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# SUBHARMONIC SOLUTIONS FOR NONLINEAR SECOND ORDER EQUATIONS IN PRESENCE OF LOWER AND UPPER SOLUTIONS

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Dedicated to Jean Mawhin on the occasion of his 70th birthday

Abstract. We study the problem of existence and multiplicity of subharmonic solutions for a second order nonlinear ODE in presence of lower and upper solutions. We show how such additional information can be used to obtain more precise multiplicity results. Applications are given to pendulum type equations and to Ambrosetti-Prodi results for parameter dependent equations.

## 1. Introduction.

The problem of existence and multiplicity of subharmonic solutions for Hamiltonian systems is classical and has been widely investigated. The first attempts in this direction seem to go back to the pioneering work by Birkhoff and Lewis [3], which provides results of local nature by perturbation-type techniques, while global (variational) methods were employed by Rabinowitz [40].

There is not an univocal definition of subharmonic solution. Indeed, the typical framework is that of a nonlinear (Hamiltonian) vector field, non-autonomous and $T$-periodic in the time variable (for some $T > 0$) and the general rule is that one has to search for $kT$-periodic solutions, with $k$ an integer number, accompanied with as much information as possible about the minimality of the period.
The proof of the existence of $kT$-periodic solutions can be performed via different methods but, as a matter of fact, considerations concerning the minimality of the period always require further investigations. From this point of view, different techniques have been proposed: for instance, proving the existence of an unbounded sequence of solutions in a context where it is possible to obtain a priori bounds for solutions of a given period [35, 40], or performing (in a variational setting) some careful estimates of the Morse indexes or of the critical levels of the solutions

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[19, 20, 44]. Summing up, it is the method itself which often suggests the most appropriate definition of subharmonic solution in the given situation.

In the special case of *planar Hamiltonian systems*, a powerful tool to detect information about the minimality of the period is based on estimates of the rotation numbers associated to (nontrivial) periodic solutions. Such rotation numbers count the essential turns of the solutions around a given point (usually the origin) the plane. With this respect, the application of the Poincaré-Birkhoff fixed point theorem turns out to be particularly useful. In fact, the so-called *twist condition* (for the associated Poincaré map, or for its iterates) can often be expressed as a gap between the numbers of revolutions associated with the solutions departing from the inner and the outer boundaries of a topological annulus; as a consequence, the fixed points whose existence is guaranteed by the Poincaré-Birkhoff theorem are naturally accompanied by an information about the rotation number of the corresponding periodic solution. Using this fact it is possible to deduce results about the minimality of the period (as well as to distinguish among solutions with the same minimal period) and, therefore, to prove sharp multiplicity results. In this framework, it is natural to call a *subharmonic solution of order* $k$, with $k$ integer, a $kT$-periodic solution which is not $lT$-periodic for every integer $l = 1, \ldots, k-1$. Such a definition of subharmonic, which corresponds to the one in [38] and was employed, for instance, in [14, 42], will be used throughout the paper.

A typical situation in which this "rotation number approach", based on the Poincaré-Birkhoff theorem, works is that of scalar undamped second order ODEs like

$$v'' + g(t, v) = 0, \tag{1.1}$$

in which the nonlinearity $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $T$-periodic in the first variable and exhibits a sublinear growth at infinity, that is to say, $g(t, x)/x \to 0$ as $x \to \pm\infty$. Actually, in several cases it is possible to show that only one-sided conditions suffice in order to obtain periodic solutions (see [21] and the references therein).

From the point of view of the rotation numbers, the sublinear growth at infinity of the nonlinearity implies that "large" solutions in the phase-plane rotate very slowly around the origin. Hence, for an arbitrarily large but fixed time interval $[0, kT]$, it is possible to prove that solutions departing sufficiently far away from the origin will be unable to perform a complete turn. On the other hand, if we are able to show that "smaller" solutions make more than one revolution (around the origin) in $[0, kT]$, then we can conclude that a twist condition is satisfied for the $k$-th iterate of the associated Poincaré map. In this manner, the $kT$-periodic solutions we find via the Poincaré-Birkhoff theorem perform *exactly one turn* around the origin in $[0, kT]$ and hence they have $kT$ as minimal period. This argument can be easily modified, whenever small solutions make a greater number of revolutions, providing for instance the existence of $kT$-periodic solutions performing $j$ turns in a time interval $[0, kT]$. The minimality of the period (at least within the class of the integer multiples of $T$, as in our definition of $k$-th order subharmonic) will be guaranteed whenever $k$ and $j$ are relatively prime integers (see [14, pp. 523-524]).

Such a kind of strategy has been proposed in a recent article [4], where the following result is proved.

**Theorem 1.1.** *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, $T$-periodic in the first variable and such that $g(t, 0) \equiv 0$. Moreover, suppose that the uniqueness and the* *global continuability for the solutions to the Cauchy problems associated with (1.1) are ensured. Finally, assume that:*

$(g_0)$ *there exists $q \in L^1(0,T)$ with $\int_0^T q(t)\,dt > 0$ such that*

$$\liminf_{x\to 0} \frac{g(t,x)}{x} \geq q(t), \quad \textit{uniformly in } t \in [0,T];$$

$(g_\infty)$ *uniformly in $t \in [0,T]$,*

$$\limsup_{x\to+\infty} \frac{g(t,x)}{x} \leq 0.$$

*Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \geq k^*$, there exists an integer $m_k$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \leq j \leq m_k$, equation (1.1) has at least two subharmonic solutions $v^{(1)}_{j,k}(t), v^{(2)}_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) with exactly $2j$ zeros in the interval $[0,kT[$.*

**Remark 1.** The proof of Theorem 1.1 relies on a version of the Poincaré-Birkhoff theorem given by W. Ding in [15] dealing with a standard annulus. Recent works [28, 26] have raised some problems about Ding's version of the theorem published in [16] (which is the most commonly quoted one). We stress that our result comes from a "safer" statement of the theorem which could be also deduced from other versions of the Poincaré-Birkhoff theorem due to Franks [22], Rebelo [41] and Qian-Torres [39], where independent proofs are given. ◁

In [4, Corollary 3.1] the conclusion about subharmonics was expressed in slightly different form, although equivalent to this one (see Remark 2 below).
We have denoted by $\mathbb{N}_0$ the set of positive integers, while by the fact that $v^{(1)}_{j,k}(t)$ and $v^{(2)}_{j,k}(t)$ do not belong to the same periodicity class we mean that each of them is not a time translation of the other one by an integer multiple of $T$, i.e., $v^{(1)}_{j,k}(\cdot) \not\equiv v^{(2)}_{j,k}(\cdot + lT)$ for $l = 1, \ldots, k-1$.
It is worth noticing that, taking $j = 1$, Theorem 1.1 in particular ensures the existence of subharmonic solutions of order $k$ for *all* $k$ large enough (and, as explained before, in this case $kT$ is the true minimal period of such solutions). As pointed out in [6, p. 428] this is a sharper conclusion with respect to those of papers using variational techniques, when typically only a sequence of solutions with minimal periods tending to infinity is provided.

**Remark 2.** We emphasize that, from the proof of the above theorem in [4] together with the estimates developed in [5, Proposition 3.2], the following estimate for $m_k$ can be deduced:

$$m_k \geq \mathscr{E}^-\left(\frac{k}{2\pi}\sup_{\xi>0}\frac{\int_0^T \min\{q(t),\xi\}\,dt}{\sqrt{\xi}}\right), \tag{1.2}$$

where we have denoted by $\mathscr{E}^-(x)$ the greatest integer strictly less then $x$, i.e.

$$\mathscr{E}^-(x) := \begin{cases} \lfloor x \rfloor & \text{if } x \notin \mathbb{N}_0, \\ x-1 & \text{if } x \in \mathbb{N}_0. \end{cases}$$

Such a quantity is strictly related to the rotation numbers of "small" (i.e., near the equilibrium point) solutions to (1.1); we refer the reader to [5, Remark 3.3] for a more detailed discussion about this point. It is obvious that $m_k \geq 1$ provided that $k$ is sufficiently large. In this manner we can find $k^*$.

We finally observe that the subharmonics of Theorem 1.1 can be counted also in a different way, by arguing as follows. Fix a positive integer $M$ and suppose that we are interested in the existence of subharmonics of order $k$, having $2j$-zeros in the time interval $[0, kT[$, for every $j = 1, \dots, M$. Since the map

$$\varphi^-(k) : \mathbb{N}_0 \ni k \mapsto \mathscr{E}^-\left(\frac{k}{2\pi} \sup_{\xi>0} \frac{\int_0^T \min\{q(t), \xi\}\, dt}{\sqrt{\xi}}\right)$$

is nondecreasing, and $\varphi^-(k) \to +\infty$ for $k \to +\infty$, we can choose the smallest positive integer $k^*(M)$ such that

$$\varphi^-(k) \geq M$$

holds. With such a choice, using Theorem 1.1 and estimate (1.2) we can conclude that *for every integer* $k \geq k^*(M)$ *and for every integer* $j$ *relatively prime with* $k$ *and such that* $1 \leq j \leq M$*, equation* (1.1) *has at least two subharmonic solutions* $v^{(1)}_{j,k}(t), v^{(2)}_{j,k}(t)$ *of order* $k$ *(not belonging to the same periodicity class) with exactly* $2j$ *zeros in the interval* $[0, kT[$. In particular, if we are able to prove that $k^*(M) = 1$ for some $M$ (usually this will occur for a not too large $M$, as $k^*(M_1) \leq k^*(M_2)$ for $M_1 \leq M_2$), we get the existence of $M$ pairs of $T$-periodic solutions, having $2, 4, \dots, 2M$ zeros in the time interval $[0, T[$. $\triangleleft$

The aim of this paper is that of showing how Theorem 1.1, which requires a "local" assumption (at zero) paired with a "global" assumption (at infinity), can be suitably used in order to produce subharmonics which are confined between a lower and an upper ($T$-periodic) solution. To be more precise, we consider a scalar undamped second order equation

$$u'' + f(t, u) = 0, \tag{1.3}$$

being $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function $T$-periodic in the first variable, and, as a first step, we prove the existence of a $T$-periodic solution to (1.3), say $u^*(t)$. Such a solution will become our "equilibrium" in order to enter in the setting of Theorem 1.1 and, accordingly, the function $q(t)$ in condition $(g_0)$ will be the outcome of a suitable linearization around $u^*(t)$. Note that only a positive average for such linearization will be required. This, in principle, allows our results to be applied to problems with sign indefinite weights (see [43] for previous results about subharmonics using average type conditions). In our applications of Section 3 and Section 4 the fundamental role for this first step is played by Mawhin's coincidence degree [24, 29], which turns out to be a particularly useful tool since it permits to localize our solution $u^*(t)$ in a quite precise manner. The more information we can obtain about the range of $u^*(t)$, the finer conditions on the function $q(t)$ in $(g_0)$ can be derived. Having in mind the dynamical interpretation of Theorem 1.1, such a construction provides the inner boundary of the annulus to which the Poincaré-Birkhoff theorem applies.

As a second step, we use upper and lower solutions techniques, in order to perform a suitable truncation on the nonlinearity, leading to a (possible one-sided) sublinear problem at infinity, namely with $(g_\infty)$ satisfied. At the level of the dynamical properties of the solutions in the phase-plane again, this second step reflects the behavior of large-amplitude solutions, providing the outer boundary of the annulus of the twist theorem.

This approach appears general enough to be performed in various different situations. For instance, one could combine the results in [27] or those in [48] to obtain multiplicity of $T$-periodic solutions which are confined between a lower and an upper solution.

The plan of the paper is the following. In Section 2 we present our main general results (Theorem 2.1 and Theorem 2.2), which produce multiple subharmonic solutions via Theorem 1.1, in a context where the existence of a $T$-periodic solution is paired with that of a lower and/or an upper solution. Some models to which Theorem 2.1 directly applies are also discussed.
In Sections 3 and 4 we propose some further applications which illustrate our approach, combining coincidence's degree theory with Theorem 2.1 and Theorem 2.2. The first one (see Section 3) deals with a pendulum type equation, while the second one (see Section 4) is addressed to the study of an Ambrosetti-Prodi type problem, concerning the number of (subharmonic) solutions in dependence of a parameter. We refer to the corresponding sections for more comments on the problems, as well as for a comparison between our results and the existing literature. Both the examples are inspired by previous works of Professor Jean Mawhin and his collaborators.

2. **The main results.** In this section, we prove our main results for equation

$$u'' + f(t,u) = 0, \tag{2.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $T$-periodic in the first variable, with $T > 0$ fixed. For simplicity, we will always assume that the following regularity assumption for $f(t,x)$ is fulfilled:

$(f)$ $\quad f(t,x)$ is continuous with continuous partial derivative $\dfrac{\partial f}{\partial x}(t,x)$.

As a preliminary result, we state a corollary which can be deduced in a standard way from Theorem 1.1 of the Introduction.

**Proposition 1.** *Suppose that the global continuability for the solutions to (2.1) is ensured and that*

$$\limsup_{x\to+\infty} \frac{f(t,x)}{x} \le 0, \quad \textit{uniformly in } t \in [0,T]. \tag{2.2}$$

*Moreover, assume that (2.1) has a $T$-periodic solution $u^*(t)$ such that*

$$\int_0^T \frac{\partial f}{\partial x}(t,u^*(t))\, dt > 0. \tag{2.3}$$

*Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \ge k^*$, there exists an integer $m_k$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \le j \le m_k$, equation (2.1) has at least two subharmonic solutions $u^{(1)}_{j,k}(t), u^{(2)}_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) such that, for $i = 1,2$, $u^{(i)}_{j,k}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0,kT[$.*

*Proof.* We set

$$g(t,x) = f(t,x+u^*(t)) - f(t,u^*(t))$$

and we consider the equation

$$v'' + g(t,v) = 0. \tag{2.4}$$

Since $g(t,0) \equiv 0$, this is an equation like in Theorem 1.1 and, as it is clear, $v(t)$ is a solution of (2.4) if and only if $u(t) = u^*(t) + v(t)$ is a solution of (2.1). As a consequence, the global continuability for the solutions to (2.4) holds. Moreover, it is easy to see that $v(t)$ is a subharmonic of order $k$ of (2.4) if and only if $u(t)$ is a subharmonic of order $k$ of (2.1); hence, to conclude we just have to show that the other assumptions of Theorem 1.1 are satisfied. The regularity of $f(t,x)$ ensures that the uniqueness for the solutions to the Cauchy problems holds; moreover, it is easy to see that (2.2) implies condition $(g_\infty)$ of Theorem 1.1. Finally, we check that $(g_0)$ holds true with the choice $q(t) = \frac{\partial f}{\partial x}(t, u^*(t))$. Indeed, by the Langrange theorem we can write

$$\frac{g(t,x)}{x} = \frac{f(t, x + u^*(t)) - f(t, u^*(t))}{x} = \frac{\partial f}{\partial x}(t, \xi(t,x)),$$

for a suitable $\xi(t,x)$ such that $|\xi(t,x) - u^*(t)| \leq |x|$. Letting $x \to 0$, in view of the uniform continuity of $\frac{\partial f}{\partial x}(t,x)$ on compact subsets, we get

$$\frac{\partial f}{\partial x}(t, \xi(t,x)) \to \frac{\partial f}{\partial x}(t, u^*(t)) = q(t), \quad \text{uniformly in } t \in [0,T]$$

and hence $g(t,x)/x \to q(t)$ for $x \to 0$ as desired. □

**Remark 3.** In view of the proofs of Theorem 2.1 and Theorem 2.2, which apply Proposition 1 via a truncation argument, it is crucial to observe that the conclusion of Corollary 1 still persists if we only assume that $f(t,x)$ is locally Lipschitz continuous in $x$ (in order to ensure the uniqueness for the solutions to the Cauchy problems) together with the existence of the continuous partial derivative $\frac{\partial f}{\partial x}(t,x)$ for every $(t,x)$ with $x \in [u^*(t) - \delta, u^*(t) + \delta]$, being $\delta > 0$ a (possibly small) constant, independent on $(t,x)$. ◁

We are now ready to state and prove our first main result.

**Theorem 2.1.** *Let us suppose that:*

$(a_1)$ *there exists a $T$-periodic solution $u^*(t)$ of (2.1) satisfying (2.3),*

$(a_2)$ *there exists a $T$-periodic function $\alpha(t)$, of class $C^2$, and such that*

$$\alpha''(t) + f(t, \alpha(t)) \geq 0, \quad \text{for every } t \in \mathbb{R}; \tag{2.5}$$

$(a_3)$ *there exists a $T$-periodic function $\beta(t)$, of class $C^2$, and such that*

$$\beta''(t) + f(t, \beta(t)) \leq 0, \quad \text{for every } t \in \mathbb{R}. \tag{2.6}$$

*Assume, moreover, that*

$$\alpha(t) < u^*(t) < \beta(t), \quad \text{for every } t \in \mathbb{R}. \tag{2.7}$$

*Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \geq k^*$, there exists an integer $m_k$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \leq j \leq m_k$, equation (2.1) has at least two subharmonic solutions $u^{(1)}_{j,k}(t), u^{(2)}_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{j,k}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$\alpha(t) \leq u^{(i)}_{j,k}(t) \leq \beta(t), \quad \text{for every } t \in \mathbb{R}. \tag{2.8}$$

*Proof.* Define the function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting

$$\tilde{f}(t,x) = \begin{cases} f(t, \alpha(t)) & \text{for } x \leq \alpha(t) \\ f(t,x) & \text{for } \alpha(t) < x \leq \beta(t) \\ f(t, \beta(t)) & \text{for } \beta(t) < x \end{cases}$$

Being $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, $\tilde{f}(t,x)$ is well defined, $T$-periodic in the first variable, continuous and locally Lipschitz continuous in $x$. By the regularity assumption $(f)$ on $f(t,x)$, there exists $\frac{\partial \tilde{f}}{\partial x}(t,x) = \frac{\partial f}{\partial x}(t,x)$ in a neighborhood of $(t, u^*(t))$. Hence, it is clear that $\tilde{f}(t,x)$ satisfies all the conditions of Proposition 1, taking into account also Remark 3.

All we need to conclude is the following.

Claim: *if $u(t)$ is a $kT$-periodic solution of*

$$u'' + \tilde{f}(t,u) = 0 \tag{2.9}$$

*such that $u(t) - u^*(t)$ has at least one zero in $\mathbb{R}$, then*

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \text{for every } t \in \mathbb{R}.$$

We prove that $u(t) \leq \beta(t)$, the other inequality being analogous. Let us suppose by contradiction that, for some $t^* \in \mathbb{R}$, $\beta(t^*) < u(t^*)$ and define $[t^-, t^+]$ to be the largest interval containing $t^*$ and such that $\beta(t) < u(t)$ for every $t \in ]t^-, t^+[$. Since $u^*(t) < \beta(t)$ and, by $kT$-periodicity, $u(t) - u^*(t)$ has at least one zero in every interval of length $kT$, we deduce that $-\infty < t^- < t^+ < +\infty$. We clearly have

$$(u - \beta)(t^-) = (u - \beta)(t^+) = 0, \tag{2.10}$$

$$(u - \beta)'(t^-) \geq 0 \geq (u - \beta)'(t^+); \tag{2.11}$$

moreover, for every $t \in ]t^-, t^+[$,

$$\begin{aligned} (u - \beta)''(t) &= u''(t) - \beta''(t) \geq \tilde{f}(t, u(t)) - f(t, \beta(t)) \\ &= f(t, \beta(t)) - f(t, \beta(t)) = 0. \end{aligned}$$

Together with (2.10) and (2.11), this implies that $(u - \beta)(t) = 0$ for every $t \in ]t^-, t^+[$, a contradiction. □

**Remark 4.** A $T$-periodic function satisfying the inequality (2.5) (resp., (2.6)) is usually referred to as a (classical) *lower solution* (resp., *upper solution*) for the $T$-periodic problem associated with equation (2.1) (more weaker concepts of lower/upper solutions could be introduced as well [12]).
Notice that in Theorem 2.1 we assume the existence of a $T$-periodic solution $u^*(t)$ satisfying (2.7). It is a classical fact that between an order pair of lower and upper solutions (that is, between a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ satisfying $\alpha(t) \leq \beta(t)$) a $T$-periodic solution of the equation always exists (see, for instance, [12]). Nevertheless, to our purposes we need to know explicitly $u^*(t)$ because of the assumption (2.3), and the existence of $\alpha(t)$ and $\beta(t)$ only plays the role of an additional information which allows to perform a suitable truncation. With this respect, it can be noticed that this fact also allows us to provide a localization of the subarmonic solutions produced (see (2.8)), which indeed lie in a vertical strip of the phase plane determined by $\alpha(t)$ and $\beta(t)$. Observe moreover, that, if $\alpha(t)$ is a solution, then the uniqueness for the solutions of the Cauchy problems further implies that $\alpha(t) < u^{(i)}_{j,k}(t)$ for every $t \in [0,T]$. Analogously, $u^{(i)}_{j,k}(t) < \beta(t)$, whenever $\beta(t)$ is a solution. ◁

**Remark 5.** Another fact that it may be worth mentioning here is related to the instability property of the periodic solutions obtained via the lower-upper solutions techniques. Indeed, according to [45] (see also [11] for previous work in this direction), under the assumptions of Theorem 2.1 and when at least one between $\alpha(t)$

and $\beta(t)$ is not a solution to (2.1), there exists a $T$-periodic solution to (2.1), satisfying (2.7), which is *unstable*, simultaneously in the past and in the future. On the other hand, according to the results in [36, 37], our condition (2.3) is not incompatible with the *stability* of the solution $u^*(t)$ itself. This is not a contradiction. In fact, in our setting $\alpha(t)$ and/or $\beta(t)$ may well be solutions of (2.1) (indeed, this will occur in some of the applications of Section 3 and Section 4). Moreover, we require the existence of a $T$-periodic solution $u^*(t)$ satisfying (2.3), but this does not prevent the existence of other (possibly unstable) $T$-periodic solutions. $\lhd$

The next result (Theorem 2.2) is a variant of Theorem 2.1 in which we assume solely the existence of an upper solution $\beta(t)$. For this theorem we suppose that *the global continuability for the solutions to* (2.1) *is ensured.* A dual result, in presence of a lower solution $\alpha(t)$, can be obtained as well.

**Theorem 2.2.** *Assume* $(a_1)$ *and* $(a_3)$ *of Theorem 2.1 and suppose that*

$$u^*(t) < \beta(t), \qquad \text{for every } t \in \mathbb{R}.$$

*Then the same conclusion of Theorem 2.1 holds, with* (2.8) *replaced by*

$$u^{(i)}_{j,k}(t) \le \beta(t), \qquad \text{for every } t \in \mathbb{R}.$$

*Proof.* The proof is similar to that of Theorem 2.1, using the truncation

$$\tilde{f}(t,x) = \begin{cases} f(t,x) & \text{for } \ x \le \beta(t) \\ f(t,\beta(t)) & \text{for } \ \beta(t) < x \end{cases}$$

and Proposition 1 again. □

**Remark 6.** The hypothesis of global continuability for the solutions of (2.1) is needed only to ensure that such a property holds for the solutions of the truncated equation

$$u'' + \tilde{f}(t,u) = 0. \tag{2.12}$$

Notice that this request was not made in Theorem 2.1 because in the corresponding proof we performed a two-sided truncation, obtaining $\tilde{f}(t,x)$ bounded and thus having guaranteed the continuability of the solutions for equation (2.9). With this respect, we could replace the global existence hypothesis of Theorem 2.2 by other conditions which produce the same effect for the solutions of (2.12). For instance, a simple possible alternative condition could be the following:

*there exist three constants* $A, B, d > 0$ *such that*

$$|f(t,x)| \le A|x| + B, \qquad \textit{for every } x \le -d \textit{ and for every } t \in [0,T].$$

It is rather easy to produce examples of differential equations satisfying the above one-sided growth assumption and possessing, at the same time, solutions which are not globally defined in $\mathbb{R}$. Other, more refined conditions could be given as well (see, [9, 17, 25]). $\lhd$

We conclude this section with a few examples in which our results immediately apply. It is worth mentioning that both the equations (2.13) and (2.16) considered below deal with nonlinearities $f(t,x)$ which are odd functions in the $x$-variable. Accordingly, solutions alway occur in pairs $(u(t), -u(t))$ and we cannot exclude, in the situation, for instance, of Theorem 2.1, that $u^{(1)}_{j,k}(t) \equiv -u^{(2)}_{j,k}(t)$. This fact, however, is purely accidental and does not affect the conclusion of Poincaré-Birkhoff twist theorem asserting, in general, the existence of pairs of fixed points. Indeed, after Example 1 and Example 2 we will propose some straightforward generalizations to second order equations without any symmetry conditions.

**Example 1.** Consider a nonlinear frictionless unforced simple pendulum with a periodically moving support (which can be equivalently described as a pendulum with a stationary support in a space with a periodically varying constant of gravity [13]). Mechanical models of this type lead to differential equations of the form

$$u'' + a(t)\sin u = 0, \tag{2.13}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous and $T$-periodic function. In this case we can apply Theorem 2.1, by taking $u^*(t) \equiv 0$, $\alpha(t) \equiv -\pi$ and $\beta(t) \equiv \pi$. With this choice of $u^*(t)$, we have condition (2.3) satisfied if and only if $\bar{a} := T^{-1}\int_0^T a(t)\,dt > 0$. Therefore the following result holds.

**Corollary 1.** *Suppose that $\bar{a} > 0$. Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \geq k^*$, there exists an integer $m_k$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \leq j \leq m_k$, equation (2.13) has at least two subharmonic solutions $u^{(1)}_{j,k}(t), u^{(2)}_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{j,k}(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$-\pi < u^{(i)}_{j,k}(t) < \pi, \qquad \text{for every } t \in \mathbb{R}.$$

Notice that, in view of the remark after Theorem 1.1, the following lower bound for $m_k$ can be given:

$$m_k \geq \mathscr{E}^-\left(\frac{kT\bar{a}}{2\pi(\max a(t))^{1/2}}\right).$$

Such an estimate, for the particular case of $a(t) \equiv (\text{constant} =)a > 0$, leads to a sharp inequality. Indeed, in this case $2\pi/\sqrt{a}$ is the limit period for the periodic orbits approaching the origin, which is also the infimum of the fundamental periods of the periodic solutions of the pendulum. Therefore, in a fixed time interval $[0, kT]$, the "small" (periodic) solutions to (2.13) make at least $\mathscr{E}^-(kT\sqrt{a}/2\pi)$ turns around the origin. Of course, in this particular case, we already know how to find subharmonic solutions for the autonomous equation $u'' + a\sin u = 0$ by direct continuity arguments on the time maps. Nevertheless, our result applies to an arbitrary weight function $a(t)$, under the only condition that $\bar{a} > 0$.

The case in which $\bar{a} < 0$ follows again from Theorem 2.1, by starting from $u^*(t) \equiv \pi$, $\alpha(t) \equiv 0$ and $\beta(t) \equiv 2\pi$ (see [33] for a similar remark). In this situation the solutions we find rotate around $(\pi, 0)$ in the phase plane, with the dynamics of an inverted pendulum.

We point out that, even if Corollary 1 applies to the unforced pendulum type equation (2.13), we do not need many special features of the sine function (in particular, its oddness and the fact that it has zero mean value in a period). Indeed, with the same argument, we can derive an application of Theorem 2.1 to an equation of the form

$$u'' + a(t)g(u) = 0, \tag{2.14}$$

with $g : \mathbb{R} \to \mathbb{R}$ a $2L$-periodic function of class $C^1$ and such that

$$g(0) = 0, \quad g'(0) > 0. \tag{2.15}$$

Indeed, from the periodicity of $g(x)$ and (2.15), we know that $g(x)$ vanishes at some points in $]-2L,0[$ as well as at some points in $]0,2L[$. Hence the constants

$$L_- := \max\{x<0 : g(x)=0\}, \quad L_+ := \min\{x>0 : g(x)=0\},$$

are well defined and, by construction,

$$g(x)<0,\ \forall\, x\in\,]L_-,0[ \quad \text{and} \quad g(x)>0,\ \forall\, x\in\,]0,L_+[\,.$$

Similarly as before, we can enter in the setting of Theorem 2.1 via the positions $u^*(t)\equiv 0$, $\alpha(t)\equiv L_-$ and $\beta(t)\equiv L_+$. Again, condition (2.3) is satisfied whenever $\bar{a} := T^{-1}\int_0^T a(t)\,dt > 0$, since $q(t) = \frac{\partial f}{\partial x}(t,u^*(t)) = a(t)g'(0)$. With these positions one can easily obtain a variant of Corollary 1 to equation (2.14). Note that with this approach, the periodicity of the potential $G(x) := \int_0^x g(\xi)\,d\xi$ is not required. For other results concerning (2.14) with $g(x)$ having a periodic potential, see [32, 44].

**Example 2.** As a second example, we consider a model studied by Belmonte-Beitia and Torres [1] arising from the search of some special solutions of a nonlinear Schrödinger equation. The equation under consideration takes the form

$$u'' + \mu u - p(t)u^3 = 0, \tag{2.16}$$

where $\mu > 0$ and $p:\mathbb{R}\to\mathbb{R}$ is a strictly positive, continuous and $T$-periodic function (to compare with the notation in [1], we have $\mu = -2\lambda$ and $p(t) = 2g(x)$, with $t = x$). Following [1] we introduce the constants

$$\xi_1 := \sqrt{\frac{\mu}{\min p(t)}}, \quad \xi_2 := \sqrt{\frac{\mu}{\max p(t)}}\,.$$

A simple computation shows that $\xi_2$ is a constant lower solution and $\xi_1$ is a constant upper solution for equation (2.16). By symmetry, also the constant functions $-\xi_1$ and $-\xi_2$ are, respectively, a lower and an upper solution for the same equation. A direct application of Theorem 2.1 can then be given with the following choices: $u^*(t)\equiv 0$, $\alpha(t)\equiv -\xi_1$ and $\beta(t)\equiv \xi_1$. Observe that the average condition (2.3) is automatically satisfied, since $\mu>0$. Therefore the following result holds.

**Corollary 2.** *With the above positions, there exists $k^*\in\mathbb{N}_0$ such that, for every integer $k\geq k^*$, there exists an integer $m_k$ such that, for every integer $j$ relatively prime with $k$ and such that $1\leq j\leq m_k$, equation (2.16) has at least two subharmonic solutions $u^{(1)}_{j,k}(t), u^{(2)}_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) such that, for $i=1,2$, $u^{(i)}_{j,k}(t)$ has exactly $2j$ zeros in the interval $[0,kT[$ and*

$$-\xi_1 \leq u^{(i)}_{j,k}(t) \leq \xi_1, \qquad \text{for every } t\in\mathbb{R}. \tag{2.17}$$

If we like to improve estimate (2.17), we can take advantage of the fact, as proved in [1], there exist two $T$-periodic solutions to (2.16), say $\rho_-(t)$ and $\rho_+(t)$, such that, for every $t\in[0,T]$,

$$-\xi_1 \leq \rho_-(t) \leq -\xi_2, \quad \xi_2 \leq \rho_+(t) \leq \xi_1$$

With this further information, we can apply Theorem 2.1 with $\alpha(t) = \rho_-(t)$ and $\beta(t) = \rho_+(t)$ and obtain the sharper estimate

$$\rho_-(t) < u^{(i)}_{j,k}(t) < \rho_+(t).$$

Moreover, as in Example 1, we can provide a lower bound for $m_k$ as follows:

$$m_k \geq \mathscr{E}^-\left(\frac{kT\sqrt{\mu}}{2\pi}\right). \tag{2.18}$$

Again, we have not used the oddness of the nonlinearity and our result easily extends to the more general equation

$$u'' + \mu u - p(t)g(u) = 0, \tag{2.19}$$

being $g : \mathbb{R} \to \mathbb{R}$ a $C^1$-function such that $g(x)x > 0$ for $x \neq 0$, and

$$g'(0) = 0, \quad \lim_{x \to \pm\infty} \frac{g(x)}{x} = +\infty.$$

Arguing like in [1] we can prove the existence of a maximal negative $T$-periodic solution $\rho_-(t)$ and a minimal positive $T$-periodic solution $\rho_+(t)$ (this follows using a lower/upper solutions technique). Now we can apply Theorem 2.1 with the positions $u^*(t) \equiv 0$, $\alpha(t) = \rho_-(t)$ and $\beta(t) = \rho_+(t)$. As before, the average condition (2.3) is automatically satisfied since $\mu > 0$. In this manner, we can extend Corollary 2 to the nonsymmetric case of equation (2.19).

In the next section we consider some more general situations which are related to Example 1 and Example 2.

## 3. Harmonic and subharmonic solutions for forced pendulum-type equations.

In this section, we propose to develop a consequence of Theorem 2.1 which is suited for possible applications to forced pendulum-type equations. As ideal model for our investigation, we consider the equation

$$u'' + \mu \sin u = e(t), \tag{3.1}$$

where $\mu > 0$ and, for simplicity, we suppose that $e : \mathbb{R} \to \mathbb{R}$ is a continuous and $T$-periodic function. For what follows it is also convenient to split $e(t)$ as

$$e(t) = \bar{e} + \tilde{e}(t), \quad \text{with} \quad \bar{e} := \frac{1}{T}\int_0^T e(t)\,dt.$$

It is a classical fact [32, 34] that integrating both sides of (3.1), a necessary condition for the existence of $kT$-periodic solutions to (3.1) (for $k \in \mathbb{N}_0$) is that $\bar{e} \in [-\mu, \mu]$. On the other hand, it is known that, if

$$\bar{e} \in \,]-\mu, \mu[\,,$$

then for every sufficiently small $\tilde{e}(t)$ there exist at least two $T$-periodic solutions (geometrically distinct, i.e., non differing by a multiple of $2\pi$). Precisely, considering the two solutions of the equation

$$\mu \sin x = \bar{e}, \quad \text{with } x \in \,]\pi, \pi],$$

namely

$$x_0 := \arcsin(\bar{e}/\mu), \qquad x_1 := \begin{cases} \pi - x_0 & \text{if } x_0 \geq 0, \\ -\pi - x_0 & \text{if } x_0 < 0, \end{cases}$$

one can prove (via degree theory) that, whenever $\tilde{e}(t)$ is small, $T$-periodic solutions of (3.1) exist near $x_0$ and $x_1$, respectively. General results which guarantee such a claim can be found, for instance, in [7, 23, 31]. Usually, the smallness of $\tilde{e}(t)$ is expressed in terms of its $L^1$-norm on $[0, T]$ or of the oscillation of some of its primitives (and, clearly, such a bound depends on $\mu$: the smaller is $\mu$, the larger upper bound for $\tilde{e}(t)$ is available).

In such a framework, our aim is to prove a result about subharmonic solutions. More in general, starting from an autonomous equation of the form

$$u'' + \mu h(t,u) = 0,$$

for which we assume the existence of three constant solutions $N^-, 0, N^+$, with $N^- < 0 < N^+$ (which, in some sense, mimic the three consecutive constant solutions $-\pi, 0, \pi$ of $u'' + \mu \sin u = 0$), we are going to show the existence of infinitely many subharmonic solutions for the perturbed equation

$$u'' + \mu h(t,u) = e(t), \tag{3.2}$$

provided that $e(t)$ is sufficiently small. We will make the convenient assumption that

$$\bar{e} = 0; \tag{3.3}$$

accordingly, we will denote by $\mathcal{E}(t)$ the unique $T$-periodic function such that

$$\mathcal{E}''(t) = e(t), \quad \text{and} \quad \|\mathcal{E}\|_\infty = \frac{1}{2}\mathrm{Osc}(\mathcal{E}),$$

being $\mathrm{Osc}(\mathcal{E}) := \max_{t\in[0,T]} \mathcal{E}(t) - \min_{t\in[0,T]} \mathcal{E}(t)$. Our main result of this section is the following.

**Theorem 3.1.** *Assume that $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $T$-periodic in the first variable and with continuous partial derivative $\frac{\partial h}{\partial x}(t,x)$, and that $e : \mathbb{R} \to \mathbb{R}$ is a continuous and $T$-periodic function satisfying (3.3).*
*Suppose that, for suitable real numbers $N^-, A, B, N^+$ and a constant $\delta > 0$ satisfying,*

$$N^- + \delta \leq A < 0 < B \leq N^+ - \delta,$$

*the following conditions hold true:*

$$h(t,N^-) = h(t,0) = h(t,N^+) = 0, \quad \text{for every } t \in \mathbb{R}; \tag{3.4}$$

$$h(t,x)(x - N^\pm) \leq 0, \quad \text{for every } t \in \mathbb{R} \text{ and } x \in [N^\pm - \delta, N^\pm + \delta]; \tag{3.5}$$

$$h(t,x)x > 0, \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \,]A,0[\,\cup\,]0,B[\,; \tag{3.6}$$

$$\frac{\partial h}{\partial x}(t,x) > 0, \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \,]A,B[\,. \tag{3.7}$$

*Finally, assume that*

$$\mathrm{Osc}(\mathcal{E}) < \min\{\delta, -A, B\}. \tag{3.8}$$

*Then there exists $\mu_* > 0$ such that, for every $\mu \in \,]0,\mu_*]$:*

*i) there exists a $T$-periodic solution $u^*(t)$ to (3.2) such that*

$$A < u^*(t) < B, \quad \text{for every } t \in \mathbb{R};$$

*ii) there exist two $T$-periodic solutions $u^-(t), u^+(t)$ to (3.2) such that*

$$N^\pm - \delta < u^\pm(t) < N^\pm + \delta, \quad \text{for every } t \in \mathbb{R};$$

*iii) there exists $k^*_\mu \in \mathbb{N}_0$ such that, for every integer $k \geq k^*_\mu$ there exists an integer $m_{k,\mu}$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \leq j \leq m_{k,\mu}$, equation (3.2) has at least two subharmonic solutions $u^1_{j,k}(t), u^2_{j,k}(t)$ of order $k$ (not belonging to the same periodicity class) such that $u^{(i)}_{j,k}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0,kT[$ and*

$$u^-(t) < u^{(i)}_{j,k}(t) < u^+(t), \quad \text{for every } t \in \mathbb{R}.$$

*Proof.* First, define

$$M = \max_{t\in[0,T],x\in[A,B]} |h(t,x)|$$

and choose $\mu_*, R > 0$ so that

$$\frac{\mu_* MT^2}{12} < R < \min\left\{-\frac{A}{2} - \|\mathcal{E}\|_\infty, \frac{B}{2} - \|\mathcal{E}\|_\infty\right\}. \tag{3.9}$$

Such a choice is possible, in view of (3.8). Now, fix $\mu \in ]0, \mu_*]$.

We first show that the conclusion at point $i)$ holds true, by proving that there exists a $T$-periodic solution $x^*(t)$ to

$$x'' + \mu h(t, x + \mathcal{E}(t)) = 0, \tag{3.10}$$

such that

$$A + \|\mathcal{E}\|_\infty < x^*(t) < B - \|\mathcal{E}\|_\infty, \quad \text{for every } t \in \mathbb{R}.$$

Setting $u^*(t) = x^*(t) + \mathcal{E}(t)$, this implies the conclusion.
Our argument is closely related to [50]. Let $C_T$ be the Banach space of the continuous and $T$-periodic functions $x : \mathbb{R} \to \mathbb{R}$, with the norm $\|x\|_\infty := \max_{t\in[0,T]} |x(t)|$; moreover, for $x \in C_T$, set $\bar{x} := \frac{1}{T}\int_0^T x(t)\,dt$ and $\tilde{x}(t) := x(t) - \bar{x}$. Define the open set

$$\Omega = \left\{x \in C_T \mid \bar{x} \in \left]\frac{A}{2}, \frac{B}{2}\right[, \|\tilde{x}\|_\infty < R\right\}$$

and observe first of all that, if $x \in \overline{\Omega}$, then relation (3.9) implies that

$$A + \|\mathcal{E}\|_\infty < x(t) < B - \|\mathcal{E}\|_\infty, \quad \text{for every } t \in \mathbb{R}. \tag{3.11}$$

We are going to show, via coincidence degree's theory, that equation (3.10) has a solution $x^* \in \overline{\Omega}$. In view of [29], or [31, Theorem 2.4], this is true if:

a) for every $\alpha \in ]0,1[$, the equation

$$x'' + \alpha\mu h(t, x + \mathcal{E}(t)) = 0, \tag{3.12}$$

has no $T$-periodic solutions $x \in \partial\Omega$;

b) it holds that

$$\int_0^T \mu h\left(t, \frac{A}{2} + \mathcal{E}(t)\right)\,dt \neq 0 \neq \int_0^T \mu h\left(t, \frac{B}{2} + \mathcal{E}(t)\right)\,dt;$$

c) it holds that

$$\deg_{\mathrm{B}}\left(x \mapsto \int_0^T \mu h(t, x + \mathcal{E}(t))\,dt, \left]\frac{A}{2}, \frac{B}{2}\right[, 0\right) \neq 0.$$

Conditions b) and c) follow from (3.6). Indeed, suppose that $s = B/2$. In this case we have $0 < s + \mathcal{E}(t) < B$, so that $h(t, s + \mathcal{E}(t)) > 0$ for every $t \in [0,T]$. Therefore, $\int_0^T \mu h(t, s + \mathcal{E}(t))\,dt > 0$. Similarly, one can check that $\int_0^T \mu h(t, s + \mathcal{E}(t))\,dt < 0$ for $s = A/2$. As consequence, b) holds and c) is satisfied with degree equal to one.
For what concerns condition a), observe first of all that whenever $x(t)$ is a $T$-periodic solution to (3.12) $(0 < \alpha < 1)$ such that (3.11) holds true, in view of the Sobolev inequality and since $x(t) + \mathcal{E}(t) \in [A,B]$, we have

$$\begin{aligned}\|\tilde{x}\|_\infty^2 &\leq \frac{T}{12}\int_0^T x'(t)^2\,dt = -\frac{T}{12}\int_0^T x''(t)\tilde{x}(t)\,dt = \\ &= \frac{T}{12}\int_0^T \alpha\mu h(t, x + \mathcal{E}(t))\tilde{x}(t)\,dt \leq \frac{\mu_* MT^2}{12}\|\tilde{x}\|_\infty < R\|\tilde{x}\|_\infty.\end{aligned}$$

From such a priori bound we conclude that in order to show condition a) it is sufficient to prove that (3.12) has no $T$-periodic solutions $x(t)$ such that $\bar{x} = \frac{A}{2}$ or $\bar{x} = \frac{B}{2}$ with $\|\tilde{x}\|_\infty < R$. Assume to the contrary that this is the case and, just to fix the ideas, that $\bar{x} = \frac{B}{2}$. Then

$$\|\mathcal{E}\|_\infty < \frac{B}{2} - R < x(t) = \bar{x} + \tilde{x}(t) < \frac{B}{2} + R < B - \|\mathcal{E}\|_\infty,$$

a contradiction in view of (3.6) (just integrate (3.12) and divide by $\alpha > 0$ to get $0 = \int_0^T h(t, x(t) + \mathcal{E}(t))\,dt$).

The conclusion at point *ii)* follows in a direct way from the lower/upper solution technique. Indeed, in view of (3.8) and (3.5), it is easy to verify that the functions

$$\alpha^-(t) = N^- - \frac{\delta}{2} + \mathcal{E}(t), \quad \beta^-(t) = N^- + \frac{\delta}{2} + \mathcal{E}(t)$$

are, respectively, a lower and an upper solution to (3.2). Hence, the existence of a $T$-periodic solution $u^-(t)$ to (3.2) satisfying, for $t \in [0, T]$,

$$N^- - \delta < N^- - \frac{\delta}{2} + \mathcal{E}(t) \le u^-(t) \le N^- + \frac{\delta}{2} + \mathcal{E}(t) < N^- + \delta,$$

follows immediately. A symmetric argument shows the existence of $u^+(t)$.

Finally, the conclusion at point *iii)* follows from Theorem 2.1, with the choice $\alpha(t) = u^-(t)$ and $\beta(t) = u^+(t)$. Indeed, (2.3) follows from (3.7), since $A < u^*(t) < B$ for every $t \in [0, T]$. $\square$

When applied to the forced pendulum equation (3.1), Theorem 3.1 immediately gives the following:

**Corollary 3.** *Assume that $e : \mathbb{R} \to \mathbb{R}$ is a continuous and $T$-periodic function satisfying (3.3) and*

$$\mathrm{Osc}(\mathcal{E}) < \frac{\pi}{2}\,. \tag{3.13}$$

*Then there exists $\mu_* > 0$ such that, for every $\mu \in ]0, \mu_*]$:*

*i) there exists a $T$-periodic solution $u^*(t)$ to (3.1) such that*

$$-\frac{\pi}{2} < u^*(t) < \frac{\pi}{2}, \quad \text{for every } t \in \mathbb{R};$$

*ii) there exist a $T$-periodic solutions $u^-(t)$ to (3.1) such that*

$$-\frac{3}{2}\pi < u^-(t) < -\frac{\pi}{2}, \quad \text{for every } t \in \mathbb{R};$$

*iii) there exists $k_\mu^* \in \mathbb{N}_0$ such that, for every integer $k \ge k_\mu^*$ there exists an integer $m_{k,\mu}$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \le j \le m_{k,\mu}$, equation (3.1) has at least two subharmonic solutions $u_{j,k}^1(t), u_{j,k}^2(t)$ of order $k$ (not belonging to the same periodicity class) such that $u_{j,k}^{(i)}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$u^-(t) < u_{j,k}^{(i)}(t) < u^-(t) + 2\pi, \quad \text{for every } t \in \mathbb{R}.$$

*Proof.* It is enough to apply Theorem 3.1 with the positions: $N^- = -\pi, A = -\frac{\pi}{2}, B = \frac{\pi}{2}, N^+ = \pi$ and $\delta = \frac{\pi}{2}$. Notice that here, in view of the $2\pi$-periodicity of $\sin x$, the solution lying near $\pi$ is just $u^+(t) = u^-(t) + 2\pi$. $\square$

Note that, according to (3.9), in this case an estimate for $\mu_*$ can be given, in the sense that our result is true for

$$\mu < \frac{6}{T^2}\left(\frac{\pi}{2} - \mathrm{Osc}(\mathcal{E})\right).$$

We stress that, in spite of an enormous amount of literature dealing with the existence of $T$-periodic solution to the forced pendulum equation (3.1) (see for instance the survey [32]), much fewer results are available for subharmonic solutions [20, 44, 47]. In particular, [20, 44] provide generic-type results under suitable nondegeneracy conditions on the associated energy functionals, while [47] is closer to the spirit of Corollary 3, showing the existence of infinitely many subharmonics for $e(t)$ small in the $L^2$-norm. Even if our assumption (3.13) is in general not comparable with the ones in [47], Corollary 3, as usual when trying to make a comparison between variational methods and the Poincaré-Birkhoff theorem, provides a sharper conclusion for what concerns the multiplicity, minimal period and localization of the subharmonics.

Existence results for pendulum-type equations based on upper bounds on $\mathrm{Osc}(\mathcal{E})$ have been previously obtained. For instance,using topological degree methods, the condition

$$\mathrm{Osc}(\mathcal{E}) \leq \pi, \tag{3.14}$$

paired with other assumptions, has been used to obtain existence results (see [10, 30]). Our condition (3.13) is clearly more restrictive than (3.14); however, it allows to obtain a multiplicity result for subharmonic solutions as well.

4. **Harmonic and subharmonic solutions for parameter dependent second order equations.** In this final section, we propose to study, via Theorem 2.2, the existence of harmonic and subharmonic solutions for scalar second order parameter dependent equations of the form

$$u'' + g(u) = \lambda + e(t), \tag{4.1}$$

being $g : \mathbb{R} \to \mathbb{R}$ a $C^1$-function, $e : \mathbb{R} \to \mathbb{R}$ a continuous and $T$-periodic function with

$$\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt = 0 \tag{4.2}$$

and $\lambda \in \mathbb{R}$ a parameter. Observe that, in this setting, (4.2) is not restrictive, up to relabelling $e(t)$ and $\lambda$.
Throughout the section, given a continuous $T$-periodic function $u : \mathbb{R} \to \mathbb{R}$, we set (as before) $\bar{u} := \frac{1}{T}\int_0^T u(t)\,dt$, $\tilde{u}(t) := u(t) - \bar{u}$ and $\|u\|_{L^p_T} := \left(\int_0^T |u(t)|^p\,dt\right)^{1/p}$ (for $p = 1, 2$).

Our goal is to find, for every $\lambda$ in a suitable interval, a $T$-periodic solution $u_\lambda(t)$ of (4.1) which plays the role of the $u^*(t)$ in Theorem 2.2. More precisely, we will look for some $u_\lambda(t)$ such that the following property is satisfied:

($\mathscr{P}$) *There exists $k^*_\lambda \in \mathbb{N}_0$ such that, for every integer $k \geq k^*_\lambda$ there exists an integer $m_{k,\lambda}$ such that, for every integer $j$ relatively prime with $k$ and such that $1 \leq j \leq m_{k,\lambda}$, equation (4.1) has at least two subharmonic solutions $u^{(1)}_{j,k,\lambda}(t), u^{(2)}_{j,k,\lambda}(t)$ of order $k$ (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{j,k,\lambda}(t) - u_\lambda(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

As in Section 3, coincidence degree will be the main tool to find and localize (in dependence of $\lambda$) such $u_\lambda(t)$. To this aim, we present a preliminary result (see Lemma 4.1 below) which deals with the case when $g(x)$ is strictly increasing on a half-line. Our result, although technical, permits a unifying treatment of the forthcoming applications.

**Lemma 4.1.** *Assume that there exists $d \in \mathbb{R}$ such that $g(x)$ is strictly increasing on $[d,+\infty[$ (respectively, on $]-\infty,-d]$). Moreover, suppose that there exists $K>0$ such that:*

- *for every $\lambda \in [g(d), g(+\infty)[$ (resp. $\lambda \in ]g(-\infty), g(-d)]$), for every $\alpha \in ]0,1[$ and for every $T$-periodic solution $u(t)$ of*
$$u'' + \alpha g(u) = \alpha(\lambda + e(t)) \tag{4.3}$$
*satisfying $u(t) > d$ (resp. $u(t) < -d$) for every $t \in \mathbb{R}$, it holds that*
$$\|u'\|_{L^1_T} < K.$$

*Then there exists $\lambda^* \in [g(d), g(+\infty)[$ (resp. $\lambda^* \in ]g(-\infty), g(-d)]$) such that, for every $\lambda \in [\lambda^*, g(+\infty)[$ (resp. $\lambda \in ]g(-\infty), \lambda^*]$), there exists a $T$-periodic solution $u^*_\lambda(t)$ of (4.1) such that $u^*_\lambda(t) > d$ (resp. $u^*_\lambda(t) < -d$) for every $t \in \mathbb{R}$. Moreover, for $\lambda \to g(+\infty)$, it holds that*
$$u^*_\lambda(t) \to +\infty, \quad \textit{uniformly in } t \in \mathbb{R} \tag{4.4}$$
*(resp. $u^*_\lambda(t) \to -\infty$ uniformly in $t$, for $\lambda \to g(-\infty)$).*

*Proof.* For every $\lambda \in [g(d), g(+\infty)[$ let us denote by $x_\lambda$ the unique real number in $[d,+\infty[$ such that $g(x_\lambda) = \lambda$. Clearly, $x_\lambda \to +\infty$ for $\lambda \to g(+\infty)$. Let us choose now $\lambda^* \geq g(d)$ so large that $x_{\lambda^*} - K > d$ and fix $\lambda \in [\lambda^*, g(+\infty)[$.
Let $C_T$ be the Banach space of the continuous and $T$-periodic functions $u : \mathbb{R} \to \mathbb{R}$, with the norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$, and, for $a, b \in \mathbb{R}$ with $a < b$, consider the open set
$$\Omega(a,b) = \{u \in C_T \mid a < u(t) < b, \text{ for every } t \in \mathbb{R}\}.$$
We are going to show that equation (4.1) has a solution $u^*_\lambda \in \overline{\Omega}$ for
$$\Omega := \Omega(a,b) \quad \text{with } \; a := x_\lambda - K, \; b := x_\lambda - K.$$
By [29], or [31, Theorem 2.4], this is true if:

a) for every $\alpha \in ]0,1[$, the equation (4.3) has no $T$-periodic solutions $u(t)$ such that $a \leq u(t) \leq b$ for every $t \in \mathbb{R}$ and $u(\tilde{t}) \in \{a,b\}$ for some $\tilde{t} \in [0,T]$;

b) it holds that
$$\int_0^T (g(a) - \lambda - e(t))\,dt \neq 0 \neq \int_0^T (g(b) - \lambda - e(t))\,dt;$$

c) it holds that
$$\deg_{\mathrm{B}}\left(x \mapsto \int_0^T (g(x) - \lambda - e(t))\,dt, \, ]a,b[\,, 0\right) \neq 0.$$

Indeed, conditions b) and c) follow easily from the fact that $g(x)$ is strictly increasing on $[d,+\infty[$ and the choice of $\lambda$. We prove condition a). Let $u(t)$ be a $T$-periodic solution of (4.3) such that $u(t) \geq x_\lambda - K > d$; then, for every $s, t \in [0,T]$,
$$|u(s) - u(t)| = \left|\int_t^s u'(\tau)\,d\tau\right| \leq \int_0^T |u'(t)|\,dt < K. \tag{4.5}$$

On the other hand, integrating equation (4.3) and dividing by $\alpha > 0$, we get

$$\frac{1}{T}\int_0^T g(u(s))\,ds = \lambda,$$

which implies that, for some $t^* \in [0,T]$, $g(u(t^*)) = \lambda$. Since $u(t^*) > d$, we get $u(t^*) = x_\lambda$. But by assumption we know that $u(\tilde{t}) \in \{x_\lambda - K, x_\lambda + K\}$ for some $\tilde{t} \in [0,T]$. Hence

$$|u(t^*) - u(\tilde{t})| = K,$$

in contradiction with (4.5). Finally, (4.4) follows from the fact that $x_\lambda - K \le u^*_\lambda(t) \le x_\lambda + K$, since $x_\lambda \to +\infty$.
The proof of the symmetric case follows a similar argument. □

Our first application deals with a problem previously considered by Cid-Sanchez, Ward and Bereanu-Mawhin [2, 8, 46]. The general framework is that of a bounded nonlinearity $g(x)$ satisfying the basic assumption:

$$g(x) > 0, \quad \text{for every } x \in \mathbb{R} \qquad \text{and} \qquad \lim_{|x|\to+\infty} g(x) = 0 \tag{4.6}$$

In such a situation, the results in [2, 8, 46] guarantee the existence of two $T$-periodic solutions for small positive values of the parameter $\lambda$. Our goal is to show that, adding an asymptotic condition on the derivative $g'(x)$, such harmonic solutions can be localized in a precise manner, and they are accompanied by the existence of infinitely many subharmonic solutions with prescribed nodal properties.

**Theorem 4.2.** *Assume (4.6) and suppose that there exists $d > 0$ such that:*

$$g'(x) > 0, \quad \text{for every } x < -d \qquad \text{and} \qquad g'(x) < 0, \quad \text{for every } x > d. \tag{4.7}$$

*Set $M = \max_{x\in\mathbb{R}} g(x)$. Then there exists $\lambda^* \in (0, M]$ such that, for every $\lambda \in ]0, \lambda^*]$:*

- *i) there exists a unique $T$-periodic solution $u^+_\lambda(t)$ of (4.1) such that $u^+_\lambda(t) > d$ for every $t \in \mathbb{R}$;*
- *ii) there exists a $T$-periodic solution $u^-_\lambda(t)$ of (4.1) such that $u^-_\lambda(t) < -d$ for every $t \in \mathbb{R}$;*
- *iii) property ($\mathscr{P}$) holds with respect to $u^-_\lambda(t)$, with*

$$u^{(i)}_{j,k,\lambda}(t) < u^+_\lambda(t), \qquad \text{for every } t \in \mathbb{R}.$$

*Moreover, for $\lambda \to 0^+$,*

$$u^+_\lambda(t) \to +\infty, \qquad \text{and} \qquad u^-_\lambda(t) \to -\infty$$

*uniformly in $t \in \mathbb{R}$.*

*Proof.* We split our arguments into three steps.

First of all, we prove that, for $\lambda > 0$ sufficiently small, there exists a unique $T$-periodic solution $u^+_\lambda(t)$ of (4.1) such that $u^+_\lambda(t) > d$ for every $t \in \mathbb{R}$. To this aim, let us denote by $x_\lambda$ the unique real number in $[d, +\infty[$ such that $g(x_\lambda) = \lambda$; moreover, let $\mathcal{E}(t)$ be the unique $T$-periodic function such that $\mathcal{E}''(t) = e(t)$ and $\int_0^T \mathcal{E}(t)\,dt = 0$. For $\lambda > 0$ small enough, we have $x_\lambda > d + 2\|\mathcal{E}\|_\infty$; accordingly, choose $m, M > 0$ such that

$$d < m - \|\mathcal{E}\|_\infty \le m + \|\mathcal{E}\|_\infty \le x_\lambda \le M - \|\mathcal{E}\|_\infty.$$

An appropriate choice can be $m := x_\lambda - \|\mathcal{E}\|_\infty$ and $M := x_\lambda + \|\mathcal{E}\|_\infty$ . It is easy to see that the $T$-periodic functions

$$\alpha^+(t) = m + \mathcal{E}(t), \qquad \beta^+(t) = M + \mathcal{E}(t)$$

are such that $d < \alpha^+(t) < \beta^+(t)$ and are, respectively, a lower and an upper solution for equation (4.1). The existence of a $T$-periodic solution $u_\lambda^+(t) > d$ follows then from the lower/upper solution method. Moreover, by the same estimates it follows that $u_\lambda^+(t) \to +\infty$ (uniformly in $t$).
The uniqueness of $u_\lambda^+(t)$ follows from a direct argument using $g'(x) < 0$ for $x > d$.

Secondly, we show that the condition at point *ii*) holds true. To this aim, we are going to use Lemma 4.1. Indeed, $g(x)$ is strictly increasing on $]-\infty, -d]$. Moreover, for $\lambda \in ]0, M]$, $\alpha \in ]0, 1[$ and every $T$-periodic solution $u(t)$ of (4.3) satisfying $u(t) < -d$ for every $t \in \mathbb{R}$, it holds that

$$\|u''\|_{L^1_T} \leq 2M + \|e\|_{L^1_T}.$$

Letting $t^* \in [0, T]$ an instant such that $u'(t^*) = 0$, the previous relation implies that, for every $t \in [0, T]$,

$$|u'(t)| = \left| \int_{t^*}^{t} u''(s)\, ds \right| \leq 2M + \|e\|_{L^1_T},$$

so that the assumption of Lemma 4.1 is satisfied by choosing $K > (2M + \|e\|_{L^1_T})T$.

Finally, the conclusion at point *iii*) follows from Theorem 2.2, with the choice $u^*(t) = u_\lambda^-(t)$ and $\beta(t) = u_\lambda^+(t)$. Indeed, (2.3) follows from (4.7) since $u_\lambda^-(t) < -d$ for every $t \in \mathbb{R}$. □

Notice that the uniqueness of $u_\lambda^-(t)$ is not guaranteed, in general. However, it can be achieved by adding some condition on $g(x)$ (for instance we could suppose that $0 < g'(x) < \left(\frac{2\pi}{T}\right)^2$ for every $x < -d$).

Our second application deals with a classical situation first considered by Fabry, Mawhin and Nkashama in [18]. Basically, we have in mind to consider the case when $g(x)$ satisfies:

$$\lim_{|x| \to +\infty} g(x) = +\infty \tag{4.8}$$

In such a situation, the results in [18] ensure the existence of two $T$-periodic solutions to (4.1) for large values of $\lambda$. Here, again, adding conditions on $g'(x)$ we provide further information about the localization of such $T$-periodic solutions, as well as the existence of subharmonic solutions with prescribed nodal properties. It is worth noticing that results providing the existence of subharmonic solutions for (4.1) in case when (4.8) is satisfied can be deduced from [42, Theorem 10]. We postpone more comments about this point after the statement of our result.

**Theorem 4.3.** *Assume* (4.8) *and suppose that there exist* $d > 0$ *and* $0 < l < \left(\frac{2\pi}{T}\right)^2$ *such that*

$$g'(x) < 0, \ \text{ for every } x < -d \quad \text{ and } \quad 0 < g'(x) \leq l, \ \text{ for every } x > d. \tag{4.9}$$

*Then there exists* $\lambda^* > 0$ *such that, for every* $\lambda \geq \lambda^*$*:*

- *i) there exists a unique* $T$*-periodic solution* $u_\lambda^-(t)$ *of* (4.1) *such that* $u_\lambda^-(t) < -d$ *for every* $t \in \mathbb{R}$*;*
- *ii) there exists a unique* $T$*-periodic solution* $u_\lambda^+(t)$ *of* (4.1) *such that* $u_\lambda^+(t) > d$ *for every* $t \in \mathbb{R}$*;*

*iii) property ($\mathscr{P}$) holds with respect to $u_\lambda^+(t)$, with*

$$u_\lambda^-(t) < u_{j,k,\lambda}^{(i)}(t), \qquad \text{for every } t \in \mathbb{R}.$$

*Moreover, for $\lambda \to +\infty$,*

$$u_\lambda^+(t) \to +\infty, \qquad \text{and} \qquad u_\lambda^-(t) \to -\infty$$

*uniformly in $t \in \mathbb{R}$.*

*Proof.* Similarly as in the proof of Theorem 4.2, we split our arguments into three steps.

The existence and uniqueness of a $T$-periodic solution $u_\lambda^-(t)$ of (4.1) such that $u_\lambda^-(t) < -d$ for every $t \in \mathbb{R}$ follows in a similar way as in the proof of point $i$) of Theorem 4.2 (using upper and lower solutions techniques like in [18] and the fact that $g'(x) < 0$ for $x < -d$).

To show the conclusion at point $ii$), we use again Lemma 4.1; indeed, $g(x)$ is strictly increasing on $[d, +\infty[$ with $g(+\infty) = +\infty$. For simplicity of notations, we set $g_\lambda(x) = g(x) - \lambda$. Assume now that $\lambda \geq g(d)$, $\alpha \in ]0, 1[$ and $u(t)$ is a $T$-periodic solution $u(t)$ of (4.3) such that $u(t) > d$ for every $t \in \mathbb{R}$. Multiplying equation (4.3) by $\tilde{u}(t)$ and integrating, we get

$$\begin{aligned}\int_0^T u'(t)^2\, dt &= \alpha \int_0^T g_\lambda(u(t))\tilde{u}(t)\, dt - \alpha \int_0^T e(t)\tilde{u}(t)\, dt \\ &= \alpha \int_0^T (g_\lambda(u(t)) - g_\lambda(\bar{u}))\tilde{u}(t)\, dt - \alpha \int_0^T e(t)\tilde{u}(t)\, dt.\end{aligned}$$

On the other hand, since $u(t), \bar{u} > d$ and $g(x)$ is strictly increasing on $[d, +\infty[$, Lagrange's theorem implies that

$$g_\lambda(u(t)) - g_\lambda(\bar{u}) = g'_\lambda(\xi(t))(u(t) - \bar{u}) = g'_\lambda(\xi(t))\tilde{u}(t)$$

for a suitable $\xi(t) > d$. Hence we get

$$\|u'\|^2_{L^2_T} \leq l\|\tilde{u}\|^2_{L^2_T} + \|e\|_{L^1_T}\|\tilde{u}\|_\infty,$$

which implies, in view of the Sobolev and Wirtinger inequalities, that

$$\left(1 - l\left(\frac{T}{2\pi}\right)^2\right)\|u'\|^2_{L^2_T} \leq \left(\frac{T}{12}\right)^{1/2}\|e\|_{L^1_T}\|u'\|_{L^2_T}.$$

In conclusion, $\|u'\|_{L^2_T}$ is bounded so that $\|u'\|_{L^1_T}$ is bounded too and the assumption of Lemma 4.1 are satisfied.
The uniqueness of $u_\lambda^+(t)$ comes from the fact that $0 < g'(x) < \left(\frac{2\pi}{T}\right)^2$ for $x > d$, by a direct argument.

Finally, the conclusion at point $iii$) follows from the dual version of Theorem 2.2, with the choice $u^*(t) = u_\lambda^+(t)$ and $\alpha(t) = u_\lambda^-(t)$. Indeed, according to Remark 6, the global continuability for the solutions to the modified equation

$$u'' + \tilde{f}(t, u) = 0,$$

where $f(t, x) = g(x) - \lambda - e(t)$ and

$$\tilde{f}(t, x) = \begin{cases} f(t, x) & \text{for } \; x \geq u_\lambda^-(t) \\ f(t, u_\lambda^-(t)) & \text{for } \; x < u_\lambda^-(t) \end{cases},$$

is guaranteed since $|g'(x)|$ is uniformly bounded for $x > d$. Moreover (2.3) follows from (4.9) since $u_\lambda^+(t) > d$ for every $t \in \mathbb{R}$. $\square$

**Remark 7.** As already anticipated, the problem of the existence of subharmonic solutions for (4.1) in case when (4.8) is satisfied has been analyzed in [42]. In particular, the following result can be deduced from [42, Theorem 10].

**Proposition 2** (From [42])**.** *Assume* (4.8)*; moreover, suppose that:*

$$0 < \liminf_{x\to+\infty} g'(x) \leq \limsup_{x\to+\infty} g'(x) < +\infty.$$

*Then for every $r$ there exists $k^*_r$ such that, for every $k \geq k^*_r$ there exists $\lambda^*_{r,k}$ such that, for $\lambda > \lambda^*_{r,k}$, equation* (4.1) *has at least $2r$ subharmonics of order $k$.*

Trying to make a comparison between such a result and our Theorem 4.3, the following facts may be emphasized.

- The assumption on the derivative $g'(x)$ considered in Theorem 4.3 does not prevent the possibility that $\liminf_{x\to+\infty} g'(x) = 0$, which is excluded in [42, Theorem 10].
- For a fixed (large) value of $\lambda$, the subharmonics found in Theorem 4.3 have a sharper nodal characterization with respect to the ones in [42, Theorem 10]. Indeed, we find subharmonic solutions rotating around the fixed harmonic solution $u^+_\lambda(t)$, whose existence is not considered in [42].

$\lhd$

**Remark 8.** Combining our argument with [49, Theorem 2.1], the same conclusion of Theorem 4.3 can be proved assuming, instead of (4.9), the following condition:

*there exist $d, k > 0$ such that $g'(x) < 0$ for $x < -d$ and*

$$\lim_{x\to+\infty} g'(x) = k \neq \left(\frac{2\pi m}{T}\right)^2, \qquad \forall m \in \mathbb{N}_0.$$

We point out that a nonresonance condition is really needed to prove that $u^+_\lambda(t) \to +\infty$ for $\lambda \to +\infty$. To see this, consider the equation

$$u'' + k|u| = \lambda + \sin(\sqrt{k}t), \tag{4.10}$$

with $k = \left(\frac{2\pi m}{T}\right)^2$ for some $m \in \mathbb{N}_0$.
We claim that, for every $\lambda > 0$, equation (4.10) has no positive $T$-periodic solutions. Indeed, suppose that a positive $T$-periodic solution $u(t)$ exists. Then, multiplying (4.10) by $\sin(\sqrt{k}t)$ and integrating on $[0, T]$, we get

$$\int_0^T u''(t)\sin(\sqrt{k}t)\,dt + k\int_0^T u(t)\sin(\sqrt{k}t)\,dt = \int_0^T \sin^2(\sqrt{k}t)\,dt;$$

integrating twice by parts the first term on the left-hand side, we find

$$0 = \int_0^T \sin^2(\sqrt{k}t)\,dt,$$

a contradiction.
The above example can be easily modified in order to show that, given a constant $L \geq 0$, there are no $T$-periodic solutions $u(t)$ with $u(t) > -L$, for all $t$, when $\lambda > 0$. $\lhd$

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