L^p Microlocal properties for vector weighted pseudodifferential operators with smooth symbols

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Preprint

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Abstract. The authors introduce a class of pseudodifferential operators, whose symbols satisfy completely inhomogeneous estimates at infinity for the derivatives.
Continuity properties in suitable weighted Sobolev spaces of L^p type are given and L^p microlocal properties studied.

Mathematics Subject Classification (2000). Primary 35S05; Secondary 35A17.

Keywords. pseudodifferential operators, weight vector, weighted Sobolev spaces, microlocal properties.

1. Introduction

A vector weighted pseudodifferential operator is characterized by a smooth symbol which in general satisfies the estimates:

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq c_{\alpha,\beta} m(\xi) \Lambda(\xi)^{-\alpha}. \tag{1.1}$$

Here $m(\xi)$ is a suitable positive continuous weight function, which indicates the "order" of the symbol, and $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$ is a weight vector that estimates the decay at infinity of the derivatives; see the next Definition 2.6.
The paper must be considered in the general framework given by the symbol classes $S^\lambda(\phi, \varphi)$ and $S(m, g)$, introduced respectively by R. Beals [1] and L. Hörmander [10], [11]. Particularly we follow here the approach of Rodino [13], where a generalization of the Hörmander smooth wave front set is given, and Garello [4], where the extension to the inhomogeneous microlocal analysis for weighted Sobolev singularities of L^2 type is performed.
In a previous work [7] we studied continuity and microlocal properties of quasi homogeneous L^p type, for pseudodifferential operators of zero order, whose symbol

satisfies the decay estimates at infinity:

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta} \prod_{j=1}^{n} \langle \xi \rangle_M^{-\frac{\alpha_j}{m_j}}, \tag{1.2}$$

where $M = (m_1, \dots, m_n) \in \mathbb{N}^n$, $\min\limits_{1 \le j \le n} m_j = 1$ and $\langle \xi \rangle_M = \sqrt{1 + \sum_{j=1}^{n} \xi_j^{2m_j}}$.
Considering now the classical Hörmander symbol classes $S^m_{\rho,\delta}$, see [11, Ch. 18], it is a matter of fact that the quasi-homogeneous symbols, characterized by (1.2), satisfy in natural way the condition

$$\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in S^0_{\rho,0}, \quad \gamma \in \{0,1\}^n, \tag{1.3}$$

for suitable $0 < \rho < 1$. The corresponding pseudodifferential operators are L^p bounded, see Taylor [14, Ch. XI, Sect. 4]. Note that the estimate (1.3) follows essentially from the fact that, $\langle \xi \rangle_M^{\frac{1}{m_j}} \ge |\xi_j|$, for any $j = 1, \dots, n$.
Assuming that a similar condition is satisfied by the components of the weight vector in (1.1) (see Definition 2.1), in Section 2 we obtain a family of weight vectors which define, via (2.17), a class of L^p bounded pseudodifferential operators.
In the study of the microlocal properties of these operators, the main problem arises from the lack of any homogeneity of the weight vector $\Lambda(\xi)$; this does not allow us to use in a suitable way conic neighborhoods in $\mathbb{R}^n_\xi$, as done in the classical definition of the Hörmander wave front set, see [11] and the quasi-homogeneous generalization given in [7]. Following now the approach in [13], [4] and [8], suitable neighborhoods of sets $X \subset \mathbb{R}^n_\xi$ are introduced in Section 4; they allow us to derive in Section 5 useful microlocal properties.
Finally, in Section 6 the microlocal results are expressed in terms of m-filter of Sobolev singularities, following the approach in [4].

2. Vector weighted symbol classes

Definition 2.1. A vector valued function $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$, $\xi \in \mathbb{R}^n$, with positive continuous components $\lambda_j(\xi)$ for $j = 1, \dots, n$, is a weight vector if there exist positive constants C, c such that for any $j = 1, \dots, n$:

$$c\langle \xi \rangle^c \le \lambda_j(\xi) \le C\langle \xi \rangle^C; \tag{2.1}$$

$$\lambda_j(\xi) \ge c|\xi_j|; \tag{2.2}$$

$$c \le \frac{\lambda_j(\eta)}{\lambda_j(\xi)} \le C \quad \text{when} \quad \sum_{k=1}^{n} |\xi_k - \eta_k| \lambda_k(\eta)^{-1} \le c. \tag{2.3}$$

As usual we denote, for $\xi \in \mathbb{R}^n$: $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$

Definition 2.2. A positive real continuous function $m(\xi)$ is an admissible weight, associated to the weight vector $\Lambda(\xi)$, if for some positive constants N, C, c

$$m(\eta) \leq C\, m(\xi)\, (1 + |\eta - \xi|)^N\,; \tag{2.4}$$

$$\frac{1}{C} \leq \frac{m(\eta)}{m(\xi)} \leq C \quad \text{when} \quad \sum_{k=1}^{n} |\xi_k - \eta_k| \lambda_k(\eta)^{-1} \leq c. \tag{2.5}$$

We say that a vector valued function $\Lambda(\xi) = (\lambda_1(\xi), \ldots, \lambda_n(\xi))$ is *slowly varying* if condition (2.3) is satisfied. Analogously, a function $m = m(\xi)$ satisfying condition (2.5) is said to be *slowly varying* with respect to the weight vector Λ, while m is *temperate* when condition (2.4) holds true.
Considering respectively $\xi = 0$ and $\eta = 0$ in (2.4) it follows that $\frac{1}{C}\langle\xi\rangle^{-N} \leq m(\xi) \leq C\langle\xi\rangle^N$.

We say that two weights $m(\xi)$, $\tilde{m}(\xi)$ are equivalent, and write $m(\xi) \asymp \tilde{m}(\xi)$, if $c \leq \dfrac{m(\xi)}{\tilde{m}(\xi)} \leq C$, for some positive constants c, C. Again $\tilde{\Lambda}(\xi) = (\tilde{\lambda}_1(\xi), \ldots, \tilde{\lambda}_n(\xi))$ is equivalent to $\Lambda(\xi)$, if $\tilde{\lambda}_j(\xi) \asymp \lambda_j(\xi)$, for any $j = 1, \ldots n$. It is trivial that $\tilde{m}(\xi)$ and $\tilde{\Lambda}(\xi)$ are respectively admissible weight and weight vector.
Moreover set $m(\xi) \approx m(\eta)$ if $c \leq \dfrac{m(\eta)}{m(\xi)} \leq C$, for some positive constants c, C.

Example.
1. Consider $\langle\xi\rangle_M = \left(1 + \sum_{j=1}^n \xi_j^{2m_j}\right)^{1/2}$ *quasi-homogeneous polynomial* , where $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$, and $\min\limits_{1\leq j\leq n} m_j = 1$. Then $\Lambda_M(\xi) = \left(\langle\xi\rangle_M^{1/m_1}, \ldots, \langle\xi\rangle_M^{1/m_n}\right)$ is a weight vector.
2. For any positive continuous function $\lambda(\xi)$ satisfying (2.1) and the slowly varying condition
$$\lambda(\eta) \approx \lambda(\xi), \text{ when} \sum_{j=1}^{n} |\eta_j - \xi_j| \left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j|\right)^{-1} \leq c, \text{ for some}\, c, \mu > 0, \tag{2.6}$$
the vector $\Lambda(\xi) := \left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_1|, \ldots, \lambda(\xi)^{\frac{1}{\mu}} + |\xi_n|\right)$ is a weight vector, see [8, Proposition 1] for the proof. In such frame emphasis is given to the *multi-quasi-homogeneous polynomials* $\lambda_{\mathcal{P}}(\xi) = \left(\sum_{\alpha\in V(\mathcal{P})} \xi^{2\alpha}\right)^{1/2}$, where $V(\mathcal{P})$ is the set of the vertices of a *complete Newton polyedron* $\mathcal{P}$ as introduced in [9], see also [2]; in this case, the value μ in the definition of $\Lambda(\xi)$ and in condition (2.6) is the *formal order* of $\mathcal{P}$.
3. Any positive constant function on $\mathbb{R}^n$ is an admissible weight associated to every weight vector $\Lambda(\xi)$.
4. For any $s \in \mathbb{R}$, the functions $\langle\xi\rangle_M^s$, $\lambda(\xi)^s$ are admissible weights for the weight vectors respectively defined in 1. and 2.

Remark 2.3. Consider a function $\lambda(\xi)$ satisfying the slowly varying condition (2.6). Since $|\xi-\eta|^\mu \le c\lambda(\eta)$ implies $\lambda(\eta) \le C\lambda(\xi) \le C\lambda(\xi)\,(1+|\xi-\eta|)^\mu$, using moreover (2.1), we obtain that $\lambda(\xi)$ satisfies the temperance condition (2.4) with constant $N=\mu$.

Proposition 2.4. *For $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$ weight vector, the function:*

$$\pi(\xi) = \min_{1\le j\le n} \lambda_j(\xi), \quad \xi \in \mathbb{R}^n \tag{2.7}$$

is an admissible weight associated to $\Lambda(\xi)$ and it moreover satisfies (2.6)*.*

Proof. In view of (2.2) and (2.7), the assumption $\sum_{k=1}^{n} |\xi_k - \eta_k|\,(\pi(\eta) + |\eta_k|)^{-1} \le c$ directly gives $\sum_{k=1}^{n} |\xi_k - \eta_k| \lambda_k(\eta)^{-1} \le \tilde{c}$, where $\tilde{c} > 0$ depends increasingly on c. Then for suitably small c, we obtain from the slowly varying condition (2.3) and some $C > 0$: $\frac{1}{C}\lambda_j(\xi) \le \lambda_j(\eta) \le C\lambda_j(\xi)$, for any $j = 1, \dots, n$. It then follows: $\frac{1}{C}\pi(\xi) = \frac{1}{C}\min_j \lambda_j(\xi) \le \pi(\eta) = \min_j \lambda_j(\eta) \le C\min_j \lambda_j(\xi) = C\pi(\xi)$. Thus $\pi(\xi)$ satisfies (2.6) and in the same way we can prove that it fulfils (2.5). Then by means of the previous remark we conclude the proof. □

Lemma 2.5. *If m, m' are admissible weights associated to the weight vector $\Lambda(\xi)$, then the same property is fulfilled by mm' and $1/m$.*

Proof. $m(\eta) \le Cm(\xi)\,(1+|\xi-\eta|)^N \iff 1/m(\xi) \le C1/m(\eta)\,(1+|\xi-\eta|)^N$; then interchanging ξ and η we immediately obtain that $1/m$ is temperate. The remaining part of the proof is then trivial. □

Definition 2.6. For Ω open subset of $\mathbb{R}^n$, $\Lambda(\xi)$ weight vector and $m(\xi)$ admissible weight, the symbol class $S_{m,\Lambda}(\Omega)$ is given by all the smooth functions $a(x,\xi) \in C^\infty(\Omega\times\mathbb{R}^n)$, such that, for any compact subset $K \subset \Omega$ and $\alpha, \beta \in \mathbb{Z}_+^n$, there exists $c_{\alpha,\beta,K} > 0$ such that:

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta,K}\, m(\xi)\Lambda(\xi)^{-\alpha}, \quad \xi \in \mathbb{R}^n \tag{2.8}$$

where, with standard vectorial notation, $\Lambda(\xi)^\gamma = \prod_{k=1}^n \lambda_k(\xi)^{\gamma_k}$.

$S_{m,\Lambda}(\Omega)$ turns out to be a Fréchet space, with respect to the family of natural semi-norms defined as the best constants $c_{\alpha,\beta,K}$ involved in the estimates (2.8).

Remark 2.7. Let $\Lambda(\xi)$ be the weight vector according to Definition 2.1 then:

1. considering now the constants C, c in (2.1) and N in (2.4), the following relation with the usual Hörmander [11] symbol classes $S^m_{\rho,\delta}(\Omega)$, $0 \le \delta < \rho \le 1$, is trivial:
$$S_{m,\Lambda}(\Omega) \subset S^N_{c,0}(\Omega)\,. \tag{2.9}$$

2. If m_1, m_2 are admissible weights such that $m_1 \le Cm_2$, then $S_{m_1,\Lambda}(\Omega) \subset S_{m_2,\Lambda}(\Omega)$, with continuous imbedding. In particular the identity $S_{m_1,\Lambda}(\Omega) = S_{m_2,\Lambda}(\Omega)$ holds true, as long as $m_1 \asymp m_2$.
   When the admissible weight m is an arbitrary positive constant function, the symbol class $S_{m,\Lambda}(\Omega)$ will be just denoted by $S_\Lambda(\Omega)$ and $a(x,\xi) \in S_\Lambda(\Omega)$ will be called a *zero order symbol*.
3. Since for any $k \in \mathbb{Z}_+$ the admissible weight $\pi(\xi)^{-k}$ is less than $C^k\langle\xi\rangle^{-ck}$, then for m admissible weight we have
$$\bigcap_{k\in\mathbb{Z}+} S_{m\pi^{-k},\Lambda}(\Omega) \subset \bigcap_{N\in\mathbb{Z}_+} S^{-N}_{1,0}(\Omega) =: S^{-\infty}(\Omega)\,.$$
   On the other hand $a(x,\xi) \in S^{-\infty}(\Omega)$ means that
$$\sup_{x\in K} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le c_{\alpha,\beta}\langle\xi\rangle^{\mu-|\alpha|}, \qquad \text{for any } \mu\in\mathbb{R},\, K \subset\subset \Omega\,. \tag{2.10}$$
   Recall now that, for suitable $N, C > 0$, $m(\xi) \ge \frac{1}{C}\langle\xi\rangle^{-N}$, $\pi(\xi) \le C\langle\xi\rangle$ and $\lambda_j(\xi) \le C\langle\xi\rangle^C$. Then setting, for any fixed $\alpha \in \mathbb{Z}^n_+$ and arbitrary $k \in \mathbb{Z}_+$, $\mu = -N - k - (C-1)|\alpha|$ in (2.10), we obtain $|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)| \le c_{\alpha,\beta} m(\xi)\pi(\xi)^{-k}\Lambda(\xi)^{-\alpha}$, for suitable $c_{\alpha,\beta}$, that is $a(x,\xi) \in S_{m\pi^{-k},\Lambda}(\Omega)$ for any $k \in \mathbb{Z}$. Then
$$\bigcap_{k\in\mathbb{Z}_+} S_{m\pi^{-k},\Lambda}(\Omega) \equiv S^{-\infty}(\Omega). \tag{2.11}$$
4. Using (2.8), (2.4), (2.1) and (2.2), it immediately follows that for any $\alpha,\gamma \in \mathbb{Z}^n_+$, $K \subset\subset \Omega$,
$$\sup_{x\in K} |\xi^\gamma \partial_\xi^{\alpha+\gamma} a(x,\xi)| \le M_{\alpha,\gamma,K}\langle\xi\rangle^{N-c|\alpha|}$$
   with some positive constant $M_{\alpha,\gamma,K}$. Then $S_{m,\Lambda}(\Omega) \subset M^N_{c,0}(\Omega)$. Here $M^r_{\rho,0}(\Omega)$, $0 < \rho \le 1$, are the symbol classes defined in [14] given by all the symbols $a(x,\xi) \in S^r_{\rho,0}(\Omega)$ such that for any $\gamma \in \{0,1\}^n$, $\xi^\gamma\partial_\xi^\gamma a(x,\xi) \in S^r_{\rho,0}(\Omega)$.

By means of the arguments in [12] Proposition 1.1.6 and [1] jointly with Remark 2.7 we obtain the following asymptotic expansion.

Proposition 2.8. *Given a weight vector $\Lambda(\xi) = (\lambda_1(\xi),\dots,\lambda_n(\xi))$, let $\pi = \pi(\xi)$ be the admissible weight defined by (2.7). For any sequence of symbols $\{a_k\}_{k\in\mathbb{Z}_+}$, $a_k(x,\xi) \in S_{m\pi^{-k},\Lambda}(\Omega)$, there exists $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ such that for every integer $N \ge 1$:*
$$a(x,\xi) - \sum_{k<N} a_k(x,\xi) \in S_{m\pi^{-N},\Lambda}(\Omega)\,. \tag{2.12}$$
Moreover $a(x,\xi)$ is uniquely defined modulo symbols in $S^{-\infty}(\Omega)$.

We write
$$a(x,\xi) \sim \sum_{k=0}^\infty a_k(x,\xi), \tag{2.13}$$
if for every $N \ge 1$ (2.12) holds.

Proposition 2.9. *Any admissible weight $m(\xi)$ admits an equivalent smooth admissible weight $\tilde{m}(\xi) \in S_{m,\Lambda}(\mathbb{R}^n) = S_{\tilde{m},\Lambda}(\mathbb{R}^n)$.*

Proof. For fixed $\varepsilon > 0$, in the set of smooth compactly supported functions $C_0^\infty(\mathbb{R}^n)$, consider a non negative $\varphi(\zeta)$ such that $|\zeta_j| \leq \varepsilon$ for every $j = 1, \dots, n$ in $\operatorname{supp}\varphi(\zeta)$ and $\varphi(\zeta) = 1$ when $|\zeta_j| \leq \frac{\varepsilon}{2}$, $j = 1, \dots, n$. Taking now the weight vector $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$, we set:

$$\Phi(\xi, \eta) := \varphi\left(\frac{\xi_1 - \eta_1}{\lambda_1(\eta)}, \dots, \frac{\xi_n - \eta_n}{\lambda_n(\eta)}\right).$$

Notice now that in the support of $\Phi(\xi, \eta)$ one has $|\xi_j - \eta_j| \leq \varepsilon\lambda_j(\eta)$, for any $j = 1, \dots, n$, and $\Phi(\xi, \eta)$ is identically equal to 1 when $|\xi_j - \eta_j| \leq \frac{\varepsilon}{2}\lambda_j(\eta)$. Then, assuming $\varepsilon < \frac{c}{2n}$ and ξ, η in $\operatorname{supp}\Phi(\xi, \eta)$, (2.3) assures that, for some $C > 0$, $H = \varepsilon C$:

$$c \leq \frac{\lambda_j(\eta)}{\lambda_j(\xi)} \leq C \quad \text{and} \quad |\xi_j - \eta_j| \leq H\lambda_j(\xi), \quad j = 1, \dots, n.$$

The same is true when $\Phi(\xi, \eta) = 1$ by changing H with $\tilde{H} = \dfrac{\varepsilon}{2}C$. Then

$$\int \Phi(\xi, \eta)\, d\eta \leq \|\varphi\|_\infty \int \chi_{B(\xi)}(\xi - \eta)\, d\eta = (2H)^n \|\varphi\|_\infty \prod_{j=1}^n \lambda_j(\xi);$$

$$\int \Phi(\xi, \eta)\, d\eta \geq \int \chi_{\tilde{B}(\xi)}(\xi - \eta)\, d\eta = (2\tilde{H})^n \prod_{j=1}^n \lambda_j(\xi).$$

Here $\chi_{B(\xi)}$ and $\chi_{\tilde{B}(\xi)}$ are the characteristic functions respectively of the cube $B(\xi) = \prod_{j=1}^n [-H\lambda_j(\xi), H\lambda_j(\xi)]$ and $\tilde{B}(\xi) = \prod_{j=1}^n \left[-\tilde{H}\lambda_j(\xi), \tilde{H}\lambda_j(\xi)\right]$. It then follows that $\int \Phi(\xi, \eta)\, d\eta \asymp \prod\limits_{j=1}^n \lambda_j(\xi)$. Set now:

$$\tilde{m}(\xi) = \int m(\eta)\Phi(\xi, \eta)\Pi_{j=1}^n \lambda_j(\eta)^{-1}\, d\eta. \tag{2.14}$$

Since for $\varepsilon < \frac{c}{2n}$ and any $j = 1, \dots, n$, $|\xi_j - \eta_j| \leq \varepsilon\lambda_j(\eta)$ in $\operatorname{supp}\Phi(\xi, \eta)$, it follows from (2.3) and (2.5), $m(\eta) \approx m(\xi)$ and $\lambda_j(\eta) \approx \lambda_j(\xi)$, for any $j = 1, \dots, n$, then $\tilde{m}(\xi) \asymp m(\xi)$. Moreover $\tilde{m}(\xi)$ is obviously smooth and for any $\alpha \in \mathbb{Z}_+^n$:

$$\partial^\alpha \tilde{m}(\xi) = \int m(\eta)\partial_\zeta^\alpha \varphi\left(\frac{\xi_1 - \eta_1}{\lambda_1(\eta)}, \dots, \frac{\xi_n - \eta_n}{\lambda_n(\eta)}\right) \prod_{j=1}^n \lambda_j(\eta)^{-\alpha_j - 1}\, d\eta. \tag{2.15}$$

Since $\operatorname{supp}\partial_\zeta^\alpha \varphi \subset \operatorname{supp}\varphi$, we obtain, for some positive constant M_α:

$$|\partial^\alpha \tilde{m}(\xi)| \leq M_\alpha \tilde{m}(\xi)\Lambda(\xi)^{-\alpha}, \tag{2.16}$$

which concludes the proof. □

Thanks to the relations with the Hörmander symbol classes (2.9), we can define for $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ the pseudodifferential operator

$$a(x,D)u := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, d\xi\,, \quad u \in C_0^\infty(\Omega). \tag{2.17}$$

Op $S_{m,\Lambda}(\Omega)$ denotes the class of all the pseudodifferential operators with symbol in $S_{m,\Lambda}(\Omega)$.
Any symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ defines by means of (2.17) a bounded linear operator: $a(x,D) : C_0^\infty(\Omega) \mapsto C^\infty(\Omega)$, which extends to a linear operator from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$. Let now $\widetilde{\mathrm{Op}} S_{m,\Lambda}(\Omega)$ be the class of *properly supported* pseudodifferential operators, that is the operators which map $C_0^\infty(\Omega)$ to $\mathcal{E}'(\Omega)$ and the same happens for their transposed.
For any $a(x,\xi) \in S_{m,\Lambda}(\Omega)$, there exists $a'(x,\xi) \in S_{m,\Lambda}(\Omega)$ such that $a'(x,D)$ is properly supported and $a'(x,\xi) \sim a(x,\xi)$, that is $a'(x,\xi) - a(x,\xi) \in S^{-\infty}(\Omega)$.

Proposition 2.10 (symbolic calculus). *Let $m(\xi)$, $m'(\xi)$ be admissible weights associated to the same weight vector $\Lambda(\xi)$ and $a_1(x,D) \in \widetilde{\mathrm{Op}}\, S_{m,\Lambda}(\Omega)$, $a_2(x,D) \in$ Op $S_{m',\Lambda}(\Omega)$. Then $a_1(x,D)a_2(x,D) = b(x,D)$, where $b(x,\xi) \in S_{mm',\Lambda}(\Omega)$, and*

$$b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha} \partial_\xi^\alpha a_1(x,\xi) D_x^\alpha a_2(x,\xi)\,, \qquad D^\alpha := (-i)^{|\alpha|}\partial^\alpha\,. \tag{2.18}$$

Consider now $m(\xi) \equiv 1$, then by means of the arguments in Remark 2.7(4), $S_\Lambda(\Omega)$ is contained in the Taylor class $M^0_{c,0}(\Omega)$, for suitable $0 < c < 1$. Let us recall that a symbol $a(x,\xi)$ belongs to $M^0_{c,0}(\Omega)$, if $\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in S^0_{c,0}(\Omega)$ for all multi-indices $\gamma \in \{0,1\}^n$. Then applying the arguments in Taylor [14], see also [6, Theorem 4.1, Corollary 4.2], the following property immediately follows.

Proposition 2.11. *If $a(x,\xi) \in S_\Lambda(\Omega)$, then, for any $1 < p < \infty$*

$$a(x,D) : L^p_{\mathrm{comp}}(\Omega) \mapsto L^p_{\mathrm{loc}}(\Omega).$$

3. Weighted Sobolev spaces

Consider the class of global symbols $S_{m,\Lambda}$ given by the smooth functions $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ which satisfy

$$\sup_{x\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta} m(\xi)\Lambda(\xi)^{-\alpha}, \quad \xi \in \mathbb{R}^n. \tag{3.1}$$

Then the pseudodifferential operators in Op$S_{m,\Lambda}$ defined by (2.17) map continuously $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and may be extended to bounded linear operators from $\mathcal{S}'(\mathbb{R}^n)$ into itself. Thanks again to the arguments in Remark 2.7, every pseudodifferential operator with zeroth order symbol in S_Λ, maps continuously $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The symbolic calculus in Proposition 2.10 is true again for $a(x,D) \in$ Op $S_{m,\Lambda}$ and $b(x,D) \in$ Op $S_{m',\Lambda}$. Moreover Proposition 2.9 assures that

$m(\xi)$ admits an equivalent weight which is a symbol in $S_{m,\Lambda}$. Without loss of generality, from now on we consider $m(\xi) \in S_{m,\Lambda}$. We can then define for $1 < p < \infty$ the weighted Sobolev space of L^p type:

$$H^p_m := \{u \in \mathcal{S}'(\mathbb{R}^n), \quad \text{such that} \quad m(D)u \in L^p(\mathbb{R}^n)\}\,. \tag{3.2}$$

H^p_m may be equipped in natural way by the norm $\|u\|_{p,m} := \|m(D)u\|_{L^p}$ and it then realizes to be a Banach space (Hilbert space in the case $p = 2$). With standard arguments it can be proved that $\mathcal{S}(\mathbb{R}^n) \subset H^p_m \subset \mathcal{S}'(\mathbb{R}^n)$, with continuous embeddings and moreover $\mathcal{S}(\mathbb{R}^n)$ is dense in H^p_m, $1 < p < \infty$. For any open subset $\Omega \subset \mathbb{R}^n$ the following local spaces may be introduced:

$$H^p_{m,\mathrm{comp}}(\Omega) = \mathcal{E}'(\Omega) \cap H^p_m; \tag{3.3}$$

$$H^p_{m,\mathrm{loc}}(\Omega) = \{u \in \mathcal{D}'(\Omega) \text{ such that, for any } \varphi \in C^\infty_0(\Omega),\ \varphi u \in H^p_m\}\,. \tag{3.4}$$

It is now trivial that $C^\infty(\Omega) \subset H^p_{m,\mathrm{loc}}(\Omega)$ for any $1 < p < \infty$.

Proposition 3.1. *Consider m, m' admissible weights and $a_1(x,\xi) \in S_{m',\Lambda}$, $a_2 \in S_{m',\Lambda}(\Omega)$, then for any $p \in]1,\infty[$ we have:*

$$a_1(x,D) : H^p_m \mapsto H^p_{m/m'}; \tag{3.5}$$

$$a_2(x,D) : H^p_{m,\mathrm{comp}}(\Omega) \mapsto H^p_{m/m',\mathrm{loc}}(\Omega). \tag{3.6}$$

If moreover $a(x,D)$ is a properly supported operator in $\widetilde{\mathrm{Op}}S_{m',\Lambda}(\Omega)$ then:

$$a(x,D) : H^p_{m,\mathrm{comp}}(\Omega) \mapsto H^p_{m/m',\mathrm{comp}}(\Omega); \tag{3.7}$$

$$a(x,D) : H^p_{m,\mathrm{loc}}(\Omega) \mapsto H^p_{m/m',\mathrm{loc}}(\Omega). \tag{3.8}$$

Proof. Since m/m' is an admissible weight, the symbolic calculus in Proposition 2.10 assures that

$$[m/m'](D)a_1(x,D)u = m(D)[1/m'](D)a_1(x,D)[1/m](D)m(D)u, u \in \mathcal{S}(\mathbb{R}^n)$$

and $m(D)[1/m'](D)a_1(x,D)[1/m](D)$ is a pseudodifferential operator with symbol in S_Λ. Thanks now to the L^p continuity of $\mathrm{Op}\,S_\Lambda$, we obtain:

$$\|a_1(x,D)u\|_{m/m'} = \|\,[m/m'](D)a_1(x,D)u\|_{L^p} \le K\|m(D)u\|_{L^p} = K\|u\|_{p,m}.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in H^p_m, then $a_1(x,D)$ extends univocally to a bounded linear operator from H^p_m to $H^p_{m/m'}$.
By standard arguments the proof applies to (3.6),(3.7), (3.8). □

Definition 3.2 (elliptic symbols). A symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ is elliptic if for any compact $K \subset\subset \Omega$ there exist $c_K > 0$ and $R_K > 0$ such that:

$$|a(x,\xi)| \ge c_k m(\xi), \quad x \in K, \quad |\xi| \ge R_K\,. \tag{3.9}$$

Proposition 3.3 (parametrix). *Let $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ be a elliptic symbol. Then a properly supported operator $b(x,D) \in \widetilde{\mathrm{Op}}S_{1/m,\Lambda}(\Omega)$ exists such that:*

$$b(x,D)a(x,D) = Id + \rho(x,D), \tag{3.10}$$

where $\rho(x,\xi) \in S^{-\infty}(\Omega)$ and Id *denotes the identity operator.*

See [12, Theorem 1.3.6] for the proof of the above result.

Proposition 3.4 (Regularity of solution to elliptic equations). *Consider* $p \in]1, \infty[$, *the admissible weights* $m(\xi)$, $m'(\xi)$, *and the* $m'-$*elliptic symbol* $a(x,\xi) \in S_{m',\Lambda}(\Omega)$. *Then for every* $u \in \mathcal{E}'(\Omega)$ *such that* $a(x,D)u \in H^p_{m/m',\mathrm{loc}}(\Omega)$, *we have* $u \in H^p_{m,\mathrm{comp}}(\Omega)$. *If* $a(x,D)$ *is properly supported, then* $u \in H^p_{m,\mathrm{loc}}(\Omega)$ *for every* $u \in \mathcal{D}'(\Omega)$ *such that* $a(x,D)u \in H^p_{m/m',\mathrm{loc}}(\Omega)$.

Proof. Thanks to Proposition 3.3, there exists $b(x,D) \in \widetilde{\mathrm{Op}}\, S_{1/m',\Lambda}(\Omega)$, such that $b(x,D)a(x,D) = I + \rho(x,D)$, with $\rho(x,\xi) \in S^{-\infty}(\Omega)$. Since $\rho(x,D)$ is a regularizing operator and $a(x,D)u \in H^p_{m/m',\mathrm{loc}}(\Omega)$, we can conclude from (3.6) that $u = b(x,D)\,(a(x,D)u) - \rho(x,D)u \in H^p_{m,\mathrm{loc}}(\Omega)$. □

4. Microlocal Properties of pseudo-differential operators with symbols in $S_{m,\Lambda}(\Omega)$

Definition 4.1. A symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ is microlocally elliptic in a set $X \subset \mathbb{R}^n_\xi$ at the point $x_0 \in \Omega$ if there are positive constants c_0, R_0 such that

$$|a(x_0,\xi)| \geq c_0 m(\xi), \quad \text{when} \quad \xi \in X, \quad |\xi| > R_0\,. \tag{4.1}$$

The Λ-neighborhood of a set $X \subset \mathbb{R}^n$ with length $\varepsilon > 0$ is defined to be the open set:

$$X_{\varepsilon\Lambda} := \bigcup_{\xi^0 \in X} \left\{|\xi_j - \xi^0_j| < \varepsilon \lambda_j(\xi^0), \quad \text{for}\, j = 1, \dots, n\right\}. \tag{4.2}$$

Moreover for $x_0 \in \Omega$ we set:

$$X(x_0) := \{x_0\} \times X, \quad X_{\varepsilon\Lambda}(x_0) := B_\varepsilon(x_0) \times X_{\varepsilon\Lambda}, \tag{4.3}$$

where $B_\varepsilon(x_0)$ is the open ball in Ω centered at x_0 with radius ε.
Noticing that $\Lambda(\xi)$ is a weight vector according to [13], the following properties of Λ-neighborhoods can be immediately deduced from [13, Lemma 1.11] (see also [8] for an explicit proof). For every $\varepsilon > 0$ a suitable ε^* (depending only on ε and Λ), satisfying $0 < \varepsilon^* < \varepsilon$, can be found in such a way that for every $X \subset \mathbb{R}^n$:

1. $(X_{\varepsilon^*\Lambda})_{\varepsilon^*\Lambda} \subset X_{\varepsilon\Lambda}$;
2. $(\mathbb{R}^n \setminus X_{\varepsilon\Lambda})_{\varepsilon^*\Lambda} \subset \mathbb{R}^n \setminus X_{\varepsilon^*\Lambda}$

In view of [13, Lemma 1.10], one can also prove that for arbitrary $\varepsilon > 0$ and $X \subset \mathbb{R}^n$ there exists a symbol $\sigma = \sigma(\xi) \in S_\Lambda$ such that $\operatorname{supp}\sigma \subset X_{\varepsilon\Lambda}$ and $\sigma(\xi) = 1$ if $\xi \in X_{\varepsilon'\Lambda}$, for a suitable ε', $0 < \varepsilon' < \varepsilon$ depending only on ε and Λ. Moreover for every $x_0 \in \Omega$ there exists a symbol $\tau_0(x,\xi) \in S_\Lambda(\Omega)$ such that $\operatorname{supp}\tau_0 \subset X_{\varepsilon\Lambda}(x_0)$ and $\tau_0(x,\xi) = 1$, for $(x,\xi) \in X_{\varepsilon^*\Lambda}(x_0)$, with a suitable ε^* satisfying $0 < \varepsilon^* < \varepsilon$.

Proposition 4.2. *If a symbol $a(x,\xi) \in \mathcal{S}_{m,\Lambda}(\Omega)$ is microlocally elliptic in $X \subset \mathbb{R}^n_\xi$ at the point $x_0 \in \Omega$, there exists a suitable $\varepsilon > 0$ such that* (4.1) *is satisfied in $X_{\varepsilon\Lambda}(x_0)$, that is for suitable constants $C, R > 0$*

$$|a(x,\xi)| \geq Cm(\xi), \quad \textit{for}\,(x,\xi) \in X_{\varepsilon\Lambda}(x_0), \quad |\xi| > R\,. \tag{4.4}$$

Proof. Let the symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ be microlocally elliptic in $X \subset \mathbb{R}^n$ at the point $x_0 \in \Omega$ and let $\xi^0 \in X$ be arbitrarily fixed. Since Ω is open, a positive ε^* can be found in such a way that $\overline{B}_{\varepsilon^*}(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon^*\} \subset \Omega$; for $0 < \varepsilon < \varepsilon^*$ and $(x,\xi) \in X_{\varepsilon\Lambda}(x_0)$, a Taylor expansion of $a(x,\xi)$ about (x_0,ξ^0) gives

$$a(x,\xi) - a(x_0,\xi^0) = \sum_{j=1}^{n} (x^j - x_0^j)\partial_{x^j} a(x_t,\xi^t)dt + (\xi_j - \xi_j^0)\partial_{\xi_j} a(x_t,\xi^t)dt\,, \tag{4.5}$$

where it is set $(x_t,\xi^t) := ((1-t)x_0 + tx, (1-t)\xi^0 + t\xi)$ for a suitable $0 < t < 1$. Since $|\xi_j^t - \xi_j^0| = |t||\xi_j - \xi_j^0| < \varepsilon\lambda_j(\xi^0)$ and $|x_t^j - x_0^j| = |t||x^j - x_0^j| < \varepsilon$, from (2.8) there exists $C^* > 0$, depending only on ε^*, such that

$$|a(x,\xi) - a(x_0,\xi^0)| \leq \sum_{j=1}^{n} \varepsilon C^* m(\xi^t) + \varepsilon\lambda_j(\xi^0) C^* m(\xi^t)\lambda_j^{-1}(\xi^t)\,. \tag{4.6}$$

In view of (2.3), (2.5), $\varepsilon > 0$ can be chosen small enough such that

$$\frac{1}{C}\lambda_j(\xi^0) \leq \lambda_j(\xi^t) \leq C\lambda_j(\xi^0)\,, \quad \frac{1}{C}m(\xi^0) \leq m(\xi^t) \leq Cm(\xi^0)\,, \qquad 1 \leq j \leq n\,, \tag{4.7}$$

for a suitable constant $C > 1$ independent of t and ε. Then (4.6), (4.7) give

$$|a(x,\xi) - a(x_0,\xi^0)| \leq \hat{C}\varepsilon m(\xi^0)\,, \tag{4.8}$$

with a suitable constant $\hat{C} > 0$ independent of ε.
Let the condition (4.1) be satisfied by $a(x,\xi)$ with positive constants c_0, R_0. Provided $0 < \varepsilon < \varepsilon^*$ is taken sufficiently small, one can find a positive R, depending only on R_0, such that $|\xi| > R$ and $|\xi_j - \xi_j^0| < \varepsilon\lambda_j(\xi^0)$ for all $1 \leq j \leq n$ yield $|\xi^0| > R_0$; indeed, from (2.1)

$$|\xi - \xi^0| \leq \sum_{j=1}^{n} |\xi_j - \xi_j^0| < \varepsilon \sum_{j=1}^{n} \lambda_j(\xi^0) \leq nC\varepsilon^*(1 + |\xi_0|)^C\,,$$

and then

$$|\xi| \leq |\xi^0| + |\xi - \xi^0| \leq |\xi^0| + nC\varepsilon^*(1 + |\xi^0|)^C\,.$$

Hence, it is sufficient to choose R such that $R > R_0 + nC\varepsilon^*(1 + R_0)^C$.
Since $|\xi^0| > R_0$, the microlocal ellipticity of $a(x,\xi)$ yields

$$|a(x_0,\xi^0)| \geq c_0 m(\xi^0)\,; \tag{4.9}$$

then (4.8) and (4.9) give for $(x,\xi) \in X_{\varepsilon\Lambda}(x_0)$ and $|\xi| > R$

$$|a(x,\xi)| \geq |a(x_0,\xi^0)| - |a(x,\xi) - a(x_0,\xi^0)| \geq (c_0 - \hat{C}\varepsilon)m(\xi^0) \geq \frac{c_0}{2}m(\xi^0)\,, \tag{4.10}$$

up to a further shrinking of $\varepsilon > 0$. From (4.10), the condition (4.4) follows at once, by using that $m(\xi) \approx m(\xi^0)$. $\square$

Definition 4.3. We say that a symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ is rapidly decreasing in $\Theta \subset \Omega \times \mathbb{R}^n$ if there exists $a_0(x,\xi) \in S_{m,\Lambda}(\Omega)$ such that $a(x,\xi) \sim a_0(x,\xi)$ and $a_0(x,\xi) = 0$ in Θ

Theorem 4.4. *For every symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ that is microlocally elliptic in $X \subset \mathbb{R}^n$ at a point $x_0 \in \Omega$ there exists a symbol $b(x,\xi) \in S_{1/m,\Lambda}(\Omega)$ such that the associated operator $b(x,D)$ is properly supported and*

$$b(x,D)a(x,D) = Id + c(x,D), \tag{4.11}$$

where $c(x,\xi) \in S_\Lambda(\Omega)$ is rapidly decreasing in $X_{r\Lambda}(x_0)$ for a suitable $r > 0$.

Proof. We follow the same arguments used for the proof of [8, Theorem 1]. By Proposition 4.2, there exists $\varepsilon > 0$ such that $a(x,\xi)$ is microlocally elliptic at $X_{\varepsilon\Lambda}(x_0)$. Let $\tau_0(x,\xi)$ be a symbol in $S_\Lambda(\Omega)$ such that $\tau_0 \equiv 1$ on $X_{\varepsilon'\Lambda}(x_0)$, for a suitable $0 < \varepsilon' < \varepsilon$, and $\operatorname{supp} \tau_0 \subset X_{\varepsilon\Lambda}(x_0)$. We define $b_0(x,\xi)$ by setting

$$b_0(x,\xi) := \begin{cases} \dfrac{\tau_0(x,\xi)}{a(x,\xi)} & \text{for} \quad (x,\xi) \in X_{\varepsilon\lambda}(x_0), \\ 0 & \text{otherwise}. \end{cases} \tag{4.12}$$

Since $a(x,\xi)$ satisfies (4.4), with suitable constants C, R, $b_0(x,\xi)$ is a well defined C^∞–function on the set $\Omega \times \{|\xi| > R\}$. For $k \geq 1$, the functions $b_{-k}(x,\xi)$ are defined recursively on $\Omega \times \{|\xi| > R\}$ by

$$b_{-k}(x,\xi) := \begin{cases} -\displaystyle\sum_{0<|\alpha|\leq k} \tfrac{1}{\alpha!}\partial_\xi^\alpha b_{-k+|\alpha|}(x,\xi)\frac{D_x^\alpha a(x,\xi)}{a(x,\xi)}, & \text{for} \quad (x,\xi) \in X_{\varepsilon\lambda}(x_0), \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

The $b_{-k}(x,\xi)$ can be then extended to the whole set $\Omega \times \mathbb{R}^n$, multiplying them by a smooth cut-off function that vanishes on the set of possible zeroes of the symbol $a(x,\xi)$; for each $k \geq 0$, the extended b_{-k} is a symbol in $S_{\pi^{-k}/m,\Lambda}(\Omega)$. In view of the properties of the symbolic calculus, a symbol $b \in S_{1/m,\Lambda}(\Omega)$ can be chosen in such a way that

$$b \sim \sum_{k\geq 0} b_{-k}\,,$$

and $b(x,D)$ is a properly supported operator. By construction, the symbol of $b(x,D)a(x,D)$ is equivalent to $\tau_0(x,\xi)$; thus the symbol of $b(x,D)a(x,D) - \mathrm{Id}$ belongs to $S_\Lambda(\Omega)$ and is rapidly decreasing in $X_{r\Lambda}(x_0)$ for $0 < r \leq \varepsilon'$. ☐

Proposition 4.5. *For $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, $1 < p < \infty$, the following properties are equivalent:*

i) there exists an operator $a(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ whose symbol is microlocally elliptic in X at the point x_0, such that $a(x,D)u \in H^p_{m,\mathrm{loc}}(\Omega)$;

ii) there exist a symbol $\sigma = \sigma(\xi) \in S_\Lambda$, such that $\operatorname{supp} \sigma \subset X_{\varepsilon\Lambda}$, $\sigma(\xi) = 1$ when $\xi \in X_{\varepsilon'\lambda}$ for suitable $0 < \varepsilon' < \varepsilon$, and a function $\phi \in C_0^\infty(\Omega)$, with $\phi(x_0) = 1$, satisfying $\sigma(D)(\phi u) \in H^p_m$.

Proof. $i) \Rightarrow ii)$: Let the operator $a(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ satisfy the assumptions in $i)$; from Theorem 4.4 there exists $b(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ satisfying

$$b(x,D)a(x,D) = \mathrm{Id} + c(x,D)\,, \tag{4.14}$$

where $c(x,\xi) \in S_\Lambda(\Omega)$ is rapidly decreasing in $X_{\varepsilon\Lambda}(x_0)$ for some $0 < \varepsilon < 1$.
Let $\sigma(\xi) \in S_\Lambda$ and $\phi(x) \in C_0^\infty(\Omega)$ satisfy the conditions in $ii)$ with suitable $0 < \varepsilon' < \varepsilon$. From (4.14) we write

$$\sigma(D)(\phi u) = \sigma(D)\phi(x)b(x,D)a(x,D)u - \sigma(D)\phi(x)c(x,D)u\,. \tag{4.15}$$

Since $a(x,D)u \in H^p_{m,\mathrm{loc}}(\Omega)$, by Proposition 3.1 $\sigma(D)\phi(x)b(x,D)a(x,D)u \in H^p_m$.
As regards to the second term $\sigma(D)\phi(x)c(x,D)u$ in the right-hand side of (4.15), the operator $c(x,D)$ is known to be properly supported (since $a(x,D)$ and $b(x,D)$ are so and (4.14) holds true). Then a function $\widetilde{\phi} \in C_0^\infty(\Omega)$ can be found in such a way that

$$\phi(x)c(x,D)u = \phi(x)c(x,D)(\widetilde{\phi}u)\,. \tag{4.16}$$

Since $c(x,\xi)$ is also rapidly decreasing, there exist $c_0(x,\xi) \in S_\Lambda(\Omega)$, supported in $(\Omega \times \mathbb{R}^n) \setminus X_{\varepsilon\Lambda}(x_0)$, such that

$$\rho(x,\xi) := c(x,\xi) - c_0(x,\xi) \in S^{-\infty}(\Omega)\,. \tag{4.17}$$

Then

$$\begin{aligned}\sigma(D)(\phi(x)c(x,D)u) &= \sigma(D)(\phi(x)c(x,D)(\widetilde{\phi}u))\\ &= \sigma(D)(\phi(x)c_0(x,D)(\widetilde{\phi}u)) + \sigma(D)(\phi(x)\rho(x,D)(\widetilde{\phi}u))\,.\end{aligned} \tag{4.18}$$

We immediately get $\sigma(D)(\phi(x)\rho(x,D)(\widetilde{\phi}u)) \in \mathcal{S}(\mathbb{R}^n) \subset H^p_m$.
Now, let us set $d(x,D) := \sigma(D)\phi(x)c_0(x,D)$; from the symbolic calculus, the symbol of $d(x,D)$ enjoys the asymptotic expansion

$$d(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\sigma(\xi)D_x^\alpha(\phi(x)c_0(x,\xi))\,. \tag{4.19}$$

Since $\operatorname{supp}\sigma \subset X_{\varepsilon\Lambda}$, $\partial_\xi^\alpha\sigma(\xi) = 0$ as long as $\xi \notin X_{\varepsilon\Lambda}$. On the other hand, $c_0(x,\xi) = 0$ for $\xi \in X_{\varepsilon\Lambda}$ and $x \in B_\varepsilon(x_0)$ and $\phi(x) = 0$ for $x \notin B_\varepsilon(x_0)$. Hence for all $\alpha \in \mathbb{Z}^n_+$

$$\partial_\xi^\alpha\sigma(\xi)D_x^\alpha(\phi(x)c_0(x,\xi)) = 0\,, \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}^n\,; \tag{4.20}$$

thus $d(x,\xi) \in S^{-\infty}(\Omega)$ and we get $\sigma(D)(\phi(x)c_0(x,D)(\widetilde{\phi}u)) = d(x,D)(\widetilde{\phi}u) \in C^\infty(\Omega)$. On the other hand, the two operators $\sigma(D)$ and $\phi(x)c_0(x,D)$ have global symbols $\sigma(\xi)$ and $\phi(x)c_0(x,\xi)$ in S_Λ; then $d(x,\xi)$ is a global symbol in $S^{-\infty}$ and $d(x,D)(\widetilde{\phi}u) \in \mathcal{S}(\mathbb{R}^n) \subset H^p_m$.
$ii) \Rightarrow i)$: Let $\sigma(\xi) \in S_\Lambda$ and $\phi(x) \in C_0^\infty(\Omega)$, satisfying the assumptions in $ii)$, be such that $\sigma(D)(\phi u) \in H^p_m$. Since $S_\Lambda \subset S_\Lambda(\Omega)$, there exists $\widetilde{\sigma}(x,\xi) \in S_\Lambda(\Omega)$ such that $\sigma \sim \widetilde{\sigma}$ and $\widetilde{\sigma}(x,D)$ is properly supported. Let us set $a(x,D) := \widetilde{\sigma}(x,D)\phi(x) \in \widetilde{\mathrm{Op}}\,S_\Lambda(\Omega)$; we also have

$$a(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\widetilde{\sigma}(x,\xi)D_x^\alpha\phi(x)\,. \tag{4.21}$$

In particular, $\tau(x,\xi) := a(x,\xi) - \widetilde{\sigma}(x,\xi)\phi(x) \in S_{\pi^{-1},\Lambda}(\Omega)$ and $\rho(x,\xi) := \widetilde{\sigma}(x,\xi) - \sigma(\xi) \in S^{-\infty}(\Omega)$. Hence

$$a(x,\xi) = (\sigma(\xi) + \rho(x,\xi))\phi(x) + \tau(x,\xi) = \sigma(\xi)\phi(x) + \theta(x,\xi)\,, \tag{4.22}$$

where $\theta(x,\xi) := \rho(x,\xi)\phi(x) + \tau(x,\xi) \in S_{\pi^{-1},\Lambda}(\Omega)$. This proves that $a(x,\xi)$ is microlocally elliptic in X at x_0; indeed for $\xi \in X$ (and in view of (2.1))

$$|a(x_0,\xi)| \geq |\sigma(\xi)| - |\theta(x_0,\xi)| = 1 - C_0\pi^{-1}(\xi) \geq 1 - C_1\langle\xi\rangle^{-C}\,, \tag{4.23}$$

hence $|a(x_0,\xi)| \geq \frac{1}{2}$ when $|\xi| > R$ and $R > 0$ is taken sufficiently large. Finally, one computes

$$a(x,D)u = \widetilde{\sigma}(x,D)(\phi u) = \sigma(D)(\phi u) + \rho(x,D)(\phi u) \in H^p_{m\,\mathrm{loc}}(\Omega)\,,$$

which completes the proof. □

Definition 4.6. For $X \subset \mathbb{R}^n$, $x_0 \in \Omega$ and $p \in]1,\infty[$ we say that that $u \in \mathcal{D}'(\Omega)$ is microlocally H^p_m-regular in X at the point $x_0 \in \Omega$, and write $u \in mclH^p_m(X(x_0))$, if one of the equivalent properties in Proposition 4.5 is satisfied.

5. Microlocal Sobolev Continuity and Regularity

In the following, we will provide a microlocal counterpart of the properties of boundedness and regularity for pseudodifferential operators developed in Section 3. In the sequel, m, m' are two admissible weights associated to the same weight vector Λ.

Proposition 5.1. *Let $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $a(x,D) \in \widetilde{\mathrm{Op}}S_{m,\Lambda}(\Omega)$ be given. Then for $p \in]1,\infty[$ and $u \in mclH^p_{m'}(X(x_0))$ one has $a(x,D)u \in mclH^p_{m'/m}(X(x_0))$.*

Proof. From Proposition 4.5, there exists $b(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$, with microlocally elliptic symbol, such that $b(x,D)u \in H^p_{m',\mathrm{loc}}(\Omega)$. From Theorem 4.4 there also exists an operator $c(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ such that

$$c(x,D)b(x,D) = \mathrm{Id} + \rho(x,D)\,, \tag{5.1}$$

where $\rho(x,\xi) \in S_\Lambda(\Omega)$ is rapidly decreasing in $X_{r\Lambda}(x_0)$ for some $0 < r < 1$. Let $r^* > 0$ be such that

$$(\mathbb{R}^n \setminus X_{r\Lambda})_{r^*\Lambda} \subset \mathbb{R}^n \setminus X_{r^*\Lambda}\,, \qquad 0 < r^* < r\,, \tag{5.2}$$

and take a symbol $\tau_0(x,\xi) \in S_\Lambda(\Omega)$ satisfying

$$\operatorname{supp}\tau_0 \subset X_{r^*\Lambda}(x_0)\,, \qquad \tau_0 \equiv 1 \text{ on } X_{r'\Lambda}(x_0)\,,$$

with a suitable $0 < r' < r^*$. Finally, let $\tau(x,\xi)$ be a symbol such that $\theta_0(x,\xi) := \tau(x,\xi) - \tau_0(x,\xi) \in S^{-\infty}(\Omega)$ and $\tau(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$. One can check (see [8] for details) that $\tau(x,\xi)$ is microlocally elliptic in X at x_0; in particular, $\tau(x,\xi) =$

$\theta_0(x,\xi) \in S^{-\infty}(\Omega)$ for $(x,\xi) \notin X_{r^*\Lambda}(x_0)$.
Arguing as in the proof of [8, Theorem 2], from (5.1) we write

$$\tau(x,D)a(x,D)u = \tau(x,D)a(x,D)c(x,D)(b(x,D)u) - \tau(x,D)a(x,D)\rho(x,D)u\,. \tag{5.3}$$

Since $\tau(x,D)a(x,D)c(x,D) \in \widetilde{\mathrm{Op}}S_{m,\Lambda}(\Omega)$ and $b(x,D)u \in H^p_{m',\mathrm{loc}}(\Omega)$ then we get $\tau(x,D)a(x,D)c(x,D)(b(x,D)u) \in H^p_{m'/m,\mathrm{loc}}(\Omega)$. Moreover, it can be shown that in view of (5.2)

$$\varphi(x)\tau(x,D)a(x,D)\rho(x,D)u \in C_0^\infty(\Omega) \subset H^p_{m'/m}\,,$$

for every $\varphi \in C_0^\infty(\Omega)$, so that $\tau(x,D)a(x,D)\rho(x,D)u \in H^p_{m'/m,\mathrm{loc}}(\Omega)$ (see [8, Theorem 2]).
This proves that $\tau(x,D)a(x,D)u \in H^p_{m'/m,\mathrm{loc}}(\Omega)$ and ends the proof. □

Proposition 5.2. *For $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, let the symbol of $a(x,D) \in \widetilde{\mathrm{Op}}S_{m,\Lambda}(\Omega)$ be microlocally elliptic in X at the point x_0. Then for every $p \in]1,\infty[$ and $u \in \mathcal{D}'(\Omega)$ such that $a(x,D)u \in mclH^p_{m'/m}(X(x_0))$ one has $u \in mclH^p_{m'}(X(x_0))$.*

Proof. From Proposition 4.5, there exists an operator $b(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ microlocally elliptic in X at x_0 such that

$$b(x,D)a(x,D)u \in H^p_{m'/m,\mathrm{loc}}(\Omega)\,. \tag{5.4}$$

From Theorem 4.4 there exist $c(x,D) \in \widetilde{\mathrm{Op}}S_\Lambda(\Omega)$ and $q(x,D) \in \widetilde{\mathrm{Op}}S_{1/m,\Lambda}(\Omega)$ such that

$$c(x,D)b(x,D) = \mathrm{Id} + \rho(x,D)\,, \qquad q(x,D)a(x,D) = \mathrm{Id} + \sigma(x,D)\,, \tag{5.5}$$

with $\rho(x,\xi), \sigma(x,\xi) \in S^{-\infty}(\Omega)$ rapidly decreasing in $X_{r\Lambda}(x_0)$ for a suitable $0 < r < 1$.
Let the symbols $\tau_0(x,\xi), \tau(x,\xi) \in S_\Lambda(\Omega)$ be constructed as in the proof of Proposition 5.1. It can be proved that $\tau(x,D)u \in H^p_{m,\mathrm{loc}}(\Omega)$, by writing

$$\begin{aligned}\tau(x,D)u &= \tau(x,D)q(x,D)c(x,D)\,(b(x,D)a(x,D)u)\\ &\quad -\tau(x,D)q(x,D)\rho(x,D)a(x,D)u - \tau(x,D)\sigma(x,D)u\,,\end{aligned} \tag{5.6}$$

where the identities (5.5) have been used, and applying similar arguments as in the proof of Proposition 5.1 (see also [8, Theorem 3]). □

6. The $m-$filter of Sobolev singularities

Let $a(x,D)$ be a properly supported pseudo-differential operator with symbol $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ and $x_0 \in \Omega$. Following [4], [13], we can define, for any $x_0 \in \Omega$,

- the $m-$filter of Sobolev singularities of $u \in \mathcal{D}'(\Omega)$ by

$$\mathcal{W}^p_{m,x_0}u := \{X \subset \mathbb{R}^n\,;\, u \in mclH^p_m\left((\mathbb{R}^n \setminus X)(x_0)\right)\}\,, \quad 1 < p < \infty; \tag{6.1}$$

- the $m-$characteristic filter of $a(x,D) \in \widetilde{\mathrm{Op}} S_{m,\Lambda}(\Omega)$ by

$$\Sigma_{m,x_0} a(x,D) := \{X \subset \mathbb{R}^n \, , \, a(x,\xi) \text{ is microlocally elliptic in } \mathbb{R}^n \setminus X \text{ at } x_0\} \, . \tag{6.2}$$

It is trivial that $\mathcal{W}^p_{m,x_0}u$ and $\Sigma_{m,x_0}a(x,D)$ are filters in the sense that they are closed with respect to the intersection of a finite collection of their members and if $X \in \mathcal{W}^p_{m,x_0}u\,(\Sigma_{m,x_0}a(x,D))$ and $X \subset Y$ then $Y \in \mathcal{W}^p_{m,x_0}u\,(\Sigma_{m,x_0}a(x,D))$.
It is also straightforward to show that the results of Propositions 5.1, 5.2 can be restated as follows.

Proposition 6.1. *Assume that m, m' are two arbitrary admissible weights and let $a(x,D) \in \widetilde{\mathrm{Op}}\, S_{m,\Lambda}(\Omega)$, $x_0 \in \Omega$ and $p \in]1,\infty[$ be given. Then the following inclusions are satisfied for every $u \in \mathcal{D}'(\Omega)$:*

$$\mathcal{W}^p_{m'/m,x_0}a(x,D)u \cap \Sigma_{m,x_0}a(x,D) \subset \mathcal{W}^p_{m',x_0}u \subset \mathcal{W}^p_{m'/m,x_0}a(x,D)u\,. \tag{6.3}$$

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