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This is the author's manuscript

Original Citation:

Availability:

This version is available http://hdl.handle.net/2318/151179 since

Published version:

DOI:10.1007/s00013-014-0644-2

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08 November 2026

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This is an author version of the contribution published on:

A. Boscaggin, R. Ortega, F. Zanolin
Subharmonic solutions of the forced pendulum equation: a symplectic approach
ARCHIV DER MATHEMATIK (2014) 102
DOI: 10.1007/s00013-014-0644-2

The definitive version is available at:
http://link.springer.com/content/pdf/10.1007/s00013-014-0644-2

Subharmonic solutions of the forced pendulum equation: a symplectic approach

A. Boscaggin, R. Ortega and F. Zanolin

Abstract. Using the Poincaré-Birkhoff fixed point theorem, we prove that, for every $\beta > 0$ and for a large (both in the sense of prevalence and of category) set of continuous and T-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with $\int_0^T f(t)\,dt = 0$, the forced pendulum equation

$$x'' + \beta \sin x = f(t),$$

has a subharmonic solution of order k *for every large integer number* k. This improves the well known result obtained with variational methods, where the existence when k is a (large) prime number is ensured.

Mathematics Subject Classification (2010). Primary 34C25.

Keywords. Subharmonic solutions, Pendulum equation, Poincaré-Birkhoff theorem.

1. Introduction and statement of the result

Consider the forced pendulum equation

$$x'' + \beta \sin x = f(t), \tag{1.1}$$

where $\beta > 0$ is a real parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function with zero average, that is,

$$\int_0^T f(t)\,dt = 0. \tag{1.2}$$

From now on, the Banach space of such functions, endowed with the norm $\|f\| = \max_{t\in\mathbb{R}} |f(t)|$, will be denoted by $\widetilde{C}_T$.

The aim of this short paper is to establish the following result about the existence of subharmonic solutions of (1.1). Let us recall that a kT-periodic solution of (1.1) is called subharmonic solution of order k if it is not lT-periodic for any $l = 1, \dots, k-1$.

Theorem 1.1. *For every $\beta > 0$ there exists an open and prevalent set $\mathcal{F} \subset \widetilde{C}_T$ and for every $f \in \mathcal{F}$ there exists $k^* \geq 1$ such that the equation* (1.1) *has a subharmonic solution of order k for each integer $k \geq k^*$.*

For the definition of a prevalent set we refer to the paper by Ott and Yorke [17]. Here we just recall that a prevalent set can be considered as the analogue of a full-measure set in infinite dimension. Notice that, since prevalent sets are dense in the ambient space, the set $\mathcal{F}$ in Theorem 1.1 is a generic set, so that it is large in both senses, measure and category. The fact that many significant properties of the forced pendulum equation can be proved to hold for a generic set of forcing terms has been shown in [10] and [3], while prevalent sets were considered only more recently in [15].

The generic existence of infinitely many subharmonic solutions of (1.1) is a well known result that follows from [4] and [10]. In [4] Fonda and Willem imposed certain non-degeneracy conditions on the equation (1.1) in order to find subharmonic solutions of order k *for every large prime number* k. See also [18] for a related result. Theorem 1.1 is an improvement on the generic existence because we can find subharmonic solutions of order k *for every large integer number* k.

Our proof relies on the Poincaré-Birkhoff fixed point theorem, so that we have a symplectic counterpart of the variational result by Fonda and Willem. Let us recall that also the classical theorem of the existence of two T-periodic solutions for (1.1) can be proved both with variational arguments (Mawhin-Willem [11]) and with symplectic arguments (Franks [5], see also [9]). Here, considering subharmonic solutions, the symplectic approach gives even more precise information.

Let us finally recall that, to the best of our knowledge, the existence of infinitely many subharmonic solutions of (1.1) *for every forcing term* satisfying (1.2) is still an open problem.

The paper is organized as follows. In Section 2, we give an auxiliary result dealing with the existence of subharmonic solutions for asymptotically linear second order equations in terms of the rotation numbers of the linearizations. In Section 3, we show how to apply this argument to the forced pendulum equation, thus providing a proof of Theorem 1.1. Finally, in Section 4 some further remarks are presented.

2. Rotation numbers, Poincaré-Birkhoff theorem and subharmonic solutions

In this section, we present an auxiliary result about the existence of subharmonic solutions. It is obtained combining ideas from [2] and [7, 8].

Given a continuous and T-periodic function $q : \mathbb{R} \to \mathbb{R}$, we consider the linear equation

$$u'' + q(t)u = 0. \tag{2.1}$$

After passing to clockwise polar coordinates, that is

$$u = r\cos\theta, \qquad u' = -r\sin\theta, \tag{2.2}$$

we obtain a first order equation for the argument,

$$\theta' = q(t)\cos^2\theta + \sin^2\theta. \tag{2.3}$$

Let $\theta(t;\theta_0)$ be the unique solution of (2.3) satisfying the initial condition $\theta(0) = \theta_0$. Sometimes, to emphasize the dependence on q, we will write $\theta(t;\theta_0,q)$.

Following the ideas of Moser [12], the equation (2.3) can be interpreted as an autonomous equation on the torus and so it has an associated rotation number $\rho = \rho(q)$. Traditionally, the rotation number is defined as the limit (which exists and is independent of θ_0)

$$\rho = T\lim_{t\to+\infty}\frac{\theta(t;\theta_0)}{t},$$

but in the following we will need its characterization via the inequality

$$|\theta(nT;\theta_0) - \theta_0 - \rho n| < 2\pi, \qquad \text{for every } \theta_0 \in [0,2\pi],\ n \in \mathbb{Z}. \tag{2.4}$$

This relation could be used to define the rotation number, as well. We refer to [14] for more information on rotation numbers.

Notice also that from (2.3) we deduce that $\theta' = 1$ if $\theta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. This is useful to prove that $\rho \geq 0$.

Based on this notion and the Poincaré-Birkhoff fixed point theorem, we have the following result.

Theorem 2.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, T-periodic in the first variable and such that $g(t,0) \equiv 0$. Suppose that, uniformly in $[0,T]$,*

$$\lim_{u\to 0}\frac{g(t,u)}{u} = q_0(t), \qquad \lim_{|u|\to+\infty}\frac{g(t,u)}{u} = q_\infty(t)$$

with $q_0(t), q_\infty(t)$ continuous and T-periodic functions such that

$$\rho(q_0) > 0 \quad \text{and} \quad \rho(q_\infty) = 0.$$

Then, there exists $k^* \geq 1$ such that, for every integer $k \geq k^*$, the differential equation

$$u'' + g(t,u) = 0 \tag{2.5}$$

has at least two kT-periodic solutions having exactly two zeros in $[0,kT[$.

The uniqueness for the initial value problems associated to (2.5) implies that the two zeros of the above solutions are indeed changes of sign. Therefore these solutions are subharmonics of order k.

Proof. We write the equation (2.5) as

$$u'' + \gamma(t,u)u = 0,$$

where $\gamma : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $\gamma(t,0) = q_0(t)$ and

$$\lim_{|u|\to+\infty}\gamma(t,u) = q_\infty(t)$$

uniformly in $t \in [0,T]$. Then we pass to polar coordinates via the formula (2.2) and transform (2.5) into the first order system

$$\begin{cases} r' = -(1-\gamma(t, r\cos\theta))\dfrac{\sin(2\theta)}{2} r \\ \theta' = \gamma(t, r\cos\theta)\cos^2\theta + \sin^2\theta. \end{cases} \tag{2.6}$$

The equivalence between this system and (2.5) implies that the solution $(\Theta(t;\theta_0,r_0),\ R(t;\theta_0,r_0))$ of the initial value problem is unique and globally defined for each $\theta_0 \in [0,2\pi]$ and $r_0 > 0$. We claim that

$$\Theta(t;\theta_0,r_0) \to \theta(t;\theta_0,q_0), \qquad \text{as } r_0 \to 0^+$$

$$\Theta(t;\theta_0,r_0) \to \theta(t;\theta_0,q_\infty), \qquad \text{as } r_0 \to +\infty$$

and the convergence is uniform in $\theta_0 \in [0,2\pi]$ and $t \in [0,\tau]$. Here $\tau > 0$ is an arbitrary fixed number.

To justify the convergence as $r_0 \to 0^+$ it is enough to observe the the system (2.6) admits a continuous extension to $r = 0$. Then the property of continuous dependence is sufficient to justify the limit. The convergence as $r_0 \to +\infty$ is more delicate. Let us assume that (r_{0n},θ_{0n}) are given sequences with $r_{0n} \to +\infty$ and $\theta_{0n} \to \theta_0$ for some $\theta_0 \in [0,2\pi]$. We must prove that $\Theta(t;\theta_{0n},r_{0n})$ converges, uniformly in $t \in [0,\tau]$, to $\theta(t;\theta_0,q_0)$. To simplify the notation we set

$$R_n(t) = R(t;\theta_{0n},r_{0n}), \qquad \Theta_n(t) = \Theta(t;\theta_{0n},r_{0n}).$$

From the first equation of (2.6) we deduce that

$$r_{0n}e^{-Ct} \le R_n(t) \le r_{0n}e^{Ct}, \qquad t \in [0,\tau],$$

where C is an upper bound of $\frac{1}{2}|1-\gamma(t,u)|$. This implies that $R_n(t) \to +\infty$ uniformly in $t \in [0,\tau]$. From the second equation in (2.6) we observe that the sequence of functions $\{\Theta_n\}_{n\ge 0}$ is uniformly bounded and equicontinuous on $[0,\tau]$. Then it is enough to prove that the limit of any convergent subsequence is precisely $\theta(t;q_\infty)$. Let $\Theta_k(t)$ be a subsequence converging to a certain function $\Theta(t)$. Each $\Theta_k(t)$ satisfies the differential inequality

$$\theta' \ge -\Gamma\cos^2\theta + \sin^2\theta,$$

where Γ is an upper bound of $|\gamma(t,u)|$. Then also $\Theta(t)$ satisfies the same inequality, at least in a weak sense. This means that

$$\Theta(t) \ge \Theta(t_0) + \int_{t_0}^{t} \left[-\Gamma\cos^2\Theta(\sigma) + \sin^2\Theta(\sigma)\right] d\sigma$$

if $t, t_0 \in [0,\tau]$, $t \ge t_0$. From here it is easy to deduce that the set

$$Z = \left\{t_0 \in [0,\tau] \,:\, \Theta(t_0) = \tfrac{\pi}{2} + k\pi, \text{ for some } k \in \mathbb{Z}\right\}$$

is finite. Thus,

$$R_n(t)\,|\cos\Theta_n(t)| \to +\infty$$

for each $t \in [0,\tau] \setminus Z$. By dominated convergence we deduce that $\Theta(t)$ is a solution of

$$\theta' = q_\infty(t)\cos^2\theta + \sin^2\theta.$$

Since $\Theta(0) = \theta_0$ we conclude that $\Theta(t) = \theta(t;\theta_0,q_\infty)$.

Once the convergence has been justified we consider, for each $k \geq 1$, the map

$$M_k : (\theta_0, r_0) \mapsto (\Theta(kT;\theta_0,r_0), R(kT;\theta_0,r_0)).$$

It is an homeomorphism of $\mathbb{R}\times[0,+\infty[$ satisfying

$$\Theta(kT;\theta_0+2\pi,r_0) = \Theta(kT;\theta_0,r_0) + 2\pi$$

and

$$R(kT;\theta_0+2\pi,r_0) = R(kT;\theta_0,r_0).$$

Moreover it preserves the measure $r\,d\theta\,dr$ and the circumference $r = 0$ is invariant under M_k.

To check the twist condition we apply (2.4) and observe that

$$|\theta(kT;\theta_0,q_i) - \theta_0 - \rho_i k| < 2\pi$$

if $\theta_0 \in [0,2\pi]$, $i = 1,\infty$ and $\rho_i = \rho(q_i)$. In particular

$$\theta(kT;\theta_0,q_0) - \theta_0 > k\rho_0 - 2\pi$$

and

$$\theta(kT;\theta_0,q_\infty) - \theta_0 < 2\pi.$$

Let us fix $k^* \in \mathbb{N}$ such that $k^*\rho_0 > 4\pi$ and define, for each k,

$$\max_{\theta_0\in[0,2\pi]} (\theta(kT;\theta_0,q_\infty) - \theta_0) = \delta_k < 2\pi.$$

Then, if $k \geq k^*$ and $\theta_0 \in [0,2\pi]$,

$$\theta(kT;\theta_0,q_\infty) - \theta_0 \leq \delta_k < 2\pi < k^*\rho_0 - 2\pi \leq \theta(kT;\theta_0,q_0) - \theta_0.$$

We can apply the convergence to find numbers $0 < \alpha_k < \beta_k$ such that

$$\Theta(kT;\theta_0,\beta_k) - \theta_0 < 2\pi < \Theta(kT;\theta_0,\alpha_k) - \theta_0.$$

The Poincaré-Birkhoff fixed point theorem on a non-invariant annulus (see [2, 7, 8] and the references therein) can be applied to obtain two solutions of the system

$$\Theta(kT;\theta_0,r_0) = \theta_0 + 2\pi, \qquad R(kT;\theta_0,r_0) = r_0.$$

From the second equation of (2.6) we observe that $\theta = \frac{\pi}{2} + j\pi$, $j \in \mathbb{Z}$, implies $\theta' = 1$ and so the corresponding solution has exactly two zeros on $[0,kT[$. □

Remark 2.2. As in [2], it would be possible to prove (just by slightly refining the above argument) a sharper result. Precisely: for every $k \geq k^*$ and $1 \leq j \leq m_k$, where m_k is the largest integer strictly below the number $\frac{k\rho_0-2\pi}{2\pi}$, equation (2.5) has at least two kT-periodic solutions with exactly $2j$ zeros on the interval $[0,kT[$. These solutions will be subharmonics of order k if j and k are relatively prime.

3. The proof

At first, we present a connection between the rotation number of the linear equation (2.1) and its Floquet multipliers. Let us recall that the Floquet multipliers $\mu_i = \mu_i(q)$, $i = 1, 2$, of (2.1) are defined as the eigenvalues of the monodromy matrix

$$M = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \phi_1'(T) & \phi_2'(T) \end{pmatrix},$$

where $\phi_1(t)$, $\phi_2(t)$ are the solutions of (2.1) satisfying

$$\phi_1(0) = \phi_2'(0) = 1, \qquad \phi_1'(0) = \phi_2(0) = 0.$$

Since M is a symplectic matrix, the Floquet multipliers always satisfy the relation $\mu_1\mu_2 = 1$. The link between rotation number and Floquet multipliers which is needed later is the following: if $\rho = 0$, then μ_1 and μ_2 are real and positive. Indeed, from (2.4) it follows that $\rho = 0$ if and only if the nontrivial solutions of (2.1) have at most two zeros, and in this case [6, Lemma 4.1] gives the existence of a positive solution $w(t)$ of (2.1) such that $w(t+T) = \mu w(t)$ for a suitable $\mu > 0$ and every $t \in \mathbb{R}$. Writing $w(t) = c_1\phi_1(t) + c_2\phi_2(t)$, with $c_1^2 + c_2^2 \neq 0$, we see that μ is an eigenvalue of M with eigenvector $(c_1, c_2)^T$. Hence, $\mu_1 = \mu$ and $\mu_2 = \mu^{-1}$ are both real and positive.

Let us now define $\mathcal{F} \subset \widetilde{C}_T$ as the set of forcing terms $f(t)$ such that every T-periodic solution of (1.1) is non-degenerate (that is, if $x(t)$ is a T-periodic solution of (1.1), then the Floquet multipliers of the equation $u'' + \beta\cos(x(t))u = 0$ satisfy $\mu_i \neq 1$, $i = 1, 2$). It is proved in [15] that $\mathcal{F}$ is open and prevalent. We claim here the following:

($\star$) *for every $f \in \mathcal{F}$, there exists a T-periodic solution $\varphi(t)$ of* (1.1) *such that $\rho(\beta\cos\varphi) > 0$.*

Once such a claim is established, the proof of Theorem 1.1 easily follows from Theorem 2.1. Indeed, with the change of variables

$$u(t) = x(t) - \varphi(t),$$

equation (1.1) is transformed into the equation

$$u'' + \beta\left(\sin(u + \varphi(t)) - \sin(\varphi(t))\right) = 0, \tag{3.1}$$

that is, an equation of the type (2.5), with

$$g(t, u) = \beta\left(\sin(u + \varphi(t)) - \sin(\varphi(t))\right).$$

Moreover, it is easy to see that

$$q_0(t) = \beta\cos(\varphi(t)) \quad \text{and} \quad q_\infty(t) \equiv 0.$$

Since

$$\rho(q_0) > 0 \quad \text{and} \quad \rho(q_\infty) = 0,$$

Theorem 2.1 applies, giving the existence of a subharmonic solution $u_k(t)$ of order k whenever k is large enough. Hence, $x_k(t) = u_k(t) + \varphi(t)$ is kT-periodic, solves the pendulum equation (1.1) and, since lT ($l \in \mathbb{N}$) is a period for $\varphi(t)$,

$$x_k(t + lT) \not\equiv x_k(t), \qquad \text{for every } l = 1, \dots, k-1,$$

that is, $x_k(t)$ is a subharmonic solution, of order k, of (1.1).

So, let us prove ($\star$). At first, we find a T-periodic solution $\varphi(t)$ of (1.1) satisfying $\gamma_T(\varphi) = 1$, where γ_T denotes the fixed point index of the (isolated) fixed point $\varphi(0)$ of the Poincaré map associated with (1.1). This is shown in [16, Section 3]. Let μ_i be the Floquet multipliers of $u'' + \beta\cos(\varphi(t))u = 0$. Since $f \in \mathcal{F}$ they satisfy $\mu_i \neq 1$, $i = 1, 2$, and the index can be computed as

$$\gamma_T(\varphi) = \operatorname{sign}(1 - \mu_1)(1 - \mu_2).$$

From $\gamma_T(\varphi) = 1$ and $\mu_1\mu_2 = 1$ we conclude that μ_1 and μ_2 cannot lie in $]0, \infty[$. In consequence $\rho(\beta\cos\varphi) > 0$.

4. Miscellaneous remarks

4.1. The need for k^*

In the framework of Theorem 1.1 it is not possible to find subharmonic solutions of order k for each $k \geq 1$. To illustrate this assertion we will select $\beta > 0$ and $k \geq 1$ satisfying

$$\beta < \left(\frac{2\pi}{kT}\right)^2. \tag{4.1}$$

We will prove the existence of $\delta > 0$ such that, if $\|f\| < \delta$, every kT-periodic solution of (1.1) is T-periodic.

To prove this claim we first observe that for the autonomous equation ($f \equiv 0$) the minimal period of any closed orbit is greater than $\frac{2\pi}{\sqrt{\beta}} > kT$. Next we introduce some notation. The Banach space of kT-periodic functions of class C^r will be denoted by C^r_{kT}, the associated norm induces the uniform convergence of all derivatives up to order r. Given $x \in C^2_{kT}$ we define

$$d_k(x) = \min_{n\in\mathbb{Z}} \|x - n\pi\|_{C^2_{kT}}.$$

This is the distance of $x(t)$ to the constant solutions of $x'' + \sin x = 0$. The proof is organized in two steps.

Step 1 Given $\epsilon > 0$ there exists $\delta > 0$ such that if $\|f\| < \delta$ then every kT-periodic solution $\varphi(t)$ of (1.1) satisfies $d_k(\varphi) < \epsilon$.

Assume that f_n is a sequence of forcings with $\|f_n\| \to 0$ and let $\varphi_n(t)$ be a kT-periodic solution of $x'' + \beta\sin x = f_n(t)$. By a standard compactness argument we can extract a subsequence $\varphi_j(t)$ such that for appropriate integers k_j the sequence $\varphi_j(t) + 2\pi k_j$ converges in C^2_{kT} to some solution $\varphi(t)$ of the autonomous equation. This solution is kT-periodic and so it cannot correspond to any closed orbit. Hence $\varphi(t)$ is a constant solution, that is $\varphi(t) = n\pi$ for some $n \in \mathbb{Z}$.

Step 2 There exist positive numbers ϵ_* and δ_* such that if $\|f\| < \delta_*$ and $d_k(\varphi) < \epsilon_*$, where $\varphi(t)$ is a kT-periodic solution of (1.1), then $\varphi(t)$ is also T-periodic.

For each $l \geq 1$ consider the smooth operator

$$\Phi_l : C^2_{lT} \to C^0_{lT}, \qquad \Phi_l(x) = x'' + \beta \sin x.$$

Thanks to the condition (4.1) we can apply the inverse function theorem at $x = 0$ and $x = \pi$ for $l = 1$ and $l = k$. Finally we observe that Φ_1 and Φ_k coincide on $C^2_T \subset C^2_{kT}$.

4.2. Abundance of subharmonic solutions

Using the refinement of Theorem 2.1 suggested in Remark 2.2 it is possible to obtain a multiplicity of subharmonics of (1.1) of order k if k is large enough, by providing the existence of kT-periodic solutions of (3.1) with different nodal behavior. Now we discuss another aspect: is it possible to ensure that the two kT-periodic solutions (with two zeros in $[0, kT[\,)$ found in Theorem 2.1 give rise to *two* geometrically distinct (i.e., not differing by an integer multiple of 2π) subharmonics of (1.1)? The answer is yes if we change the set $\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the set of forcings such that, for each $l \geq 1$, the lT-periodic solutions of (1.1) are non-degenerate. This set is prevalent and generic, as follows from [16]. A simple proof can be obtained from [15] replacing T by lT in the proof of the main theorem. It $f \in \widetilde{\mathcal{F}}$ the two solutions given by Theorem 2.1 can be chosen with opposite index and so they cannot be geometrically equivalent. Also, they are not in the same periodicity class (i.e., they cannot be obtained the one from the other via a translation in time by an integer multiple of T).

4.3. More general nonlinearities

Theorem 1.1 is still valid for the equation

$$x'' + s(x) = f(t),$$

where $s : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function of class C^1 satisfying $\int_0^{2\pi} s(x)\,dx = 0$ and such that the set $\{x \in \mathbb{R} : s(x) = 0\}$ is totally disconnected. After a couple of observations it is clear that the proof presented above for the forced pendulum also works in this more general case. First we notice that the conditions on the function $s(x)$ are sufficient to apply Theorem 3.1 in [15]. The second remark is that the discussion in [16, Section 3] is valid for any function $s(x)$ that is of class C^1, periodic and has zero average.

4.4. Periodic solutions of the second kind

We have dealt with periodic solutions of the first kind, meaning that $x(t + kT) = x(t)$. Solutions of the second kind satisfy

$$x(t + kT) = x(t) + 2\pi N$$

with $N \in \mathbb{Z} \setminus \{0\}$. These solutions exist for arbitrary $k \geq 1$ and $N \neq 0$ as a consequence of Moser's formulation of Aubry-Mather theory [13, Theorem 2.2.2]. See also [1] for a different approach.

4.5. Rotation number and Morse index

A completely equivalent point of view could be obtained by considering the Morse index $m = m(q)$ of (2.1) instead of the rotation number. Indeed, it is well-known that $\rho > 0$ if and only if $m \geq 1$. This provides a connection between Theorem 2.1 and the results in [7, 8]. The proof in Section 3 shows that, for every $f \in \mathcal{F}$, there exists a T-periodic solution $\varphi(t)$ of (1.1) having Morse index greater or equal than 1. This means that $\varphi(t)$, as a critical point of the action functional, cannot be a local minimizer.

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