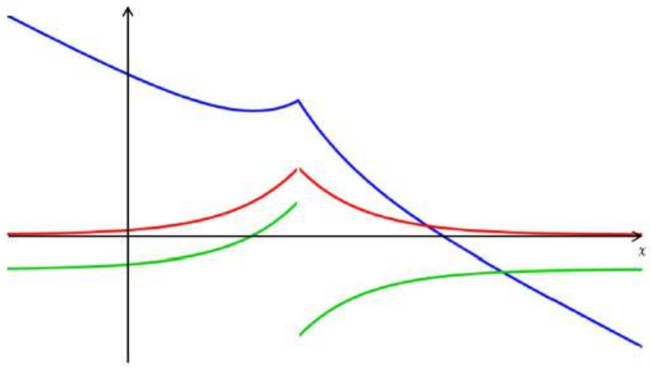
We chose this problem especially because it stimulates the adoption of a global perspective. Indeed, the pointwise perspective alone is not enough to reach the correct combination of functions. This is because the maximum point of the blue function corresponds to the zero of both the red function and the green one. Thus, by activating the relation *l-p* between the maximum of the function and the zero of its derivative in a pointwise-pointwise perspective, the students are not able to establish if the derivative of the blue function is the red or the green one. A global perspective is needed to see the following correspondences:

- the positive sign of the blue function with the increasing variation of the red one;
- the positive/negative sign of the green function with the increasing/decreasing variation of the blue one.

Notice that the choice of the graphical register of representation and the absence of axis scale are aimed to support and foster the adoption of a global perspective.

Finally, a logic link of the obtained relations allows to conclude that the green function is the derivative of the blue one that, in turns, is the derivative of the red one. The problem is solved: f = red; f' = blue; f'' = green.

Problem 2



TASK 2 - The drawing shows the graphs of three functions. Determine the right combination of them: a function f; its derivative: f'; the derivative of f': f''.

This problem is inspired from Thomas's research, but it is specifically constructed to have discontinuous functions and non-differentiable points. In particular, the three functions have been drawn with the software GeoGebra, by taping the following algebraic equations:

$$y = e^{-|x|} - \frac{x}{2} + 1; \qquad y' = \begin{cases} -e^{-x} - \dfrac{1}{2}, & \text{if } x > 0 \\ e^{x} - \dfrac{1}{2}, & \text{if } x < 0 \end{cases}; \qquad y'' = e^{-|x|}$$

As in Problem 1, the students are not required to find the algebraic expressions of the involved functions. Indeed, only the three graphs are given and their equations are hidden. As before, the main semiotic resources are the drawing in the graphical register in the question and the written argumentation in the answer.

This problem is especially constructed to encourage the adoption of a local perspective. Indeed, the pointwise perspective alone does not allow to guess the correct combination of functions. This is because in correspondence to the corner in the blue function both the red and the green one are not continuous. So, if the *l-l* relation between the non-differentiable points of a function and the discontinuity points of its derivative is activated in a pointwise-pointwise perspective, the students cannot decide which function between the red and the green one is the derivative of the blue function. Three different ways are then possible.

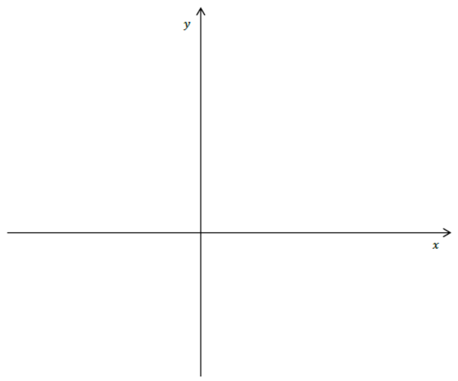
- Adopting a local perspective and basing the reasoning on the *l-p* relation between the slope of the tangent to the blue graph and the ordinate of the graph of its derivative; noticing that the gradient of the left/right tangent to the blue graph in the corner is positive/negative and it corresponds to a positive/negative ordinate of the green function.

- Activating a global perspective and noticing the *g-g* correspondence between the increasing/decreasing/increasing variation of the blue function and the positive/negative/ positive sign of the green one.

- Exploiting the *l-p* correspondence between the relative minimum of the blue function and the zero of the green one.

The global relation linking the function variation to the derivative sign leads to the conclusion that the red function is the derivative of the green one. As in Problem 1, notice that the graphical choices in the statement are aimed to support and foster the adoption of a global perspective.
Finally, logically linking the obtained relations one finds that the red function is the derivative of the green one that, in turns, is the derivative of the blue one. So, f = blue, f' = green, and f'' = red.

Problem 3



TASK 3 - Constructing a graphical problem for this task: "The drawing shows the graphs of three functions. Determine the right combination of them: a function f; its derivative: f'; the derivative of f': f''".

The third request is designing the problem, after having learnt how to solve it. This task is useful to reinforce the process of reasoning and check it again. We are interested in the approach chosen by the students. It can be graphical and purely analytic, but also algebraic based on functions whose algebraic expressions are known.

The semiotic resource exploited here is mainly the drawing in the graphical register at the level of both question and answer.

Expected praxeologies

Problems 1 and 2

Problems 1 and 2 have the same type of task: given three graphs in the same Cartesian reference system, determining the right combination of them: a function f, its derivative f' and the derivative of f', which is f''.
As far as the techniques are concerned, we distinguish two different methods.

1. The *analytic method* is based on the properties of the functions which are directly deduced from the properties of the graphs.

2. The *tangent method* is based on the estimation of the slope/gradient of the tangent to the function to be differentiated, in order to guess the value of its derivative.

Then, one needs to logically concatenate the obtained relations to reach the solution.
The technologies that justify the analytic method are the relations formulated in Table 5.2, while the technology for the tangent method is the correspondence between the slope/gradient of the tangent to the function in a point x_0 and the value of its derivative in x_0.
The common theory behind all these technologies is the definition of derivative as the gradient of the tangent line to the function in the point of tangency (through the limit of the incremental ratio).

Problem 3

Task 3 is somehow the converse of tasks 1 and 2. Indeed, it is required to draw three graphs in the same Cartesian reference system in order to have a function f, its derivative f' and the derivative of f', which is f''.
We expect two different techniques.

1. The application of the *analytic method* or the *tangent method* depending on the approach chosen by the group to solve Problems 1 and 2.

2. The application of an *algebraic method* that develops in the following steps: choosing a function whose algebraic expression and graphical representation are known; differentiating them through the differentiation formulas; drawing the graphs of the three obtained functions.

The technology behind techniques 1 and 2 are the same as those of Problems 1 and 2. Instead, for the algebraic method, the differentiation formulas applied to differentiate f and f' guarantee that the chain f-f'-f'' is correct.
The theory behind the technology of the analytic method or the tangent method is always the definition of the derivative as the gradient of the tangent line to the function in the point of tangency (through the limit of incremental ratio). Instead, behind the technology of the algebraic method, we mostly have the algebraic operations and rules involving derivatives, with their related proofs.

5.3.3 *A posteriori* analysis

Let us start with an introduction to precise how the a posteriori analysis is made. First of all, we focus on the activated praxeologies and we search for the expected methods we have distinguished. Obviously, we take into particular account also methods and behaviours that we have not predicted. We choose at least one example for each method, in order to analyse it more in depth. When available, we prefer analysing one of the videotaped group, of which we have more data. Then, our analysis concentrates on the perspectives activated by the students in every phase of the method they develop. Moreover, we are interested in the semiotic resources the students use to support their perspectives. In particular, are the given semiotic resources well-exploited in order to activate the expected perspective on the involved functions? Do the students need to introduce other semiotic resources to adopt this perspective? Do they assume an unexpected perspective and does it lead them to a mistake?

M.'s students

Let us recall that M. has worked deeply on the graph of the derivative function, since the first lesson about the derivative notion (see the subparagraph "Elaboration of a technology, passing through the graphical technique" in Paragraph 4.3.3). She has immediately spoken about derivative of second order, so her students are expected to know the relation between f properties and f'' properties. Moreover, the students are familiar with correspondences such as "if the function increases, then the derivative is positive" and so on. Thus, in our expectations, the analytic method should be the most employed.
In Table 5.3, you find a general overlook on the work done by the different groups (A, B, C, D, E, F, G and H). We do not analyze Problem 3 because no group manages to approach it within the hour of activity in classroom. Nonetheless, they solve it as homework. At a quick glance, we could observe that six group over eight solve it graphically, while two over eight work algebraically, starting from the equation and the graph of known functions (e.g. sine, cosine, straight line, parabola, ...). These two groups are medium-low or low-level group. Probably, if left free, they feel more self-confident in resorting algebra: calculations reassure them of doing a good job.

	Method		Perspectives		
	analytic	tangent	pointwise	global	local
Problem 1	all		all	all	A
Problem 2	all	$\sim$B,$\sim$C,$\sim$G	A,B,C,E,G,H	all	A,B,C,D,F,G

Table 5.3 - M.'s students and activity 1. The sign $\sim$ indicates that the method/perspective is partially used by the group in solving the corresponding problem.

At a technical level, all the students use the analytic method, as expected. Two groups over eight resort the tangent method only to deal with the corner in Problem 2. In particular, it happens when they have to explain the type of discontinuity in the green and red graphs. This kind of justification is part of the students' general local remarks, along with some considerations about the asymptotic behaviour of the presented functions. Six groups over eight use such argumentation to support their solving process. The local perspective is much more worked in Problem 2, triggered by the presence of points of discontinuity and non-differentiability. Nonetheless, one group (group A of medium-high level) makes some local remarks about the asymptotic end behaviour also within Problem 1. Some groups (e.g. D and F) justify their solution to Problem 2 only through local and global argumentation. They do not need to highlight pointwise aspects.

Analytic method

Let us consider group A as an example. This group has been also video-recorded. They only base their reasoning on the properties of functions they deduce from the given graphs. In particular, for approaching the first problem they employ the relations we list below. Especially when their conjectures are wrong, we support them with extracts from the text (see Fig. 5.3.1) and with video transcriptions.

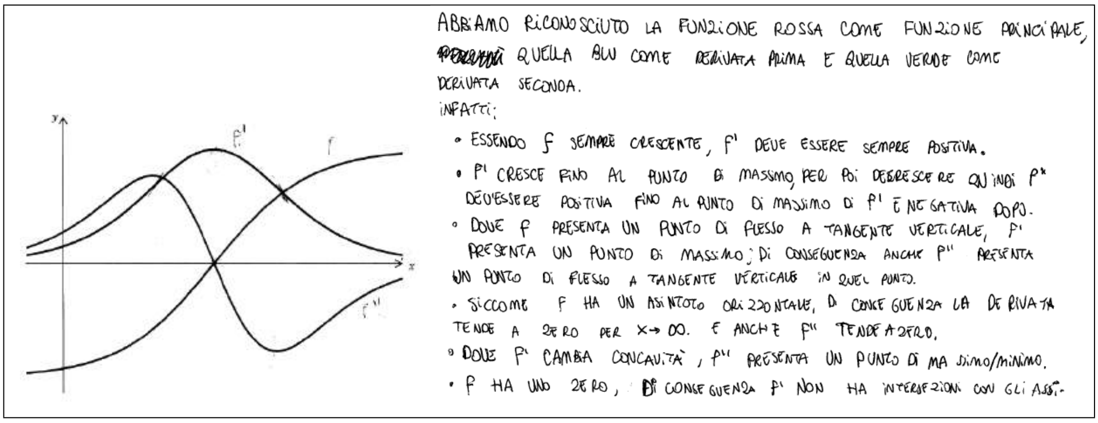


Figure 5.3.1 - Group A's solution and justification to Problem 1.

- Correct *g-g* relation between the variation of the function and the sign of its derivative.

  > "Since f always increases, f' must be always positive." [5th line, Fig. 5.3.1]
  > "f' increases till the maximum point, then it decreases. So f'' must be positive till the maximum point of f' and negative after." [6th-7th lines, Fig. 5.3.1]

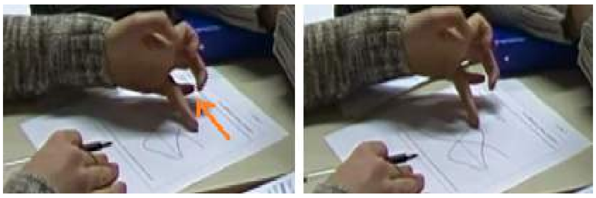
- Correct or partially correct *l-l* relation between the inflection point of the function and maximum point of its derivative.

  > "Where f presents an inflection point with vertical tangent, f' presents a maximum point;" [8th-9th lines, Fig. 5.3.1]

"Where f' changes concavity, f'' presents a point of maximum/minimum" [13th line, Fig. 5.3.1]

- Incorrect *l-l* relation between the maximum point of a function and the inflection point of its derivative.

  [...] "f' presents a maximum point; as a consequence, also f'' presents an inflection point with vertical tangent at that point." [9th-10th line, Fig. 5.3.1]

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
1	S1: *"Here f has an inflection point"* (she moves her finger vertically in a neighbourhood of the zero of f, see Fig. 5.3.2) *"and here there is a maximum point"* (she moves her finger horizontally in a neighbourhood of the maximum of f', see Fig. 5.3.3) [...] *"We know that the coefficient of the tangent at that point is infinite, I believe..."*	local	speech indicators + gestures	f and f'
2	S2: [...] *"And so, as a consequence, also f'' presents an inflection point, since f' has a maximum point, I think!"*	pointwise	speech indicators	f' and f''
Figures	 **Figure 5.3.2** - S1's local gesture to reproduce the vertical tangent.  **Figure 5.3.3** - S1's local gesture to reproduce the horizontal tangent.			

- Correct *l-l* relation between horizontal asymptotes of a function and horizontal asymptotes of its derivative.

  "Since f has an horizontal asymptote, consequently the derivative goes to zero as $x \to \infty$. And also f'' goes to zero." [11th-12th lines, Fig. 5.3.1]

	What the students say and do	Perspectives	Semiotic resources	on f, f', f''
3	S2: *"You see, as it goes to infinity, the main function [the red one] becomes more and more feeble, the coefficient"* (he points to the blue graph right tail) *"goes towards zero."*	local	speech indicators	f and f'
4	S1: *"And so, as x goes to infinity, f goes to... infinity..."* (she moves horizontally the pen on the red graph, as in Fig. 5.3.4)	local	speech indicators + iconic gesture	f
5	S2: *"Yes, but f' that is the derivative represents the value of the angular coefficient of the tangent line. It goes towards zero because, you see, it tends more and more to become horizontal, even if it won't never become horizontal, I think!"* (he keeps his hand horizontal, as in Fig. 5.3.5)	local	speech indicators + iconic gesture	f' and f
6	S1: *"How can we write it?"*			
7	S2: *"Where $f(x)$ goes to..."*	local	speech indicators	f
8	S1: *"... has an horizontal asymptote, the tangent is horizontal... It has an horizontal asymptote."* (She keeps her hand horizontal, as in Fig. 5.3.6)	local	speech indicators + iconic gesture	f
9	S2: *"Does it have it? It goes slowly towards infinity. I don't know, I don't know!"*	local	speech indicators	f
10	S3: *"No, to me, no."*			
11	S1: *"It works, if you think about it, because if you have an horizontal tangent, you have the coefficient more and more towards zero."* (she keeps again her hand horizontal)	local	speech indicators + iconic gesture	f and f'
12	S2: *"Yes, that's true. Since f goes towards its horizontal asymptote, as a consequence the derivative will assume values more and more close to zero."*	local	speech indicators	f and f'

Figures

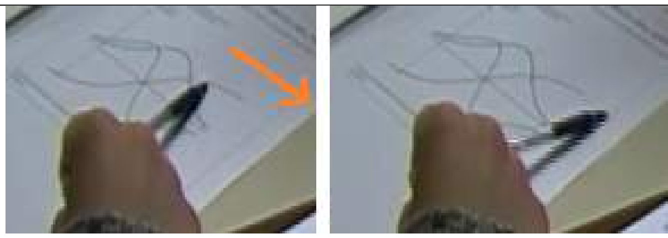


Figure 5.3.4 - S1's local gesture on the graph to lengthen it in the plane.

Figure 5.3.5 - S2 keeps his hand horizontal as he says "it becomes horizontal" [*5*].

Figure 5.3.6 - S1 keeps her hand horizontal as she says "it has an horizontal asymptote" [*8*].

- Incorrect *p-p* relation between zeros of the function and zeros of its derivative.

> "f has a zero, consequently f' has no intersections with axis." [last line, Fig. 5.3.1]

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
13	S1: *"Ah! we have to say something about the zeros! [...] It works: f has a zero and f' has no zeros."*	pointwise	speech indicators	f and f'
14	S2: *"And consequently f'' has one zero?"*	pointwise	speech indicators	f''
15	S1: *"No, it is not so. Let's write it only for f and f'."*			
16	S2: *"How does it work?"*			
17	S1: *"Well, maybe it works only for polynomial functions... for the powers x^n... the derivative has one zero less than the function."*	pointwise	speech indicators	f' and f

We can observe that the most part of the mistakes involve the local perspective. Indeed, it is the freshest perspective on functions for the students.

The wrong conjectures can have different origins. Firstly, a graph misreading in a neighbourhood of a point: it happens when S1 sees the vertical tangent in the inflection point of f, for example [*1*] (Fig. 5.3.2). Secondly, a logical duality of properties which are not dual: establishing that an inflection point in the function entails a maximum point in its derivative leads S2 to conclude that f'' has an inflection point, since f' has a maximum point [*1-2*]. Thirdly, the misleading idea to relate two similar properties, such as zeros of f and zeros of f' [*13-17*].
Nevertheless, this group has been the only one who notices the local asymptotic properties of the involved functions [*3-12*], bring them as justification of the solution. Their formulation has been supported by the use of speech and iconic gestures together.

Tangent method

Three groups over eight (two of high level and one of low level) partially adopt the tangent method, integrating it with the analytic method. It occurs in solving Problem 2 when they have to explain some local aspects of the involved functions.
High-level group B justifies the asymptotic behaviour of the red and the green functions, starting from that of the blue one. An extract of written production is shown in Fig. 5.3.7:

> "Finally, observing the behaviour towards ∞, we see that:
> - where f has an oblique asymptote, f' has an horizontal asymptote
> - where f has an horizontal asymptote, f' goes to 0
> It occurs because as f tends to the asymptote, the angular coefficient of the tangent goes to a determinate value, consequently also the derivative (which describes the trend of the angular coefficient of the tangent) goes to this value."

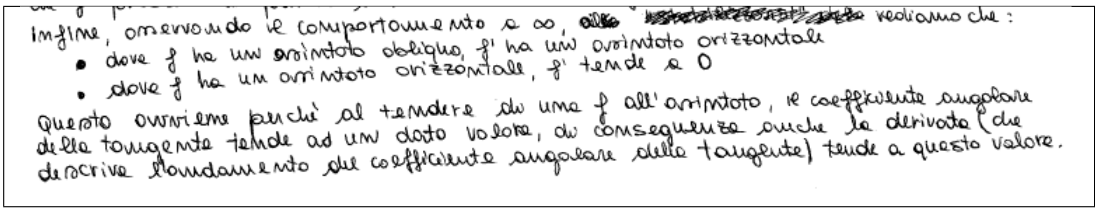


FIGURE 5.3.7 - EXTRACT OF GROUP B'S JUSTIFICATION TO THE SOLUTION OF PROBLEM 2.

Group C, composed of low level students, makes some considerations about the tangent line. They focus on the left and on the right of the corner of the blue function, in order to justify the discontinuities of the red and the green graphs. Let us read some local sentences, taken from their written argumentation (Fig. 5.3.8).

> "The blue curve is $f(x)$ because where this curve presents a corner $f'(x)$ (the green curve) presents a jump discontinuity (because, as in $f(x)$ one finds two different values of the tangents, the same must happen also for $f'(x)$). In

$f''(x)$ we find an empty point because in $f'(x)$ the tangents are parallel. Moreover, by applying the properties of the derivative functions, one can check that where $f(x)$ is increasing $f'(x)$ is positive and where $f(x)$ has upward concavity $f''(x)$ is positive."

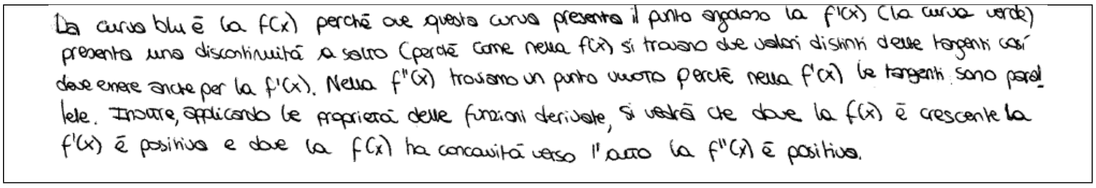

La curva blu è la f(x) perchè ove questa curva presenta il punto angoloso la f'(x) (la curva verde) presenta una discontinuità a salto (perchè come nella f(x) si trovano due valori distinti delle tangenti così deve essere anche per la f'(x). Nella f''(x) troviamo un punto vuoto perchè nella f'(x) le tangenti sono parallele. Inoltre, applicando le proprietà delle funzioni derivate, si vedrà che dove la f(x) è crescente la f'(x) è positiva e dove la f(x) ha concavità verso l'alto la f''(x) è positiva.

FIGURE 5.3.8 - EXTRACT OF GROUP C'S JUSTIFICATION TO PROBLEM 2 SOLUTION.

Finally, high-level group G, who has also been videotaped, provides a correct classification of the types of discontinuity in the derivative function, in relation with the types of non-differentiable point of the function. Part of their discussion is analysed below [*1-3*] and their written conclusions are shown in Fig. 5.3.9:

- "The discontinuity points of $f'(x)$ are non-differentiable points of $f(x)$.
- The corners of $f(x)$ correspond to the jump discontinuity of $f'(x)$.
- The cusps of $f(x)$ are II order discontinuities of $f'(x)$.
- The inflections points with vertical tangent are II order discontinuities of $f'(x)$."

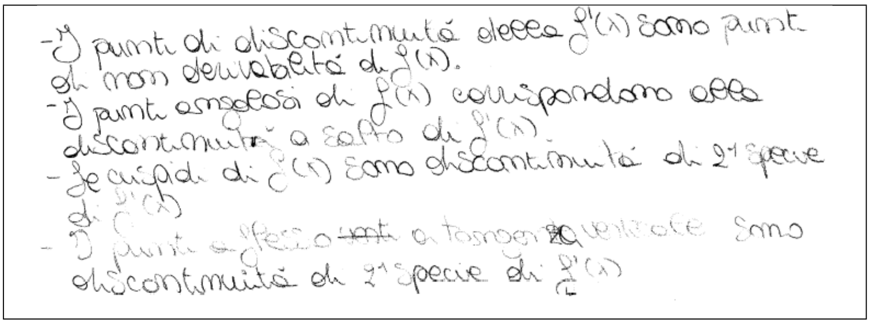

- I punti di discontinuità ~~della~~ f'(x) sono punti di non derivabilità di f(x).
- I punti angolosi di f(x) corrispondono alla discontinuità a salto di f'(x).
- Le cuspidi di f(x) sono discontinuità di 2ª specie di f'(x)
- I punti di flesso ~~vert~~ a tangente verticale sono discontinuità di 2ª specie di f'(x)

FIGURE 5.3.9 - EXTRACT OF GROUP G'S JUSTIFICATION TO THE SOLUTION OF PROBLEM 2.

S1 explains to S2 (who has missed that lesson) the difference between a cusp and an inflection point with vertical tangent, by taking into account the example of $f(x) = \sqrt[3]{x}$. He tries to give a graphical interpretation of its inflection point in terms of gradient. Notice that S1 speaks about "angular coefficient of the function", implicitly referring to the angular coefficient of the tangent line to the function.

	What the students say and do	Perspectives	Semiotic resources	on f, f', f''
1	S1: "*Here*" (he points to the inflection point of $y = \sqrt[3]{x}$ in the sketch he has drawn, see Fig. 5.3.10) "*the angular coefficient of it is infinite, isn't it?*"	local	graph	f
2	S2: "*Yes.*"			
3	S1: "*So, your derivative... your angular coefficient is initially positive: it goes to zero, it goes up, it goes upper and upper, it goes to infinity... Asymptote... Then, it goes again to minus infinity, in this way, it gets again to go to zero. So it should be an hyperbola.*" (He accompanies his words by drawing the sketch in Fig. 5.3.11)	global and local	speech indicators + graph	f'

Figures

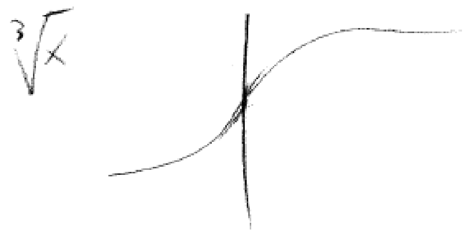


Figure 5.3.10 - S1 draws a sketch of the graph of $y = \sqrt[3]{x}$.

Figure 5.3.11 - S1 draws a sketch of the derivative.

Actually, S1's sketch of the derivative (Fig. 5.3.11), described in line [*3*] is a wrong graphical interpretation. Indeed, the function $f(x) = \sqrt[3]{x}$ he chooses is always increasing and its derivative always positive. After the vertical asymptote the derivative graph must not start at $-\infty$, but at $+\infty$. Despite this mistake, it is interesting that the student chooses the graph as the semiotic resource to explain to his classmate what a vertical inflection point means for the derivative. Notice that S1's considerations, while he is drawing the derivative graph and describing it, are mainly global and local, without any pointwise contribution.

Influences of M.'s praxeology

All groups mainly base their reasoning on the properties of the given graphs. In effect, M. has given a great important to the graphical construction of the derivative and its relations with the graph of the starting function. We consider it as a strong influence of M.'s praxeology.
It is not by chance that some students resort the tangent method when they have to establish new relations. It is the case of the high-level groups B and G, coping respectively

with the asymptotic behaviour and the non-differentiable and discontinuous points of the involved functions. These are local considerations autonomously made by the students. Whilst other local properties can be linked directly to M.'s practices in classroom. It is the case of group C who refers to the tangent to locally interpret the corner in Problem 2. We suppose it is a direct reproduction of one of M.'s practices. Indeed, working on the graph of the derivative function in classroom, she has had to deal with a corner (see line [*35*] in the subparagraph "Elaboration of a technology, passing through the graphical technique" in Paragraph 4.3.3). In that occasion, she interpreted the corner in the function as a jump of the derivative, referring to the different left and right tangents, exactly as the group C does.

M.G.'s students

Remind that M.G. has given a precise praxeology for tracing the graph of the derivative function, starting from the graph of a function (see Table 4.15 in the subpragraph "The graphical technique is shown on an example" in Section 4.4). The praxeology $OM_{f'}^{gra}$ is strongly based on an analytic method. Thus, we expect that the students choose it to solve the given problems.

A general overlook on the work done by the different groups (A, B, C, D and E) is shown in Table 5.4. Problem 3 has not been approached by any of the groups within the hour of activity in classroom. The students were free to solve it for homework, but only two groups over five have actually solved it. Thus, we do not find it significant to analyse it.

	Method		Perspectives		
	analytic	tangent	pointwise	global	local
Problem 1	all		all	all	
Problem 2	all	∼E	A,D,E	all	E

Table 5.4 - M.G.'s students and activity 1. The sign ∼ indicates that the method/perspective is partially used by the group in solving the corresponding problem.

As expected, all the students employ the analytic method. Three groups (A, C, E) over five employ the technique of the study of function, which the teacher has shown in classroom working on the type of task $T_{f'}$ in a graphical register. Groups B and D, instead, base on general pointwise and global remarks about the involved graphs without expressly use the study of function scheme. Only high-level group E makes local considerations about the tangent within Problem 2. In particular, it occurs when they have to explain the type of discontinuity in the green and red graphs. This justification is the single local remark made by M.G.'s students during this activity. No consideration has been made about the asymptotic behaviour of the given functions. This lack seems to be a natural consequence of the fact that the teacher in classroom has made no reference to the relation between the asymptotic behaviour of f and f'. Consequently, the local perspective does not turn out to be so worked. Not even in Problem 2 where the points of discontinuity and of non-differentiability have been read at the most in a pointwise perspective (except for high-level group E).

Analytic method

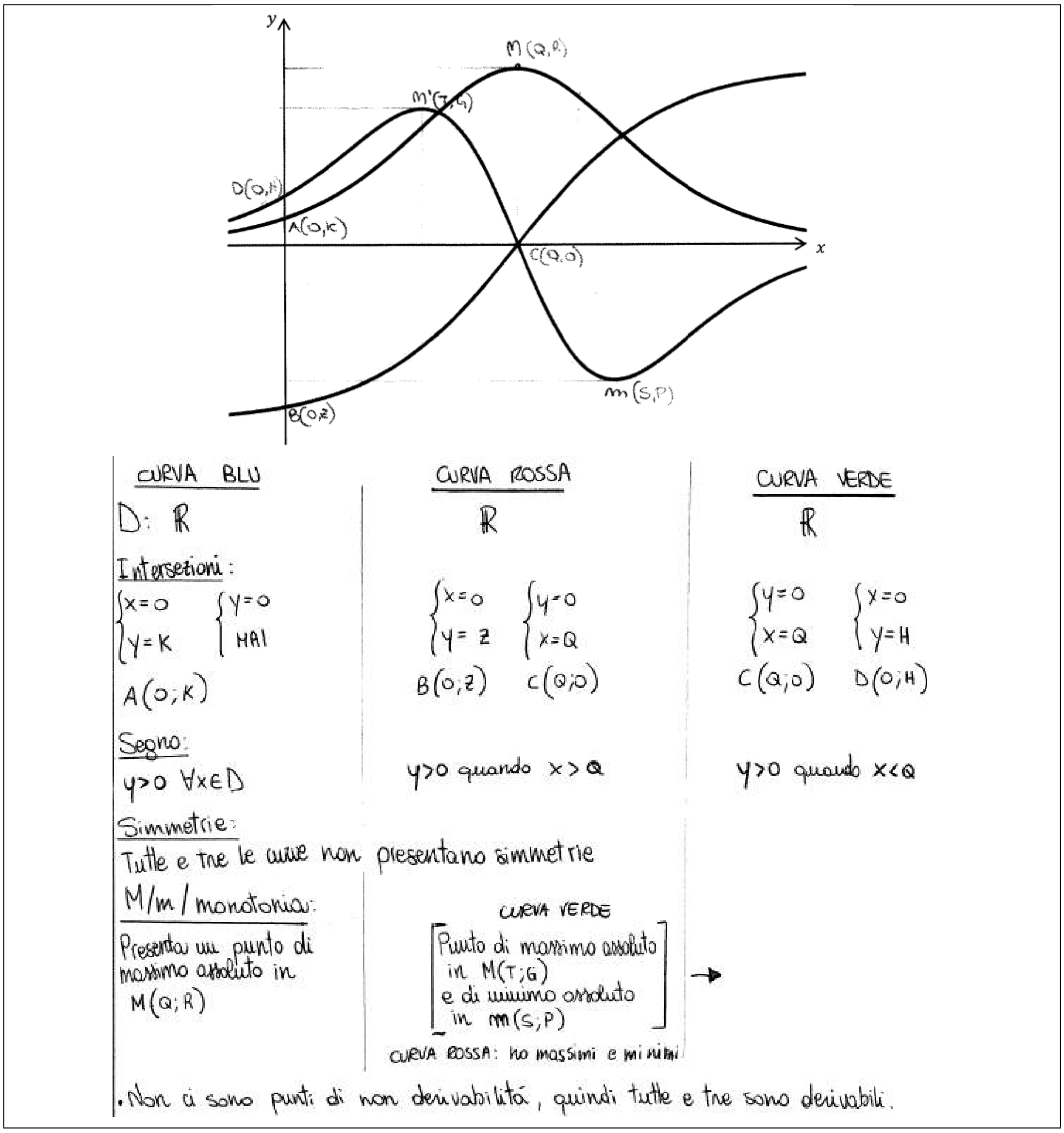


FIGURE 5.3.12 - GROUP A'S FIRST APPROACH TO PROBLEM 1.

Low-level group A blindly uses the technique of the study of function they have seen in classroom. S1 suggests: "Our reasoning has to be similar to the last things we have studied." S2 completes "The deducible graphs" and opens her notebook. They resume the technique from their lesson notes, which are even incorrect somewhere. They think

to study the three functions at the same time, giving names to the most of the relevant points in the graphs and to their coordinates (see Fig. 5.3.12).
Let us analyse a significant moment in the group discussion that leads to the table in Fig. 5.3.12.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
1	S2: "*The blue curve has a maximum point here.*" (She points to the blue maximum, Fig. 5.3.13)	pointwise	pointing gesture	f
2	S1: "*Absolute.*"	global	speech indicators	f
3	S2: "*Ok. Absolute.*"	global	speech indicators	f
4	S3: "*And here is an undefined point?*"	pointwise	speech indicators	f
5	S1: "*It might be a chance, but probably this point is on the same...*" (he traces an imaginary vertical line passing through M and C, as in Fig. 5.3.14) "*They are aligned.*" (They give to M the same abscissa of C)	pointwise	speech indicators + gesture + symbols	f and f'
6	S1: "*Maybe it is not by chance that the other functions cut in this point.*" (He points to C) "*We know that all depend on one of them that is the main one, but this intersection point here and also here can have a meaning.*" (He points to the intersections of the red and the blue curves, then to that of the red and the green curves)	pointwise	speech indicators + pointing gestures	all

Figures

FIGURE 5.3.13 - S2 POINTS TO THE MAXIMUM OF THE BLUE GRAPH [*1*].

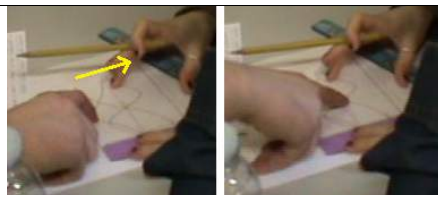


FIGURE 5.3.14 - S1 NOTICES THAT M AND C ARE VERTICALLY ALIGNED [*5*].

In this extract the group perspective on the involved function is mainly pointwise. They observe that the blue graph has an absolute maximum point M [*1-3*], but then they read it in a pointwise perspective: it is aligned with point C [*5*] (Fig. 5.3.14). This

perspective is strengthened by some other pointwise considerations and gestures on the involved graphs [*6*]. S1 imagines that the meaningless intersections between the given graphs can have a sense.
The group does not actually try to exploit the study of function technique. Indeed, after having completed the table in Fig. 5.3.12, they turn the page and never try to find relations between the three columns representing the properties of the blue, the red and the green functions.
Instead they try to directly apply the relations written on their notes. It consists of a blind application of rules; a technique without the supporting technology. Unfortunately S2's notes contains some errors [*7*] that misleads the group in finding the solution.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
7	S2: "*The intersections with x-axis of the function... are the maxima and minima of the derivative. So, since this one*" (she generally indicates the blue curve) "*<u>has no intersection with axis</u>, it cannot be the function.*"	pointwise	speech <u>indicators</u>	f
8	Others: "*Mm... Yes.*"			
9	S1: "*What are the steps for graphical deducing? Look at the intersections, sign,...?*"			
10	S2: "*The stationary points of f are the <u>intersections with x-axis</u> of f'.*"	pointwise	speech <u>indicators</u>	f'
11	S1: "*What do we mean with stationary points? The <u>intersections with x and y</u>?*"	pointwise	speech <u>indicators</u>	f
12	S2: "*The stationary points are the <u>intersections</u>...*"	pointwise	speech <u>indicators</u>	f
13	S1: "*So, for example C and D?*"	pointwise	symbols + graph	f
14	S2: "*We find the stationary points by imposing f' equal to zero... So wherever it is null.*"	pointwise	oral symbols	f'
15	S1: "*Yes, I know.*"			
16	S2: "*Ah! The points where the derivative function <u>cuts x-axis</u> are the stationary points of the starting function.*"	pointwise	speech <u>indicators</u>	f'
17	S1: "*Ok. So with stationary we mean maximum and minimum.*"			

18	S2: "*Where* <u>*it stands*</u>*.*"	local	speech <u>indicators</u>	f

They initially reverse the relation between the function and its derivative [*7*]. It is due to the incorrect relation S2 has noted on her notebook: "*The intersections with x-axis of $f(x)$ are the maximum and minimum points of $f'(x)$*". By following this property S2 excludes the blue one from being the starting function. After she reads the property in its correct formulation [*10*], but they do not really know what "stationary" means [*11-13*]. They make confusion between the pointwise perspective that is not on the function but on the derivative, as they correctly recall after [*14*]. Then, S1 associates the adjective "stationary" to the maximum and minimum points of a function [*17*] and not to its zeros. S2 locally adds that the stationary points are where the function stands [*18*]. The long discussion the group has had on this property shows us that, even if the used name "stationary" evokes a local feature of a function, it might not be enough to really activates such a perspective. More frequently, a stationary point is read in a pointwise perspective.
Anyway, they use the relation zeros-stationary points in the two directions, as S1 says "*the intersections with x-axis give the maxima and minima*" without specifying the subjects. It leads them to the wrong conclusion that the red function is the derivative of the blue function.
The teacher, who is walking through the classroom, asks them if everything is fine. They explain their reasoning and when they quote the wrong relation between zeros of the function and stationary points of its derivative, the teacher intervenes by noticing that this property is not valid, but it is the converse. She suggests to use the textbook as source, instead of their notebook. Thus, S1 reads on page 441: "*The sign of the derivative function is positive in the intervals in which $f(x)$ is increasing*". He proposes to consider the sign of f' and the variation of f. He works keeping the textbook on the side of the text of Problem 1 (see Fig. 5.3.15). Here is how the group, led by S1, comes to the conclusion that the green curve is the derivative of the blue one.

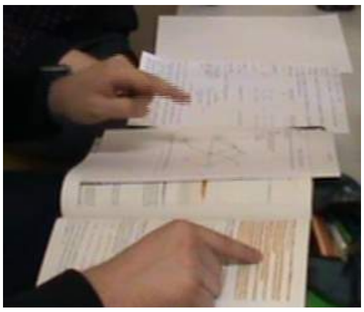


Figure 5.3.15 - S1 resumes theory from the textbook and directly tries to apply it to Problem 1.

	What the students say and do	Perspectives	Semiotic resources	on f, f', f''
19	S1: "*The blue one is increasing till M.*"	global	speech indicators	f
20	T: "*Yes.*"			
21	S2: "*In this interval the green function is positive, the red one is negative. Then viceversa, when it [the blue] is decreasing it [the derivative] must be negative, the red one is positive and the green one is negative.*"	global	speech indicators + pointing gestures on graphs	f, f'
22	T: "*So?*"			
23	S2: "*It leads us to say that the blue one is the function and the green one is its derivative.*"			
24	T: "*It can be so. But now I have to verify that the red one is actually the derivative of second order of the blue one or the derivative of first order of the green one.*"			

Group A abandons the pointwise perspective, finally exploiting the global potential of the given graphs. S1's global interventions [*19* and *21*] lead the group to the right conclusion [*22*].

Another group, medium-level Group C, follows the technique employed by the teacher in classroom step by step. The scheme given by the teacher is clearly recognizable. Differently from group A, they support it with the correct technology. In Fig. 5.3.16 you can find their answer to Problem 1.

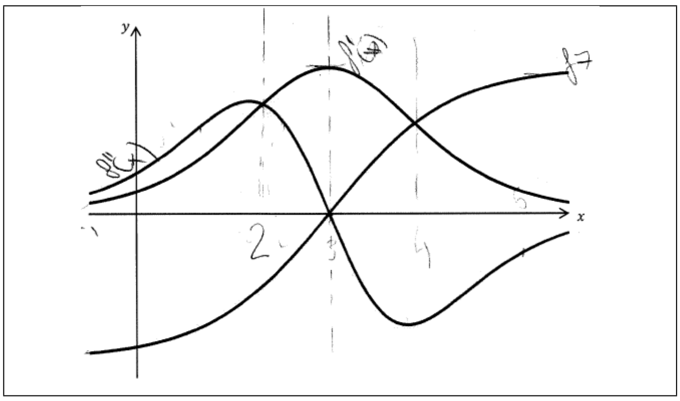


FIGURE 5.3.16 - GROUP C'S ANSWER TO PROBLEM 1.

Group C introduces numerical references on x-axis, precisely $x = -1, 2, 3, 5, 7$. Notice that $x = 2$ and $x = 4$ correspond respectively to the intersections of the green and the blue graphs and of the red and the blue graphs. These abscissas are not meaningful to the purpose of the study of function. They adopt a pointwise perspective on the given graphs, at least in the initial phase of the reasoning.
Group C adopts a local perspective on a function to get pointwise information about its derivative (see Fig. 5.3.17 and Fig. 5.3.18).

- "Let us suppose by trial and error that the function f is the red one, let us search for its stationary points in order to find the intersections with x-axis for $f'(x)$ and let us give them arbitrary values.
  $f'(x) = 0$

  $\searrow x = -1, x = 7$ stationary points for$f$
  points of intersection with x-axis for$f'(x)$"

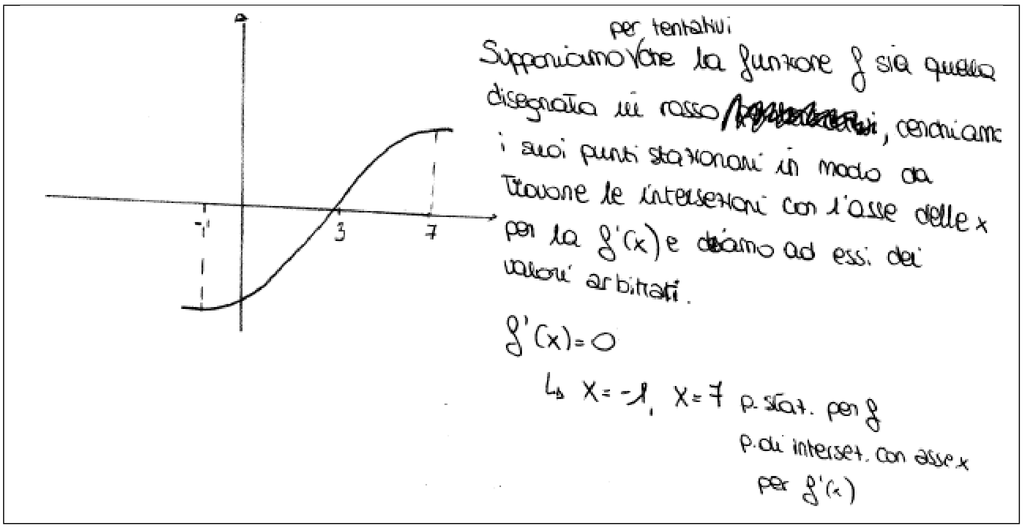


FIGURE 5.3.17 - EXTRACT FROM GROUP C'S SOLUTION AND JUSTIFICATION TO PROBLEM 1.

- "Let us impose $f''(x) = 0$ to find the stationary points of $f'(x)$ that correspond to the intersection points of $f''(x)$ with x-axis.
  $f''(x) = 0$

  $\searrow x = 3$ points of intersection with x-axis for$f''(x)$
  stationary points for$f'(x)$"

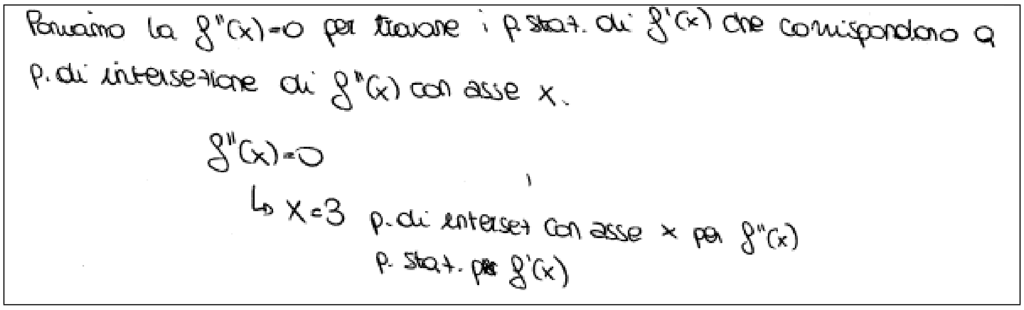
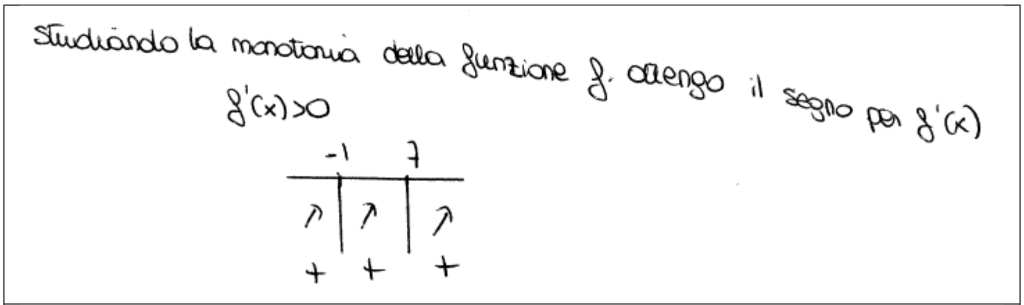


Figure 5.3.18 - Extract from Group C's solution and justification to Problem 1.

They deduce global information about the derivative of a function by studying its global properties (see Fig. 5.3.19, 5.3.20 and 5.3.21).

- "By studying the variation of the function $f(/f')$, I get the sign for $f'(x)(/f''(x))$
(signs scheme)"



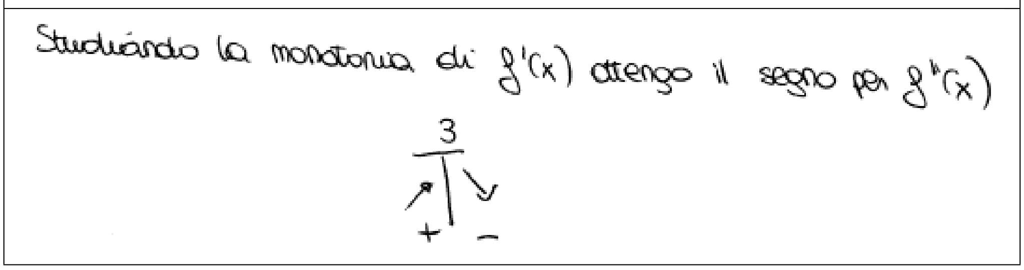


Figure 5.3.19 and 5.3.20 - Extracts from Group C's solution and justification to Problem 1.

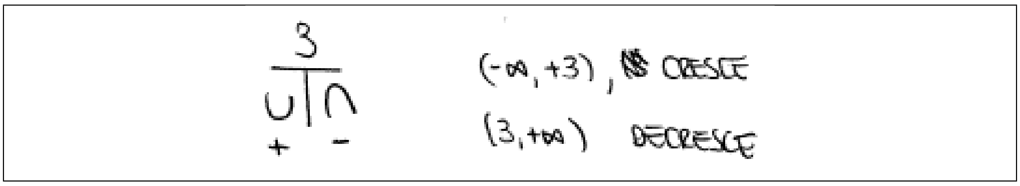
- 

Figure 5.3.21 - Extracts from group C's solution and justification to Problem 1.

They also relate local properties of the function with local properties of its derivative (see Fig. 5.3.22).

- "Let us find the points where f changes concavity that correspond to search for max/min for $f'(x)$
  $f''(x) = 0$
  $\searrow x = 3 =$ maximum point"

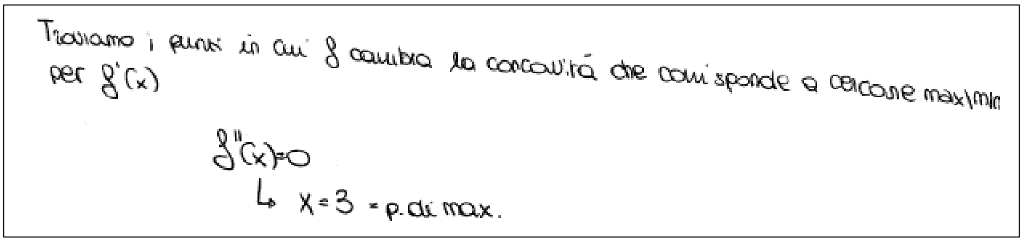

Troviamo i punti in cui f cambia la concavità che corrisponde a cercare max\min per f'(x)
f''(x)=0
↳ x=3 = p. di max.

Figure 5.3.22 - Extract from group C's solution and justification to Problem 1.

Beside words, symbols and schemes, the students use graphical resources. Indeed their goal is drawing f' graph (see Fig. 5.3.23, on the left), provided that f graph is the red one. In this way, they find the matching graph among the given ones. They make the same to deduce f'' graph (see Fig. 5.3.23, on the right).

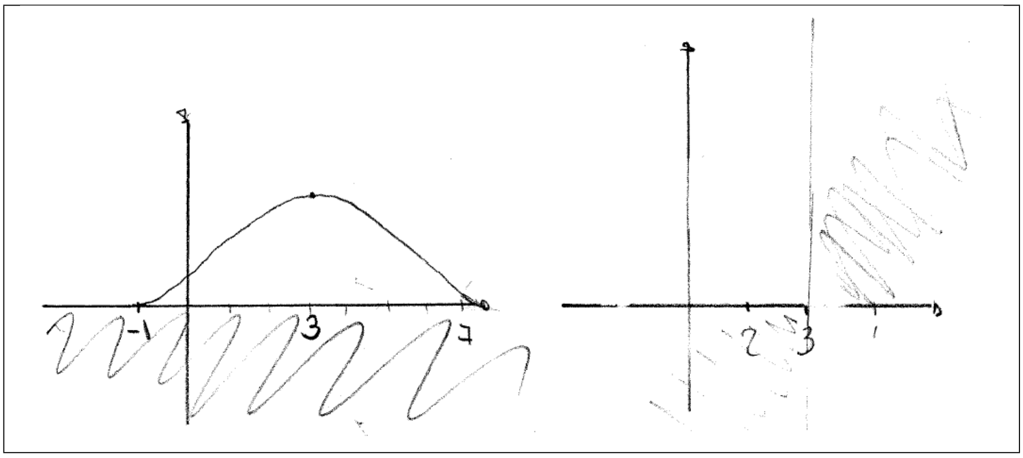


Figure 5.3.23 - Extracts from Group C's solution and justification to Problem 1.

Tangent method

The only group that makes some local considerations about the involved functions is the high-level group E. They are triggered by the presence of non-differentiable and discontinuous points in Problem 2. The students call p the blue function, v the red

function and g the green one. They correctly establish that $p' = g$ and $g' = v$.
As a surplus, they make some final observations about the involved functions in x_0, which is point of non-differentiability for p and of discontinuity for g and v. They refer to "m of the tangent in the neighbourhood of x_0". Their words (see Fig. 5.3.24) initially describe what they see on the given graphs:

> "x_0 in function p is a cusps, then a non-differentiable point
> We can say that in function g (first derivative) x_0 is a jump, the values m-tangent in the neighbourhood of x_0 are equal.
> In function v (second derivative) x_0 is a discontinuity which is possible to eliminate. The values of m-tangent in the neighbourhood of x_0 are equal but opposite."

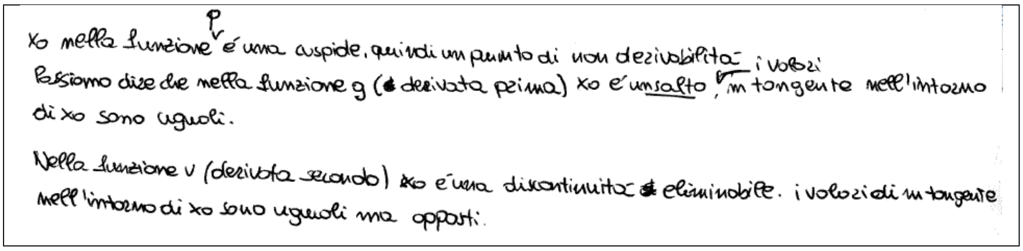

Xo nella funzione p è una cuspide, quindi un punto di non derivabilità
Possiamo dire che nella funzione g (derivata prima) xo è un salto, i valori m tangente nell'intorno di xo sono uguali.
Nella funzione v (derivata seconda) xo è una discontinuità eliminabile. i valori di m tangente nell'intorno di xo sono uguali ma opposti.

FIGURE 5.3.24 - EXTRACT FROM GROUP E'S SOLUTION AND JUSTIFICATION TO PROBLEM 2.

Group E looks at the given graphs in a local perspective and describes what happens to the gradient of the tangent in a neighbourhood of the point x_0. They verify that "the values m-tangent in the neighbourhood of x_0 are equal" for the green function and "equal but opposite" for the red one.
Then the group asks me if their solution is correct. In return, I ask them if they are able to justify why in the derivative of first order there is a jump with equal m-tangent values and in the derivative of second order there is a hole with equal but opposite m-tangent values. I give them a little help by observing that in the minimum point of the blue function m-tangent is zero and correspondingly the ordinate of the green derivative function is exactly zero.
They grasp this suggestion about the way of looking at the graphs: local on the function and pointwise on the derivative. Thus, their final observations base on the correspondence between the value of the gradient of the tangent to a function and the value taken by its derivative at the same abscissa (see Fig. 5.3.25). Such relation is also stressed on the given graphs (see Fig. 5.3.26).

> "k is a minimum. The angular coefficient is equal to zero and, as a matter of fact, the intersection point with axis [x] y is zero, whereas the angular coefficient in the neighbourhood of x_0^- is positive; [$1m$ on the blue graph, see Fig. 5.3.26] so in the neighbourhood of x_0^- in its first derivative y has the same value as the angular coefficient [1 on y-axis for the green graph, see Fig. 5.3.26] while in the neighbourhood of x_0^+ m-tang is negative [$-2m$ on the blue

graph, see Fig. 5.3.26] so in the neighbourhood of x_0^+ [in] its first derivative y is equal to m. [-2 on y-axis for the green graph, see Fig. 5.3.26] In function g the two coefficients are equal because the two letters [grammatical error: straight lines] are //, therefore in its derivative $g' = v$ in the neighbourhood $x_0^{\pm}$ y is the same.

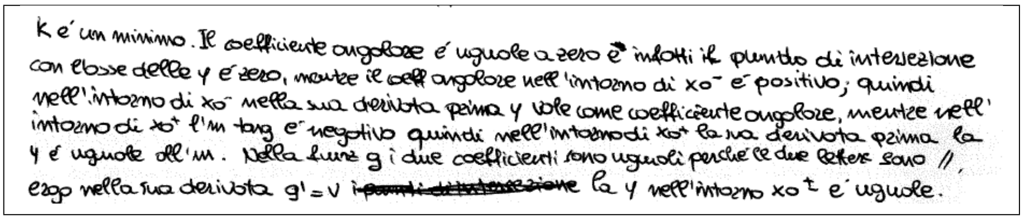

k é un minimo. Il coefficiente angolare é uguale a zero è infatti il punto di intersezione con l'asse delle y é zero, mentre il coeff angolare nell'intorno di xo⁻ é positivo; quindi nell'intorno di xo⁻ nella sua derivata prima y vale come coefficiente angolare, mentre nell'intorno di xo⁺ l'm tang é negativo quindi nell'intorno di xo⁺ la sua derivata prima la y é uguale all'm. Nella funz g i due coefficienti sono uguali perchè le due lettere sono //, ergo nella sua derivata g' = v la y nell'intorno xo± é uguale.

Figure 5.3.25 - Extract from group E's solution and justification to Problem 2.

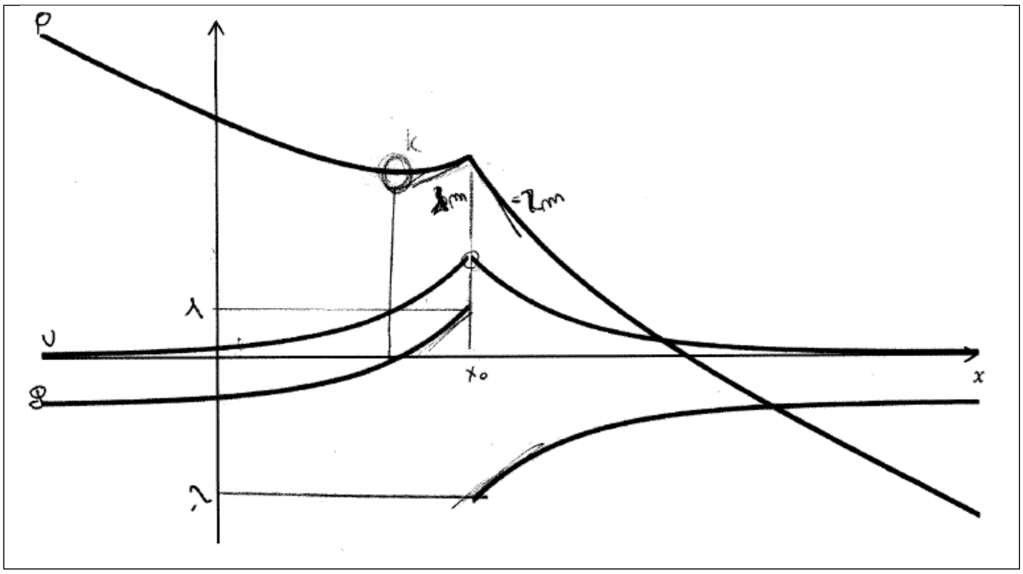


Figure 5.3.26 - Extract from group E's solution and justification to Problem 2.

Influences of M.G.'s praxeology

We find it relevant that more than half of M.G.'s students try to employ step by step the technique used by the teacher. We think that one of the reasons of this behaviour relies on the way the teacher has given $OM_{f'}^{gra}$ in classroom. The study of function technique has not been discussed or found out by the students. As a consequence, the majority of them find it easy to apply it step by step as an infallible method, even though they do not handle the technology. No group tries to make new or original considerations, except for group E that anyway has been guided by my indication. Except for this group, all the local considerations consist of the repetition of the properties noted during the lesson. Thus, we are not able to say if the students really look at the given function in a neighbourhood of a point or if they are repeating what they have listened and noted.

V.'s students

Before analysing how V.'s students approach the activity, we want to recall that V. has only briefly spoken about the graph of the derivative function. She has only introduced it through an example in GeoGebra (see the subparagraph "Some hints about a graphical technique" of Paragraph 4.5.3). The derivative graph was automatically created by the software while a point and the tangent in that point moved on the graph of a given function f. So the graph of f' has been shown as locus of the points having the same abscissa of the corresponding point on f, and the angular coefficient of the tangent to f in that point as ordinate.
Moreover, in the classroom V. has not spoken about the derivative of second order yet. It is not a prerequisite of the activity, since in the text the function f'' is presented as "the derivative of f'". Nevertheless, as a consequence, V.'s students have no direct information about the relation between the sign of f'' and the concavity/convexity of f. In Table 5.5, we give a general overlook on the work done by the different groups (A, B, C, D, E and F).

	Method			Perspectives			
	analytic	tangent	*algebraic*	point.	*univ. p.*	global	local
Problem 1	A,B,C,E	D	*F,∼E*	A,B,D	*∼E,F*	B,D,E,F	C,D
Problem 2	A,B,C	D	*∼F*	A,B,D	*∼F*	A,B,C,D,E	∼B,C,D,∼F
Problem 3	A,C	D		A,C,D		A,C,D	

Table 5.5 - V.'s students and activity 1. The sign ∼ indicates that the method/perspective is partially used by the group in solving the corresponding problem. The columns in *italic* are the unexpected ones.

At a technical level, the most part of the students use an analytic method to solve the problems. Nonetheless, it is also relevant that a group (D) chooses the tangent method and a group (F) employs an unexpected algebraic method. Only the three high-level groups (A, C and D) complete the whole activity within the given hour. The low-level group (E) manages to solve only Problem 1, while the two medium-level groups B and F get to approach Problem 2 and group B solves it. The most part of the groups base on pointwise and global properties of the involved functions. Only two groups (C and D) over six adopt a local perspective and in particular group C explicitly uses it in their reasoning.

Analytic method

Let us consider group A as an example. This group has been also videotaped. They only base their reasoning on the properties of functions they deduce from the given graphs. In particular, for approaching the first problem they use the *l-p* relation between the maximum of a function and the zero of its derivative (see Table 5.2) but they exploit it only in a pointwise-pointwise perspective. This fact leads them to detect a wrong combination of functions f-f'-f''. It follows the transcription of their utterances accompanied by their gestures. In this extract, students S1 and S2 are explaining their reasoning to S3, who joins them a few minutes later. Notice that f, f' and f'' in the last column refers not to the correct combination, but to the chain the students have in mind.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
1	S1: "*We have to decide which is f, which is the derivative of f' and which is the second derivative of f.*"			
2	S3: "*Ah, ok.*"			
3	S2: "*Let me explain to you what we have thought. The green one...*"			
4	S1: "*Wait. Before the blue one. The blue one is the function, because you see there is a crossing of x*" (he points with the pen to the intersection of the green and the red functions on x-axis, Fig. 5.3.27) "*which does not correspond to the zero of any other function but it. Instead, here*" (he points to the maximum of the blue graph, Fig. 5.3.27) "*it is zero because it is a maximum and it is aligned with the two intersections of x of the two derivatives.*"	pointwise	speech indicators + pointing gestures	f, f'
5	S1: "*This...*" (he points to the red graph with the pen, Fig. 5.3.28) "*We can say, this is f'.*"	global	continuous pointing gesture	f'
6	S2 and S3: "*Yes.*"			
7	S2: "*And this*" (he generally points to the green graph with the pen) "*this is f'', I thought, because here it has the zeros...*" (he points to the maximum of the green function, Fig. 5.3.29)	pointwise	speech indicators + pointing gestures	f''
8	S1: "*But it has no correspondence here.*" (he points to x-axis in correspondence to the maximum of the green function, Fig. 5.3.29)	pointwise	pointing gesture	f''

9	S2: "*And also here.*" (he points to the minimum of the green function) "*But I have no crossing with x here.*" (S1 and S2 point together to x-axis in correspondence to the maximum of the green function, Fig. 5.3.29)	pointwise	speech indicators + pointing gestures	f''
10	S2: "*So, f, f' and f''.*" (He generally points respectively to the blue function, the red one and the green one)	global	pointing gestures	all

Figures

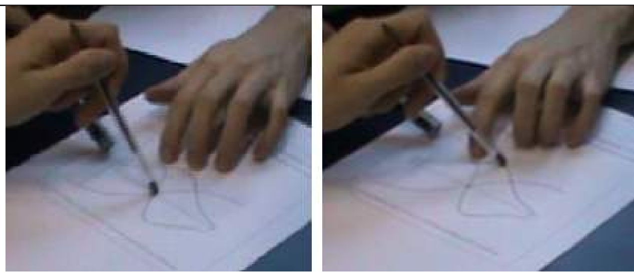


Figure 5.3.27 - With pointwise pointing gestures, S1 and S2 detect the *l-p* relation between the blue graph and the other two.

Figure 5.3.28 - With a global continuous pointing gesture, S1 follows the red graph with his pen.

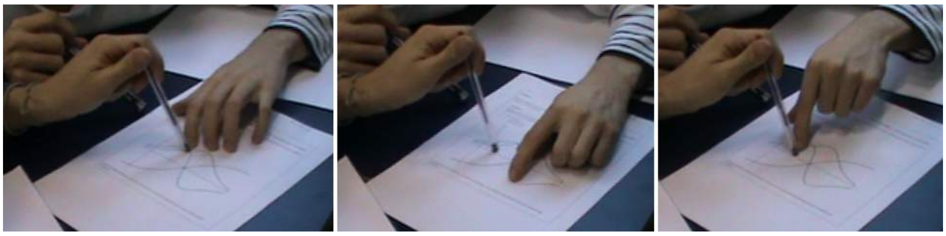


Figure 5.3.29 - With pointwise pointing gestures, S1 and S2 imagine the relation between the green graph and another hypothetical one.

In Fig. 5.3.30 we find a copy of their written justification:

"Dear Lorenzo and Francesca,
we noticed that to the maximum of f(blue) corresponds the intersections of the axis of abscissas of f(red) and f(green). So, these last two are the derivatives of f(b); now, reasoning on them we can [say] that the maximum and the minimum of f(v) do not correspond to any intersection of f(r), therefore f(r) is the first derivative and f(v) is the second derivative."

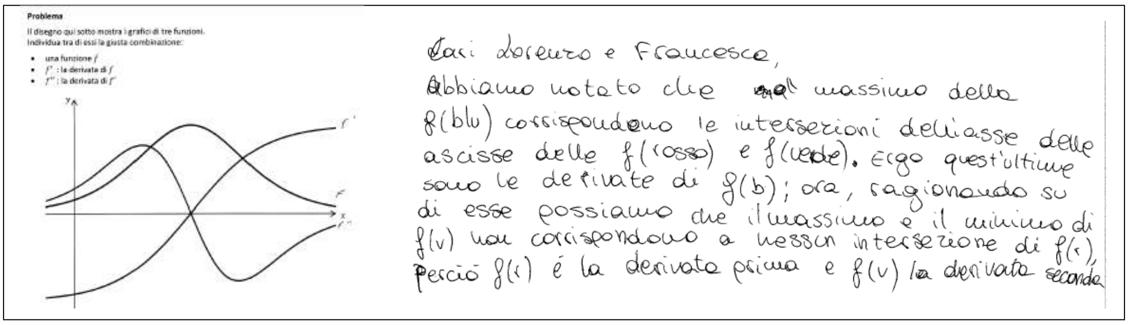


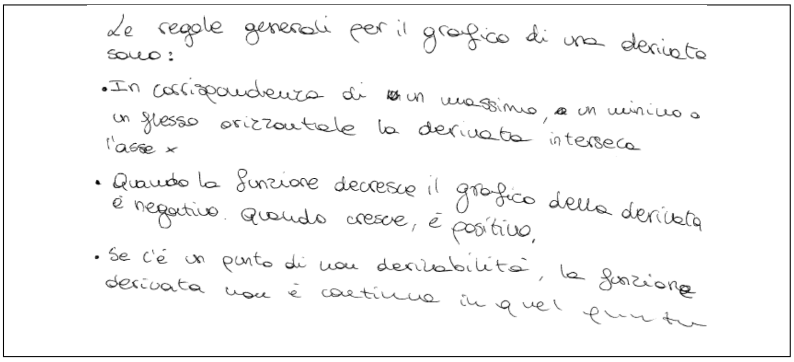
FIGURE 5.3.30 - SOLUTION TO PROBLEM 1 AND ITS JUSTIFICATION BY GROUP A.

The students of group A base their reasoning in particular on the relation between the minimum/maximum points of a function and the zeros of its derivative. This relation is supposed to be a combination of a local perspective on f and a pointwise perspective on f'. Unfortunately, the students exploit it only in a pointwise way: the absolute maximum of the blue graph corresponds to the intersections of the red and the green ones with x-axis. It is a relation point to point, as the pointing gestures made by S1 underline [*4*], (Fig. 5.3.27). It leads them to say that the blue function must be the starting function. The same happens when S2 argues the the green function must be the second derivative, because the absolute maximum and absolute minimum of the green graph do not correspond to any intersection on x-axis of other functions. The used *l-p* relation is detected with pointwise pointing gestures on the involved functions [*7-9*], (Fig. 5.3.29). This pointwise-pointwise reading of the relation leads them to say that the green function has no derivative.
Not only in the utterances [*1-10*], but also in the written justification (Fig. 5.3.30), we find frequently the terms "crossing", "intersection", "zeros" that reveal a pointwise perspective.
In this problem, group A does not exploit the potentiality of the graph as a global semiotic resource. Only S1 makes a global continuous gesture on the red graph (Fig. 5.3.28), but he does not know how to interpret it. He simply says: *"This... We can say, this is f'."* [*5*], without further global checking.

In solving Problem 2 the students understand their mistake. They firstly notice the *l-l* correspondence between the non-differentiable point in the blue graph and the discontinuous points in the red and the green graphs. Actually, the relation they identify is pointwise-pointwise. Nevertheless, in this case, they also observe global properties of the given graphs, in particular if they increase or decrease. Thus, they rely on the *g-g* relation between the function variation and the derivative sign. Employing it globally, they get to the right solution. These relationships between f and f' are summarized in the group written argumentation (Fig. 5.3.31).

"The general rules for the graph of a derivative are:

- In correspondence to a maximum, a minimum or an horizontal inflection point, the derivative intersects x-axis.
- When the function is decreasing the graph of the derivative is negative. When [the first] is decreasing, [the other] is positive.
- If there is a non-differentiable point, the derivative function is not continuous in that point."



Le regole generali per il grafico di una derivata sono:

- In corrispondenza di un massimo, un minimo o un flesso orizzontale la derivata interseca l'asse x
- Quando la funzione decresce il grafico della derivata è negativo. Quando cresce, è positivo.
- Se c'é un punto di non derivabilità, la funzione derivata non è continua in quel punto

Figure 5.3.31 - General *analytic method* proposed by group A to decide if a graph represents the derivative of another one.

Also group C develops an analytic method. Beside the g-g relation between the sign of the derivative and the variation of the function, in their written solution and justification of Problem 1 (Fig. 5.3.32) and Problem 2 (Fig. 5.3.33) we find explicit local considerations.

> "Dear Lorenzo and Francesca,
> we reflected upon your problem, and we got to this solution, that f is the red function, f' is the blue one and f'' is the red one. The reason is the following: f' at the right and left ends is approaching 0 (because f in the same points tends to become horizontal, and so $m = 0$) in the maximum point of f', the function f starts to decrease (inflection point). f'' instead at the ends is approaching 0 for the reason previously expressed, and the maximum and minimum points of f'' represent the inflection points of f'. Good luck."

Also in the resolution and written justification of Problem 2 (Fig. 5.3.33), group C bases on local properties of the involved graphs in order to decide the right combination of functions. We translate below only the local considerations we find in their written argumentation (Fig. 5.3.33).

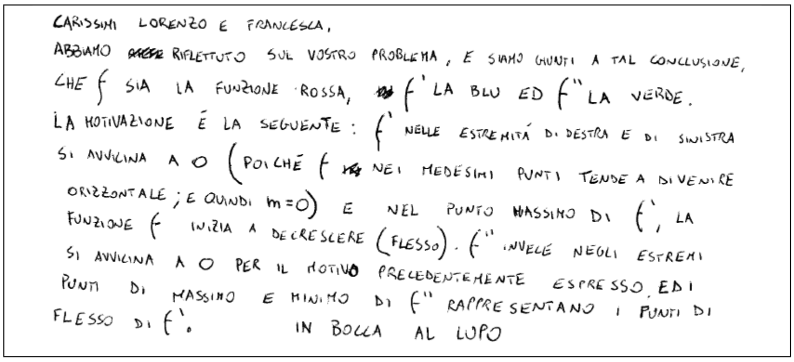

CARISSIMI LORENZO E FRANCESCA,
ABBIAMO ~~...~~ RIFLETTUTO SUL VOSTRO PROBLEMA, E SIAMO GIUNTI A TAL CONCLUSIONE, CHE f SIA LA FUNZIONE ROSSA, f' LA BLU ED f'' LA VERDE.
LA MOTIVAZIONE É LA SEGUENTE: f' NELLE ESTREMITÁ DI DESTRA E DI SINISTRA SI AVVICINA A O (POICHÉ f NEI MEDESIMI PUNTI TENDE A DIVENIRE ORIZZONTALE; E QUINDI $m=0$) E NEL PUNTO MASSIMO DI f', LA FUNZIONE f INIZIA A DECRESCERE (FLESSO). f'' INVECE NEGLI ESTREMI SI AVVICINA A O PER IL MOTIVO PRECEDENTEMENTE ESPRESSO. ED I PUNTI DI MASSIMO E MINIMO DI f'' RAPPRESENTANO I PUNTI DI FLESSO DI f'.
IN BOCCA AL LUPO

Figure 5.3.32 - Solution to Problem 1 and justification by group C.

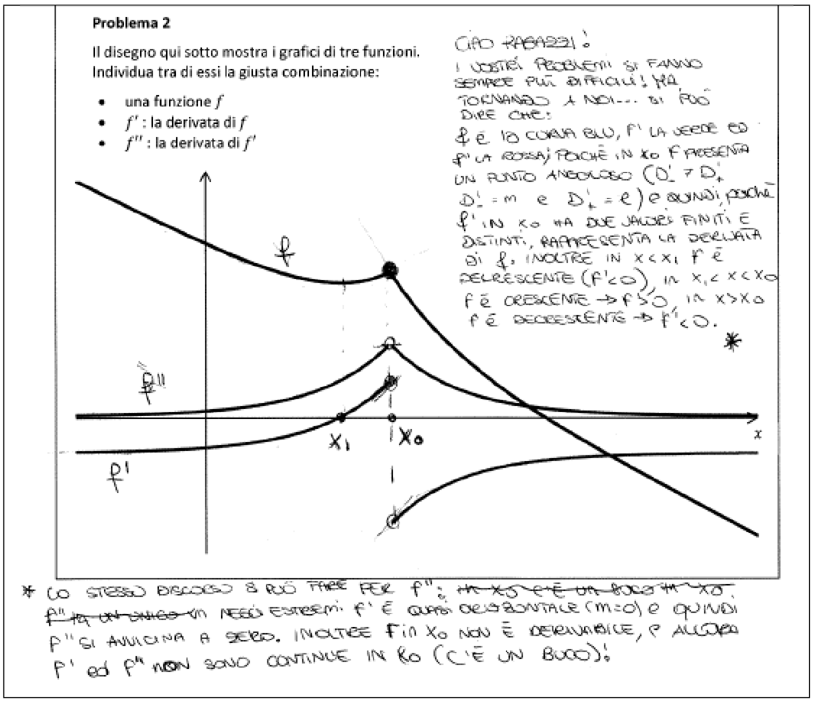


Figure 5.3.33 - Solution to Problem 2 and written justification by group C.

"[...] f is the blue curve, f' is the green one and f'' the red one; because in x_0 f presents a corner ($D'_- \neq D'_+ \quad D'_- = m$ and $D'_+ = l$), and then, since f' in x_0 has two finite and different values, [the green f'] represents the derivative of f.
[...] The same remark can be made for f'': at the ends f' is almost horizontal ($m = 0$) and so f'' is approaching zero. Moreover, f in x_0 is not differantiable, then f' and f'' aren't continuous in x_0 (there is a hole)!"

These students make some local remarks about the horizontal asymptotes, the maximum, minimum and inflection points and the non-differentiable and discontinuous points.

They are marked by the use of specific speech indicators, which are underlined in the text. They all reveal a local perspective on the involved functions.

- Expressions such as "is approaching 0", "tends to become horizontal", "is almost horizontal" stress the great importance the students give to the end behaviour of the involved functions. They are all linked to the *l-l* relation between asymptote in the function and horizontal asymptote in the derivative.

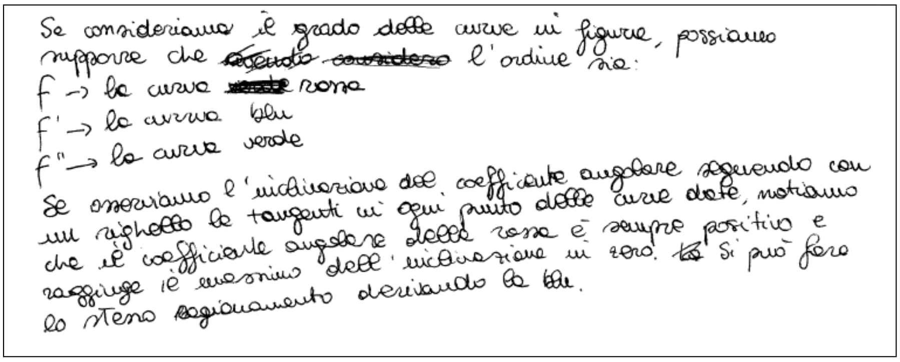
- Expressions such as "starts to decrease" referred to the function after a maximum point underline that the relation between the maximum/minimum points of a function and the inflection points of its derivative is used locally.

- Sentences such as "f' in x_0 has two finite and different values", "f in x_0 is not differantiable", "not continuous in x_0" together with symbolic expressions like "$D'_- \neq D'_+ \quad D'_- = m$ and $D'_+ = l$" are signs of a local focus on the involved functions. Moreover, another resource is used to activate the local perspective: they draw the two different tangents in the corner of the blue curve and the two parallel tangents to the green curve in the jump point.

Tangent method

We take group D as an example of application of the tangent method. This group has not been video-recorded while solving Problems 1 and 2, but we have their written productions. Nonetheless, we have a short video of their resolution of Problem 3. From the justifications given in the written production, we observe that their reasoning is grounded on the tangent line changing on the graph. We find confirmation of that also in the short video. In particular, for solving the first problem they use the relation between the gradient of the tangent line to the function and the value of the derivative. Their work is implicitly local on f. Let us firstly examine their written justification (Fig. 5.3.34).

> "If we consider the degree of the curves in the figure, we can suppose that the order is:"
> $f \rightarrow$ the red curve
> $f' \rightarrow$ the blue curve
> $f'' \rightarrow$ the green curve
> If we observe the inclination of the angular coefficient by following with a ruler the tangent lines in each point of the given curves, we notice that the angular coefficient of the red one is always positive and it reaches the greatest inclination in zero. The same reasoning can be made to differentiate the blue one."



Se consideriamo il grado delle curve in figura, possiamo supporre che ~~secondo considero~~ l'ordine sia:
f → la curva rossa
f' → la curva blu
f'' → la curva verde
Se osserviamo l'inclinazione del coefficiente angolare seguendo con un righello le tangenti in ogni punto delle curve date, notiamo che il coefficiente angolare della rossa è sempre positivo e raggiunge il massimo dell'inclinazione in zero. Si può fare lo stesso ragionamento derivando la blu.

FIGURE 5.3.34 - SOLUTION TO PROBLEM 1 AND ITS JUSTIFICATIONS BY GROUP D.

In Problem 2, they summarize their method as follows:

> "Observe the angular coefficient: if it increases the derivative will increase, if it decreases f' decreases. [...]
> RULE: Follow the angular coefficient and notice non-differentiable points."

Moreover, in the short video at our disposal, we have confirmation of their technique with the pencil. The pencil lies on the paper and is moved as if it were the tangent line to the graph they want to differentiate. We propose below the transcript of the students working on Problem 3, accompanied by their gestures with the pencil on the paper. Notice that S1 and S2 follow the graph they want to differentiate but the subject of their speech is already the derivative function. Thus, in the utterances such as "it starts..." or "it goes up..." the subject "it" is the derivative of the function on which they are moving the pencil.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
1	S1: "*It starts positively.*"	local	speech indicators	f'
2	S2: "*Here it starts positive*" (he positions the pencil tangent to graph of the function they want to differentiate, as in Fig. 5.3.35(a))	local local	continuous gesture + speech indicators	f f'
3	S2: (He moves slowly the pencil on the graph they want to differentiate) "*It goes up, it goes up...*"	local global	continuous gesture + speech indicators	f f'
4	S1: "*There it starts to go down.*" (S2's pencil is passing through the inflection point, Fig. 5.3.35(b))	local local	continuous gesture + speech indicators	f f'

5	S1: "*There <u>it is zero</u>.*" (S2's pencil is getting to the maximum point, Fig. 5.3.35(c))	local pointwise	continuous gesture + speech <u>indicators</u>	f f'

Figures

Figure 5.3.35(a), 5.3.35(b) and 5.3.35(c) - S2 is moving his pencil on the graph they want to differentiate.

The students of group D base their reasoning on the estimation of the value and the variation of the tangent gradient. They implicitly adopt a local perspective on the function they want to differentiate and, at the same time, they make explicit local, global or pointwise considerations on its derivative. While the pencil moves keeping tangent to the graph of f (Fig. 5.3.35), the students imagine what happens to the derivative values. Thanks to local gestures on f, they deduce some properties of f' that are:

- local such as "it starts positively" [*1-2*] or "it starts to go down" [*4*];
- global such as "it goes up" [*3*];
- pointwise such as "it is zero" [*5*].

In this way, the students manage to identify most of the relations between properties of f and f' we have summed up in Table 5.2. They do not need to make the properties of f explicit, because they rely on the direct correspondence between the properties of the gradient of the tangent and the properties of the derivative. Group D stresses this direct correspondence in their written method: "if it [the angular coefficient] increases the derivative will increase, if it [the angular coefficient] decreases f' decreases". What they find is the relation between the concavity/convexity of f and the variation of f'. However, they do not need to make explicit considerations about the concavity of f. That's because they are directly relating the variation of the gradient with the variation of the derivative.
In conclusion, the students of group D exploit the global potentiality of the given graphs, but for this purpose they have to introduce a tool. This tool is the tangent line, iconically represented by a ruler or a pencil. Through the introduction of this intermediary, the students grasp direct information about the derivative of each of the graphs. So their method turns to be pseudo-analytic, because it does not highlight properties of a function that are particularly relevant in order to discover something about the derivative. It is a correct technique that does not really lead the students to adopt a certain perspective on the function that has to be differentiated.

Unexpected algebraic method

Surprisingly, one of the videotaped groups carries out an unexpected technique to solve Problem 1. The text provides only the graphs without any algebraic expression or any specific reference on the Cartesian axis. Nevertheless, group F tries to guess the equations of the given curves in the form $y = f(x)$, in order to recreate a known situation, where they know how to behave. This is the reason why we call their technique "algebraic method". Actually, as stressed above in our a priori analysis, we expected this method as one of the possible resolutions of Problem 3. We find very interesting to find it applied to solve Problem 1. The main perspective activated here is the universal pointwise one. Indeed, every formula the group employs is valid for each x belonging to $\mathbb{R}$. Let us comment their discussion in order to solve the problem. We can distinguish two parallel attempts: the former is algebraic and universal pointwise; the latter is analytic but strictly pointwise. The first method prevails on the second one because of the greater security provided by calculations against the uncertain and disconnected pointwise properties observed directly on the graphs.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
1	S1: "*Well, if you have $f(x)$... To me, this*" (he points to the red graph) "*is minus cosine for example. [...] We should have the derivative of cosine that is sine.*"	global	graph	f
2	S2: "*To me, rather than a cosine, that stuff could be similar to the arctangent.*"	global	graph	f
3	S1: "*Mmm... Great! But the arctangent passes through...*"			
4	S2: "*The origin. Yes, with a translation it's not difficult but...*"			
5	S3: "*What is the derivative of the arctangent?*"			
6	S1: "*The derivative of the arctangent...*"			
7	S2: "*1 over ...*"			
8	S1: [...] "*It is 1 over 1 plus squared x, isn't it?*"	global(=univ. pointwise)	symbolic exp.	f'

9	S1: [...] "*Wait, what about thinking like this... Here*" (he points to the maximum of the blue function, Fig. 5.3.36(a)) "*it [the blue function] has a maximum. Let's use the maximum and minimum points. Because you see there are lots of picks. Here [the blue] we have a maximum, here [the green] they are two and here [the red] we have nothing. To me, the red is the function, the blue is the derivative and the green one is the second derivative.*"	pointwise	pointing gesture	f'
10	S3: "*So, if we graphically represent* $\frac{1}{1+x^2}$*?*"	global(=univ. pointwise)	symbolic exp.	f'
11	S1: "*Listen, I have an idea!*" (he traces a vertical line to link the maximum of the blue graph to the zero of the red one, Fig. 5.3.36) "*If we take x_0 of $f(x)$ where y = 0...*"	pointwise	sketch + symbols	f, f'
12	S3: "*Prof? The graph of 1 over something?*"			

Figures

Figure 5.3.36(a), 5.3.36(b) and 5.3.36(c) - S1 draws a vertical line passing through the blue maximum and the zero of the red and green graphs.

The first idea that the group has is to globally guess the analytic expression of the red function (which accidentally is the right starting function) [*1-2*]. The students S1 and S2 do so by comparing the form of the red graph to another one they have in mind (e.g. cosine or arctangent graphs). The students S1 and S3 have two different approaches. S3 immediately starts to do calculations supposing that the analytic expression of the red function is $f(x) = \arctan x$. In line [*8*] it appears for the first time a generic variable x. From this moment on, S3 works in a universal pointwise perspective on the symbolic expression of $f'(x)$. S1, instead, tries to find a graphical rule to detect f and f' but he only makes pointwise considerations on each single graphs without being able to link them [*9* and *11*]. His pointing gesture and his sketch (Fig. 5.3.36) on the given graphs remain pointwise at the level of the maximum point or of the zeros. He does not get to formulate the relation between the maximum of a function and the zero of its derivative. S3 asks the teacher's suggestion about the graph of a function reciprocal [*12*]. She wants to study and draw the graph of $f'(x) = \dfrac{1}{1+x^2}$ [*10*] in order to verify if it is similar to the blue or to the green one. The teacher stresses that they are not required to find the analytic expression of the represented functions. Nonetheless, S3 supported by the other components of the group (S1 abandons the graphical approach for a while) continues on her paper. She gets to the graph in Fig. 5.3.37, which the group recognize similar to the blue one.

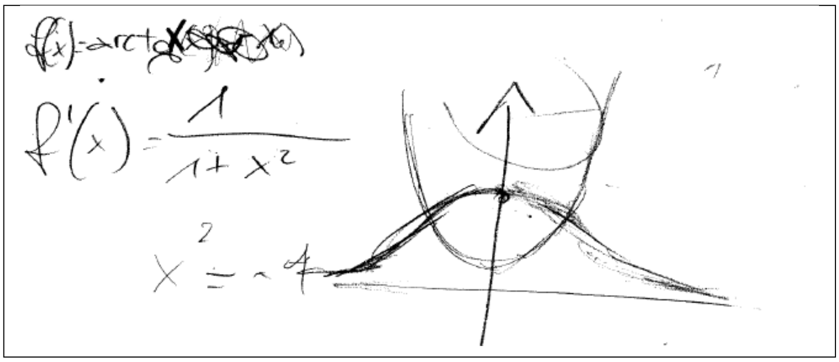


FIGURE 5.3.37 - DERIVATIVE GRAPH DEDUCED BY GROUP F.

Then, applying the quotient differentiation formula, she computes the expression of $f''(x) = \dfrac{-2x}{(1+x^2)^2}$. At this point the students make a mistake in the study of the stationary points of f''. Indeed, they study the sign of f'' instead of the sign of its derivative ($f^{(3)}$). Since they find a maximum in $x = 0$, they stop confused. The teacher intervenes again.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
13	T: "*Try to justify the answer basing on the graphs, do not search for an analytic expression.*"	global	graph	all
14	S1: "*Ok.*"			
15	S2: "*Well, no justification based on the graphs comes to my mind...*"			
16	S1: "*To me, we must reason here*" (he points to the blue maximum and to the zeros of the other functions) "*I would write $f(x_0) = 0$, then $f'(x_0)$ must be what?*"	pointwise	pointing gesture + symbols	f, f'
17	S3, S2: "y_0"			
18	S3, S1: "*1*"			
19	S2: "*How can you say that it is 1?*"			

20	S1: "*No. It's true. Ok, let's take y_0. And then $f''(x_0)$?*"	pointwise	symbols	f''
21	S2: "*Its value is 0 again.*"			

The teacher intervenes at the level of the semiotic resources that the students are using: she advises to abandon the algebraic register in favor to the graphical one [*13*]. Through pointwise gestures and symbols, S1 proposes again his graphical approach [*16*] but his conclusion is simply a list of strictly pointwise properties on f, f' and f'' in the abscissa x_0 [*16-21*]. No relation is stated between them. When the teacher returns to check the group work, they explain to her their reasoning based on the algebraic expressions of the curves [*1-12*].

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f', f''**
22	T: "*So, if it is the way through which you understand it, write it.*"			
23	S1: "*Let's write it.*"			
24	S2: "*But it doesn't work!*"			
25	S1: "*Why not? It works very well!*"			
26	S2: "*We have calculated the maximum and minimum points and they aren't correct.*"			
27	T: [...] "*Have you differentiated three times?*"	global(=univ. pointwise)	speech	all
28	S1: "*Twice.*"			
29	S2: "*We compute the derivative and we compute the maximum and minimum points...*"			
30	S3: "*No! We are so stupid! This*" (she refers to the sign scheme of $\frac{-2x}{(1+x^2)^2}$) "*is the maximum of that one*" (she points to the blue one) "*We have to find the derivative of this function to obtain its maximum and minimum points!*"	global	scheme	f''

The teacher accepts the students' justifications and, differently from before [*13*], she does not try to convince them to change the semiotic resource [*22*]. She simply checks with them all the calculations. Thanks to the question-doubt the teacher poses [*27*], S3 notices that their technique lacks of a part. We can read all the steps of their method directly through their words (Fig. 5.3.38, 5.3.39(a) and 5.3.39(b)).

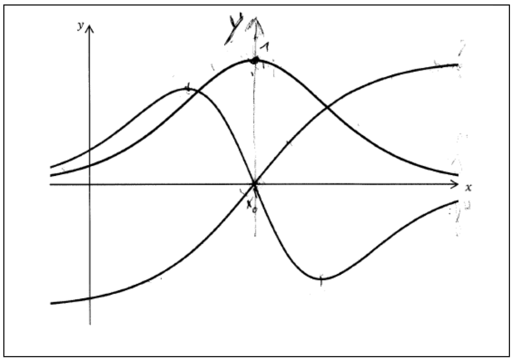
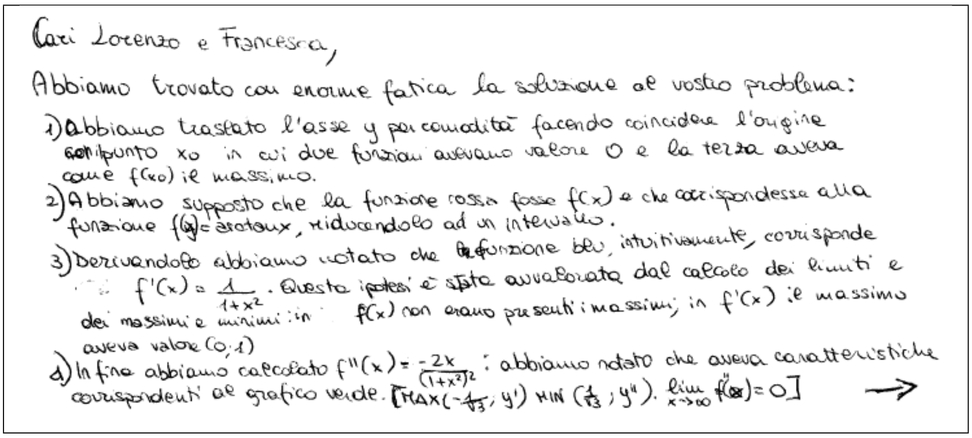


Figure 5.3.38 - Solution to Problem 1 by group F.



Cari Lorenzo e Francesca,

Abbiamo trovato con enorme fatica la soluzione al vostro problema:

1) Abbiamo traslato l'asse y per comodità facendo coincidere l'origine col punto x_0 in cui due funzioni avevano valore 0 e la terza aveva come $f(x_0)$ il massimo.

2) Abbiamo supposto che la funzione rossa fosse $f(x)$ e che corrispondesse alla funzione $f(x) = \arctan x$, riducendolo ad un intervallo.

3) Derivandolo abbiamo notato che la funzione blu, intuitivamente, corrisponde $f'(x) = \frac{1}{1+x^2}$. Questa ipotesi è stata avvalorata dal calcolo dei limiti e dei massimi e minimi: in $f(x)$ non erano presenti i massimi; in $f'(x)$ il massimo aveva valore $(0;1)$

4) Infine abbiamo calcolato $f''(x) = \frac{-2x}{(1+x^2)^2}$: abbiamo notato che aveva caratteristiche corrispondenti al grafico verde. [MAX$(-\frac{1}{\sqrt{3}}, y')$ MIN$(\frac{1}{\sqrt{3}}; y'')$. $\lim_{x\to\infty} f''(x) = 0$] $\rightarrow$

Figure 5.3.39(a) - Solution to Problem 1 and its justifications by group F (recto).

"Dear Lorenzo and Francesca,
we found with difficulty the solution to your problem.

1. We translated y-axis for convenience, making the origin coincide with x_0 where two functions had value 0 and the third one had $f(x_0)$ as maximum.
2. We supposed that the red function was $f(x)$ and that it corresponded to the function $f(x) = \arctan(x)$, restricted to an interval.
3. By differentiating it, we noticed that the blue function intuitively corresponds [to] $f'(x) = \dfrac{1}{1+x^2}$. This hypothesis was validated through the calculation of limits and maximum/minimum points: in $f(x)$ there were no maximum point; in $f'(x)$ the maximum point had value $(0;1)$.
4. Finally, we calculated $f''(x) = \dfrac{-2}{(1+x^2)^2}$: we noticed that its characteristics were correspondent to those of the green graph. [MAX$(-\dfrac{1}{\sqrt{3}}; y')$

$\mathrm{MIN}(\frac{1}{\sqrt{3}}; y')$. $\lim_{x\to\infty} f''(x) = 0]$"

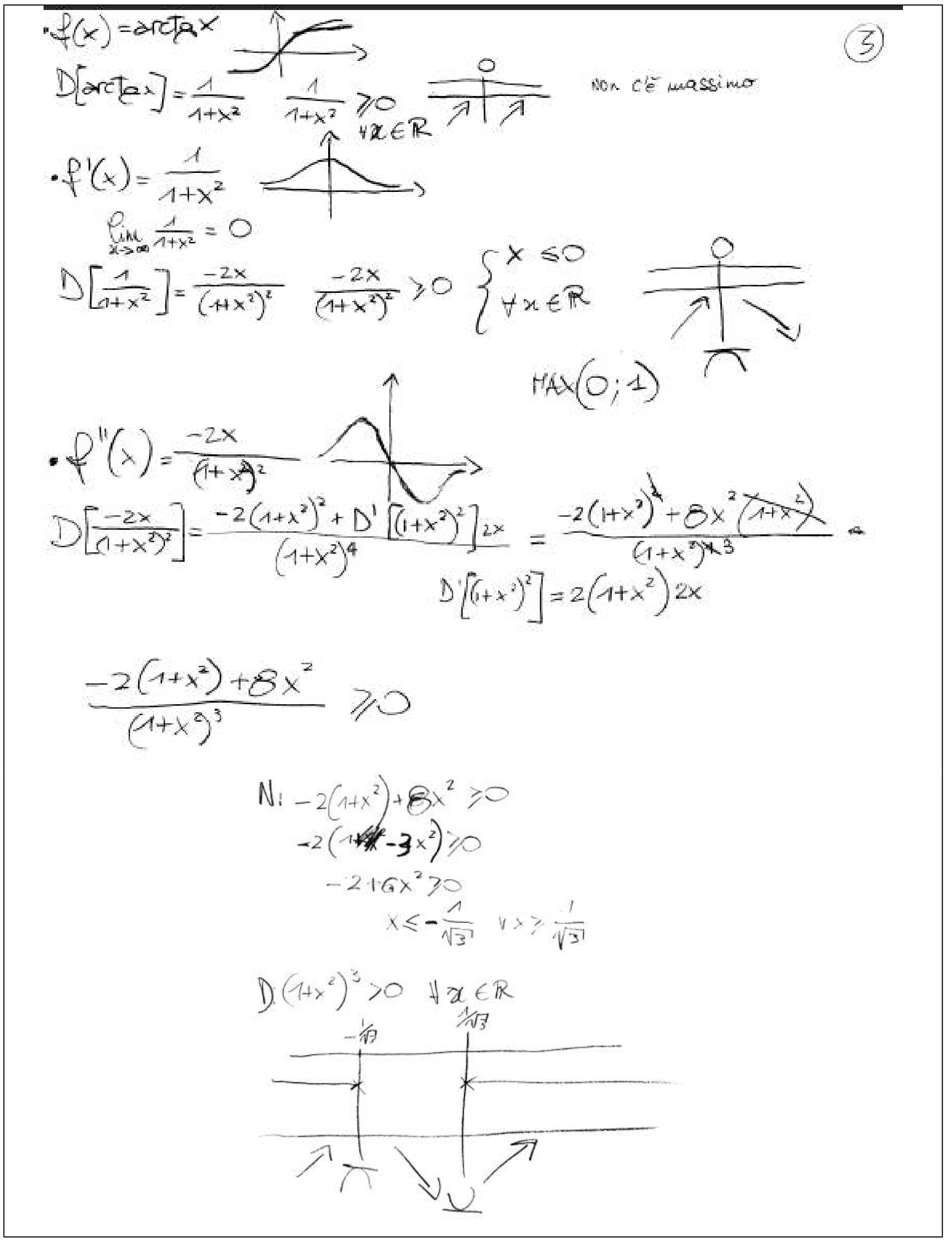


Figure 5.3.39(b) - Solution to Problem 1 and its justifications by group F (verso).

The students of group F do not exploit the global potentiality of the given graphs, although indirectly the expression and the formulas they use are universal pointwise. Actually, they do not get to grasp any property of the represented functions, and they do not formulate any relation between the properties of the graphs. Their algebraic method entails the activation of several semiotic resources. Beside the graphs, they use symbolic expressions, symbols, sketches and schemes. To justify their solution, this group trusts in algebraic calculations. The fact of using the differentiation formulas and of studying the derivative sign is their warrant of correctness. Unfortunately, as they realize with the second problem, there exists very few cases in which this technique can be used. In very few cases, indeed, you are able to recognize the analytic expression of a curve under the form $y = f(x)$. Moreover, you have to guess the starting function, otherwise lots of your calculations go wasted. Also the risk to commit computation errors or to make confusion among the analytic expressions is really high. Further, the students can automatically follow this method, thanks to the confidence given by algebraic formulas, without being actually capable to interpret what they find. In conclusion, this technique does not really lead the students to adopt a certain perspective on the involved functions. Not even a universal pointwise perspective is effectively activated, because it usually remains implicit in the algebraic calculations.

Influences of V.'s praxeology

The presence of the tangent method is not a case. Indeed, V. has given a great importance to the tangent line definition within the praxeology developed in the classroom. The relation between the derivative concept and the tangent gradient has not been simply presented as a property. It has been the starting point for constructing the technique to find the derivative of a function (see the subparagraph "Construction of the technological-theoretical block for $\mathcal{T}_{m_{tg}}$" in Paragraph 4.5.2).
The local perspective that the teacher has introduced in the previous lessons seems not to be very strong in the students' productions. Only group D, using the tangent as a tool, seems to implicitly exploit a local perspective on the function they want to differentiate. But such an approach does not become a way to explicitly highlight local properties on this function. Also in solving Problem 2, which has been especially designed to foster a local perspective, only group C bases on some local remarks.

5.4 Activity 2

Activity 2 is specifically designed around the condition of tangency and its algebraic translation.

5.4.1 Description

As the first proposed activity, also activity 2 lasts one hour. It is composed of two problems. The students are required to solve them and to give a written justification of their reasoning.
The first problem is given in two possible versions, A and B, whereas the second problem is the same for all (see "Scheda A" and "Scheda B" in Appendix D). Problem 1 is based on the following type of task: determining the equation of a function that intersects x-axis in two given points, knowing the tangent lines in these two points. This type of task

is proposed in two different formulations: graphical (version A) and symbolic (version B). In Problem 2, instead, the students are supposed to find the algebraic condition for the tangency of two curves. More precisely, they have to determine the exact value of a real parameter k for which a function g, whose equation depends on k, is tangent to another given function f, and then find the coordinates of the tangency point. These are two types of task and the second depends on the first one.
General prerequisites are

- functions, in particular in the algebraic register of representation;
- pointwise properties of functions, especially the zeros;
- elementary functions $y = mx + q$, $y = ax^2$ and $y = \ln x$ and related properties;
- parameters and study of their variation.

Specific prerequisites are

- the concept of derivative of a function in a point;
- the equation of the tangent to a function in a point;
- local properties of functions, such as the continuity and the differentiability in a point.

5.4.2 *A priori* analysis

As for the activity 1, our analysis is based on the three lenses:

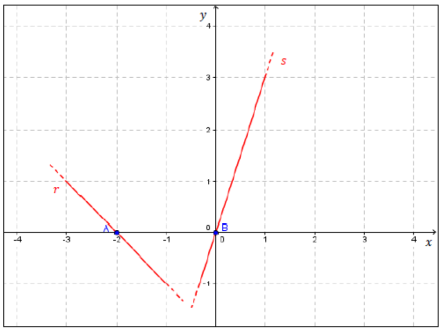
1. the praxeologies adopted in order to solve the proposed problems;
2. the perspectives assumed on the involved functions;
3. the semiotic resources activated in order to solve the activity.

These three great components influence the design of both the problems. They are grounded on the definition of derivative as the gradient of the tangent line:

$$f'(x_0) = m_{\text{tg in } x_0}. \tag{5.1}$$

The relation (5.1) is local on f and, at the same time, pointwise on f'. Thus, it is a *l-p* relation. This shift of perspectives by passing from a function to another may represent a difficulty. Another delicate fact may be that students are usually asked to calculate the derivative of a given f in a certain point x_0 in order to find the gradient (and then the equation) of the tangent line in that point. Here, the type of task is somehow the dual one: known the gradient of the tangent in x_0, finding $f'(x_0)$ (and then the analytic expression of f).
In this activity, we are interested in the algebraic formulation and use of the relation (5.1). Therefore, on the contrary of activity 1, we do expect an algebraic work on the involved functions.

Problem 1

About the function f you know that its graph - passes through the points A and B; - is tangent in A to the straight line r and in B to the straight line s. Write a possible equation $y = f(x)$ for this function. 	About the function f you know that - $x_1 = -2$ and $x_2 = 0$ are among its zeros; - the curve whose equation is $y = f(x)$ is tangent in x_1 to the straight line $r : x + y + 2 = 0$ and in x_2 to the straight line $s : 3x - y = 0$. Write a possible equation $y = f(x)$ for this function.

This problem is inspired by some similar problems which are proposed on Sasso's textbook (Sasso, 2012). Some pointwise information (about the zeros) and local information (about the slope) are given on an unknown function f. Version A of Problem 1 provides these information in the graphical register: in a Cartesian reference system, two points A and B are detected on x-axis and two straight lines r and s are drawn through these points. A grid is visible and the scale is specified on the axis. In the text all the indications have a descriptive role, in the natural language register, with reference to the figure. Thus, the semiotic resources used in version A are the graphical drawing together with a written explication. Version B of Problem 1 uses the symbolic resource to give the information about f: two zeros of f are written in symbols "x_i = value" and the equations of the two tangent lines r and s are given in the form $ax + by + c = 0$. Nonetheless, the same semiotic resource is expected in the solving process. It essentially consists of algebraic symbols in order to

1. find a generic analytic expression of f: $f(x) = ax^3 + bx^2 + cx + d$, with a, b, c and $d \in \mathbb{R}$;

2. write the given conditions
$$f(-2) = 0 \quad \text{and} \quad f(0) = 0$$
$$m_r = f'(-2) = -1 \quad \text{and} \quad m_s = f'(0) = 3;$$

3. find the generic analytic expression of the derivative of f': $f'(x) = 3ax^2 + 2bx + c$;

4. exploit the given conditions and solve the system among them

$$\begin{cases} 0 &= -8a + 4b - 2c + d \\ 0 &= d \\ -1 &= 12a - 4b + c \\ 3 &= c \end{cases} \qquad \begin{cases} a &= 1/2 \\ b &= 5/2 \\ c &= 3 \\ d &= 0 \end{cases}$$

5. express the final result: the analytic expression for the involved function $y = \frac{1}{2}x^3 + \frac{5}{2}x^2 + 3x$.

We chose this problem because it stimulates the shift from a local information on f (the gradient of the tangents $m_r = -1$, $m_s = 3$) to a pointwise information on f' ($f'(-2) = -1$ and $f'(0) = 3$). Notice that the use of the relation (5.1) is not indicated or evident in the statement. It certainly represents a great cognitive jump, which is essential to solve the problem.
Therefore, let us come to the reason why we proposed two versions of this problem. The question we posed was: is there a semiotic resource that best helps this shift in perspectives, by passing from f to f'? On the one hand, the graphical drawing resource may help the students imagining what kind of function may satisfy the given conditions. We expect that they easily exclude the parabola (see step 1), for example. The students are also easily fostered to trace a graph for this function. The fact of drawing the graph so that it is tangent to some drawn lines may help the students activating the *l-p* relation (5.1)? On the other hand, the symbolic resource may foster the students to think more "algebraically". It might be immediately clear for them that a solution in the symbolic register is needed. Nevertheless, within this context, is the *l-p* relation (5.1) more immediate? We are interested in investigating these questions at all levels, so the two versions of Problem 1 has been given in a uniform way to high-level groups, medium-level groups, low-level groups.

Expected praxeologies in solving Problem 1

The type of task "determining the equation of a function that intersects x-axis in two given points, knowing the tangent lines in these two points" is given in two different formulations. However, the technique for solving it is the same: it consists in the steps 1-5 which are summarized above.
The difference occurs at the technological level, namely in step 2. Let us explain the technology step by step. First of all, only two zeros of the function f are given. Nevertheless, f can not be a parabola in the form $y = ax^2 + bx + c$, because at the points A and B, which would be symmetrical with respect to the vertical axis, the tangent lines r and s are not symmetrical. For this reason, we need a function of at least third order. It will have another zero x_3, though it is not given among the data.
We expect two different justifications in the second step, at the level of interpretation and algebraic conversion of the given conditions. The students who have version A are expected to do some graphical work to find the coordinates of the points A and B, and

the gradient of the straight lines r and s. Since A and B belong to the graph of f, with a conversion from the graphical to the algebraic register, the students obtain $f(x_A) = y_A$ and $f(x_B) = y_B$. To graphically determining the gradient of r and s, they have to choose two points on each of them. Since on r they have already A and on s they have already B, it is enough to choose a second point on each straight line, by using the grid and the scale on the axis. Then, by applying the formula $m = \dfrac{\Delta y}{\Delta x}$, they finally find m_r and m_s.
Instead, with version B, students are expected to give to the symbolic condition "x_i is a zero of f" an algebraic interpretation as $f(x_i) = 0$. Then, to know the gradient of r and s, it is sufficient to find the explicit form $y = mx + q$ of their equations.
Following different ways, the students get the four algebraic conditions: $f(-2) = 0$, $f(0) = 0$, $m_r = -1$, $m_s = 3$. By recalling the relation (5.1), they can finally interpret $m_r = -1$ and $m_s = 3$ as $f'(-2) = -1$ and $f'(0) = 3$.
Further, the differentiation formula for the elementary function $x \to x^n$ and the theorem on the sum of derivatives guarantee the calculation of the derivative f' in step 3.
Then, among all the functions with the generic analytic expression of f (see step 1), the one that satisfies simultaneously all the conditions (see step 2) will be the function we are searching for. This is the reason why we impose and solve the system in step 4.
This system has a unique solution, since it is composed of four equations in four parameters a, b, c and d. This is the justification for obtaining, by replacement in the generic analytic expression of f (step 5), a unique equation $y = f(x)$.
The supporting theory is quite wide. We have polynomial functions theory, analytic geometry (e.g. coordinates and straight lines), the definition of the derivative in a point as the gradient of the tangent line to the function in that point (relation (5.1)), differentiation formulas and theorems (with proof), systems theory.

Notice that the solution to this problem is not unique and that no condition about continuity or differentiability is given about f in the text. Thus, also other resolution processes are possible. For example, it is possible to define a piecewise function, which can be continuous or not. The banal example is the piecewise function composed of the two given straight lines. A more articulated answer involves a piecewise function composed of two parabolas in the form $y = ax^2 + bx + c$ respectively tangent to the two given straight lines at the two given points.

Problem 2

We say that two curves are tangent in one of their common point if and only if they have the same tangent line in that point. Establish for which exact value of the real parameter $k > 0$, the curves with equations

$$f(x) = kx^2 \qquad \text{and} \qquad g(x) = \ln x$$

are tangent. What are the coordinates of the tangency point?

This problem is inspired by french researches, namely Gueudet and Vandebrouck's study about technologies and evolution of teachers' practices (Gueudet & Vandebrouck,

2011). A first remark is that the students have not studied the tangency between two curves yet (see Table 5.1). Thus, the geometrical definition is given as introduction to the problem. By solving this task, the students are supposed to carry out, on their own, the algebraic tangency condition.
The data consist of a family of parabolas, with vertex in the origin $f(x) = (f_k(x)) = kx^2$, and the function $y = \ln x$. The index k for the family of functions does not appear to avoid further complications. The first question is determining the exact value of k for which the curves having the given equations are tangent. The word "exact" is underlined because the students know the bisection process for approximating the abscissa of an intersection point. The adjective exact should foster them to think about another way that allows them to find the value of k without any uncertainty.
Notice that it is necessary to move from a geometrical frame to a functional frame. This change may be supported by the availability of the analytic expressions of the functions f and g that represent the curves. The *l-p* relation (5.1) has to be applied on each representative function, f and g. Further, the two obtained information must be put together, finding a unique pointwise relation, which depends only on the derivatives of these functions in the abscissa of tangency x_0. While determining k, one gets also x_0. So, the second question consists simply of finding the related ordinate.
From a semiotic point of view, the symbols resource prevails both in the statement and in the solution.
We chose this problem because of the interesting shift in perspectives we have just described above. The important fact is that the students have to do this shift alone, without the teacher's help, but with all the knowledge and competences that the teacher has built in the classroom.

Expected praxeologies in solving Problem 2

The types of task are two: determining the exact value of the parameter k for which a function f, whose equation depends on k, is tangent to a given function g; finding the coordinates of the tangency point. The technique is based on the combined use of the relation (5.1) on both f and g. First of all, it is needed to express the algebraic conditions for which the two curves

1. pass through the same point of abscissa x_0:

$$f(x_0) = g(x_0) \Rightarrow kx_0^2 = \ln x_0;$$

2. are tangent at x_0:

$$f'(x_0) = m_{\text{tg in } x_0} = g'(x_0) \Rightarrow 2kx_0 = \frac{1}{x_0}$$

Then, by imposing and solving the system between the conditions (1) and (2), one obtains $k = \dfrac{1}{2e}$ and $x_0 = +\sqrt{\dfrac{1}{2k}} = \sqrt{e}$. By replacing $x_0 = \sqrt{e}$ in the analytic expression of f or

g, one gets the ordinate of the tangency point $y_0 = \frac{1}{2}$.
The technology behind the condition (1) relies on the fact that two functions pass through the same point $P(x_0, y_0)$ if and only if P belongs to both their graphs. Thus, $f(x_0) = y_0$ and $g(x_0) = y_0$. For the transitive property, one obtains the condition (1): $f(x_0) = g(x_0)$.
The warrant for the condition (2) is the combined recall and application of the relation (5.1) with the definition given in the text for two tangent curves. The tangent in x_0 has to be the same for both the functions. In particular, the gradient $m_{\text{tg in } x_0}$ is the same. It means that $f'(x_0) = m_{\text{tg in } x_0}$ and $g'(x_0) = m_{\text{tg in } x_0}$. For the transitive property, one obtains the condition (2): $f'(x_0) = g'(x_0)$. It is necessary to differentiate the power function $x \to x^2$ and the logarithmic one $x \to \ln x$ and then to replace x with x_0. With two equations in two unknowns (k and x_0) the system is determinate and it allows to find at the same time the exact value of the parameter and the abscissa of the tangency point.
The theory of Problem 2 includes some elements of analytic geometry (e.g. coordinates and straight lines), the definition of the derivative in a point as the gradient of the tangent line to the function in that point (relation (5.1)), differentiation formulas, systems theory.

5.4.3 *A posteriori* analysis

Let us explain how the a posteriori analysis is conducted. First of all, we focus on the activated praxeologies and in particular on the justifying discourse that accompanies the implemented method. We are interested in how the students make the cognitive jump in recalling and applying the relation $f'(x_0) = m_{\text{tg in } x_0}$ (5.1). Although we ask the students to write down their justifications, they usually delegate the justifying role to symbolic manipulations and computations. In the written data at our disposal, the warrant for the problem solution are the algebraic symbolic steps themselves. Therefore, we prefer to analyse the videotaped groups, since we have access to the justifying speech which arises from the discussion.
In the videos analysis, we focus on the perspectives activated by the students, especially when they have to use the *l-p* relation (5.1). Moreover, we are interested in the semiotic resources the students use to support their perspectives. In particular, do the different semiotic resource used in the first problem have some influences in how the students activate the *l-p* relation (5.1)?

M.'s students

Table 5.6 provides a general overlook on the work done by the groups A, B, C, D, E, F, G and H.

	Use of the relation (5.1)		Semiotic resources
Problem 1	system	A,B,D,E,F,G,H	graph, symbols
		C	symbols
	piecewise f		
Problem 2	system	A,B,G	graph, symbols
		C,D,E,F,H	symbols

Table 5.6 - M.'s students and activity 2.

Notice that all the groups manage to use the relation $f'(x_0) = m_{\text{tg in } x_0}$ (5.1) to solve Problem 1. They all impose it as a condition of a system of 4 equations in 4 unknowns. Only two groups over eight make a first attempt with a non-cubic function. Group A tries with a parabola, whereas group H tries with a homographic function making it pass through A and B and be tangent to r and s.
As far as the semiotic resources are concerned, the four groups having the graphical version of Problem 1 (D, E, F and G) do not seem to make any attempt of drawing a function in the given graphical situation. They do not seem to exploit this graphical resource. On the contrary, groups A, B and H having the symbolic version of Problem 1 find it useful to draw the situation in a graphical environment. Within the resolution of Problem 2, where the students are left free, only three groups over eight try to give a graphical representation on the proposed task.
M.'s students seem to be used to work in a symbolic register without necessarily resorting the graph. The presence or the absence of the graph in the text does not lead to really different reasoning processes.

We are going to analyse the two videotaped groups (A and G), and in particular the moment of recalling and application of the relation (5.1). We are interested in the involved perspectives and the semiotic resources which help its activation. To refer to each component of the groups, we keep using the same students' names (S1, S2 and S3) of Activity 1 (see the subparagraph "M.'s students" in Paragraph 5.3.3).

The medium-level group A works on the symbolic version of Problem 1. They approach its resolution by tracing the Cartesian axis, but actually they do not draw anything in this plane. They immediately move on to algebraically translate the given conditions, and they start from the tangency ones "the curve whose equation is $y = f(x)$ is tangent in x_1 to the straight line $r : x + y + 2 = 0$ and in x_2 to the straight line $s : 3x - y = 0$". Students S1 and S2 collaborate in the algebraic formulation of these two tangency conditions. Let us analyse a brief extract from the video.

	What the students say and do	Perspectives	Semiotic resources	on f, f'
1	S2: "*It's sufficient to do the derivative and to impose it equal to -1 so that we get the angular coefficient.*"			
2	S1: "*Wait...*"			
3	S2: "*The derivative in x.*" (He points his left index downwards, as in Fig. 5.4.1)	pointwise	speech indicators + gesture	f'
4	S1: "*Exactly. So y' in... what is the point? It would be: first derivative in the point must be equal to angular coefficient.*"	pointwise	speech indicators + symbols	f'
5	S2: "*x_2 is equal to 0 and the angular coefficient will be...* 3." (S1 writes "$y'(-2) = -1$" and "$y'(0) = 3$", see Fig. 5.4.2)	pointwise	symbols	f'

Figures

Figure 5.4.1 - S2's pointwise gesture to accompany the words "the derivative in x" [*3*].

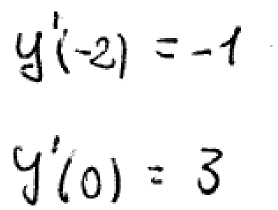


Figure 5.4.2 - S1 writes in symbols the given tangency conditions.

In this first part of the discussion about Problem 1, S2 immediately proposes the technique: to differentiate and then to impose the derivative equal to the gradient of the given tangent line [*1*]. Then, he adds "in x" [*3*] and specifies his pointwise perspective on f' with a pointwise pointing gesture (Fig. 5.4.1). It recalls M.'s gesture for the derivative in x_0, when she distinguished a particular x_0 from the generic x.
The algebraic translation of the tangency conditions comes quickly (Fig. 5.4.2), but how and on what function applying them require a more deep reflection from the group.
When they wonder of what function $y = f(x)$ taking into account, S1 sketches the situation in the Cartesian plane. She writes $y = x(x + 2)$, but S3 suggests to consider a generic function, so she writes $y = ax^2 + bx + c$. After a general remark I made about the text with all the students, namely that x_1 and x_2 are f zeros but not the only ones, they restart the calculations with a cubic function $y = ax^3 + bx^2 + cx + d$.
We can notice that they consider a graphical situation, but they do not really exploit it. The main activated semiotic resource is the symbolic writing.

It happens also in approaching Problem 2: S1 draws the given functions in a Cartesian plane (Fig. 5.4.3), but then they solve the task only through symbolic steps.

In particular, it is S1 that after having drawing the graphs opens the discussion on the tangency condition for two curves.

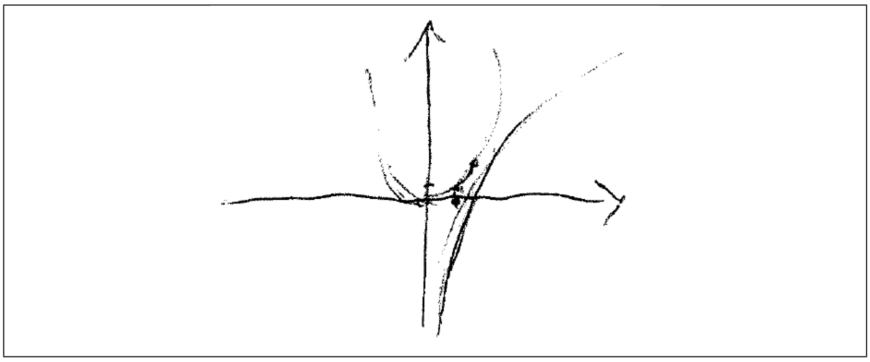


Figure 5.4.3 - Group A's graphical conversion of Problem 2.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
6	S1: "*Practically, their first derivatives must have the same value <u>in the point</u>. Because they must have the same tangent line. But we do not know <u>in what point they meet</u> each other, since it depends on how k changes. [...] In a point that we do not know yet... it is their <u>intersection point</u>.*"	pointwise	speech <u>indicators</u>	f', g', f and g
7	S2: "*You have to know the tangency point...*"			
8	S1: "*We have to find <u>their common point</u>, and then to impose that <u>in that point</u> the tangent is the same.*"	pointwise	speech <u>indicators</u>	f and g
9	S2: [...] "*The coefficients*" (he points to the derivatives S1 has found, Fig. 5.4.4) "*have to be equal <u>for a point</u>.*"	pointwise	speech <u>indicators</u>	f' and g'
10	S1: "*Well, let's start imposing them equal <u>in x</u>.*" (She writes $2kx = \frac{1}{x}$, then $2kx^2 - 1 =$ 0) "*So this is the first condition. The other one is this.*" (She writes $kx^2 - \ln x = 0$)	global(=univ. pointwise)	speech <u>indicators</u> + symbols	f', g',f and g

Figures

$$y' = 2kx$$
$$y' = \frac{1}{x}$$

Figure 5.4.4 - Extract from group A's solution and justification to Problem 2.

The group works directly via symbols on the expression in x of the functions and the derivative functions. They make some pointwise considerations about the fact that they do not know the abscissa of the common point of f and g [*6-9*]. S1 proposes a universal pointwise perspective on the involved functions and on their derivatives [*10*]. Indeed, she suggests to start using their algebraic expressions in x.
We can observe that not even in this second problem the students need to resume a local perspective on the involved functions. All passes through algebraic symbols.
They find $kx^2 = 1/2$ from the first condition and substitute it in the second one. They correctly obtain the coordinates of the common point $(\sqrt{e}, \frac{1}{2})$ and S1 checks if it is possible on the graph. So the graph is used by this group with a role of control.

High-level group G works on the graphical version of Problem 1. It allows them to say immediately that the solution cannot be a parabola. S2 says "*It cannot be a parabola, otherwise they [the straight lines r and s] would have the same...the same thing... if it was a parabola.*". He supports his words by posing his hands symmetrically on the sheet (see Fig. 5.4.5).

Figure 5.4.5 - S2's symmetrical gesture of the hands to justify why the solution cannot be a parabola.

Then, the group concentrates on the meaning of the given conditions. They come to find the relation (5.1).

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
1	S1: "*We have that here the derivative is -1...$-x$...*" (He points to A)	pointwise	gesture	f'
2	S2: "-1."			
3	S1: "*And here*" (he generally points to the straight line s) "*is -3, no! $+3$.*"			
4	S3: "*Here where? What do you indicate with "here"?*"			

5	S2: "*The angular coefficient is this one*" (S1 and S2 retraces simultaneously s, as in Fig. 5.4.6) "*so the derivative* <u>*in that point*</u> *is* 3."	pointwise pointwise	gesture speech <u>indicators</u>	f f'
6	S2: "*So we said that the derivative in A is* $-x$" (he writes "$y'(A) = -x$") "*Let's write 1? The number?*"	global(=univ. pointwise)	symbols	f'
7	S1: "*There you have to put* -1."	pointwise	symbols	f'
8	S2: "*Let's do so. Let's write D...*" (he writes "$D'(A) = -1$" and "$D'(B) = +3$", see Fig. 5.4.7)	pointwise	symbols	f'

Figures

Figure 5.4.6 - S1 and S2's common gesture on the tangent line s.

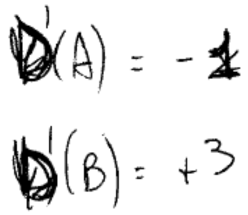


Figure 5.4.7 - Symbolic expression of the relation (5.1).

The group perspective on the involved function is pointwise, without any local implication, since the tangency condition is immediately translated in terms of derivative [*1*]. The algebraic conditions on the derivative function gets a pointwise connotation for the students [*6-8*] (Fig. 5.4.7).
In order to exploit these algebraic conditions (Fig. 5.4.7), S2 proposes to use the incremental ratio. Thus, they impose the system in Fig. 5.4.8.

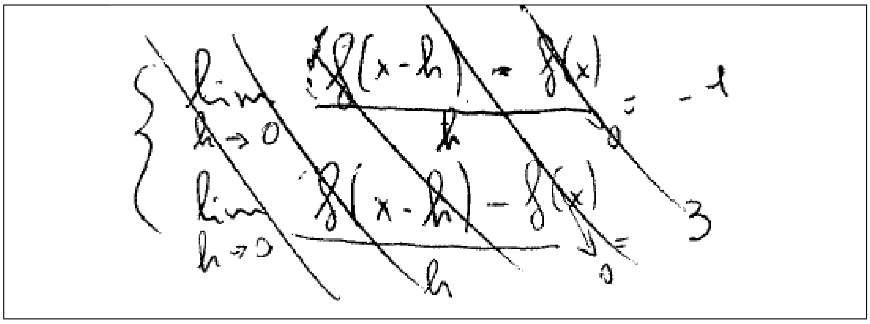


Figure 5.4.8 - S2's system using the limit of the incremental ratio.

We can notice that the perspective of $S2$ is still universal pointwise, as in his comment [*6*]. The incremental ratio indeed are calculated in the generic abscissa x. This way leads them in a more and more intricate and meaningless manipulation of symbols.
S1 finally proposes to write the function as $y = ax^3 + bx^2 + cx$ since it must pass through the origin. Then they impose the pointwise condition on f, by making it pass through the point $(-2, 0)$, and the pointwise constraint on the derivative, by imposing $f'(0)$

equal to 3. However, instead of imposing the forth pointwise condition on the derivative, namely $f'(-2) = -1$, they suppose that the function has an inflection point in $(-2, 0)$. They impose that the first derivative has a maximum. They are locally reasoning on the function f and on its derivative f', but they do not get the real behaviour of the function in the neighbourhood of -2.
The teacher, who is following the group's reasoning, expresses a doubt [*9*].

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
9	T: "*There is a thing that doesn't convince me. You have decided to have an inflection point here, but it is an inflection point that not necessarily...*" (She retraces the tangent with her finger) "*You haven't imposed that this inflection point has necessarily this tangent.*"	local	gesture	f
10	S1: "*No, I haven't imposed it. It must be* $f'(-2)$*...*"	pointwise	symbols	f'
11	T: "*If you obtain* -1*, you got it. But I'm not so sure...*"	pointwise	symbols	f'

It is the teacher who makes the students notice that their local reasoning on f in the neighbourhood of -2 is not the required one [*9-11*]. When the students verify if the obtained function satisfies the local condition on f, through the pointwise condition on f', namely $f'(-2) = -1$, they find an incongruity.
Afterwards, they correctly solve the problem by imposing a system of only pointwise conditions on f and its derivative.

Also in Problem 2, group G has the capability to get very quickly to the algebraic condition to apply, but they get lost in algebraic calculations, which are often useless and meaningless. In this case, they immediately say that "*the derivatives of* $\ln x$ *and of* kx^2 *must be equal*" and establish a first relation between k and x: $2kx^2 = 1$. Then, they notice that they "*have to find the point in which they meet each other [...] the intersection between them*". They write this constraint as $\ln x = kx^2$. Nevertheless, they try to solve this equation both algebraically, introducing an exponential, and graphically (see the extracts in Fig. 5.4.9). S1 surrenders because he is not able to obtain x.
The group manages to solve the task thanks to another teacher's intervention. She makes them aware that they already have the two required conditions and that it is enough to put them together.
The example of high-level group G shows us that often it is not sufficient adopting the right perspective on the involved functions. Moreover, it is not enough converting correctly a pointwise, global or local information in an algebraic form. It is necessary developing a good dialectics among the arisen perspectives to give a global sense to the

obtained algebraic conditions.

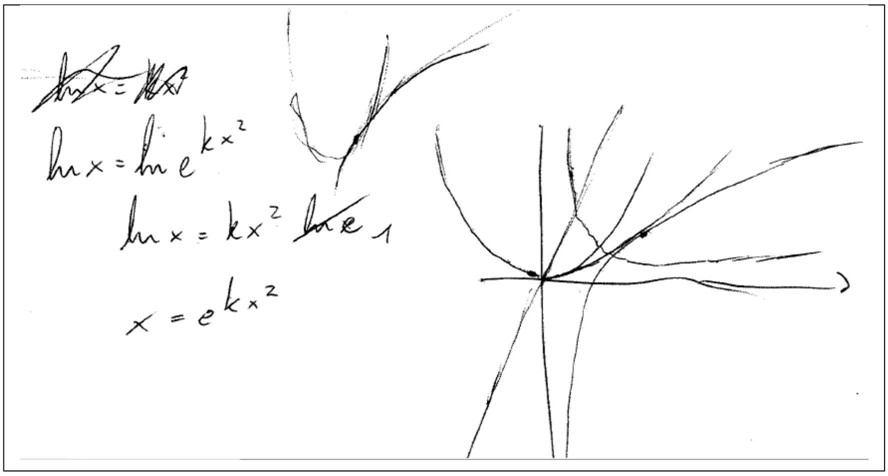


Figure 5.4.9 - Extracts from group G's solution and justification to Problem 2.

Influences of M.'s praxeology

M.'s students do not need any local consideration on f to activate on the derivative function both a universal pointwise and a pointwise perspective. We notice that often, when they algebraically write an equality in x using the analytic expressions of the involved functions and their derivatives, they can see x both as a universal pointwise sign or as a pointwise sign, according to their goal. They use other semiotic resources to specify if x is actually a special x. For example, group A uses a gesture which recalls M.'s gesture to indicate a specific point x_0. Group G refers to the graph in order to specify that the values of x in which they are calculating the derivative are particular values, even if they continue to use the sign x. It might be due to the way in which M. has introduced the shift from the pointwise sign x_0 to the generic sign x, talking about derivative function (see subparagraph "Elaboration of the algebraic technique, starting from the technological speech" in Paragraph 4.3.3). It was a matter of syntactical writing, for convenience. The students seems to have internalized the capability of moving from a particular x or a generic x depending on what they are searching for.

M.G.'s students

Table 5.7 provides a general overlook on the work done by the groups A, B, C, D and E.

	Use of the relation (5.1)		Semiotic resources
Problem 1	system	A,B,C,E	graph, symbols
		D	symbols
	piecewise f		
Problem 2	system	$\sim$B	graph
		$\sim$C,D,E	symbols

Table 5.7 - M.G.'s STUDENTS AND ACTIVITY 2. THE SIGN $\sim$ DENOTES THAT THE CORRESPONDING TECHNIQUE/RESOURCE HAS BEEN PARTIALLY USED BY THE GROUP TO SOLVE THE PROBLEM.

All the students have solved Problem 1, but only the two high-level groups D and E have completed the solution of Problem 2. The technique used by every group for solving the first task consists of the system. The symbolic version of Problem 1 was given to groups A, C and D. Only group D relies exclusively on symbols, whereas groups A and C draw the graphical situation. Thus, there is no real difference between those who have had the graphical version and those who have had the symbolic one. Almost all feel the need for supporting their reasoning with a graphical resource.
The three low- and medium-level groups A, B and C have firstly approached Problem 1 through a parabola. High-level groups D and E have directly started with a cubic function.
As for the second problem, group B has only explored it graphically, group C has symbolically sketched it out correctly, and groups D and E have symbolically solved it. The symbolic resolution prevails for this second problem.

We are going to analyse group A that is the one we could videotape. They work on the symbolic version of Problem 1. They find some difficulties in solving it, so they do not manage to approach Problem 2. S1, S2, S3 and S4 denote the same students as before in Activity 1 (see the subparagraph "M.G.'s students" in Paragraph 5.3.3).

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
1	S1: "*If it is tangent it means that it touches...*" (He generally points to the straight line r)			
2	S2: "*In that point it touches the straight line.*" (She points to x_1 on x-axis, Fig. 5.4.10)	pointwise	speech indicators + pointing gesture	f
3	S1: "*In this point? Are we sure?*" (S2 reads again the text)	pointwise	speech indicators	f

4	S2: "*So it must pass through both x_1 and x_2.*"	pointwise	speech indicators + symbols	f
5	S2: "*Exactly.*"			
6	S1: "*The curve is tangent in that point to the two straight lines. So it means that it cannot do... it cannot pass through... Then it probably does so...*" (He imagines to trace f tangent to r, Fig. 5.4.11)	pointwise local	speech indicators gesture on graph	f

Figures

Figure 5.4.10 - S2's pointing gesture on the graph.

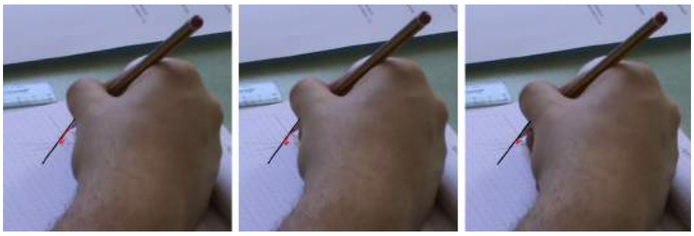


Figure 5.4.11 - S1 tries to locally trace an imaginary curve which is tangent to r in x_1.

The students deduce from the given data the pointwise information for f: it has to cut x-axis in the given abscissas [*1-5*]. They also put the data in a graphical frame. With a local gesture (Fig. 5.4.11), S1 imagines how the curve could pass through x_1. He starts to wonder what the tangency condition means. This local gesture is made without any accompanying speech that make the perspective explicit. It occurs also later [*18*], when the teacher intervenes in the group discussion about a possible parabola respecting the given conditions.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
7	T: "*You have supposed that...*"			
8	S2: "*The curve is a parabola, and we are searching for the parameters.*"			

9	T: "*Ok. So you have a parabola that <u>passes through two points</u>. <u>Try to draw it</u>.*"	pointwise global	speech <u>indicators</u> graph	*f*
10	S1: "*Very ugly eh! However, it has to be tangent in that point... so... It is horrible!*" (He sketches it, see Fig. 5.4.12)	global	graph	*f*
11	T: "*Do you feel uncomfortable while drawing?*"			
12	S1: "*In what sense?*"			
13	S2: "*No.*"			
14	T: "*No?*"			
15	S1: "*Is it yes?*"			
16	S2: "*No, if you don't feel so.*"			
17	S1: "*Because, well... We know that it is tangent.*"			
18	T: "*Let me see how you move your hand.*" (He retraces the curve as in Fig. 5.4.11) "*You try to <u>flatten yourself on the tangency</u>...*"	local	gesture + speech <u>indicators</u>	*f*
19	S1: "*Do you feel any unease in doing it with your hand? You have two tangent lines a little bit crooked. [...] Where it would be the vertex, if the parabola passes through* -2 *and* 0*?*"			
20	S1: "*In* $x = -1$*.*"			
21	T: "*And how can you make <u>a vertex in</u>* $\underline{-1}$ *there?*"	pointwise	graph + speech <u>indicators</u>	*f*
22	S1: "*Yes... it's true.*"			

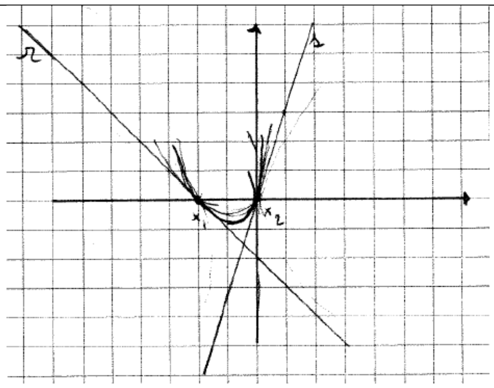


Figure 5.4.12 - S1's global sketch of the parabola.

The teacher wants them to draw the parabola in order to make them aware of its asymmetry [*9*]. However, the graphical work does not reveal to the students what she expects. S1 draws a parabola with vertical axis tangent to s in the origin, but not tangent in the same way to r in $(-2, 0)$ [*10*] (see Fig. 5.4.12). Nonetheless, they do not feel uncomfortable in drawing it [*11-16*]. S1 traces it again locally in a neighbourhood of x_1. At the same time, the teacher locally comments "*you try to flatten yourself*" [*18*]. Despite this local incongruity, the students do not really see any problem. Thus the teacher uses a pointwise consideration about the vertex in order to make them aware that this parabola does not fit with the given conditions [*19-22*].
Finally, the group chooses a cubic function and they impose the pointwise conditions making it pass through $(-2, 0)$ and $(0, 0)$. As for the tangency condition, they take their notes to revise the lesson about m-tangent. I read with them the sentence "m-tangent is the derivative of the function in x_0" they found on their notes. I finally help them in algebraically translating, within the example, this sentence "m-tangent is the derivative of the function $f(x)$ in the abscissa x of the tangency point."

Let us show you also the case of high-level group E whom written production is interesting because of the given justifications. They work on the graphical version of Problem 1. We can deduce from their written solution that they start supposing $y = ax^3 + bx^2 + cx + d$ and they impose the system. There is no further argumentation, only the algebraic computations and manipulations. The interesting fact is that, after all the calculation, in order to show and to verify that $y = \frac{1}{2}x^3 + \frac{5}{2}x^2 + 3x$ is the correct answer to the problem, they graphically study it. Thus, this group uses the graphical resource they have in the text with the role of control. They actually study only the domain, the zeros and the sign of $f(x)$ (see Fig. 5.4.13) and they put all the information on the graph given in the text, in order to see if their solution is compatible with the given situation (see Fig. 5.4.14).

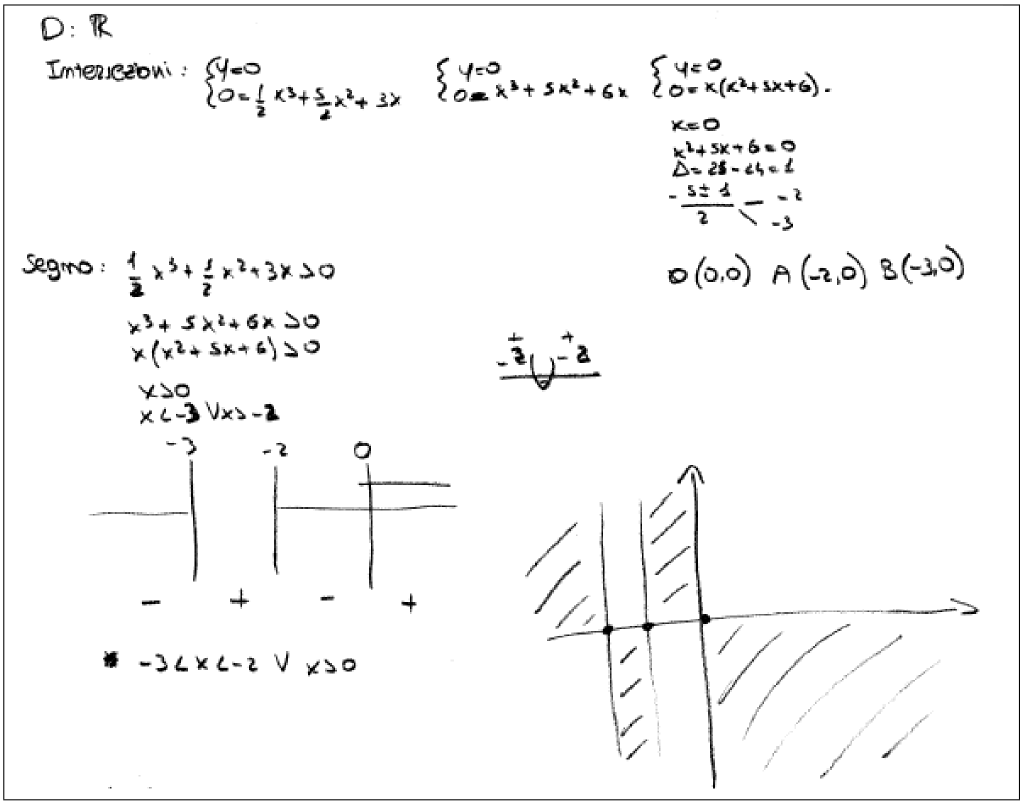


FIGURE 5.4.13 - EXTRACT FROM GROUP E'S JUSTIFICATION TO PROBLEM 1.

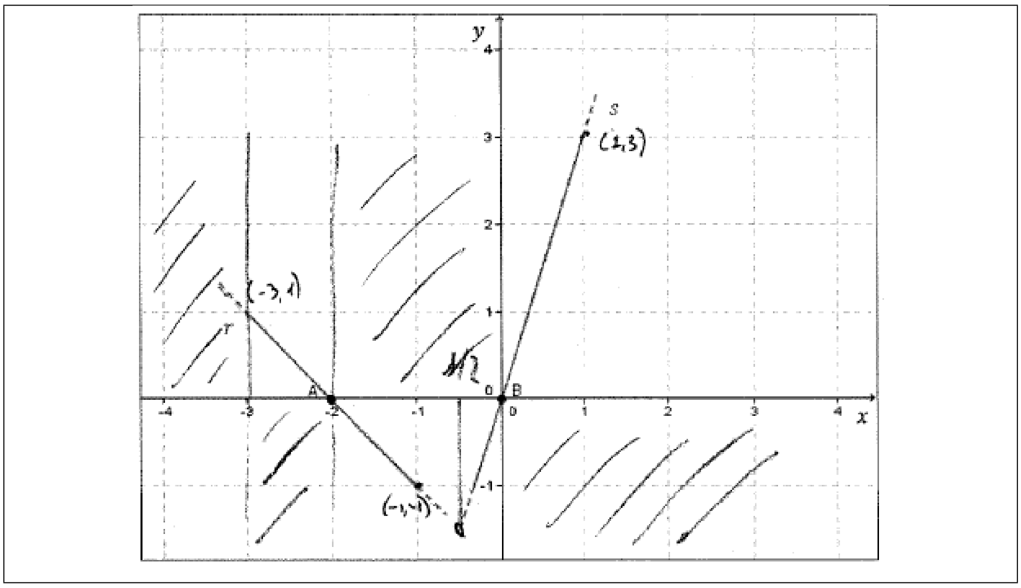


FIGURE 5.4.14 - EXTRACT FROM GROUP E'S JUSTIFICATION TO PROBLEM 1.

The perspectives they adopt on the function are then pointwise and global. The global perspective is activated in the sense of universal pointwise within the system calculation

and in the sense of global on intervals within the graphical verification.
It does not occur the same for Problem 2, where the students do not resort the graph, but work directly in a symbolic way. Nonetheless, they give a verbal explanation of their solving process (see Fig. 5.4.15). They write:

> "By imposing the system between the 2 functions we found their common point.
> By imposing the system between the 2 derivatives we find the tangency point, which we imposed to be equal to the common point.
> From this, we got x and we replaced [it]."

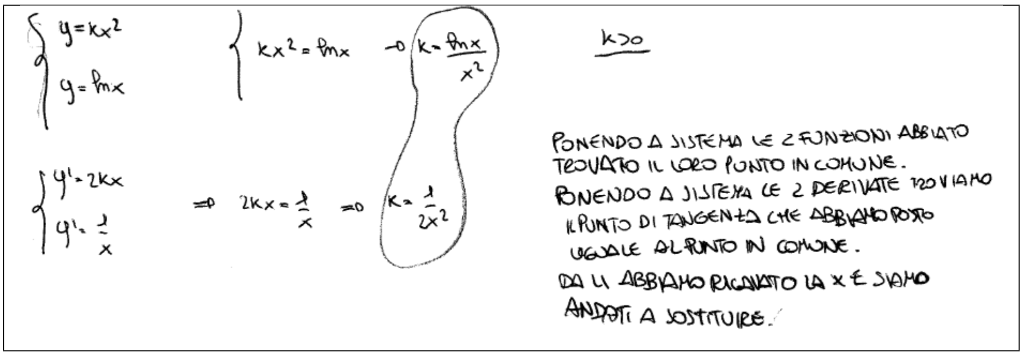


Figure 5.4.15 - Extract from group E's solution and justification to Problem 2.

From the symbols in the left part of the text, we can deduce a starting universal pointwise perspective on the involved functions and on their derivatives. Thanks to the written argumentation, however, we can detect a pointwise way to read the obtained symbolic relation, in particular the second one which is the *l-p* relation (5.1). In their words, x is a particular abscissa: the abscissa of the common point and at the same time of the tangency point. The relation is read only in a pointwise perspective. No local consideration on f and g seems to have helped finding the relation.

Influences of M.G.'s praxeology

M.G.'s low-level students seem to have difficulties in recalling and applying the relation (5.1). We think that their disorientation could be due to the fact of having seen m-tangent within different contexts, with different and apparently sequential expressions. Nonetheless, they try to make local considerations on the function, accompanied by gestures on the graph they have drawn to analyse the given symbolic situation. But such an implicit local perspective does not make them recall the praxeology seen in classroom on m-tangent. With high-level students, instead, who have grasped the relation (5.1), correctly recalled and applied it, it results a capability of writing the formula in a universal pointwise way, but then to read it in a pointwise perspective. The universal pointwise sign x in the symbolic equation involving the derivative becomes the particular wanted abscissa. It is the abscissa of the common point and at the same time of the tangency point. It may be an influence of M.G.'s praxeology when she speaks about derivative function (see subparagraph "The algebraic technique is shown on an example" in Paragraph 4.4.3). Indeed, she has stressed the fact that the calculation of the derivative in a generic point x gives a function of x, an expression depending on x. Nevertheless, she has also highlighted that this expression depending on x can be "measured" on a specific value of x, by substitution. Another M.G.'s influence that seems relevant involves the graphical method chosen by group E students to verify their solution. Indeed, M.G. has given a great importance to the graphical differentiation and study of function in the previous lessons (see subparagraph "The graphical technique is shown on an example" in Paragraph 4.4.3).

V.'s students

In the table below (Table 5.8), we give a general overlook on the work done by the different groups (A, B, C, D, E and F).

	Use of the relation (5.1)		Semiotic resources
Problem 1	system	A,B,E,F	graph, symbols
	piecewise f	C,D	
Problem 2	system	A,B,C,D	graph, symbols
		F	symbols

Table 5.8 - V.'s students and activity 2.

We can notice that all the groups manage to employ the relation $f'(x_0) = m_{\text{tg in } x_0}$ (5.1) for solving Problem 1. Two of them define a piecewise function: they are two high-level groups. Group C's solution is the composition of an exponential decreasing function and a parabola, whereas group D provides the composition of the two given straight lines as a solution. All the groups, except for group C, make a first attempt with a parabola passing through A and B and tangent to r and s.
The difference between the two statements (graphical or symbolical) does not lead to really different reasoning processes. Indeed, those who have not the graph find it useful to draw it as a first step in solving the problem. So all the groups use graphs and symbols as semiotic resources to solve the problem.
As for Problem 2, instead, there is a group who does not use the graph as a support, but only the symbols. The resolution of this problem turns to be quicker than that of Problem 1, probably because the students have already mobilized the relation (5.1) in solving Problem 1 and they have only to adapt it.

We are going to analyse the two videotaped groups (A and F), in order to have insights in their justifying arguments. We are particularly interested in the moment of recalling and application of the relation (5.1), and in the involved perspectives and semiotic resources which help its activation. To refer to each component of the groups, we keep using the same students' names (S1, S2 and S3) of Activity 1 (see the subparagraph "V.'s students" in Paragraph 5.3.3).

High-level group A works on the graphical version ("Scheda A") of Problem 1. They firstly approach it through a parabola, although one of the students notices that the four given conditions are too many for finding the three parameters a, b and c. They consider the algebraic translation of the tangency conditions. They evaluate the $\Delta = 0$ technique and they even propose to impose the distance between the point and the function equal to zero. Such suggestions derive from the conics-related praxeology.
After a brief discussion (of about seven minutes) S2 makes another proposal, which involves the derivative. Let us examine their speech.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
1	S2: "*Let's do as if it was its derivative in the point.*" (He points to , referring to the tangent, Fig. 5.4.16)	pointwise pointwise	speech indicators + pointing gesture	f' f
2	S3: "*Mm... Eh!... It can work, it's true! We are not sure that it is a parabola, but it can work. Because, you have to say: "the function passes through that...and the first derivative of that stuff is...""*			
3	S2: "*At the point -2 the derivative...*"	pointwise	speech indicators	f'
4	S3: "*At the point -2 the derivative is equal to...*"	pointwise	speech indicators	f'
5	S2: "*-1.*"			
6	S3: "*Yes. Because we know it. Well, it can work.*"			
7	S1: "*The derivative at -2 of the function?*" (He retraces a segment of the tangent with his pen)	pointwise local	speech indicators + continuous gesture	f' f
8	S3: "*Yes. It's -1.*"			
9	S2: [...] "*I was wondering about the derivative... Because one could say: "$f'(-2)$ equal to -1"... "$f'(0)$ equal to 3""*	pointwise	oral symbols	f'
10	S3: "*Yes.*"			
11	S1: [...] "*You are talking about the derivative here, right?*" (He traces with his pen an hypothetical piece of function that is locally tangent to the straight line s, Fig. 5.4.17)	local	iconic gesture	f

12	S3: "*Yes.*"			
13	S1: "*And you are saying that the derivative of the function*" (with his pen, he traces again an hypothetical piece of function tangent to s, Fig. 5.4.17) "*is equal to this straight line more or less.*" (He retraces s)	local	iconic gestures	f
14	S3: "*Yes, exactly. <u>At the point</u>.*"	pointwise	speech <u>indicators</u>	f'
15	S2: "*So, I would have written "$f'(-2) = -1$" and "$f'(0) = 3$".*" (He writes down the two symbolic equivalences, Fig. 5.4.18) "*If I remember correctly.*"	pointwise	written symbols	f'

Figures

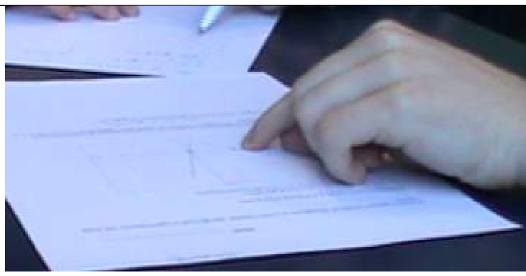


Figure 5.4.16 - S2's pointwise pointing gesture which helps recalling the relation (5.1)

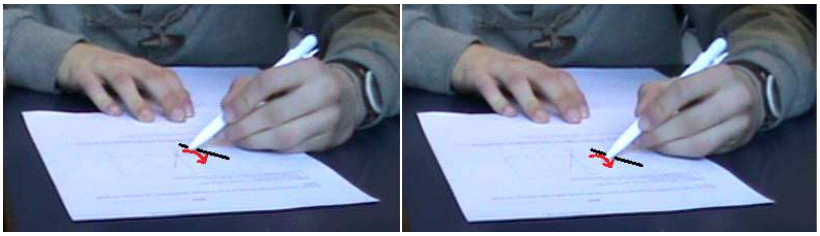


Figure 5.4.17 - S1's local iconic gesture on an hypothetical f.

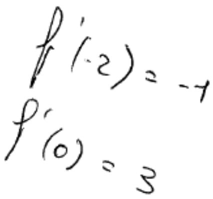


Figure 5.4.18 - S2 writes down the relation (5.1) in symbols.

As we can notice, it is S2's pointwise pointing gesture on the given graph (Fig. 5.4.16) that helps the relation (5.1) to emerge [*1-5*]. Thus, at this stage, this *l-p* relation is recalled in a pointwise perspective. Then, S1's local gesture, retracing a segment of the tangent, introduces a certain local perspective on f [*7*]. Even when S2 starts to express into symbols the relation (5.1) [*9*], it's always S1 that stresses the local character on

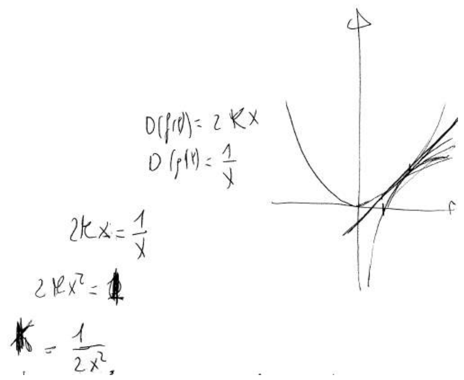
f [*10* and *13*]. This local perspective is implicit in the iconic gesture he makes on an hypothetical function f (Fig. 5.4.17) tangent to the given straight line. To this implicit local perspective on f, S3 relates the pointwise perspective on f', by adding "at the point" [*14*]. This dialectics of perspectives leads S2 to write the relation (5.1) in symbols [*15*] (Fig. 5.4.18). We can conclude that this group has managed the *l-p* relation (5.1) by adopting the right perspectives to read and apply it. Their justifying speech has been implicitly local on f and explicitly pointwise on f'. Moreover, the given graph has been exploited as a support to locally imagine how the function f could behave in the neighbourhood of the tangency point.
We can make another remark on S2's utterance: "*If I remember correctly*" [*15*]. It results evident that S2 is trying to recall the relation (5.1) as if it has been taught during the previous V.'s lessons (see Paragraph 4.5.2).
Another moment in which the influence of the praxeology constructed in the classroom is particularly strong is a S2's further utterance: "*Be careful that we said that the tangency is only local*". He says so by making a circle around point A with his finger. With his statement, S2 wants to stress the fact that, on the left of point A, the function f "*could do what it wants*". By discussing on S2's affirmation, the students realize that f is not necessarily a parabola. So, they try with a cubic function and solve the system of four equations in four unknowns. They made some errors in computation and, each time, they use a graphical application on their smartphones to check if their solution respects the given conditions. It is the teacher that in the end helps them to find the error in the system computation.

The resolution of Problem 2, instead, requires them less time, probably because they have just recalled and applied the relation (5.1). They immediately know how adapt it to the new problem [*16*], but its symbolic application is not so immediate [*17-27*].

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
16	S1: "*Let's calculate the derivatives of both and impose them to be equal, so that they have the same tangent line. So, you calculate the derivative of $f(x)$ that is $2kx$, right?*" (He writes $D[f(x)] = 2kx$, see Fig. 5.4.19 on the left)	global(=univ. pointwise)	symbols	f'
17	S3: "*Yes.*"			
18	S1: "*The derivative of the second one, that is $g(x)$, is equal to 1 over x.*" (He writes $D[g(x)] = \frac{1}{x}$, see Fig. 5.4.19 on the left) "*And then $2kx = \frac{1}{x}$.*" (He finds k, see Fig. 5.4.19 on the left)	global(=univ. pointwise)	symbols	f', g'

19	S1 draws f and g in a Cartesian reference system (see Fig. 5.4.19 on the right)	global	graph	f, g
20	S3 (reading again the text of the problem): *"You have to impose it [the condition] in that point."*	pointwise	speech indicators	f', g'
21	S3: *"Our error is here. We impose the derivatives to be always equal. So, whether they have the same variation or..."*	global	speech indicators	f', g' f, g
22	S3: *"In a point that belongs to both of them."*	pointwise	speech indicators	f, g
23	S1: *"Yes, but how can we do the intersection between kx^2 and $\ln x$?"*	pointwise	speech indicators	f, g
24	S3: *"We have to find this point, because after we can say that the derivative of $f(x)$ in that point and the derivative of $g(x)$ in that point are equal. Otherwise, it makes no sense."*	pointwise	speech indicators	f', g'
25	S1: *"Yes, you should do $kx^2 = \ln x$."* (He writes it down) *"But I do not know how to solve it."*	global(=univ. pointwise)	symbols	f, g
26	S2: [...] *"What about imposing the system between the two?"* (S1 writes and solve the system of the two conditions, Fig. 5.4.20)			
27	S3: [...] *"So, you are solving the system between what we said, namely they must have a common point, and the fact that the derivatives must be equal in that point?"*	pointwise	speech indicators	f, g, f', g'
28	S1, S2: *"Yes."*			

Figures



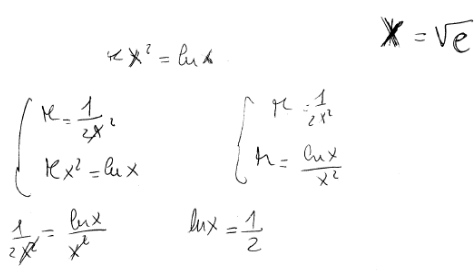


Figure 5.4.19 - Group A imposes the tangency condition and traces a graph.

Figure 5.4.20 - Group A imposes and solves the system between the two conditions.

The tangency condition involving derivatives is immediately proposed by S1 in words [*16*]. Also its translation into symbols is not difficult for the students [*17-18*] (see Fig. 5.4.19 on the left). What blocks them is the realization that they have found k as a function of x. S1 draws the two functions in a Cartesian reference system [*19*] (Fig. 5.4.19 on the right), but this global perspective does not help them. It is S3 that makes the attention of the group shift to a pointwise perspective on the point of contact [*20*]. The equivalence between the derivatives they have expressed into symbols, $2kx = \frac{1}{x}$, is globally conceived as universal pointwise, namely as something that is valid for all x. S3 makes it explicit by saying "*we impose the derivatives to be always equal*" [*21*]. Moreover, he adds a global consideration: "*whether they have the same variation or...*" [*21*]. But he stresses that what they need is a different pointwise perspective in order to find the common point between the curves [*22* and *24*]. S1 interprets, in a pointwise way, what S3 wants to find as the "intersection" between f and g [*23*]. Nevertheless, the symbolic expression he writes, $kx^2 = \ln x$, still has a universal pointwise character [*25*] for him. The situation is unblocked by S2's proposal to establish the system between the two conditions [*26*]. The technique is finally summed up in words by S3 who adopts a pointwise perspective on all the involved functions [*27*]. The group succeeds in establishing a pointwise technique, which involves the derivatives, for solving the problem of the tangency between two curves. Without passing through local considerations on the given curves, they immediately transfer the justifying role to the symbolic expressions. They finally succeeds in overcoming the universal pointwise view that the symbolic expressions and algebraic manipulations usually and implicitly convey.

Medium-level group F works on the symbolic version of Problem 1. Immediately after having read the text, the student S2 proposes to draw the graphical situation. Every student draws the situation in a Cartesian reference system on his proper sheet (Fig. 5.4.21). Thus, also these students' reasoning eventually relies on the graph. As for group A, their initial approach bases on a parabola, in spite of S1's remark that the two tangents are not symmetrical.

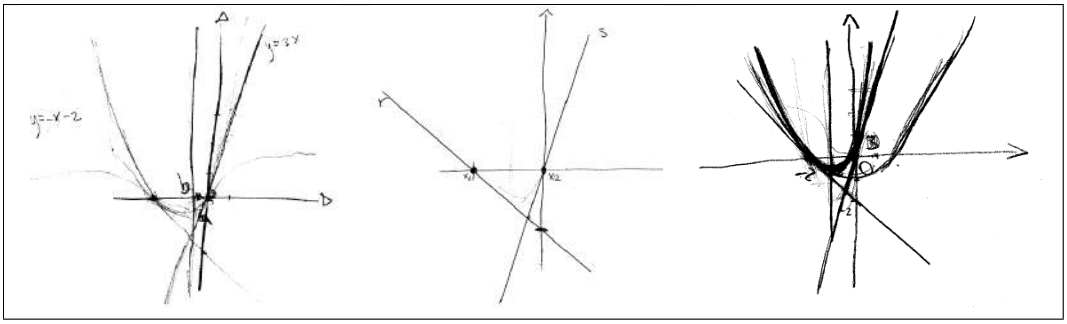


FIGURE 5.4.21 - GRAPHICAL DRAWINGS BY S1, S2 AND S3.

The group discusses about how to express the tangency condition and they upfront involve the derivative of f.

	What the students say and do	Perspectives	Semiotic resources	on f, f'
1	S1: "*If I have a function*" (he makes a generic gesture in the air) "*that is tangent to a straight line*" (he points to B, Fig. 5.4.22) "*It means that m is the angular coefficient of this straight line.*"	pointwise	pointing gesture	f
2	S2: "*Yes, and m is the derivative.*"			
3	S1: "*So,* 3 *will be the derivative... and* -1 *the derivative in the second point.*"			
4	S1: [...] (after having read the text again) "*It means that the derivative of* $f(x)$ *is equal to a function... How did we do to find it in a point? Derivative in the point.*"	global $\rightarrow$ pointwise	speech indicators	f' f
5	S2: "*It's a mess to write.*"			
6	S1: "*For example, if I have* $f(x) = x^2$ *its derivative is* $2x$*, right?*" (He writes "$D[f(x)] = 2x$")	global(=univ. pointwise)	symbols	f'
7	S2: "*Yes.*"			
8	S1: "*But in what point is this?*" (He points to $2x$) "*I don't remember this thing... But to me, we need to have it in the point... which are* -2 *and* 0*.*"	pointwise	speech indicators	f'
9	S1: [...] "*We have to impose the system between derivative of* $f(x)$ *equal to* 3 *and derivative of* $f(x)$ *equal to* -1*.*" (He writes "$D[f(x)] = 3$" and "$D[f(x)] = -1$", as in Fig. 5.4.23)	global(=univ. pointwise)	symbols	f'

10	S2: "*Yes, but we have only one derivative.*"	global(=univ. pointwise)	speech	f'
11	S2: "*Ah! Wait! We have to write like this: derivative of* f *of* -2 *is equal to* -1 *and derivative of* f *of* 0 *is equal to* 3." (She writes "$D[f(-2)] = -1$" and "$D[f(0)] = 3$", as in Fig. 5.4.24)	pointwise	symbols	f'
12	S1: "*It's not stupid at all!*"			
13	S2: "*We have to find a function which satisfies that.*"			

Figures

Figure 5.4.22 - S1's pointwise pointing gesture to speak about the tangency condition.

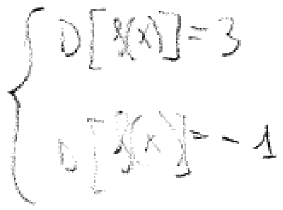


Figure 5.4.23 - S1's relation (5.1) in symbols.

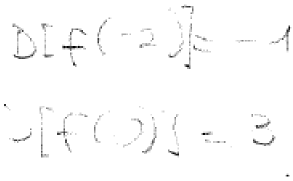


Figure 5.4.24 - S2's relation (5.1) in symbols.

After four minutes of discussion, S1 starts to reflect aloud about the tangency condition [*1*]. His words do not actually add anything to what it's obvious for the group and have no effective link with the tangency condition. However, he focus the group attention on the angular coefficient of the given straight lines. Indeed, S2 recalls that m is related to the derivative of f [*2-3*]. S1 wants to write in symbols the property they are stating, but he realizes that the derivative of the function f is a function itself [*4*]. So, he has a global object, and he wants it to become pointwise. He makes the example of the parabola $y = x^2$ whose derivative is the function $2x$ [*6*]. This is exactly the example the teacher made in the previous lessons to speak about the derivative function. Thus, we can observe that the students are trying to recall that technological part of the previously constructed praxeology. S1 proposes two different symbolic equivalences for $D[f(x)]$ [*9*] (see Fig. 5.4.23). In a universal pointwise perspective, they seem incongruous for the students [*10*]. It is S2 that suddenly realizes how they can indicate the dependence on the tangency point. She replaces x with the values -2 and 0 [*11*] (see Fig. 5.4.24), and

the two conditions appear now compatible [*12-13*].
The group F makes no local consideration about the function f, but gets to the *l-p* relation (5.1) with a symbolic work on the derivative function f' that is firstly universal pointwise (for each x) and then pointwise (at the tangency point).

They initially try to apply the found relation to a generic parabola $y = ax^2 + bx + c$. Then, they think of a composition of two parabolas, but they do not succeed in managing the relation (5.1). Afterwards, S1 uses a graphical application on his smartphone in order to check if a cubic function can satisfy the given conditions. He inserts a random equation $y = -(x^3 + 2x^2) + 2$ that looks compatible, with proper adaptations, to the requests. They start modifying the coefficients by trial and error. At this moment, the teacher intervenes to make them notice that the software has given them an idea, but it is better to think how to exploit it than empirically trying. She gives the input to count how many conditions they have and what do they allow them to find. S1 proposes to use $ax^3 + bx^2 + cx + d = y$. Imposing the system and solving it turns out to be very simple for the students. Once they have the solution, S1 draws it with his application together with the given straight lines. It gives them a confirmation of their computations.

As for group A, Problem 2 requires less time to be solved. The students do not even need to have a graphical support. It follows the transcription of what they say.

	What the students say and do	**Perspectives**	**Semiotic resources**	**on f, f'**
14	S1: "*Well, derivative of* $\ln x$ *that is equal to* $\frac{1}{x}$, *derivative of* kx^2 *that is equal to* $2kx$. *Now, I impose them to be equal.*"	global(=univ. pointwise)	symbols	f', g'
15	S2: "*Yes, you impose the derivatives to be equal, and then?*"			
16	S1, S3: "$2kx^2 = 1$, $k = \frac{1}{2x^2}$." (S2 writes it down)	global(=univ. pointwise)	symbols	f', g'
17	S1: "*And now you have to find x in order to find the correct k. So you have to find x of the <u>intersection point</u>. Stop, let's do the <u>intersection point</u>.*"	pointwise	speech <u>indicators</u>	f, g

Again the students starts from a universal pointwise perspective on the derivatives of the involved functions [*14* and *16*]. Then they move on to consider them in a pointwise perspective [*17*] for searching for the point of tangency. After, the students find k as a function of x also from the equivalence $kx^2 = \ln x$. They impose the two expressions found for k to be equal and they get x (see Fig. 5.4.25).

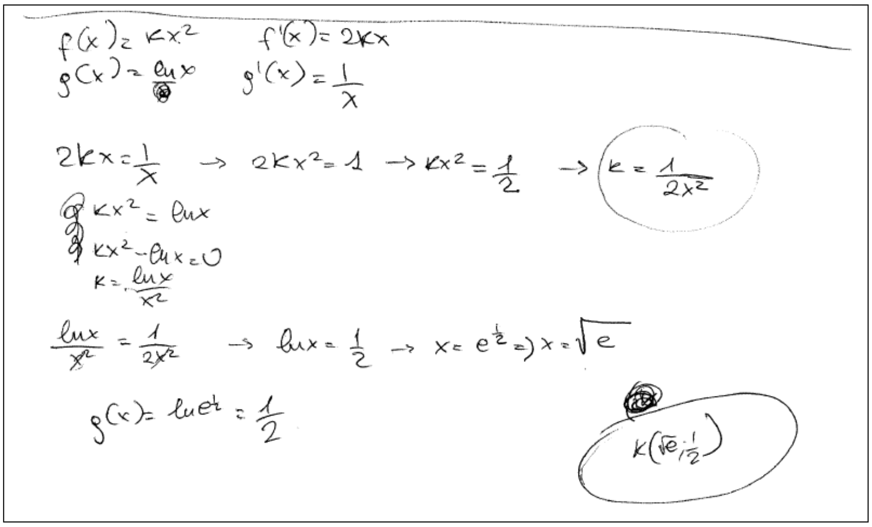


Figure 5.4.25 - Group F's solution to Problem 2.

Influences of V.'s praxeology

V.'s students seem to well manage the conception of the derivative as a function of x. They show to have a global perspective on algebraic writing such as "$D[f(x)] = 3$" or "$D[f(x)] = 2kx$", by reading them under a universal pointwise lens. They seem to have internalized the idea of derivative function as derivative "in any x", "for all x". It can be a consequence of the fact that V. has introduced it giving immediately to x_0 a universal pointwise role (see subparagraph "Elaboration of an algebraic technique" in Paragraph 4.5.3). As for the local perspective on f, some students, as those of group A, recall that the tangency is a local property. It helps them in drawing pieces of function in such a way that they are tangent to the given straight lines. Thus, the local perspective on f mainly shows through gestures on the graph (given or not by the text).

Chapter 6

Conclusions and implications

The main aim of this thesis was to investigate the presence and the role of the local perspective on functions in the secondary teaching of Calculus concepts. In particular, we chose to investigate the practices related to the notion of derivative. Indeed, this is one of the first concepts that make the local perspective intervene in the study of a function. Moreover, differentiating a function one obtains the derivative function, and also working on it involves pointwise, global and local properties.
We presented and discussed the local definitions of differentiable function given at the university, within the scholarly mathematics (Chapter 1, see particularly Paragraph 1.2.2). Framing our study in the Chevallard's Anthropological Theory of the Didactic, we focused on the didactic transposition of the derivative notion in the secondary school teaching. We outlined our theoretical framework by coordinating three theoretical tools (Chapter 2): the praxeology (Chevallard, 1999), the perspectives on functions (Vandebrouck, 2011a, 2011b), the semiotic bundle (Arzarello, 2006). Through this networked framework, we analysed how the derivative notion is didactically transposed in the Italian intended, implemented and attained curricula. As for the intended curriculum (Chapter 3), we took into account the national guidelines, two of the most widespread textbooks and one recent final examination in mathematics, with specific regard to scientific high schools. As for the implemented curriculum (Chapter 4), we observed three case studies of three teachers introducing the derivative concept and the derivative function in their grade 13 classrooms. To have a small insight in the effect of the implemented curriculum on the attained one (Chapter 5), we proposed and analysed two activities for the students of the three observed classrooms.
We recall that our methodology bases on case studies (three teachers and three classrooms), so on qualitative data. Thus, also our conclusions are of qualitative kind.

With the aim of studying the presence and the role of the local perspective on functions in the didactic transposition of the derivative notion in the secondary teaching, our research problem has been built around the following research question:

(RQ) How does the local perspective intervene in the development of derivative-related

praxeologies in the secondary school?

Thus, within all the different contexts (e.g., guidelines, textbooks, final examination, teachers and students in classroom), our main concern has constantly been to identify the intervention of the local perspective and to stress how it works in dialectic with the other perspectives and through the used semiotic resources. The heterogeneous data we have analysed have so acquired a common factor. In certain situations we found that the local perspective on functions is absent or not hinted. In other contexts instead it is present, sometimes implicitly, some others explicitly.

In this chapter, we will try to answer question *(RQ)* through a comparison and discussion of the results obtained from the analysis (Chapters 3-4-5). To make this, we will be guided by our research sub-questions:

(RQ.1) What role is given to the local perspective on functions in the secondary teaching of the derivative?

(RQ.2) How do teachers construct the derivative-related praxeologies with and for their students?

(RQ.1+2) What role do teachers give to the local perspective on functions in the construction of such derivative-related praxeologies?

(RQ.3) In which ways different praxeologies developed in classroom can affect the students' praxeologies, in terms of local perspective?

Through the discussion of the results, we will try not only to answer our research questions, but also to evaluate our networked theoretical framework as an analysis tool and to provide some useful implications on teaching.

6.1 Answering to our research questions

From the analysis of the intended and implemented curricula, we can make some conclusive remarks about the importance and the activation of the local perspective in the teaching practices involving the derivative concept. In Paragraph 6.1.1, we address the question *RQ.1*, while in Paragraph 6.1.2 the questions *RQ.1* and *RQ.1+2*.

6.1.1 The local perspective on functions in the intended curriculum

The national guidelines for scientific high schools give few indications about how to teach the derivative notion. First of all, they specify what concept can be anticipated in grade 11-12, namely that of speed of variation. Then, for grade 13 they list the differentiability along with other important properties of infinitesimal calculus, such as continuity and integrability. Finally, they cover further concepts in which the derivative intervene, especially the differential equations. These are all regulations about which notions to deal with and when to teach them, rather than how to do it. The only methodological indication concerns to avoid a particular training in computational techniques or very articulated exercises. A further recommendation is to give to infinitesimal calculus the role of fundamental conceptual tool in describing and modelling phenomena. Therefore, the local perspective, whose activation on functions can be enhanced by the work with the derivative, is neither mentioned nor hinted.

The analysed textbooks (Sasso, 2012; Bergamini et al., 2013) propose to approach the derivative notion as the limit of the incremental ratio of the function, for the increment h of the independent variable tending to 0. This is the didactic transposition of the DEF. 1, given and discussed within the scholarly mathematics in Paragraph 1.2.2. More precisely, the transposition consists in the following phases.

a. Introducing the problem of the tangent to a generic curve in a point, with possible recall of the techniques used for the conics.

b. Illustrating, usually with a graphical support, a new dynamical idea on a generic curve: considering two points on it, the secant line passing through them and making the distance between the points become smaller and smaller.

c. Defining the derivative as the gradient of the tangent line, obtained in the way illustrated at point b.

d. Operationally, increasing the abscissa of the tangency point of h, making the ratio between the consequent increment of the dependent variable over h, and finally establishing the limit as h goes to zero.

Thus, the formal definition of the derivative of a function in a point comes to be defined as the gradient of the tangent, that in turn is defined as the "limit of secants". The local perspective on the function represented by the generic curve intervenes in the phases b and d. In phase b, the employed semiotic resources are graph and words, and the local dimension is implicitly conveyed through terms of movement, in particular of approaching (e.g., "getting closer and closer", "moving towards") or change (e.g., "becoming smaller and smaller", "getting tangent"). The local dimension remains implicit in the terms used to describe a static supporting graph. Normally, the technique consists of drawing two points on the graph of the function, but, even though they are chosen closer and closer - in different juxtaposed drawings or in the same drawing but with progressive index - it is difficult to convey a local perspective on the involved function. Indeed, the two points appear as two distinct and rather distant points on the curve, anyhow separated by a non-negligible interval. Thus, the pointwise and global perspectives on the function are enhanced by the graphical resource, while the written description invites the reader to imagine the two points getting closer and the interval between them getting smaller. Then, the adoption of a local perspective on the function is left to the capability of the reader to establish and interpret the relationships between the different semiotic resources composing the semiotic bundle *words+graph*. The last phase (phase d) forces the local dimension in a sudden way, in the sense that, after having resorted a pointwise and global perspective on the function, the symbol $\lim_{h \to 0}$ is introduced, and the semiotic bundle therefore becomes *words+graph+symbols*. This semiotic bundle potentially, but rather implicitly, contains the elements to activate a local perspective.

As for the introduction of the derivative function, the symbol $\lim_{h \to 0}$ implicitly continues to retain a local dimension on the function. It is thanks to the graphical resource (especially in the proposed exercises) that the work on the derivative function can activate a local perspective. The local dimension is efficiently conveyed by the semiotic bundle *words+graph*, provided that the verbal resource gives a local interpretation of the graph (e.g., at a maximum/minimum or inflection point, at a non-differentiable point) through its reading in a neighbourhood of the point (i.e., on the left/right of it). In this case, the local perspective on the function becomes explicit.

In the final examination we analysed, we have to recognize that the curricular recommendation of making the derivative work not only as an object but particularly as a tool (in the sense of Douady, 1986) is fully respected. However, to solve the proposed tasks involving the derivative, it is not required any particular local consideration on the given functions. Actually, the local character of the differentiability property is not evaluated. Nevertheless, we noticed that the integral concept makes it intervene in the resolution of a task.

We can conclude that, within the intended curriculum, the local feature of the differentiability property is implicitly recognized, little worked, but sometimes required in the tasks the student is expected to be able to solve. The local perspective on functions is potentially activable by a subject who avails of this material. Nonetheless, we think that it might be difficult for a low or medium-level student to fully activate it alone, without any kind of mediation. The textbooks, for example, propose to intertwine some semiotic resources for approaching the derivative, but the relationships between them are left to the reader. In presence of local properties, such as differentiability, we can detect a sort of **gap** between the system of knowledge that the student has previously acquired, basically made of pointwise and global techniques (and sometimes technologies) and the adoption of a local perspective to properly understand those local properties. The role of the teacher, then, reveals essential to bridge the gap. She can set the stage as the textbooks do, but in addition she can make the potential activation of the local perspective become actual.

6.1.2 The local perspective on functions in the implemented curriculum

From the analysis of the three case studies observed in classroom (teachers M., M.G. and V.), we can distinguish two different didactic transpositions of the derivative concept.

We refer to the type of task $T_{tangent}$: determining the equation of the tangent line to a generic function in a point. M. and M.G. worked out a process which is very similar to the textbooks' one (they use Sasso, 2012). We will denote it DT1, which stands for Didactic Transposition 1. On the contrary, V. elaborated a personal didactic transposition of the concept which detaches from the textbooks' one (she has adopted Bergamini et al., 2013). The latter can be seen as a direct didactic transposition of the scholarly mathematics. We will denote it DT2, which stands for Didactic Transposition 2. In Table 6.1 we briefly summarize DT1 and DT2 main phases and detect in what they differ, in terms of local perspective activation.

DT1 of the derivative *(DT of the DT of def1)* Gradient of the tangent line defined as limit position of a sequence of secants	➢ justifying speech centred on the secant (m_{PQ}) introduced as an intermediary, as well as *h* ➢ suggested introduction of the sign of limit: $\lim_{h \to 0} m_{PQ}$	POINTWISE GLOBAL LOCAL conveyed by symbols (*lim*) justified through terms of movement
DT2 of the derivative *(direct DT of def2)* Gradient of the tangent line defined as the best linear approximation of the function in the point	➢ justifying speech centred on the tangent definition ➢ graphical adaptation of the problem (condition to apply the property of asymptotic equivalence) ➢ technology in symbols $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{m(x-x_0)} = 1$ ➢ from which *m* is deduced	LOCAL conveyed by reasoning in a neighbourhood, inherent in the idea of best approx. The symbol *lim* is borrowed by another local praxeology

Table 6.1 - DT1 and DT2 main phases and the local perspective activation.

The DT1 consists in the teacher's didactic transposition of the approach proposed by Sasso's textbook, according to the "scholar" Def. 1 of differentiable function (see Paragraph 1.2.2). Thus, the derivative of a function in a point is the gradient of the tangent line, defined as the limit of a sequence of secants to the function. The initial phase of DT1 is pointwise and global, with a forced passage through the secant line. The technological speech is centred on the secant, its different positions and its slope m_{PQ}. In such a context, the local symbol of limit does not come spontaneously, but it has to be suggested and introduced in a guided way by the teacher. The justification is made of expressions which convey an idea of approaching (e.g., "getting closer and closer", "approaching more and more"). The teachers can use different semiotic resources to express this idea: besides graph and words, also gestures. Continuous movements of the hand or the arm can express, better than a static image could do, the approaching of the two points on the curve, or the changing of the secant into the tangent. Anyway, the teachers find themselves in the same situation provided by the textbook. The local nature of the tangent is really accomplished only with the introduction in the semiotic bundle of the symbol of limit. Once reached the symbolic writing $\lim_{h\to 0}$, the graph and the terms of approaching are suddenly abandoned. However, the symbolic work alone can not support a durable local perspective on the function. Often, it may happen that students make a flawless symbolic work on a function through limit calculation, finding local interesting outcomes, but without knowing or noticing it. An explanation of this recurring lack of interpretation can lie in the fact that generally, within the DT1, the local reading of a function is only quickly hinted in the definition of its derivative, and the local perspective is almost immediately delegated to symbols, remaining implicit in them.

The DT2, instead, is the product of a direct didactic transposition made by the teacher of the "scholar" DEF. 2', as it is explained on some university textbooks (e.g., Geymonat, 1981). The derivative of a function in a point is the gradient of the tangent line, defined as the best linear approximation of the function in a neighbourhood of the point. The notion of approximation is conveyed by the idea of zooming the function till finding a straight line. Given this concept image and concept definition of the tangent, the technological speech remains centred on the tangent line and on its local character. Graphs are used with the idea of magnifying them in a neighbourhood of a point (examples in GeoGebra can also be provided) and words and gestures support this idea. We noticed a predominance of expressions such as "best approximates", "asymptotically equivalent", "in a neighbourhood of the point" and gestures to "limit the zone". Such a semiotic bundle *speech+graph+gestures* makes explicitly activate a local perspective on the function. The passage to mathematical symbols, namely the introduction of the limit sign, comes gradually. In particular, it occurs thanks to the recall and the embedding of another praxeology, that we have denoted as $O\mathcal{M}_{\text{asymptotic equivalence}}$ (see Table 4.18). This praxeology has already been developed for solving a local type of task, namely the solution of remarkable limits such as $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$. The whole praxeology $O\mathcal{M}_{\text{asymptotic equivalence}}$ enters the developing praxeology for $\mathcal{T}_{tangent}$ on the technological-theoretical plane. This embedding both allows to shift the focus of attention on the symbolic formulation of the type of task and suggests the symbolic technique for accomplishing it. In such a technique the limit sign is already involved. Indeed, saying that two infinitesimal functions are asymptotically equivalent means setting the limit of their ratio equal to one. The introduction of the limit sign, therefore, comes more spontaneously in this didactic transposition. It is borrowed from a previous praxeology which is already local and which is based on the same idea of approximating and magnifying the function. It is not by chance indeed that the introduction of the limit is proposed by a student, in the case of V.. The only adaptation to do is a vertical translation of the x-axis so that the two quantities to compare are infinitesimal, and here the role of the teacher is extremely important. The symbolic work is strictly intertwined with the graphical one. The semiotic bundle which has supported the construction of the technological-theoretical block of the praxeology for $\mathcal{T}_{tangent}$ keeps accompanying the refinement of a proper technique.

Comparing the two didactic transpositions, we can remark that the praxeological structure of DT2 is characterized by a strong local component, explicitly conveyed through the semiotic resources of speech, graph, gestures and symbols, efficiently combined together to express the idea of approximation. There is no need to force any artificial movement, or to introduce any intermediary (e.g., the symbol h).
Behind both the techniques used in DT1 and DT2, we can recognize the *Basic Metaphor of Infinity* introduced by Lakoff and Núñez (2000) as "a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result" (Lakoff & Núñez, 2000, p. 158). More precisely, DT1 technique involves an infinite sequence of secants that, to infinity, ends with the tangent. In DT2, instead, we find the process of indefinitely zooming the curve until obtaining, to infinity, the segment tangent to it. In both didactic transpositions the *Basic Metaphor of Infinity* is eventually incorporated in the symbol of limit. However, we have noticed that the passage to the limit comes to be somehow forced in DT1, and more spontaneous in DT2. This difference leads us to conclude that DT2 seems to prompt more naturally towards the tangent as an ultimate end of an infinite process. As a consequence, DT2 seems to be more efficient in introducing the local perspective on a function, allowing to bridge the gap we perceived between the previous pointwise and global knowledge and local properties such as differentiability.

Concerning the type of task $T_{tangent}$, we want to underline a further point at the level of the didactic praxeology. Indeed, it determines the teacher's didactic transposition process to achieve DT2 as a product in the classroom. Our further remark concerns the way V. has implemented DT2 with her students. She has given a great importance to the definition of tangent, by devoting an entire lesson to discuss this topic. Such a didactic choice has entailed benefits for the students' understanding and for the strength of the developed mathematical praxeology. The collective discussion of the properties that characterize a tangent line to a generic curve has allowed the students to free themselves from pointwise and global considerations and so to "open their eyes" toward a local perspective. The theoretical component of the mathematical praxeology turns out to be firmly grounded on a local definition of tangent that the students have formulated by themselves and internalized.

Let us consider the second type of task we analysed: $T_{f'}$, that is representing the derivative function. As far as the perspectives are concerned, in this case, on the derivative function f' of f it is needed a shift from the pointwise definition of $f'(x_0)$ as $\lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ to the global definition of $f'(x)$ as a function. This shift usually occurs firstly in the algebraic-symbolic register, with the syntactic technique of replacing x_0 with x, and secondly in the graphical register.
Within the algebraic technique, the perspective on the derivative function changes from pointwise to global, in the sense of universal pointwise. This technique does not require any further local consideration on the starting function. The writing $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

retains the limit sign which implicitly conveys a local involvement of f.
Instead, by working on a graphical technique to represent the derivative function, some local implications on the starting function have been highlighted by the teachers in classroom. For instance, establishing a relationship between the zeros of f' and the stationary points of f has permitted to locally correct some inaccuracies of the graph of f. Or even, relating the inflection points of f to the stationary points of f' has led to correctly position the abscissa of maximum/minimum points of the derivative f'. Therefore, the work on the graph of the derivative has fostered the local reading of the graph of the starting function in the neighbourhood of some remarkable points (e.g., maximum/minimum and inflection points, corners).
Hence, among all the activable semiotic resources, the one that seems fundamental to support the adoption of a local perspective on functions is the graph, provided that it is accompanied by a suitable combination of speech and gestures. In this case, the resulting semiotic bundle *speech+graph+gestures* is really aimed to teach the students how to "regard locally" a graph, fostering in an efficient way a local perspective on the represented function.

6.2 Evaluation of the analysis tool

The theoretical framework of this thesis is the result of the coordination of three elements coming from three different theoretical approaches (see Section 2.2). The macro-lens is provided by the construct of praxeology from ATD (Chevallard, 1999). Moreover, a micro-analysis is based on the intertwine of the (pointwise, global, local) perspectives activated on functions (Vandebrouck, 2011a, 2011b), and the employed semiotic resources along with their mutual relationships, that form the semiotic bundle (Arzarello, 2006). In particular the coordination of the last ones is due to our initial assumption that the actual adopted perspective on a function f is revealed by all the semiotic resources used to state a certain property about f (see the sub-paragraph "The semiotic bundle and the perspectives" in Paragraph 2.2.4). This is effectively what we find in analysing the lessons. In order to detect if the teacher was trying to stress a particular perspective, we needed not only the speech, so the transcription of what she said, but especially all the other semiotic resources that she activated simultaneously for conveying that perspective. Sometimes it has been interesting to notice that not all the semiotic resources were concordant in expressing the same perspective. Emblematic, for our research, are some cases in which the teacher wants to stress a local property: she claims it, but at the same time her gestures, her drawings or the symbols she writes are enhancing a specific element involved in the statement of the property, which is pointwise or global. For instance, saying "I distance myself a little bit" and simultaneously making a quite wide gesture with the hands. Or talking about the approaching of two points on a curve using a supporting drawing, and then writing the symbol $\Delta x \to 0$ without having shown Δx decreasing on the graph.

To deal with these situations, we can introduce the idea of "**perspective potential**", as the degree of suitability to activate a certain perspective[1].

A particular technique can have a perspective potential. Both the techniques used in DT1 and in DT2 have a local potential. Specifically, the technique of approaching a point on a curve with a sequence of other points on it and the technique of zooming a graph have mainly a local potential. This is because they can both trigger a local perspective on the curve. As we have noticed in Paragraph 6.1.2, the zooming technique reveals more suitable than the other to convey a local perspective, because it avoids the artificial introduction of the secant as a sort of *deus ex machina*.

In order to activate the perspective potential of a certain technique, and consequently to foster the adoption of that perspective on the involved function, the teacher uses certain semiotic resources. Even a semiotic resource can have a perspective potential, as far as it is suitable to activate that perspective on the function. To make an example, the graph has mainly a global potential. Indeed, drawing the graph of a function entails necessarily to draw a portion of graph that has certain properties on the chosen interval (be the choice conscious or not). Concerning gestures, we can make other examples. A pointing gesture has mainly a pointwise potential, a continuous gesture has mainly a global potential, while a small circular sign or gesture has mainly a local potential.

Now, drawing on the theory of the semiotic bundle, we know that the semiotic sets do not live isolated, but in mutual relationship with other semiotic sets. In all the examples above a given semiotic resource as a certain perspective potential. We can be sure of the activation of such a potential, and so of the associated perspective, only if this semiotic resource is combined with other semiotic resources that enhance the same perspective. To make an example, the graph has a global potential, but it gets active only if another resource exalting the global aspect is efficiently combined with the graph. It can be a speech indicator (e.g., "the whole function", "for all x", "always") or perhaps it can be a continuous gesture along the graph. In a similar way, if the property one wants to stress on a graph is pointwise, he could just indicate an interesting point on it, with a pointing gesture which activates a pointwise potential, and consequently a pointwise perspective on the function. Therefore, the perspective potential of a semiotic resource is activable only thanks to the coordination of at least another resource exalting that particular perspective.

Using the idea of the "perspective potential" of a semiotic resource, we can better reformulate our initial idea, recalled at the beginning of this paragraph. It consists in our belief that a (written or uttered) claim alone may not be sufficient to detect with certainty what perspective is adopted on a function. By reformulating it in other words, the speech as a semiotic resource can have different perspective potentials, according to the use of specific speech indicators. The most interesting examples entail the local perspective. For instance, the claim "the function is non-differentiable in the point x_0" has a local potential. However, we can affirm that the local perspective on the function is really active only taking into account the other semiotic resources that the subject

[1]In particular, we will refer to pointwise, global and especially local potential, in order to stress the perspective that can potentially be enacted.

employs to support his/her speech. For instance, if the claim is accompanied by symbols like x_0^+ and x_0^-, then we can conclude that it really detects a local aspect of the function. On the contrary, if the claim is accompanied by a pointing gesture to indicate the single point x_0 on the graph, we would say that its local potential is not made explicit and remains implicit in the speech.

6.3 Implications of our research on teaching

From the analysis of the students' activities, we do not manage to find significant answers to our very deep question *RQ.3*. Perhaps, a different approach is needed with the students, in order to get really insights into their acquirement of the local perspective on functions. It opens a new direction that the research may follow in the future (see Section 6.4). Nevertheless, we select some examples from the students' productions (see the right column of the table above). Drawing on them, we detect some implications on teaching related to our previous conclusions.

The teachers should discourage the students from adopting the perspectives on functions in an isolated way while they are solving a task. Therefore, it is important that the teachers get the students used to **manage all the existing perspectives on functions**.	*Example 1: [lines 1-10] Group A of V.'s students, in Paragraph 5.3.3.* The high-level group A is solving the Problem 1 proposed in Activity 1. The students relate in a pointwise way the maximum/minimum points of a graph to the zeros of another one, deducing erroneously that the latter is the derivative of the former. All the semiotic resources, namely pointing gestures and speech indicators, they use on the graph are pointwise. A student makes a continuous gesture along one of the graphs, which has a great global potential, but he does not manage to make any global consideration.

There is a **gap** between the classroom's previously acquired knowledge and the local properties of Calculus. Thus, it is advisable to **gradually introduce the local perspective**, bridging the gap. Indeed, to obtain a durable local perspective on functions, it is necessary to educate students to it.
In particular, the notion of tangent has been often studied and discussed in Mathematics Education. It is a concept that makes more evident the gap between the old techniques and perspectives and the new local Calculus ones. This is because students have already met it in a geometrical context, with algebraic techniques developing pointwise and global perspectives on the conics. Thus, when Calculus is approached, it reveals important **not to give for granted the tangent definition**.

Example 2: [lines 1-18] Group A of M.G.'s students, in Paragraph 5.4.3.
The low-level group A is trying to solve Problem 1 of Activity 2. They attempt to make a parabola tangent to the two crooked straight lines given. This is because they recall the definition of tangent given with conics: a parabola touches a tangent and does not pass through it. The teacher intervenes to ask if in drawing such a parabola they feel somehow uncomfortable. However, despite the way the graph touches the two given tangents is locally very different, the students do not see any problem in it.

Recalling or introducing the symbol of limit often does not have the effect to introduce a local perspective, but more frequently the result is fostering the students towards tricky algebraic calculations.

Example 3: [lines 1-8] and Fig. 5.4.8 Group G of M.'s students, in Paragraph 5.4.3.
The high-level group G is solving Problem 1 of Activity 2. They algebraically express the given tangency conditions as $D'(A) = -1$ and $D'(B) = +3$, but they initially discuss if the derivative in A has to be $-x$ or -1 (since the tangent in A is $y = -x - 2$). Then, in order to exploit the found algebraic conditions, they impose the system in Fig. 5.4.8, where for both $D'(A)$ and $D'(B)$ they write $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Then, they tried to make some simplification of this writing but it does not lead anywhere.

Teachers stand in front of students with their whole persons. They teach mathematics not only speaking or writing. More or less consciously, they use a great number of semiotic resources in a multimodal way. It is recommendable then to **pay attention to all the engaged semiotic resources**, written signs on the blackboard, sketches, gestures and drawings included. Indeed, the students base on all these supporting signs to grasp the concept and to form a personal image of it.

Example 4: Fig. 5.3.17-23 Group C of M.G.'s students, in Paragraph 5.3.3.
In the solution of Problem 1 of Activity 1, Group C reproduces exactly the same graphic-symbolic scheme given by the teacher in order to draw the graph of the derivative function (see lines [10-26], in Paragraph 4.4.3).
Example 5: Fig. 5.3.35 Group D of V.'s students, in Paragraph 5.3.3.
In their technique to solve the problems of Activity 1, Group D reproduces on the given graphs the same gesture the teacher has made with the stick shadow on the graph of the parabola x^2 (see Fig. 4.5.23).
Example 6: [1-13] Group F of V.'s students, in Paragraph 5.4.3.
In solving Problem 1 of Activity 2, Group F is trying to remember how they have expressed with the teacher the derivative in a point. One of the students reconstructs the graphic-symbolic example the teacher has proposed in classroom, namely the tangent to the parabola x^2 (see lines [15-19], in Paragraph 4.5.3).

Moreover, the graph has shown a good potential to convey the local perspective on functions (local potential). However, its activation has to be properly guided by the teacher, possibly through the coordination of other semiotic resources. Specifically, the students have to be guided to reason on a graph within the neighbourhood of a point.
The idea of "**perspective potential**", elaborated in the previous paragraph, can be useful to a teacher for a double aim. On the one hand, for choosing the most suitable semiotic resources to introduce the local perspective on functions. On the other hand, for recognizing if a student has internalized a local perspective and if he is using it.

As a final remark, we would add that, even though they have been subjected to DT2, V.'s students have not shown a particularly strong local perspective on functions as we could expect. The education to local perspective needs time and work within different contexts and different types of task. To see the effects in students' reasoning, its introduction must occur before the derivative concept, starting with the limit notion and permeating all the teaching of Calculus.

6.4 Possible future developments of the research

In the case of V. we want to remind that, from the preliminary interview, a transposition like DT2 was not in her planning. Normally, she had always implemented DT1, with

an initial discussion on the tangent, especially to make the students aware that none of the previous ideas and techniques was adaptable in the context of a generic curve. Her intention was to come out with a dynamical idea to solve the problem.
However, maybe due to the fact of joining a research about the teaching practices with the derivative concept, and to the students' unexpected reaction in classroom, she tries an experimental implementation of DT2. Thus, we can say that the interaction from one side with researchers in Mathematics Education, and from the other side with the students, prepares the ground for a meta-didactic transposition process (see Paragraph 2.3.2). In particular, we have discussed with V. about the lens of perspectives after the observation and the activities in her classroom. If at the beginning V. was not so convinced about the pointwise/global/local analysis we made of the lessons, after having discussed it in more detail, we started noticing that the teacher herself began to use expressions like "introducing the local dimension" or "to better stress the local aspect".
In the last months, fostered by the fact of having another class attending the last year of scientific high school, she has revealed us her desire of rewriting her notes for the lessons. Her intention is structuring a new approach to Calculus teaching, starting from the continuity and the limits, moving on to the derivative and the integrals, that enhances the local aspect of these concepts and properties.
In the light of the meta-didactic transposition model, we can recognize that in V.'s praxeologies it is going to occur an evolution. This change has been triggered by the entrance of the local perspective as an analysis tool in Mathematics Education in the reflections about her own praxeologies. Before the observation in classroom, the local perspective as a theoretical tool was internal only to the group of researchers, but in the last meetings it has become also internal to the teacher's praxeology.
This dynamics from external to internal component has led to a first step toward a change in the professional development of the teacher. Such an unexpected result of our study could be certainly deepened in a natural future development of this research.

Furthermore, the focus of this research has been on the teachers and their praxeologies, with a small insight in the students' understanding and internalization of them. An interesting point to develop in the future could be to study in more details the meaning that the students build of the local aspect of the concepts and the properties of Calculus. Such a complementary study would consider in deep detail the cognitive implications of the adoption of the local perspective on functions.

Our research has also been presented to teachers and researchers who are or intend to become teachers educators. Indeed, we hope that our outcomes could be used to foster among mathematics teachers a deeper awareness of the local feature that the work on functions has to assume in the teaching of Calculus. Moreover, since the new Italian curriculum gives teachers a great freedom in choosing how to approach the various concepts, our desire is that they could think of possible different patterns to teach Calculus. In the case of the derivative, the didactic transposition DT2 we studied in this thesis could be, with appropriate refinement, a challenging but also powerful alternative to the traditional scheme, whose processes are sometimes obscure or artificial for students.

Bibliography

[1] Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F., Robutti, O., Sabena, C., & Soury-Lavergne, S. (2013). The meta-didactical transposition: a model for analysing teachers education programs. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 97-124). Kiel, Germany: PME.

[2] Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. Lester (Ed.) *Second Handbook of Research on Mathematics Thinking and Learning* (pp. 1011-1049). Greenwich, Connecticut: Information Age Publishing, Inc.

[3] Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME. Revista latinoamericana de investigación en matemática educativa, 9*(1), 267-300.

[4] Arzarello, F., Bazzini, L., & Chiappini, G. P. (1994). Intensional semantics as a tool to analyse algebraic thinking. *Rendiconti del Seminario Matematico dell'Università di Torino, 52*(1), 105-125.

[5] Arzarello, F., & Bussi, M. G. B. (1998). Italian trends in research in mathematical education: A national case study from an international perspective. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity. An ICMI Study* (pp. 243-262). Springer Netherlands.

[6] Arzarello, F., & Paola, D. (2007). Semiotic games: the role of the teacher. In W. Jeong-Ho, L. Hee-Chan, P. Kyo-Sik & S. Dong-Yeop (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17-24). Seoul: PME.

[7] Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: a theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.) *The mathematics teacher in the digital era* (pp. 347-372). Springer Netherlands.

[8] Bachelard, G. (1938). *La formation de l'ésprit scientifique*. Paris: Vrin.

[9] Balacheff, N., & Gaudin, N. (2010). Modeling students' conceptions: The case of function. *CBMS Issues in Mathematics Education, 16*, 207-234.

[10] Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

[11] Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education, 59*(5), 389-407.

[12] Barnard, T., & Tall, D. (1997). Cognitive units, connections and mathematical proof. In E. Pehkonen (Ed.), *Proceedings of the 21st conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 41-48). Lahti, Finland: PME.

[13] Bergamini, M., Trifone, A., & Barozzi, G. (2013). *Matematica.blu 2.0*, Vol.5, Libro digitale multimediale. Bologna: Zanichelli.

[14] Bikner-Ahsbahs, A., & Prediger, S. (2014). Introduction to Networking: Networking Strategies and Their Background. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 117-125). Springer International Publishing.

[15] Biza, I., & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. *The Journal of Mathematical Behavior, 29*(4), 218-229.

[16] Bloch, I., & Ghedamsi, I. (2005). The teaching of calculus at the transition between upper secondary school and university: factors of rupture. Communication to the Topic Study Group 12. In M. Niss (Ed.), *Proceedings of ICME 10*. Copenhagen: Roskilde University.

[17] Bos, H. J. M. (1974). Differentials, higher-order differentials and the derivative in the Leibnizian calculus. *Archive for history of exact sciences, 14*(1), 1-90.

[18] Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51-63.

[19] Boyer, C. B. (1949). *The history of the calculus and its conceptual development.* Courier Dover Publications.

[20] Boyer, C. B., & Merzbach, U. C. (2011). *A history of mathematics.* Wiley.

[21] Bramanti, M., Pagani, C. D., & Salsa, S. (2000). *Matematica: calcolo infinitesimale e algebra lineare.* Zanichelli.

[22] Brousseau, G. (1997). *Theory of didactical situations in mathematics: didactique des mathématiques (1970-1990).* Dordrecht, NL: Kluwer.

[23] Bruner, J. S. (1966). *Towards a Theory of Instruction.* New York: Norton.

[24] Carreño, E., Rojas, N., Montes, M. Á., & Flores, P. (2013). Mathematics teacher's specialized knowledge. Reflections based on specific descriptors of knowledge. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds.), *Proceedings of the 8th Conference of the European Society for Research in Mathematics Education* (pp. 2976-2984). Ankara, Turkey: ERME.

[25] Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: un exemple concret, celui de la tangente. *Recherches en didactique des mathématiques, 15*(1), 7-47.

[26] Chevallard, Y. (1985). *La transposition didactique.* Grenoble: La pensée sauvage.

[27] Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques, 12*(1), 73-112.

[28] Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques, 19*(2), 221-265.

[29] Clement, J. (1985). Misconceptions in graphing. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 369-375). Utrecht, The Netherlands: Utrecht University.

[30] Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-166). Springer Netherlands.

[31] Derouet, C., & Panero, M. (2014). *Etude comparative sur l'enseignement des fonctions dans le secondaire en France et en Italie.* Cahiers du Laboratoire de Didactique André Revuz, n.11. Institut de recherche sur l'enseignement des mathématiques, Université Paris Diderot.

[32] Dewalt, K., Dewalt, B., & Wayland, C. (1998). Participant observation. In H. R. Bernard (Ed.), *Handbook of methods in cultural anthropology* (pp. 259-299). Walnut Creek, CA: AltaMira Press.

[33] Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en didactique des mathématiques, 7*(2), 5-31.

[34] Dubinsky, E., & Harel, G. (1992b). The nature of the process concept of function – The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects on Epistemology and Pedagogy* (pp. 85-106). USA: MAA

[35] Dupont, P. (1975) *Storia del concetto di derivata.* Libreria Scientifica Cortina.

[36] Dupont, P. (1981a) *Appunti di storia dell'analisi infinitesimale.* Vol. II(1): Nascita e sviluppo. Libreria Scientifica Cortina.

[37] Dupont, P. (1981b) *Appunti di storia dell'analisi infinitesimale.* Vol. II(2): Newton e Leibniz. Libreria Scientifica Cortina.

[38] Dupont, P. (1983) *L'analisi infinitesimale negli studi preuniversitari: proposte didattiche.* Libreria Scientifica Cortina.

[39] Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives, 5*, 37-65.

[40] Elbaz, F. (1983). *Teacher Thinking. A Study of Practical Knowledge. Croom Helm Curriculum Policy and Research Series.* New York: Nichols Publishing Company.

[41] Enriques, F., & Chisini, O. (1915). *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche.* Bologna: Zanichelli.

[42] Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-254). London: Falmer Press.

[43] Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics, 61*(1-2), 67-101.

[44] Estrella, S., Kuzniak, A., Montoya-Delgadillo, E., & Vivier, L. (2014, July). El trabajo matemático en el Análisis: una approximación a los ETM en Francia y Chile (submitted). In *Proceedings of the Forth Symposium ETM*, Madrid.

[45] Fini, M. (2003). *Uno sguardo sul concetto di differenziale dalle origini ai giorni nostri: tra storia e teoria.* Working Paper n. 18.2003-Luglio, Dipartimento di Economia Politica e Aziendale dell'Università degli Studi di Milano.

[46] Fischbein, E. (1987). *Intuition in science and mathematics. An educational approach* Dordrecht: D. Reidel Publishing Company.

[47] Flores, E., Escudero-Avila, D., Montes, M., & Carrillo, J. (2014) Dos acercamientos para la caracterizacion del conocimiento que tiene un profesor acerca del aprendizaje en mathematicas (submitted). In *Proceedings of the Forth Symposium ETM* in Madrid.

[48] Fondazione Giovanni Agnelli. (2012, April). Valutare la qualità delle scuole superiori attraverso i risultati deiloro diplomati immatricolati negli atenei. Retrieved from http://www.fga.it/interventi-e-analisi/tutti-gli-interventi/dettaglio/article/valutare-la-qualita-delle-scuole-superiori-attraverso-i-risultati-dei-loro-diplomati-immatricolati-n-1.html#.VE4-myKG91Y (consulted on 10/27/2014).

[49] Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: hows and whys in the teaching of proof. *ZDM, 43*(4), 587-599.

[50] Geymonat, G. (1981). *Lezioni di Analisi Matematica 1.* Torino: Levrotto & Bella.

[51] Giraldo, V., & Carvalho, L. M. (2003). Local straightness and theoretical-computational conflicts: computational tools on the development of the concept image of derivative and limit. In M. A. Mariotti (Ed.), *Proceedings of the 3rd Conference of the European Society for Research in Mathematics Education.* Bellaria, Italy : ERME.

[52] Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

[53] Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational studies in mathematics, 67*(3), 237-254.

[54] Harel, G., & Tall, D. O. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

[55] Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

[56] Hoffkamp, A. (2011). The use of interactive visualizations to foster the understanding of concepts of calculus: design principles and empirical results. *ZDM, 43*(3), 359-372.

[57] Kuhn, T. S. (1970). *The structure of scientific revolutions.* II ed., University of Chicago Press.

[58] Kuzniak, A. (2011). L'Espace de Travail Mathématique et ses genèses. *Annales de Didactique et de Sciences Cognitives, 16*, 9–24.

[59] Kvale, S. (1996). *InterViews. An introduction to qualitative research writing.* Sage Publications, Thousand Oaks California.

[60] Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* Basic books.

[61] Legrand, M. (1993). Débat scientifique en cours de mathématiques et spécificités de l'analyse. *Repères IREM, 10*, 123-159.

[62] Leron, U., & Zaslavsky, O. (2013). Generic proving: Reflections on scope and method. *For the Learning of Mathematics, 33*(3), 24-30.

[63] Locke, J. (1690). *Essay concerning humane understanding*, it. tr. "Saggio sull'intelletto umano". Torino: UTET.

[64] Malle, G. (2000). Zwei Aspekte von Funktionen: Zuordnung und Kovariation. *Mathematik lehren, 103*, 8-11.

[65] Maraschini, W., & Palma, M. (2000). *Multi ForMat.* Moduli per il triennio del Liceo Scientifico. Torino: Paravia.

[66] Maschietto, M. (2002). *L'enseignement de l'analyse au lycee: les debuts du jeu global/local dans l'environnement de calculatrices.* Doctorat dissertation, Université Paris 7 and Università di Torino.

[67] Maschietto, M. (2008). Graphic calculators and micro-straightness: Analysis of a didactic engineering. *International Journal of Computers for Mathematical Learning, 13*(3), 207-230.

[68] Ministero dell'Istruzione, dell'Università e della Ricerca. (2010) Schema di regolamento recante indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali. D.M. 7 ottobre 2010, n. 211. Retrieved from http://nuovilicei.indire.it/content/index.php?action=lettura&id_m=7782&id_cnt=10497 (consulted on 10/27/2014).

[69] Ministero dell'Istruzione, dell'Università e della Ricerca. (2013) *Esame di Stato di Liceo Scientifico - corso sperimentale. Indirizzo: P.N.I. Tema di: MATEMATICA.* Retrieved from http://archivio.pubblica.istruzione.it/argomenti/esamedistato/secondo_ciclo/prove/2013/as2013.htm (consulted on 10/27/2014).

[70] Mullis, I. V., & Martin, M. O. (2007). TIMSS in perspective: Lessons learned from IEA's four decades of international mathematics assessments. In T. Loveless (Ed.) *Lessons learned: What international assessments tell us about math achievement* (pp.9-36). Brookings Institution Press.

[71] Nemirovsky, R., & Rubin, A. (1992). *Students' tendency to assume resemblances between a function and its derivative.* Cambridge, Massachusetts: TERC Working Paper.

[72] Newton, I. (1964). *The mathematical works of Isaac Newton. 1.* D. T. Whiteside (Ed.). Johnson Reprint Corporation. In Dupont (1981a).

[73] Páez Murillo, R. E., & Vivier, L. (2013). Teachers' conceptions of tangent line. *The Journal of Mathematical Behavior, 32*(2), 209-229.

[74] Panero, M. (2013). Teachers' practices with the derivative concept: a problematic meeting of Algebra and Calculus in secondary school. In Di Paola, B. (Eds.), *Mathematics Education in a globalized environment. Quaderni di Ricerca in Didattica (Mathematics), 23*(1), 389-399.

[75] Panero, M., Arzarello, F., & Sabena, C. (2014) Practices of Italian teachers with the derivative concept: a problematic meeting between Algebra and Analysis in secondary school (submitted). In *Proceedings of the Forth Symposium ETM* in Madrid.

[76] Peirce, C. S. (1977). *Semiotics and significs: the correspondence between Charles S. Peirce and Lady Victoria Welby.* Indiana University Press.

[77] Lombardo-Radice, L. (1966). Geometria degli indivisibili di Bonaventura Cavalieri. *Classici della Scienza*, n. 5, Unione tipografico-editrice torinese.

[78] Robert, A. (1982). L'acquisition de la notion de convergence des suites numériques dans l'enseignement supérieur. *Recherches en didactique des mathématiques, 3*(3), 307-341.

[79] Robert, A. (1998). Outils d'analyse des contenus mathématiques à enseigner au lycée et à l'université. *Recherches en didactique des mathématiques, 18*(2), 139-189.

[80] Rogalski, M. (2008). Les rapports entre local et global: mathématiques, rôle en physique élémentaire, questions didactiques. In L. Viennot (Ed.), *Didactique, épistémologie et histoire des sciences* (pp. 61-87).

[81] Sasso, L. (2012). *Nuova Matematica a colori.* Edizione BLU per la riforma. Quinto anno. Vol.5. Novara: Petrini, De Agostini Scuola.

[82] Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM, 43*(4), 457-469.

[83] Scott, J. F. (1981). *The Mathematical Work of John Wallis, DD, FRS,(1616-1703).* Chelsea Publishing Company.

[84] Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researches 15*(2), 4-14.

[85] Sierpinska, A. (1985). Obstacles épistémologiques relatifs à la notion de limite. *Recherches en didactique des mathématiques, 6*(1), 5-67.

[86] Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects on Epistemology and Pedagogy*, (pp. 25-58). USA: MAA.

[87] Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics, 22*(1), 1-36.

[88] Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics, 33*(3), 259-281.

[89] Speranza, F. (1996). "Il triangolo qualunque" è un qualunque triangolo? *L'Educazione Matematica, 17*, 13–27.

[90] Tall, D. O. (1987). Constructing the concept image of a tangent. In J. C. Bergeron *et al.* (Eds.) *Proceedings of the 11th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 69-75). Montreal: PME.

[91] Tall, D. O. (1989). Concept images, generic organizers, computers, and curriculum change. *For the learning of mathematics, 9*(3), 37-42.

[92] Tall, D. O. (1996). Functions and calculus. In A. Bishop, M. A. Clements, C. Keitel-Kreidt, J. Kilpatrick, C. Laborde (Eds.), *International handbook of mathematics education* (pp. 289-325). Springer Netherlands.

[93] Tall, D. O. (2000). Biological brain, mathematical mind and computational computers. In W.-C. Yang, S.-C. Chu & J.-C. Chuan (Eds.) *Proceedings of the 5th Asian Technology Conference in Mathematics* (pp. 3-20). Chiang Mai, Thailand.

[94] Tall, D. (2004). Thinking through three worlds of mathematics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281-288). Bergen, Norway: Bergen University College.

[95] Tall, D., Mc Gowen, M., & DeMarois, P. (2000.) The Function Machine as a Cognitive Root for the Function Concept. In M. L. Fernandez (Ed.), *Proceedings of 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 255-261). Tucson, AZ: PME-NA.

[96] Tall, D. O., & Schwarzenberger, R. L. (1978). Conflicts in the learning of real numbers and limits.*Mathematics Teaching, 82*, 44-49.

[97] Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics, 12*(2), 151-169.

[98] Unione Matematica Italiana. (2004). G. Anichini, F. Arzarello, L. Ciarrapico, & O. Robutti (Eds.), *Matematica 2003. La matematica per il cittadino. Attività didattiche e prove di verifica per un nuovo curricolo di Matematica (ciclo secondario).* Lucca: Matteoni stampatore.

[99] Vandebrouck, F. (2011a). Perspectives et domaines de travail pour l'étude des fonctions. In *Annales de Didactiques et de Sciences Cognitives, 16*, 149-185.

[100] Vandebrouck, F. (2011b). Students' conceptions of functions at the transition between secondary school and university. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2093-2102). University of Rzeszów, Poland.

[101] Van der Waerden, B. L. (1926). Zur Nullstellentheorie der Polynomideale, *Mathematische Annalen, 96*(2), 183–208. En. tr. in Schappacher, N. (2007). A historical

sketch of BL van der Waerden's work in algebraic geometry: 1926–1946. *Episodes in the history of modern algebra (1800-1950)* (pp. 245-283). Providence, RI: American Mathematical Society.

[102] Vinner, S. (1982). Conflicts between definitions and intuitions: the case of the tangent. In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 24-28). Antwerp: PME.

[103] Vinner, S. (1983). Concept definition, concept image and the notion of function. *The International Journal of Mathematical Education in Science and Technology, 14*, 293-305.

[104] Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for research in mathematics education, 20*(4), 356-366.

[105] Vollrath, H. J. (1989). Funktionales Denken. *Journal für Mathematik-Didaktik, 10*(1), 3-37.

[106] Yoon, C., Thomas, M. O., & Dreyfus, T. (2011). Gestures and insight in advanced mathematical thinking. *International Journal of Mathematical Education in Science and Technology, 42*(7), 891-901.

Appendices

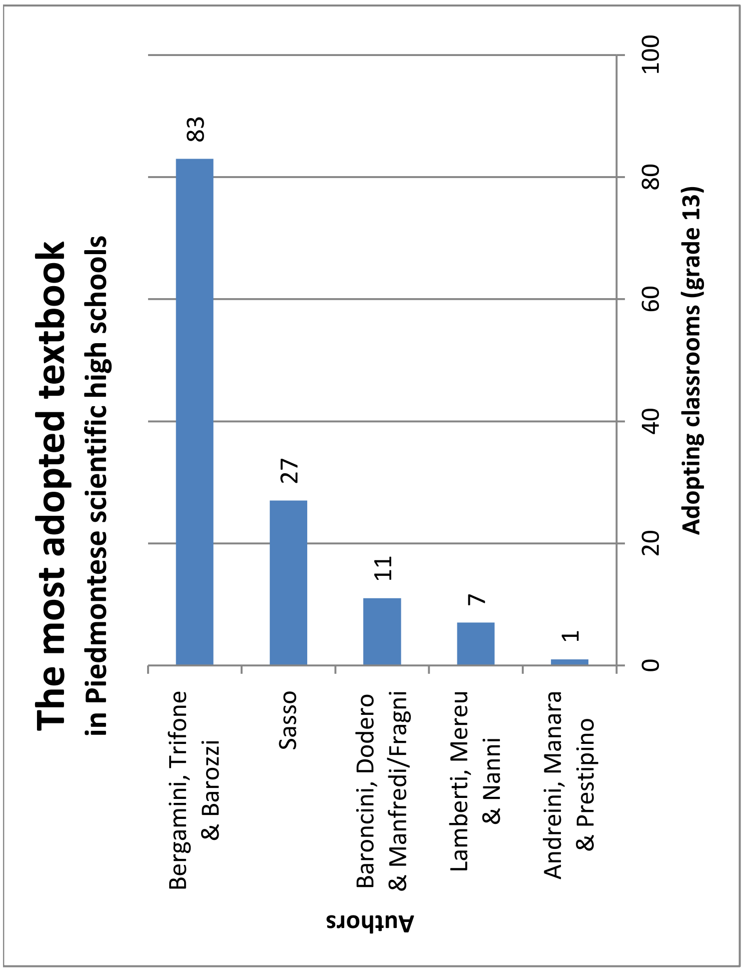
Appendix A

Choice of textbooks to analyse

Fond.Agnelli rank	School Name	Town/City	Author(s)	Title	Editor(s)	Adoptions number	Freq./ classrooms	Freq./ schools
4	**Peano**	Cuneo	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli V+W, SIGMA	Zanichelli	8	**83**	18
12	**Carlo Alberto**	Novara	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli U, V+W, SIGMA	Zanichelli	2		
13	**Amaldi**	Novi Ligure	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 3, 4, 5	Zanichelli	2		
13	**Amaldi**	Novi Ligure	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli V+W, SIGMA	Zanichelli	1		
16	**Baruffi**	Ceva	Bergamini Trifone Barozzi	Matematica.blu.2.0 vol. 5 Libro digitale multimediale	Zanichelli	1		
17	**Vercelli**	Asti	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli V+W, SIGMA	Zanichelli	5		
18	**Gobetti**	Torino	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 4, 5 Libro digitale multimediale	Zanichelli	3		
18	**Gobetti**	Torino	Bergamini Trifone Barozzi	Corso base blu di matematica vol. 5 moduli U V W alpha 1	Zanichelli	1		
19	**Bodoni**	Saluzzo	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli V+W, SIGMA	Zanichelli	3		
23	**Monti**	Chieri	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli V+W, SIGMA	Zanichelli	5		
24	**Curie**	Pinerolo	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli U, V+W, SIGMA	Zanichelli	11		
26	**Marconi**	Domodossola	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 4, 5 Libro digitale multimediale	Zanichelli	1		
27	**Spezia**	Domodossola	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 5 Libro digitale multimediale	Zanichelli	4		
30	**Antonelli**	Novara	Bergamini Trifone Barozzi	Corso base blu di matematica vol. 5 moduli U V W alpha 1	Zanichelli	6		
31	**Fermi**	Arona	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 3,4,5 Libro digitale multimediale	Zanichelli	5		
32	**Galilei**	Ciriè	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 Moduli U, V+W, SIGMA	Zanichelli	4		
33	**Catteneo**	Torino	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 5 Libro digitale multimediale	Zanichelli	6		
34	**Ferraris**	Torino	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5	Zanichelli	11		
35	**Parodi**	Acqui Terme	Bergamini Trifone Barozzi	Matematica.blu 2.0 vol. 5 Libro digitale multimediale	Zanichelli	1		
38	**Spinelli**	Torino	Bergamini Trifone Barozzi	Manuale Blu 2.0 di Matematica vol. 5 +La seconda prova di matematica ed. 2014	Zanichelli	3		

6	**Arimondi**	Savigliano	Sasso	Nuova matematica a colori, Ed. blu, vol. 5	Petrini	4	**27**	6
9	**Cocito**	Alba	Sasso	Nuova matematica a colori, Ed. blu, vol. 5	Petrini	6		
10	**Vasco**	Mondovì	Sasso	Nuova matematica a colori, Ed. blu, moduli D E F G H	Petrini	3		
18	**Gobetti**	Torino	Sasso	Nuova matematica a colori, Ed. blu, moduli G, H	Petrini	6		
28	**Cavalieri**	Verbania	Sasso	Nuova matematica a colori, Ed. blu, vol. 5	Petrini	3		
33	**Catteneo**	Torino	Sasso	Nuova matematica a colori, Ed. blu, vol. 5	Petrini	5		
25	**Avogadro**	Biella	Baroncini Dodero Manfredi	Lineamenti.Math Blu vol. 4, 5	Ghisetti & Corvi	7	**11**	3
35	**Parodi**	Acqui Terme	Baroncini Dodero Manfredi	Nuovi Lineamenti di Matematica Licei 4 + Lineamenti.Math Blu vol. 5	Ghisetti & Corvi	2		
19	**Bodoni**	Saluzzo	Baroncini Manfredi Fragni	Lineamenti.Math Blu - Modulo H	Ghisetti & Corvi	2		
5	**Ancina**	Fossano	Lamberti Mereu Nanni	Nuovo Lezioni di Matematica D, E	Etas Scuola	2	**7**	3
17	**Vercelli**	Asti	Lamberti Mereu Nanni	Nuovo Lezioni di Matematica E	Etas Scuola	3		
30	**Antonelli**	Novara	Lamberti Mereu Nanni	Nuovo Lezioni di Matematica E	Etas Scuola	2		
36	**Vallauri**	Fossano	Andreini Manara Prestipino	Pensare e fare Matematica set 3 vol.3	Etas Scuola	1	**1**	1

Note: Data of the *Istututo Superiore "Baldessano-Roccati"* (Carmagnola, TO) were not accessible on the school website. Even though it is the 18th scientific high school of the general *Fondazione Agnelli* classification, with the rank of 29, we skipped it. Thus, in order to account for 25 scientific high schools, we considered the 26th scientific high school in the list (*Liceo Scientifico "Spinelli"*, Torino).


The most adopted textbook
in Piedmontese scientific high schools
Authors
Bergamini, Trifone & Barozzi
Sasso
Baroncini, Dodero & Manfredi/Fragni
Lamberti, Mereu & Nanni
Andreini, Manara & Prestipino
83
27
11
7
1
0
20
40
60
80
100
Adopting classrooms (grade 13)

Appendix B

Scientific high school (P.N.I.) National Exam 2013

Sessione ordinaria 2013
Seconda prova scritta

Ministero dell'Istruzione, dell'Università e della Ricerca

Y557 – ESAME DI STATO DI LICEO SCIENTIFICO

CORSO SPERIMENTALE

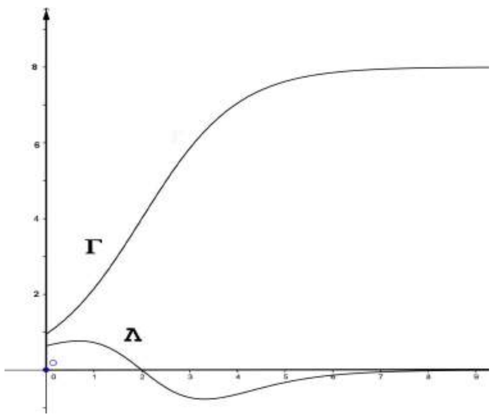
Indirizzo: PIANO NAZIONALE INFORMATICA

Tema di: MATEMATICA

Il candidato risolva uno dei due problemi e risponda a 5 quesiti del questionario.

PROBLEMA 1

Una funzione $f(x)$ è definita e derivabile, insieme alle sue derivate prima e seconda, in $[0,+\infty[$ e nella figura sono disegnati i grafici Γ e Λ di $f(x)$ e della sua derivata seconda $f''(x)$. La tangente a Γ nel suo punto di flesso, di coordinate $(2;4)$, passa per $(0;0)$, mentre le rette $y=8$ e $y=0$ sono asintoti orizzontali per Γ e Λ, rispettivamente.



1) Si dimostri che la funzione $f'(x)$, ovvero la derivata prima di $f(x)$, ha un massimo e se ne determinino le coordinate. Sapendo che per ogni x del dominio è: $f''(x) \leq f'(x) \leq f(x)$, qual è un possibile andamento di $f'(x)$?
2) Si supponga che $f(x)$ costituisca, ovviamente in opportune unità di misura, il modello di crescita di un certo tipo di popolazione. Quali informazioni sulla sua evoluzione si possono dedurre dai grafici in figura e in particolare dal fatto che Γ presenta un asintoto orizzontale e un punto di flesso?
3) Se Γ è il grafico della funzione $f(x) = \dfrac{a}{1+e^{b-x}}$, si provi che $a=8$ e $b=2$.
4) Nell'ipotesi del punto 3), si calcoli l'area della regione di piano delimitata da Λ e dall'asse x sull'intervallo $[0,2]$.

PROBLEMA 2

Sia f la funzione definita per tutti gli x positivi da $f(x) = x^3 \ln x$.

1. Si studi f e si tracci il suo grafico γ su un piano riferito ad un sistema di assi cartesiani ortogonali e monometrici *Oxy*; accertato che γ presenta sia un punto di flesso che un punto di minimo se ne calcolino, con l'aiuto di una calcolatrice, le ascisse arrotondate alla terza cifra decimale.
2. Sia P il punto in cui γ interseca l'asse *x*. Si trovi l'equazione della parabola, con asse parallelo all'asse y, passante per l'origine e tangente a γ in P.
3. Sia R la regione delimitata da γ e dall'asse x sull'intervallo aperto a sinistra $]0,1]$. Si calcoli l'area di R, illustrando il ragionamento seguito, e la si esprima in mm^2 avendo supposto l'unità di misura lineare pari a 1 *decimetro*.
4. Si disegni la curva simmetrica di γ rispetto all'asse y e se ne scriva altresì l'equazione. Similmente si faccia per la curva simmetrica di γ rispetto alla retta $y=-1$.

Ministero dell'Istruzione, dell'Università e della Ricerca

Y557 – ESAME DI STATO DI LICEO SCIENTIFICO

CORSO SPERIMENTALE

Indirizzo: PIANO NAZIONALE INFORMATICA

Tema di: MATEMATICA

QUESTIONARIO

1. Un triangolo ha area 3 e due lati che misurano 2 e 3. Qual è la misura del terzo lato? Si giustifichi la risposta.
2. Se la funzione $f(x)-f(2x)$ ha derivata 5 in $x=1$ e derivata 7 in $x=2$, qual è la derivata di $f(x)-f(4x)$ in $x=1$?
3. Si considerino, nel piano cartesiano, i punti $A(2;-1)$ e $B(-6;-8)$. Si determini l'equazione della retta passante per *B* e avente distanza massima da *A*.
4. Di un tronco di piramide retta a base quadrata si conoscono l'altezza *h* e i lati *a* e *b* delle due basi. Si esprima il volume V del tronco in funzione di *a*, *b* e *h*, illustrando il ragionamento seguito.
5. In un libro si legge: *"se per la dilatazione corrispondente a un certo aumento della temperatura un corpo si allunga (in tutte le direzioni) di una certa percentuale (p.es. 0,38%), esso si accresce in volume in proporzione tripla (cioè dell'1,14%), mentre la sua superficie si accresce in proporzione doppia (cioè di 0,76%)"*. È così? Si motivi esaurientemente la risposta.
6. Con le cifre da 1 a 7 è possibile formare $7!=5040$ numeri corrispondenti alle permutazioni delle 7 cifre. Ad esempio i numeri 1234567 e 3546712 corrispondono a due di queste permutazioni. Se i 5040 numeri ottenuti dalle permutazioni si dispongono in ordine crescente qual è il numero che occupa la 5036-esima posizione e quale quello che occupa la 1441-esima posizione?
7. In un gruppo di 10 persone il 60% ha occhi azzurri. Dal gruppo si selezionano a caso due persone. Quale è la probabilità che nessuna di esse abbia occhi azzurri?
8. Si mostri, senza utilizzare il teorema di *l'Hôpital*, che:
$$\lim_{x\to\pi}\frac{e^{senx}-e^{sen\pi}}{x-\pi}=-1$$
9. Tre amici discutono animatamente di numeri reali. Anna afferma che sia i numeri razionali che gli irrazionali sono infiniti e dunque i razionali sono tanti quanti gli irrazionali. Paolo sostiene che gli irrazionali costituiscono dei casi eccezionali, ovvero che la maggior parte dei numeri reali sono razionali. Luisa afferma, invece, il contrario: sia i numeri razionali che gli irrazionali sono infiniti, ma esistono più numeri irrazionali che razionali. Chi ha ragione? Si motivi esaurientemente la risposta.
10. Si stabilisca per quali valori $k\in R$ l'equazione $x^2(3-x)=k$ ammette due soluzioni distinte appartenenti all'intervallo $[0,3]$. Posto $k=3$, si approssimi con due cifre decimali la maggiore di tali soluzioni, applicando uno dei metodi iterativi studiati.

Durata massima della prova: 6 ore.
È consentito l'uso della calcolatrice non programmabile.
È consentito l'uso del dizionario bilingue (italiano-lingua del paese di provenienza) per i candidati di madrelingua non italiana.
Non è consentito lasciare l'Istituto prima che siano trascorse 3 ore dalla dettatura del tema.

Appendix C

Activity 1

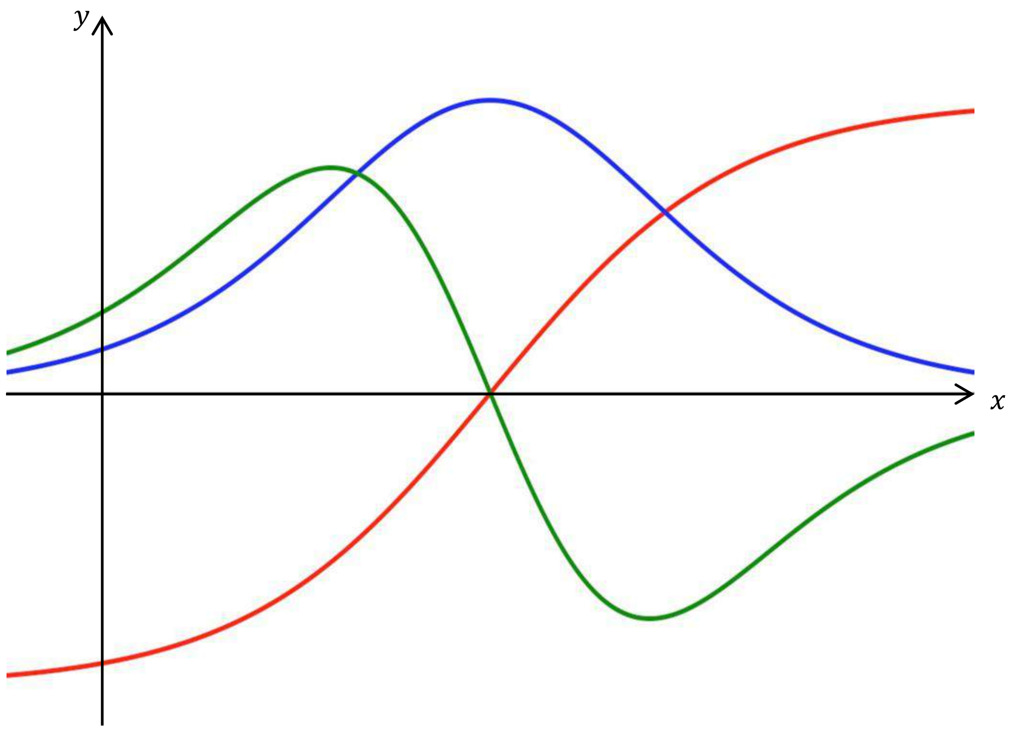
SCHEDA 1 **Nomi:** ____

Mentre studiavano per l'esame di maturità, Lorenzo e Francesca si sono imbattuti nel seguente problema, trovato in un libro di testo.

Problema

Il disegno qui sotto mostra i grafici di tre funzioni.
Individua tra di essi la giusta combinazione:

- una funzione f
- f' : la derivata di f
- f'' : la derivata di f'



Scrivete una breve lettera a Lorenzo e Francesca, in cui fornite e giustificate la vostra soluzione del problema.

SCHEDA 2 **Nomi:** ________________________________

Ecco la risposta di Lorenzo e Francesca alla vostra lettera.

Grazie mille, ragazzi!

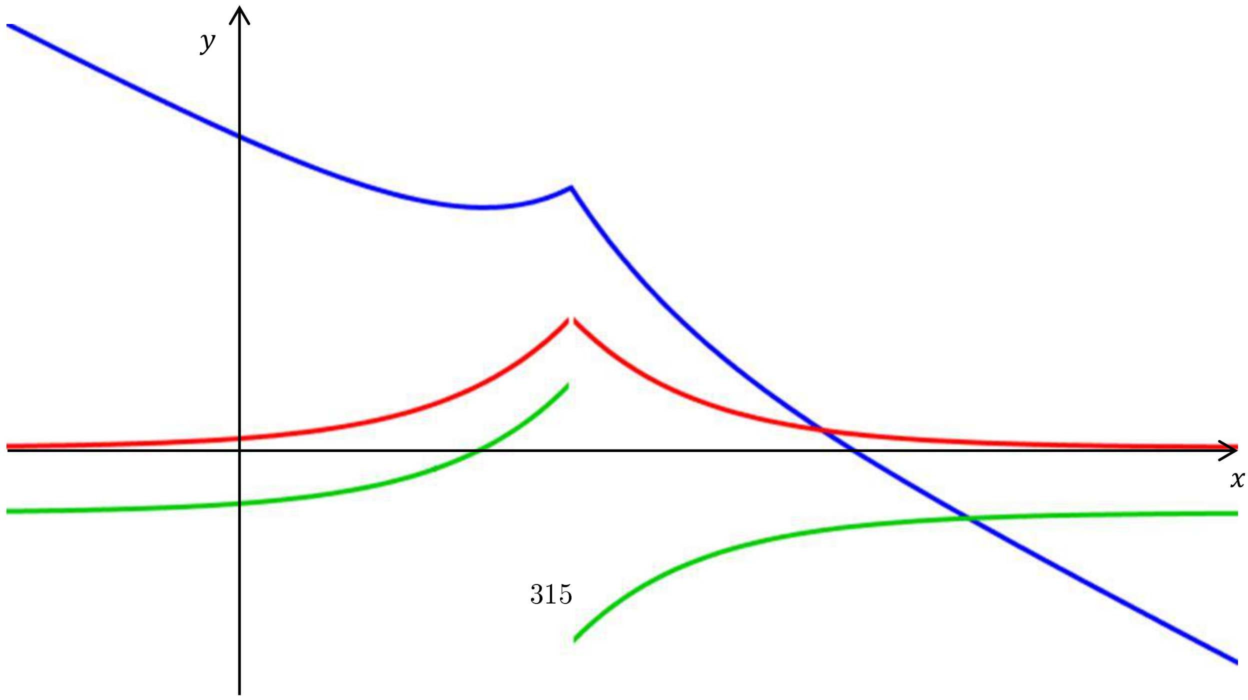
Siete stati gentilissimi, ma abbiamo ancora dei dubbi...
Nelle pagine successive, infatti, abbiamo trovato un altro problema di questo tipo.
Ci sembra un po' più difficile del primo e non siamo d'accordo sulla soluzione.
Ne riportiamo il testo qui sotto e vorremmo sapere come lo risolvereste voi.
Per evitare di disturbarvi ogni volta, abbiamo anche pensato che ci servirebbe un "metodo generale" che funzioni per tutti i problemi di questo tipo. Ci spieghiamo meglio: voi sapreste indicarci, in generale, ***quali proprietà di una funzione e della sua derivata dobbiamo considerare e come le dobbiamo mettere in relazione tra loro?***

Grazie ancora!
Lollo e Fra

Problema 2

Il disegno qui sotto mostra i grafici di tre funzioni.
Individua tra di essi la giusta combinazione:

- una funzione f
- f' : la derivata di f
- f'' : la derivata di f'



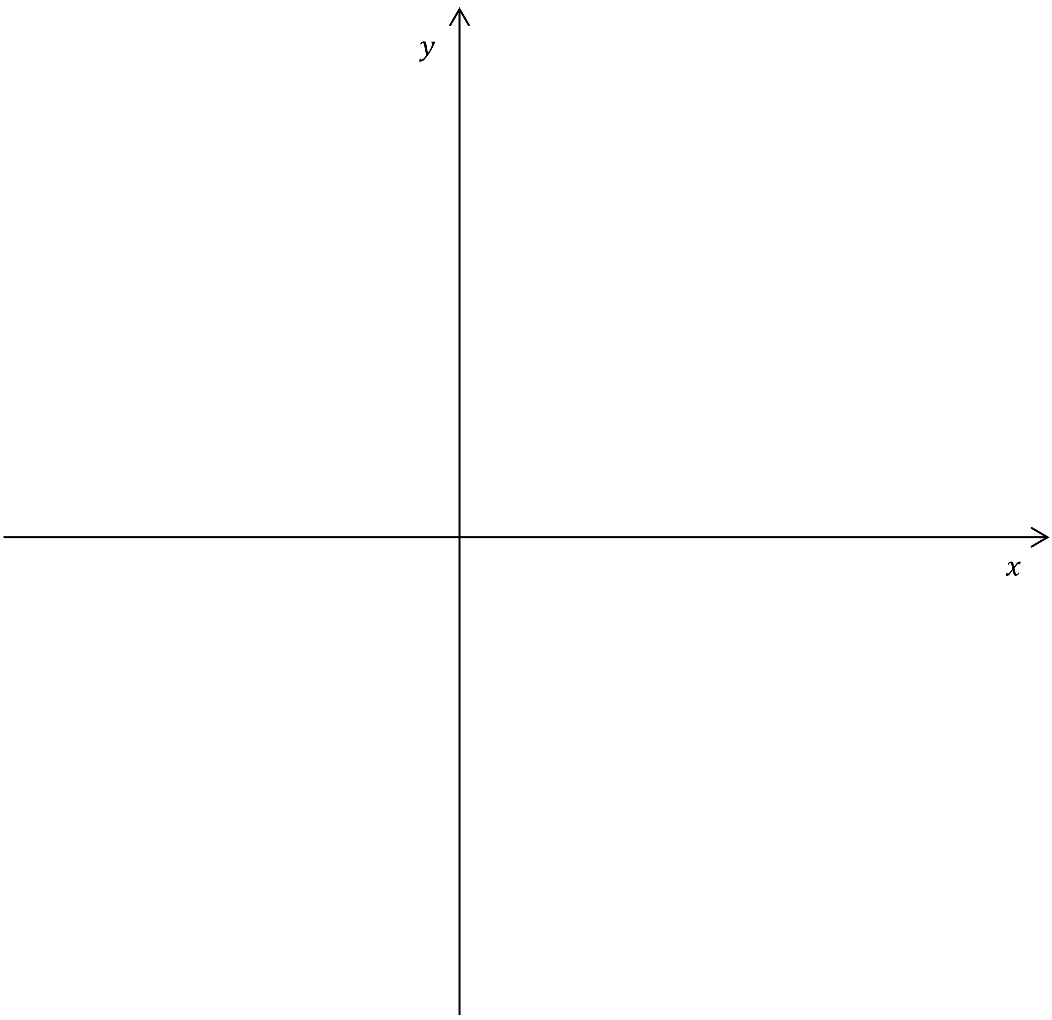
SCHEDA 3 **Nomi:** ____

Allegate alla vostra risposta a Lorenzo e Francesca un ipotetico **Problema 3***, per il quale il vostro "metodo generale" funzioni e spiegate brevemente come avete ragionato per costruire i grafici del problema.*

Problema 3

Il disegno qui sotto mostra i grafici di tre funzioni.
Individua tra di essi la giusta combinazione:

- una funzione f
- f' : la derivata di f
- f'' : la derivata di f'



Appendix D

Activity 2

SCHEDA A

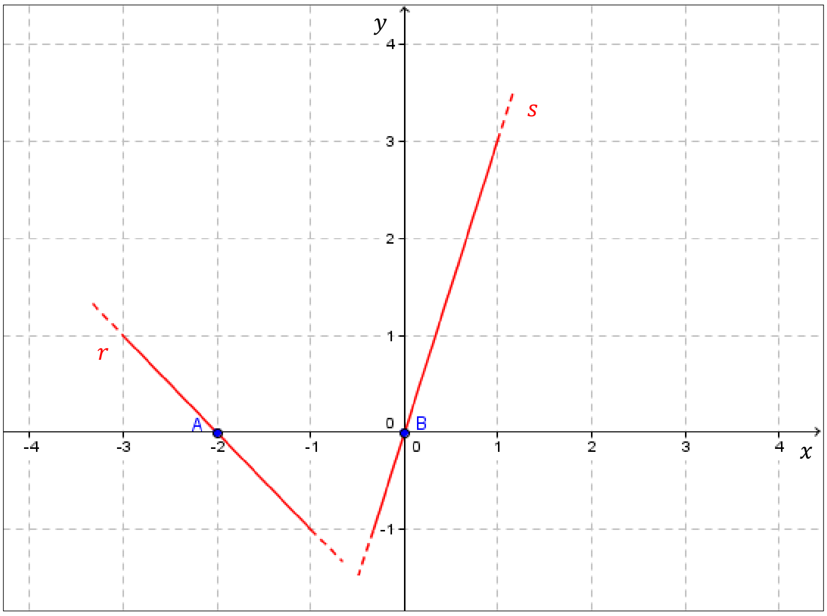
Nomi: ______________________________

Risolvete i seguenti problemi. Giustificate le vostre risposte, specificando il ragionamento che avete seguito.

Problema 1. Della funzione f si sa che il suo grafico

- passa per i punti A e B
- è tangente in A alla retta r e in B alla retta s.

Scrivi una possibile equazione $y = f(x)$ per questa funzione.



Problema 2. Due curve si definiscono *tangenti* in un loro punto comune se e solo se hanno in questo punto la stessa retta tangente. Stabilisci per quale valore esatto del parametro reale $k > 0$, le curve di equazione

$$f(x) = kx^2 \quad \text{e} \quad g(x) = \ln x$$

risultano tangenti. Quali sono le coordinate del punto di tangenza?

SCHEDA B

Nomi: ______________________________

Risolvete i seguenti problemi. Giustificate le vostre risposte, specificando il ragionamento che avete seguito.

Problema 1. Della funzione f si sa che

- $x_1 = -2$ e $x_2 = 0$ sono suoi zeri
- la curva di equazione $y = f(x)$ è tangente in x_1 alla retta $r: x + y + 2 = 0$ e in x_2 alla retta $s: 3x - y = 0$.

Scrivi una possibile equazione $y = f(x)$ per questa funzione.

Problema 2. Due curve si definiscono *tangenti* in un loro punto comune se e solo se hanno in questo punto la stessa retta tangente. Stabilisci per quale valore esatto del parametro reale $k > 0$, le curve di equazione

$$f(x) = kx^2 \quad \text{e} \quad g(x) = \ln x$$

risultano tangenti. Quali sono le coordinate del punto di tangenza?